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THE STABLE IDEALS OF A CONTINUOUS NEST ALGEBRA. II

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ABSTRACT. We continue the study of the rich family of norm-closed, automorphism invariant ideals of a continuous nest algebra. First we present a unified framework which captures all stable ideals as the kernels of limits of diagonal compressions. We then characterize when two such limits give rise to the same ideal, and we obtain detailed information of the structure of sums and intersections of ideals.

KEYWORDS: *Nest algebra, ideals, automorphism invariant.*

MSC (2000): 47L35.

1. INTRODUCTION

In [11] we studied norm-closed, automorphism-invariant ideals of continuous nest algebras, which we termed stable ideals. Our study was motivated by the fact that a number of natural examples of stable ideals of nest algebras have been identified and studied, such as the compact operators of the algebra; the Jacobson radical of the algebra, described in [13]; the strong radical (that is, the intersection of all maximal two-sided ideals) which was described in [9]; and the ideals studied by Erdos in [6]. In addition, other interesting ideals narrowly fail to be automorphism invariant for subtle reasons, specifically Larson's ideal $\mathcal{R}^\infty$ [7]. In [11] we were able to find a complete description of all stable ideals of a continuous nest algebra. In the present paper we continue this study.

In the original description of stable ideals, we identified a family of eleven exceptional minimal ideals, which we dubbed ideals of compact character, and then gave a characterization of all remaining stable ideals (see Theorem 2.4, below). In the present paper we start, in Theorem 3.18, by giving a single, unified description of the stable ideals, which brings together both the compact character and non-compact character cases. This description generalizes other characterizations given elsewhere of ideals of nest algebras as the kernels of the limits of certain diagonal expectations.

As a result of Theorem 3.18, we associate each stable ideal with a net of generalized partitions of the identity. In Section 4 we investigate when two such nets give rise to the same stable ideal, and find that the simplest condition, that the two nets be cofinal in each other, is both necessary and sufficient. In Section 5 we build on these results to give natural formulas for the quotient norm by a stable ideal, and in Section 6 we characterize the sums of stable ideals. The quotient norm formulas substantially generalize the quotient norm formula obtained by Ringrose for the Jacobson radical [13] and similar formulas found by Erdos [6] for related ideals.

It is only possible to get such detailed information about stable ideals of continuous nest algebras because of the very clear understanding we have of the automorphism groups of nest algebras in general. Two key results underlie all of the work in this paper: First Ringrose [14] showed that isomorphisms between nest algebras are necessarily spatial; that is, every isomorphism $\Phi : \operatorname{Alg}\mathcal{L} \to \operatorname{Alg}\mathcal{M}$ can be expressed as $\Phi = \operatorname{Ad}_S$, where S is a bounded, invertible operator. Then Davidson [2] gave a necessary and sufficient condition for two nest algebras to be isomorphic in terms of order-dimension invariants of the nests:

THEOREM 1.1. *Let $\mathcal{L}$ and $\mathcal{M}$ be two nests on separable Hilbert spaces and suppose there is an order-preserving bijection $\theta : \mathcal{L} \to \mathcal{M}$ with the property that*

$$\operatorname{rank}\theta(M) - \theta(L) = \operatorname{rank} M - L \quad \textit{for all } L < M \textit{ in } \mathcal{L}.$$

Then there is an invertible operator S which maps the range of each $L \in \mathcal{L}$ onto the range of $\theta(L)$. Furthermore, S can be taken to be an arbitrarily small compact perturbation of a unitary operator.

In addition, the results of [10], although not used directly in the present work, are crucial in the characterization of stable ideals from [11].

2. PRELIMINARIES

Throughout this paper, $\mathcal{N}$ will denote a continuous nest of projections on a separable Hilbert space $\mathcal{H}$. See [3] for a comprehensive introduction to nest algebras.

At the heart of our analysis of ideals of continuous nest algebras is a collection of families of submultiplicative seminorms parameterized by $\mathcal{N}$. These seminorms have their origins in the earliest work in the field; Ringrose introduced $i_N^{\pm}$ in [13] and showed that the Jacobson radical is the intersection of the kernels of these seminorms. The following lists the full set of seminorms we shall need:

DEFINITION 2.1. For $X \in \operatorname{Alg}\mathcal{N}$ and $N \in \mathcal{N}$ define:

$$e_N^+(X) = \begin{cases} \lim\limits_{M\downarrow N} \sup\limits_{N<L<M} \|(M-L)X(M-L)\|_{\mathrm{ess}} & \text{if } N < I, \\ 0 & \text{if } N = I; \end{cases} \tag{2.1}$$

$$e_N^-(X) = \begin{cases} \lim\limits_{M\uparrow N} \sup\limits_{N>L>M} \|(L-M)X(L-M)\|_{\mathrm{ess}} & \text{if } N > 0, \\ 0 & \text{if } N = 0; \end{cases} \tag{2.2}$$

$$i_N^+(X) = \begin{cases} \lim\limits_{M\downarrow N} \|(M-N)X(M-N)\|_N & \text{if } N < I, \\ 0 & \text{if } N = I; \end{cases} \tag{2.3}$$

$$i_N^-(X) = \begin{cases} \lim\limits_{M\uparrow N} \|(N-M)X(N-M)\|_N & \text{if } N > 0, \\ 0 & \text{if } N = 0; \end{cases} \tag{2.4}$$

$$j_N(X) = \begin{cases} \lim\limits_{M\downarrow N, L\uparrow N} \|(M-L)X(M-L)\|_N & \text{if } 0 < N < I, \\ i_0^+(X) & \text{if } N = 0, \\ i_I^-(X) & \text{if } N = I. \end{cases} \tag{2.5}$$

For each $N \in \mathcal{N}$ the seminorms 0, $e_N^\pm$, $i_N^\pm$, and j_N are called the *elementary seminorm functions*. If the map $(X, N) \mapsto \|X\|_N$ is defined so that for each $N \in \mathcal{N}$, $\|\cdot\|_N$ takes one of these seminorms as a value, then $\|\cdot\|_N$ is called a *diagonal seminorm function*.

REMARK 2.2. For each fixed $N \in \mathcal{N}$, the six elementary seminorms at N form a lattice under pointwise ordering (see Figure 2 of [11]). Thus, with pointwise ordering as N varies, the family of all diagonal seminorm functions is a complete lattice. A collection $\mathcal{F}$ of diagonal seminorm functions is called a *stable family of diagonal seminorm functions* if it is closed under meets and under composition with order automorphisms of $\mathcal{N}$. In other words if $\|\cdot\|_N^{(i)} \in \mathcal{F}$ $(i = 1, 2)$ and $\theta : \mathcal{N} \to \mathcal{N}$ is an order automorphism, then

$$\|\cdot\|_N^{(1)} \wedge \|\cdot\|_N^{(2)} \quad \text{and} \quad \|\cdot\|_{\theta(N)}$$

also belong to $\mathcal{F}$.

DEFINITION 2.3. We say that an operator $K \in \operatorname{Alg}\mathcal{N}$ is of *compact character* if $(N-M)K(N-M)$ is compact for all $0 < M < N < I$ in $\mathcal{N}$. Say that an ideal of $\operatorname{Alg}\mathcal{N}$ is of compact character if all its elements are of compact character.

In [11] we showed that $\operatorname{Alg}\mathcal{N}$ has exactly eleven stable ideals of compact character. These ideals (excluding 0) are listed in Figure 1 of [11]. The following theorem characterizes the stable ideals which are not of compact character, and will be the basis for all our results on stable ideals in the present paper.

THEOREM 2.4. *Let $\mathcal{I}$ be a stable ideal in a continuous nest algebra,* $\operatorname{Alg}\mathcal{N}$. *If $\mathcal{I}$ is not of compact character then there is a stable family $\mathcal{F}$ of diagonal seminorm functions such that $X \in \mathcal{I}$ if and only if, for any $\varepsilon > 0$, there is a diagonal seminorm function $\|\cdot\|_N$ in $\mathcal{F}$ such that $\|X\|_N < \varepsilon$ for all $N \in \mathcal{N}$.*

3. CHARACTERIZATION OF STABLE IDEALS

DEFINITION 3.1. Let P_1 and P_2 be two families of intervals of $\mathcal{N}$. Say that P_1 *refines* P_2, and write $P_1 \geqslant P_2$, if whenever $E \in P_1$ there is an interval $F \in P_2$ such that $E \leqslant F$.

DEFINITION 3.2. Let Ω be a set of families of intervals of $\mathcal{N}$. We call Ω a *net of intervals* if it is a directed set under the ordering of refinement. Call Ω a *stable net* if whenever θ is an order isomorphism of $\mathcal{N}$ onto itself and $P \in \Omega$ then the set

$$\theta(P) := \{\theta(E) : E \in P\}$$

also belongs to Ω.

PROPOSITION 3.3. *If Ω is a net of intervals on $\mathcal{N}$ then the set*

$$\mathcal{I} := \left\{X \in \operatorname{Alg}\mathcal{N} : \lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} = 0\right\}$$

is an ideal of $\operatorname{Alg}\mathcal{N}$. *If Ω is a stable net of intervals, then $\mathcal{I}$ is a stable ideal.*

Proof. For fixed X, the map $P \mapsto \sup_{E\in P} \|EXE\|_{\mathrm{ess}}$ is decreasing in P, and so the limit exists. Since, for any fixed interval E, $\|EXE\|_{\mathrm{ess}}$ is a submultiplicative seminorm on $\operatorname{Alg}\mathcal{N}$, then so is $\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}}$, and $\mathcal{I}$ is its kernel, which is an ideal.

If Ad_S is an automorphism of $\operatorname{Alg}\mathcal{N}$ then S induces an order isomorphism $\theta : \mathcal{N} \to \mathcal{N}$ and it is routine to prove that

$$k^{-1} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} \leqslant \sup_{E\in\theta(P)} \|E(SXS^{-1})E\|_{\mathrm{ess}} \leqslant k \sup_{E\in P} \|EXE\|_{\mathrm{ess}}$$

where $k = \|S\|\|S^{-}1\|$. It is clear from this, and the stability property of Ω, that $\mathcal{I}$ must be stable under conjugation by S. ∎

REMARK 3.4. We shall say that $\mathcal{I}$ is the ideal associated with the net Ω, or that $\mathcal{I}$ arises as the kernel of Ω.

REMARK 3.5. We should make a brief remark concerning the essential norm, $\|X\|_{\mathrm{ess}}$, which is used ubiquitously throughout this paper. As a result of Theorem 5.1 in [4], we know that for $T \in \operatorname{Alg}\mathcal{N}$,

$$\operatorname{dist}(T, \mathcal{K}(\mathcal{H})) = \operatorname{dist}(T, \mathcal{K}(\mathcal{H}) \cap \operatorname{Alg}\mathcal{N})$$

were $\mathcal{K}(\mathcal{H}) \subseteq B(\mathcal{H})$ is the set of all compact operators. Thus we shall not make any distinction between the quotient norms on $\operatorname{Alg}\mathcal{N}/(\mathcal{K}(\mathcal{H}) \cap \operatorname{Alg}\mathcal{N})$ and on $B(\mathcal{H})/\mathcal{K}(\mathcal{H})$, but shall use $\|X\|_{\text{ess}}$ to denote either, as long as $T \in \operatorname{Alg}\mathcal{N}$.

EXAMPLE 3.6. Let Ω consist of the single family, $\{0\}$. Then the associated ideal is all of $\operatorname{Alg}\mathcal{N}$. Conversely if Ω consists of the singleton $\{I\}$ then the associated ideal is $\mathcal{K}$, the compact operators in $\operatorname{Alg}\mathcal{N}$.

EXAMPLE 3.7. Let Ω consist of the single family, $\{N \in \mathcal{N} : N < I\}$, of all intervals with lower endpoint equal to 0. This is a stable net on $\mathcal{N}$ and the associated ideal is the ideal of compact character, $\mathcal{K}^-$. Similarly if Ω consists of the singleton $\{M^\perp : M > 0\}$ then the associated ideal is $\mathcal{K}^+$. Finally, if Ω contains the single family $\{N - M : 0 < M < N < I\}$ then the associated ideal is the ideal of all operators of compact character, or $\mathcal{K}^+ + \mathcal{K}^-$ (see Lemma 2.14 of [11]).

EXAMPLE 3.8. Let Ω consist of the set of all singletons $\{N\}$ for $N > 0$. This is a stable net and the associated ideal is equal to

$$E_0 := \Big\{X \in \operatorname{Alg}\mathcal{N} : \inf_{N>0} \|NEN\| = 0\Big\}.$$

The transition from the essential norm to the operator norm in this example is routine: If $\|NXN\|_{\text{ess}} < \varepsilon$ then there is a compact $K \in \mathcal{K}$ such that $\|NXN + K\| < \varepsilon$ and we can find a projection $0 < M < N$ in $\mathcal{N}$ such that $\|MKM\| < \varepsilon$ and so $\|MXM\| < 2\varepsilon$. In the same way, the set

$$E_I := \Big\{X \in \operatorname{Alg}\mathcal{N} : \inf_{N<I} \|N^\perp E N^\perp\| = 0\Big\}$$

also arises as the kernel of a stable net.

EXAMPLE 3.9. Let Ω consist of all finite partitions of $\mathcal{N}$. That is to say, each P in Ω is a finite set of pairwise orthogonal intervals which sum to the identity. Clearly this is a stable net on $\mathcal{N}$, and the associated ideal is the Jacobson radical of $\operatorname{Alg}\mathcal{N}$ [13]. The transition from the essential norm to the operator norm is similar to the last example. If $\Big\|\sum\limits_{i=1}^{m} E_i X E_i\Big\|_{\text{ess}} < \varepsilon$ then there is a compact $K \in \mathcal{K}$ such that $\Big\|\sum\limits_{i=1}^{m} E_i X E_i + K\Big\| < \varepsilon$. But since K belongs to the Jacobson radical, there is a finite partition $\{F_k\}_{k=1}^{n}$ refining $\{E_i\}$ and such that $\Big\|\sum\limits_{i=1}^{n} F_i K F_i\Big\| < \varepsilon$. Thus $\Big\|\sum\limits_{i=1}^{n} F_i X F_i\Big\| < 2\varepsilon$.

EXAMPLE 3.10. Let Ω consist of all partitions (finite or infinite) of $\mathcal{N}$. Then by [7] Ω is a net but not a stable net, and the associated ideal is Larson's ideal, $\mathcal{R}_{\mathcal{N}}^{\infty}$.

EXAMPLE 3.11. In [9] a pseudopartition was defined as a maximal (with respect to set inclusion) family of pairwise orthogonal intervals of $\mathcal{N}$. If Ω is the set of all pseudopartitions of $\mathcal{N}$ then Ω is a stable net and the associated ideal is $\mathfrak{I}_N^\infty$, which has been shown to be the strong radical of $\operatorname{Alg}\mathcal{N}$, or the intersection of all the maximal two-side ideals of $\operatorname{Alg}\mathcal{N}$.

PROPOSITION 3.12. *Let Ω and Ψ be two stable nets of intervals on $\mathcal{N}$, associated with the stable ideals $\mathfrak{I}$ and $\mathfrak{J}$, respectively. Then $\mathfrak{I} \cap \mathfrak{J}$ is the kernel of the stable net*

$$\Omega + \Psi := \{P \cup Q : P \in \Omega, Q \in \Psi\}.$$

The proof is straightforward, and is left to the reader.

COROLLARY 3.13. *All the stable ideals of compact character in* $\operatorname{Alg}\mathcal{N}$ *arise as kernels of stable nets.*

Proof. By Theorem 2.16 of [11], the stable ideals of compact character are the set of ten ideals listed in Figure 1 of [11]. Each of these ideals can be expressed as an intersection of the ideals $\mathcal{K}^+ + \mathcal{K}^-$, $\mathcal{K}^+$, $\mathcal{K}^-$, E_0, E_I, and $\mathcal{K}$. The preceding examples have shown that these all arise as the kernels of stable nets, and so the result follows from Proposition 3.12. ∎

REMARK 3.14. In what follows, let $\mathfrak{I}$ be a fixed stable ideal. For each $X \in \mathfrak{I}$ and $\varepsilon > 0$ define $P_{X,\varepsilon}$ to be the set of all intervals E of $\mathcal{N}$ for which $\|EXE\|_{\mathrm{ess}} < \varepsilon$. Finally, let $\Omega = \{P_{X,\varepsilon} : X \in \mathfrak{I}, \varepsilon > 0\}$.

LEMMA 3.15. *With the definitions above, Ω is a stable net of intervals on $\mathcal{N}$.*

Proof. First we shall show that Ω is a directed set. As is thoroughly described in [8], [10], we can find a projection F in $\mathcal{N}''$ such that F and $F^\perp$ are both algebraically equivalent to I. In other words, there are operators A, B, C, D in $\operatorname{Alg}\mathcal{N}$ such that $AB = F$, $CD = F^\perp$ and $BA = DC = I$, and these operators can be taken to be compact perturbations of partial isometries. Now let $P_{X,a}$ and $P_{Y,b}$ be collections of intervals belonging to Ω. We shall show that $P_{Z,1}$ is a refinement of $P_{X,a}$ and $P_{Y,b}$, where

$$Z := \frac{1}{a}AXB + \frac{1}{b}CXD.$$

To see this, let $E \in P_{Z,1}$ and observe that the next quantity is less than 1: $\|EZE\|_{\mathrm{ess}} = \left\|\frac{1}{a}FE(AXB)EF + \frac{1}{b}F^\perp E(CYD)EF^\perp\right\|_{\mathrm{ess}} = \max\left\{\frac{1}{a}\|EAXBE\|_{\mathrm{ess}}, \frac{1}{b}\|ECYDE\|_{\mathrm{ess}}\right\}$, so that $\|EAXBE\|_{\mathrm{ess}} < a$ and $\|ECYDE\|_{\mathrm{ess}} < b$. Now since $X = BAXBA$, it follows that $EXE = EBE(AXB)EAE$ and so

$$\|EXE\|_{\mathrm{ess}} \leqslant \|EB\|_{\mathrm{ess}}\|E(AXB)E\|_{\mathrm{ess}}\|AE\|_{\mathrm{ess}} \leqslant \|E(AXB)E\|_{\mathrm{ess}} < a$$

(with the last inequality following since A and B were taken to be compact perturbations of partial isometries). Thus $E \in P_{X,a}$ and, by the same token, $E \in P_{Y,b}$.

Next, we shall show that Ω is stable. In other words, given $P_{X,\varepsilon} \in \Omega$ and an order isomorphism θ on $\mathcal{N}$ we must show that $\theta(P_{X,\varepsilon})$ is also in Ω. Now by

Theorem 1.1, find an invertible S implementing an automorphism of $\operatorname{Alg}\mathcal{N}$ such that $SN = \theta(N)SN$ for all $N \in \mathcal{N}$. Further take S to be a compact perturbation of a unitary, $S = U + K$. Then

$$\begin{aligned}\|\theta(E)SXS^{-1}\theta(E)\|_{\mathrm{ess}} &= \|\theta(E)S(EXE)S^{-1}\theta(E)\|_{\mathrm{ess}} \leqslant \|(U+K)EXE(U+K)^{-1}\|_{\mathrm{ess}} \\ &= \|EXE\|_{\mathrm{ess}}\,.\end{aligned}$$

The same argument, using S^{-1} and θ^{-1} in place of S and θ, yields the reverse inequality and so $\|\theta(E)SXS^{-1}\theta(E)\|_{\mathrm{ess}} = \|EXE\|_{\mathrm{ess}}$. Thus $\theta(P_{X,\varepsilon}) = P_{SXS^{-1},\varepsilon}$, which belongs to Ω. ∎

The next definition and the lemma that follows establish a connection between the collections $P_{T,\varepsilon}$ of intervals and diagonal seminorm functions from Theorem 2.4.

DEFINITION 3.16. Say that a collection P of intervals of $\mathcal{N}$ is compatible with the diagonal seminorm function $\|\cdot\|_N$ if, for each $N \in \mathcal{N}$:

(i) Whenever $\|\cdot\|_N = j_N$ there are projections $G > N > L$ in $\mathcal{N}$ such that $(G-L) \in P$.

(ii) Whenever $\|\cdot\|_N = a_N^+ \vee a_N^-$ where $a_N^\pm = 0, e_N^\pm, i^\pm$ then:

(a) If $a_N^+ = i_N^+$ then there is a projection $G > N$ in $\mathcal{N}$ such that $(G - N) \in P$.

(b) If $a_N^+ = e_N^+$ then there is a projection $G > N$ in $\mathcal{N}$ such that $(G-L) \in P$ for all projections $G > L > N$ in $\mathcal{N}$.

(c) If $a_N^- = i_N^-, e_N^-$ then the analogous lower conditions hold.

LEMMA 3.17. *Let $T \in \operatorname{Alg}\mathcal{N}$ and $a > 0$. If $P_{T,\varepsilon}$ is compatible with $\|\cdot\|_N$ then $\|T\|_N \leqslant \varepsilon$ for all $N \in \mathcal{N}$. Conversely, if $\|T\|_N < \varepsilon$ for all $N \in \mathcal{N}$ then $P_{T,\varepsilon}$ is compatible with $\|\cdot\|_N$.*

Proof. Suppose $P_{T,a}$ is compatible with $\|\cdot\|_N$ and fix $N \in \mathcal{N}$. Then $A \mapsto \|A\|_N$ is one of the 10 possible seminorms listed in Figure 2 of [11]. That is to say, it is of the form $a_N(A) \vee b_N(A)$ where a_N is one of $0, e_N^+, i_N^+, j_N$ and b_N is one of $0, e_N^-, i_N^-, j_N$.

If $a_N = 0$ there is nothing to prove. Next, suppose $a_N = e_N^+$. Since $P_{T,\varepsilon}$ is compatible with $\|\cdot\|_N$, there is a $G > N$ in $\mathcal{N}$ such that $(G-L) \in P_{T,\varepsilon}$ for all $G > L > N$. Thus

$$a_N(T) \leqslant \sup\{\|(G-L)T(G-L)\|_{\mathrm{ess}} : L \in \mathcal{N}, N < L < G\} \leqslant \varepsilon.$$

Next, suppose $a_N = i_N^+$. Since $P_{T,\varepsilon}$ is compatible with $\|\cdot\|_N$, there is a $G > N$ such that $(G-N) \in P_{T,\varepsilon}$. Thus $\|(G-N)T(G-N)\|_{\mathrm{ess}} < \varepsilon$. Pick a compact operator K such that $\|(G-N)T(G-N) - K\| < \varepsilon$, and then

$$a_N(T) = i_N^+(T) = i_N^+(T-K) \leqslant \|(G-N)T(G-N) - K\| < \varepsilon.$$

The argument for the case $a_N = j_N$ is almost identical.

Conversely if $\|T\|_N < \varepsilon$ for all $N \in \mathcal{N}$ then it follows directly from the definitions of the seminorm functions that we can find intervals of $\mathcal{N}$ with the appropriate properties to belong to $P_{T,\varepsilon}$. ∎

THEOREM 3.18. *Let $\mathcal{I}$ be a non-zero stable ideal of* $\operatorname{Alg}\mathcal{N}$. *Then there is a stable net Ω of intervals of $\mathcal{N}$ such that*

$$\mathcal{I} = \Big\{ X \in \operatorname{Alg}\mathcal{N} : \lim_{P\in\Omega} \sup_{E\in\mathcal{P}} \|EXE\|_{\mathrm{ess}} = 0 \Big\}.$$

Proof. By Corollary 3.13 we need only consider the case when $\mathcal{I}$ is not of compact character. As above, let Ω be the set of collections $P_{T,\varepsilon}$ as T ranges over $\mathcal{I}$ and ε ranges over all positive values. Since Lemma 3.15 shows Ω is a stable net, it is clear that the limit $\lim_{P\in\Omega} \sup_{E\in P} \|ETE\|_{\mathrm{ess}}$ exists and is zero for all $T \in \mathcal{I}$. The main body of the theorem is to establish the converse.

Since $\mathcal{I}$ is a stable ideal, by Theorem 2.4 there is a set $\mathcal{F}$ of diagonal seminorm functions which specifies $\mathcal{I}$. Now suppose that $\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} = 0$ for some $X \in \operatorname{Alg}\mathcal{N}$. Given $\varepsilon > 0$, find a $P \in \Omega$ such that $\|EXE\|_{\mathrm{ess}} < \varepsilon$ for all $E \in P$. By definition, $P = P_{T,a}$ for some $T \in \mathcal{I}$ and $a > 0$. By rescaling T we may as well assume $P = P_{T,\varepsilon}$. Since $T \in \mathcal{I}$, find a diagonal seminorm function $\|\cdot\|_N$ in F such that $\|T\|_N < \varepsilon$ for all $N \in \mathcal{N}$. Thus $\|\cdot\|_N$ is compatible with $P_{T,\varepsilon}$ and so is compatible with $P_{X,\varepsilon}$, which contains $P_{T,\varepsilon}$. It follows by Lemma 3.17 that $\|X\|_N \leqslant \varepsilon$ for all $N \in \mathcal{N}$, and so $X \in \mathcal{I}$. ∎

4. COFINAL NETS

The net constructed in Theorem 3.18 for a general stable ideal is much larger than any of the natural nets given in the examples of the last section. This naturally raises the question of when two nets give rise to the same ideal. Knowing the answer to this question which will also be essential to proving the quotient norm formula in Theorem 5.8. Thus this section will be devoted to establishing the following result:

THEOREM 4.1. *Suppose that $\mathcal{I}_1$ (respectively $\mathcal{I}_2$) is the ideal associated with the net of intervals Ω_1 (respectively Ω_2). Then $\mathcal{I}_1 \supseteq \mathcal{I}_2$ if and only if Ω_1 is cofinal in Ω_2.*

If Ω_1 is cofinal in Ω_2 then clearly for any $T \in B(\mathcal{H})$

$$\lim_{P\in\Omega_2} \sup_{E\in P} \|ETE\|_{\mathrm{ess}} \geqslant \lim_{P\in\Omega_1} \sup_{E\in P} \|ETE\|_{\mathrm{ess}}$$

and so $\mathcal{I}_1 \supseteq \mathcal{I}_2$. To prove the converse, suppose that Ω_1 is not cofinal in Ω_2. This means that there must be a $Q \in \Omega_2$ which is not refined by any $P \in \Omega_1$. The strategy in this section will be to suppose that nevertheless $\mathcal{I}_1 \supseteq \mathcal{I}_2$ and derive a contraction.

Much of the main part of the argument in this section will rely only on combinatoric arguments concerned with the ordering of $\mathbb{R}$. It will be much more convenient to work directly with real numbers and intervals of real numbers, than with projections and intervals in $\mathcal{N}$. Thus let $\mathcal{N}$ be parameterized as $N(t)$ $(t \in [0,1])$ and for convenience take $N(t) = I$ for $t \geqslant 1$ and $N(t) = 0$ for $t \leqslant 0$. Write $E(x,y) := N(y) - N(x)$ and, if P is a collection of open intervals in $\mathbb{R}$, write $E(P)$ for the set of intervals $\{E(a,b) : (a,b) \in P\}$. Write Λ for the set of families P of open intervals in $\mathbb{R}$ for which $E(P)$ is in Ω_1. Observe that Λ is a directed set under refinement, and that if $\theta : \mathbb{R} \to \mathbb{R}$ is any order isomorphism then $\theta(P) \in \Lambda$ for all $P \in \Lambda$.

Fix Q as a collection of open intervals of $\mathbb{R}$ with the property that $E(Q) \in \Omega_2$ and no collection in Λ refines it. The proof of Theorem 4.1 will be established when we see, in Proposition 4.32, that all sufficiently large $P \in \Lambda$ refine Q.

DEFINITION 4.2. Let P be a collection of open intervals. Say that an *interval is dominated by P* if it is a subset of an interval in P.

The next four lemmas establish some basic relationships between Λ and Q. Once these facts are in place, no other reference to operator theory will be used, and the remainder of the section will be purely combinatoric.

LEMMA 4.3. *Given any sequence of pairwise disjoint intervals, none of which is dominated by Q, we can find a $P \in \Lambda$ which also does not dominate any of these intervals.*

Proof. Let such a sequence of pairwise disjoint intervals, (x_i, y_i), be given. For each i choose sequences $x_i < x_j^{(i)} < y_j^{(i)} < y_i$ with $x_j^{(i)}$ decreasing to x_i and $y_j^{(i)}$ increasing to y_i. Let $X_j^{(i)}$ be a finite rank partial isometry mapping $N(y_{j+1}^{(i)}) - N(y_j^{(i)})$ into $N(x_j^{(i)}) - N(x_{j+1}^{(i)})$ and let $X := \sum_{i,j} X_j^{(i)}$. Because all the ranges and domains are pairwise orthogonal, the sum for X converges weakly to a partial isometry in $\operatorname{Alg}\mathcal{N}$. Let $(a,b) \in Q$. For each i, since no interval of Q contains (x_i, y_i), $E(a,b)X_j^{(i)}E(a,b)$ is non-zero for only finitely many j. Further, since the (x_i, y_i) are pairwise disjoint, no interval of Q can meet more than two of them, otherwise it would have to contain one of them. Thus in fact $E(a,b)X_j^{(i)}E(a,b)$ is non-zero for only finitely many values of i and j. Hence $E(a,b)XE(a,b)$ is finite rank. Since this holds for any $(a,b) \in Q$, we conclude $X \in \mathfrak{I}_2$ and so $X \in \mathfrak{I}_1$. Thus there must be a $P \in \Lambda$ such that

$$\sup\{\|E(a,b)XE(a,b)\|_{\text{ess}} : (a,b) \in P\} < 1.$$

Clearly this is only possible if no interval in P dominates any (x_i, y_i). ∎

LEMMA 4.4. *Given any sequence of pairwise disjoint intervals (x_i, y_i) with the property that no interval (x_i, y) is dominated by Q for any $y > y_i$, then we can find a $P \in \Lambda$ which also does not dominate any interval (x_i, y) with $y > y_i$.*

Proof. The proof is very similar to Lemma 4.3. Let the sequence of pairwise disjoint intervals (x_i, y_i) be given. For each i choose sequences $x_i < x_j^{(i)} < y_i$ and $y_i < y_j^{(i)}$ with $x_j^{(i)}$ decreasing to x_i and $y_j^{(i)}$ decreasing to y_i, and let $X_j^{(i)}$ be a finite rank partial isometry mapping $N(y_j^{(i)}) - N(y_{j+1}^{(i)})$ into $N(x_j^{(i)}) - N(x_{j+1}^{(i)})$. Note that this time, the intervals $(y_{j+1}^{(i)}, y_j^{(i)})$ need not be pairwise disjoint as i, j vary, but we shall simply stipulate that the $X_j^{(i)}$ should be chosen with pairwise orthogonal initial spaces so that again $X := \sum_{i,j} X_{i,j}$ converges weakly. If $(a, b) \in Q$ then $E(a, b)X_j^{(i)}E(a, b)g$ is non-zero only if $a < y_i < b$, and this condition can be met for at most a single value of i. Since (x_i, y_i) is not dominated by (a, b), the compression must be finite rank. Thus $E(a, b)XE(a, b)$ is finite rank, and so $X \in \mathcal{I}_2 \subseteq \mathcal{I}_1$. Thus there must be a $P \in \Lambda$ such that

$$\sup\{\|E(a,b)XE(a,b)\|_{\mathrm{ess}} : (a,b) \in P\} < 1,$$

and this is only possible if no interval in P dominates any (x_i, y) with $y > y_i$. ∎

The next lemma is almost identical to the last, and the proof is left to the reader.

LEMMA 4.5. *Given any sequence of pairwise disjoint intervals (x_i, y_i) with the property that no interval (x_i, x) is dominated by Q for any $x > x_i$, then we can find a $P \in \Lambda$ which also does not dominate any interval (x_i, x) with $x > x_i$.*

REMARK 4.6. There are obvious analogues to Lemmas 4.4 and 4.5 which deal with the corresponding behavior at the upper endpoints of the intervals (x_i, y_i).

LEMMA 4.7. *For all sufficiently large $P \in \Lambda$, $\bigcup P \subseteq \bigcup Q$.*

Proof. Let $K := (\bigcup Q)^c$. By Lemma 2.4 of [9] there is an $X \in \operatorname{Alg}\mathcal{N}$ such that $\|E(a, b)XE(a, b)\|_{\mathrm{ess}} \geqslant 1$ if (a, b) intersects K and is zero otherwise (the lemma cited claims a sightly weaker result in its statement, but the construction used in fact establishes this fact). But thus $E(a, b)XE(a, b) = 0$ for $(a, b) \in Q$ and so $X \in \mathcal{I}_2 \subseteq \mathcal{I}_1$ and so there must be a $P \in \Lambda$ with $\sup\{\|E(a, b)XE(a, b)\|_{\mathrm{ess}} : (a, b) \in P\} < 1$. This shows that every (a, b) in P must be a subset of $\bigcup Q$. Clearly if $\bigcup P \subseteq \bigcup Q$ then the same is true for any $P' \in \Lambda$ that refines P. ∎

REMARK 4.8. Having established some basic relations between the collection Q and at least all sufficiently large members of the net Λ, we shall now develop a framework of properties of families of open intervals. The next few results will make no assumptions about Λ or Q and will not use any operator theory. We shall return to the context of operator algebras with Lemma 4.28.

DEFINITION 4.9. Given a collection P of open intervals and $x \in \bigcup P$, write:

$$L_P(x) := \inf\{a : x \in (a,b) \text{ for some } (a,b) \in P\},$$
$$R_P(x) := \sup\{b : x \in (a,b) \text{ for some } (a,b) \in P\}.$$

When the context is clear, we shall omit the subscript P.

REMARK 4.10. Note that $L(x)$ and $R(x)$ are increasing functions and that for all $x \in \bigcup P$

$$L(x) < x < R(x).$$

DEFINITION 4.11. A *linked list of intervals* is a sequence of intervals (a_i, b_i) which is indexed by a finite set of integers, or by one of $\mathbb{Z}$, $\mathbb{Z}^+$, or $\mathbb{Z}^-$, and has the property

$$a_i < b_{i-1} < a_{i+1} < b_i$$

for all i.

REMARK 4.12. The union of the the intervals of a linked list is always an open interval.

DEFINITION 4.13. Let P be a collection of open intervals. Say that an *interval is approximately dominated by P* if it is the union of an increasing sequence of intervals each of which is dominated by P.

REMARK 4.14. For any $x \in \bigcup P$, the intervals $(L(x), x)$ and $(x, R(x))$ are approximately dominated by P.

DEFINITION 4.15. Let P be a collection of open intervals with $\bigcup P = (a,b)$. A linked list whose union is (a,b) is called an *inner cover of* (a,b) if every interval of the list is approximately dominated by P.

DEFINITION 4.16. Let P be a collection of open intervals with $\bigcup P \subseteq (a,b)$. A linked list of intervals whose union is (a,b) is called an *outer cover of* (a,b) if every interval of P is contained in an interval of the list.

Trivially, (a,b) is itself an outer cover for (a,b), so outer covers always exist. The following lemma shows that inner covers also always exist.

LEMMA 4.17. *If P is a collection of open intervals and $\bigcup P = (a,b)$ then (a,b) has an inner cover with respect to P.*

Proof. Pick t_0 in (a,b) and inductively pick $t_k := R(t_{k-1})$ for $k > 0$ and $t_k := L(t_{k+1})$ for $k < 0$. The sequence so obtained is strictly increasing, by Remark 4.10. Continue this process for as long as these t_k are contained in (a,b). By compactness, t_k increases to b and decreases to a, for only finitely many intervals are needed to cover any $[a+\varepsilon, b-\varepsilon]$, and so t_k will be below $a+\varepsilon$ or above $b-\varepsilon$ after finitely many steps.

Now each $t_k \in (a,b) = \bigcup P$ and so belongs to an interval of P. We can pick pairwise disjoint intervals (a_k, d_k) containing t_k and each contained in an interval of P. Then choose b_k and c_k to satisfy

$$a_k < b_k < t_k < c_k < d_k.$$

By construction, each (c_k, b_{k+1}) is also contained in an interval of P. If the sequence of t_k has no greatest or smallest element then the sequence of (a_k, d_k) and (c_k, b_{k+1}) is an inner cover. If the sequence of t_k's has a final element, t_n, take the final interval of the inner cover to be (c_n, b). Likewise if the sequence of t_k's has a first element, t_m, take the first interval of the inner cover to be (a, b_m). These last two intervals need not be dominated by P, but since $L(t_m) = a$ and $R(t_n) = b$, they clearly are approximately dominated by P. ∎

Our goal is to construct outer covers for $P \in \Lambda$ and inner covers for intervals of Q which are of compatible order type (in a sense made precise in Lemma 4.26). As partial steps in that direction, the next two lemmas relate the existence of least elements of a cover to a property that can be transferred between $P \in \Lambda$ and Q using Lemmas 4.3 to 4.5.

LEMMA 4.18. *Let P be a collection of open intervals with $\bigcup P = (a, b)$. Then the following are equivalent:*

(i) *There is an $x > a$ with $L(x) = a$.*

(ii) *There is an inner cover of (a, b) with a least element.*

Proof. Suppose that $x > a$ with $L(x) = a$. By Lemma 4.17 we know that inner covers for P can be found. Suppose that (a_i, b_i) is an inner cover with no least element. Then eventually, as i decreases, $b_i < x$. Since $L(x) = a$ we know that (a, x) is approximately dominated by P and so the same is true for (a, b_i). It follows that if we delete all intervals to the left of (a_i, b_i) and replace (a_i, b_i) with (a, b_i) then we obtain a new inner cover, having a least element.

On the other hand suppose that there is an inner cover with a least element, which we can write (a, c). Since (a, c) is the increasing union of intervals which are contained in members of P clearly $L(x) = a$ for all $a < x < c$. ∎

LEMMA 4.19. *Let P be a collection of open intervals with $\bigcup P \subseteq (a, b)$. Then the following are equivalent:*

(i) *There is an $x > a$ in $\bigcup P$ with $L(x) = a$.*

(ii) *Every outer cover of (a, b) has a least element.*

Proof. Suppose that $x > a$ is in $\bigcup P$ with $L(x) = a$, and that (a_i, b_i) is an outer cover of (a, b). It follows that $x \in (a_k, b_k)$ for some k. If the outer cover has no least element then there is an interval (a_j, b_j) which lies to the left of (a_k, b_k) and is not its immediate predecessor. Such an interval must satisfy $a < a_j < b_j < x$. However since $L(x) = a$, there is an interval (u, v) in P satisfying $a < u < a_j < b_j < x < v$. But no interval of a linked list is a subset of any other interval of the list, and so (u, v) cannot be contained in any (a_i, b_i), contrary to the property of an outer cover. Thus every outer cover must have a least element.

Suppose there is no $x > a$ in $\bigcup P$ with $L(x) = a$ and aim to construct an outer cover with no least element. If it is possible to find a sequence $x_i \in (a, b) \setminus \bigcup P$ decreasing to a, then we take the cover to be (x_2, b) together with the intervals (x_{2i+2}, x_{2i-1}). Otherwise we assume that for all x sufficiently close to a, $x \in \bigcup P$ and $L(x) > a$. Then by hypothesis, one can inductively choose a sequence $x_i \in \bigcup P$, decreasing to a with the property that $x_{i+1} < L(x_i)$ for each i. We claim that the sequence of intervals $(L(x_{i+1}), x_i)$, together with $(L(x_1), b)$, is an outer cover for P. Since $L(x_{i+1}) < x_{i+1} < L(x_i) < x_i$, the intervals have the correct overlapping property to make them a linked list. Now, let an interval (c, d) in P be given and aim to show it is contained in an interval of the cover. Since x_i decreases to a, eventually $x_i < d$, and so let n be the greatest i with $d \leqslant x_i$. Thus, $x_{n+1} < d \leqslant x_n$ and so $L(x_{n+1}) \leqslant c$, so that $(c, d) \subseteq (L(x_{n+1}), x_n)$. If there were no i with $d \leqslant x_i$ then $x_1 < d$ and so $(c, d) \subseteq (L(x_1), b)$, and the claim is established. ∎

REMARK 4.20. Clearly there are natural analogues of Lemmas 4.18 and 4.19 relating the condition $R(x) = b$ to the existence of greatest elements in inner and outer covers.

COROLLARY 4.21. *Suppose that there are $x, y \in \bigcap P = (a, b)$ such that $L(x) = a$ and $R(y) = b$. Then (a, b) admits a finite inner cover.*

Proof. By Lemma 4.18 there is an inner cover with a least element. If the cover has a greatest element we are done, so suppose otherwise. By the dual of Lemma 4.18, there is another inner cover, having a greatest element, (c, b). All but finitely many of the intervals from the first cover must be contained in (c, b), and so we may form a new, finite inner cover consisting of (c, d) together with those intervals from the first cover which are not contained in (c, d). ∎

COROLLARY 4.22. *Suppose that $L(x) > a$ and $R(x) < b$ for all $x \in \bigcup P \subseteq (a, b)$. Then there is an outer cover for (a, b) with no greatest or least elements.*

Proof. By Lemma 4.19, there is an outer cover (a_i, b_i) with no least element. If this cover also has no greatest element we are done, so suppose that it has a greatest element, and so without loss let the cover be indexed by $i \in \mathbb{Z}^-$. By the dual of Lemma 4.19, there is also an outer cover with no greatest element. Likewise we shall suppose this cover has a least element and so we can list its elements as the intervals (a'_i, b'_i) for $i \in \mathbb{Z}^+$.

Clearly $a_{-1} < b_{-2} < b_{-1} = b$. Thus $b_{-1} < a'_i < b$ for all sufficiently large positive values of i. Select an i_0 such that $b_{-1} < a'_{i_0} < b$ and define (a_0, b_0) to be (a_{-1}, b'_{i_0}). Finally set $(a_i, b_i) := (a'_{i+i_0}, b'_{i+i_0})$ for $i \geqslant 1$ and we obtain a linked list which is easily seen to be an outer cover. ∎

LEMMA 4.23. *Let P be a collection of open intervals with $\bigcup P \subseteq (a, b)$. The following are equivalent:*

(i) *There is no two-element outer cover of* (a, b).
(ii) $\bigcup P = (a, b)$ *and there is an one-element inner cover of* (a, b).
(iii) *Every interval* (c, d) *with* $a < c < d < b$ *is dominated by* P.

Proof. Suppose that (a, b) admits no two-element outer covers and consider the sequence of pairs of intervals $(a, b - \frac{1}{n})$ and $(a + \frac{1}{n}, b)$. Since these two intervals can never form an outer cover, we must always be able to find an interval (a_n, b_n) in P which is not contained in either. Such an interval must contain $(a + \frac{1}{n}, b - \frac{1}{n})$. This shows that the interval (a, b) is approximately dominated by P. Hence the single interval (a, b) is an inner cover for (a, b).

Conversely, suppose $\bigcup P = (a, b)$ and (a, b) is an inner cover, and suppose for a contradiction that there is an outer cover $(a, d), (c, b)$ with $c < d$. Since (a, b) is an inner cover, (a, b) is the union of an increasing sequence of intervals dominated by P. Thus one of these intervals must contain both c and d. But this interval is supposed to be a subset of a member of P, and so be contained in one of (a, d) or (c, b), which yields a contradiction.

The equivalence of items (ii) and (iii) is an immediate consequence of the definitions. ∎

LEMMA 4.24. *Let* P *be a collection of open intervals with* $\bigcup P = (a, b)$ *and suppose there is a sequence* $x_1, \ldots, x_n$ *in* (a, b) *satisfying:*

$$a = L(x_1), \quad R(x_i) = x_{i+1} \ (i = 1, \ldots, n-1), \quad \textit{and} \quad R(x_n) = b.$$

Then:
(i) *Every outer cover for* (a, b) *has at most* $n + 1$ *elements.*
(ii) *There exists an outer cover for* (a, b) *with* $\lfloor \frac{n}{2} \rfloor + 1$ *elements.*
(iii) *There exists an inner cover for* (a, b) *with* $n + 2$ *elements.*

Proof. Suppose if possible that $(a_1, b_1), \ldots, (a_m, b_m)$ is an m-element outer cover where $m > n + 1$. Since $L(x_1) = a$, there is an interval $(c, d) \in P$ containing x_1 with $c < b_2$. The only interval of the outer cover that can contain this is (a_1, b_1), and so $x_1 \in (a_1, b_1)$.

Having now established that $x_1 \leqslant b_1$, suppose for induction that $x_k \leqslant b_k$. If it were possible that $b_{k+1} < x_{k+1}$ then, since $x_{k+1} = R(x_k)$, there would be an interval (e, f) of P containing x_k with $b_{k+1} < f$. But (e, f) must be contained in an interval (a_i, b_i) satisfying

$$a_i \leqslant e < x_k \leqslant b_k < b_{k+1} < f \leqslant b_i.$$

Thus (a_i, b_i) contains points greater than b_{k+1}. By the ordering property of linked lists, this implies that $i \geqslant k + 2$. Similarly we see (a_i, b_i) contains points smaller than b_k which, implies $i \leqslant k + 1$. From this contradiction we see by induction that $x_i \leqslant b_i$ for all $i = 1, 2, \ldots, n$. But since $R(x_n) = b$ there is an interval (c, d) in P containing x_n with $b_{m-1} < d$. The only interval of the outer cover than can contain (c, d) is (a_m, b_m). But since we have also seen $x_n \leqslant b_n$, this means that

(a_m, b_m) must meet an (a_i, b_i) with $i \leqslant n < m-1$ or, in other words, $i \leqslant m-2$. This is impossible for any linked list, and so $m \leqslant n+1$.

Next we shall construct an outer cover having $m := \lfloor \frac{n}{2} \rfloor + 1$ elements. Consider the intervals $E_i := (x_{2i-3}, x_{2i})$ for $i = 1, 2, \ldots, m$ (where we define x_i to be a for $i \leqslant 0$ and to be b for $i \geqslant n$). One readily sees that this sequence is a linked list. To see that it is an outer cover, let $(c, d) \in P$ be given and let i_0 be the smallest i for which $c < x_i$. If $i_0 = n+1$ then $x_n \leqslant c < d \leqslant x_{n+1} = b$ and so $(c, d) \subseteq E_m$. Otherwise, $1 \leqslant i_0 \leqslant n$, and either $d \leqslant x_{i_0}$, or else $c < x_{i_0} < d$, which implies $d \leqslant R(x_{i_0}) = x_{i_0+1}$. In either case, $(c, d) \subseteq (x_{i_0-1}, x_{i_0+1})$. Every such interval is contained in an E_i, so the E_i are an outer cover.

Finally let us construct an inner cover with $n+2$ elements. First take the two intervals (a, x_1) and (x_n, b). Next, since $x_n \in (a, b) = \bigcup P$, find an interval (a_n, b_n) in P that contains x_n. If necessary, adjust a_n, b_n so that $x_{n-1} < a_n < x_n < b_n < b$ and so that (a_n, b_n) is still dominated by P. Then start at x_{n-1} and work backwards, choosing intervals (a_i, b_i) dominated by P, that contain x_i and have the property $x_{i-1} < a_i < x_i$ and $a_{i+1} < b_i \leqslant x_{i+1}$. Stepping backwards, we terminate at $i = 1$, having found a list of $2 + 1 + (n-1) = n+2$ intervals. ∎

COROLLARY 4.25. *Let P be a collection of open intervals with $\bigcup P \subseteq (a, b)$. If there is a sequence of pairwise disjoint nonempty subintervals $(a_i, b_i), i = 1, 2, \ldots, k$ of (a, b) none of which are dominated by P, then (a, b) has an outer cover of size $\lceil \frac{k}{2} \rceil$.*

Proof. First add intervals to P so that in fact $\bigcup P(a, b)$, without in the process changing the fact that no interval (a_i, b_i) is dominated by P. This is easy to accomplish by, for example, ensuring that every interval added is shorter than all of the (a_i, b_i). Now if there is no $x \in (a, b)$ for which $L(x) = a$ then by Lemma 4.19 there is an infinite outer cover. Likewise there is an infinite outer cover if there is no $x \in (a, b)$ for which $R(x) = b$, and so we may suppose that $L(x_1) = a$ for some $x_1 > a$ and that $R(x) = b$ for x sufficiently close to b. Without loss, assume that the (a_i, b_i) have been indexed so that the a_i (and b_i) are strictly increasing. Since we must have $a < b_1$, we may also suppose $x_1 < b_1$.

Now recursively define $x_{i+1} := R(x_i)$ for as long as the sequence lies in (a, b). This sequence must terminate, for it is strictly increasing and cannot have a limit point inside (a, b). Thus eventually x_i must increase to a value at which $R(x_i) = b$, at which point the sequence terminates. Suppose that the sequence has n terms.

Observe that whenever $x_i \leqslant b_i$ then $x_{i+1} = R(x_i)$ cannot be greater than b_{i+1}, otherwise (a_{i+1}, b_{i+1}) would be dominated by P. Thus inductively $x_i \leqslant b_i$ for $1 \leqslant i \leqslant \min\{k, n\}$. But if $k > n+1$ then this shows that $x_n \leqslant a_{n+1} < b_{n+1} \leqslant a_{n+2} < b$, and, since $R(x_n) = b$, this shows (a_{n+1}, b_{n+1}). It follows from this contradiction that in fact $n \geqslant k-1$. By Lemma 4.24, there exists an open cover of at least $\lfloor \frac{k-1}{2} \rfloor + 1 = \lceil \frac{k}{2} \rceil$ terms. ∎

LEMMA 4.26. *Suppose that E_i ($i \in I$) and F_j ($j \in J$) are two linked lists and that $I \supseteq J$. Then there is an increasing bijection $\theta : \mathbb{R} \to \mathbb{R}$ which maps each interval E_i into an interval F_j.*

Proof. Both I and J are subsets of $\mathbb{Z}$. For each $j \in J$, let C_j be the set of $i \in I$ which are closer to j than to any other element of J. (In the event i is equidistant from two elements of J, assign it to the smaller of the two.) The C_j partition I into ranges of consecutive numbers. Each of the sets $G_j := \bigcup_{i \in C_j} E_i$ is an interval, and the collection of intervals so formed is a linked list.

Write $G_j = (a_j, b_j)$ and $F_j = (a'_j, b'_j)$. By the overlapping property of linked lists we can define an order preserving map that takes each $a_j \mapsto a'_j$ and each $b_j \mapsto b'_j$. This correspondence can be extended to a piecewise linear bijection of $\mathbb{R} \to \mathbb{R}$ mapping each G_j to F_j. ∎

DEFINITION 4.27. Let P and Q be two collections of open subintervals of (a, b). We shall say that P and Q are *order compatible* if we can find an outer cover E_i ($i \in I$) of (a, b) with respect to P and an inner cover F_j ($j \in J$) of (a, b) with respect to Q that satisfy $I \supseteq J$.

We shall now return to the analysis of the two fixed stable ideals $\mathcal{I}_1$ and $\mathcal{I}_2$, with the associated net Λ and fixed family of open intervals Q. Recall in Lemma 4.7 we saw that $\bigcup P \subseteq \bigcup Q$ for all sufficiently large $P \in \Lambda$. We shall fix the following notation for the remainder of this section. Write (a_i, b_i) ($i \in \mathbb{N}$) for a fixed enumeration of the connected components of $\bigcup Q$. If P is any collection of open intervals satisfying $\bigcup P \subseteq \bigcup Q$ we shall write $P^{(i)}$ for the set of all members of P which are subsets of (a_i, b_i).

Our main goal in the remainder of this section is to show, in Proposition 4.32, that all sufficiently large $P \in \Lambda$ refine Q. To do this, we shall find a P with the property that $\bigcup P \subseteq \bigcup Q$ and use the machinery of the last few lemmas to show each $P^{(i)}$ refines $Q^{(i)}$. The main intermediate result is to show in Lemma 4.31 that we can ensure all the $P^{(i)}$ are order compatible with the corresponding $Q^{(i)}$. The next couple of lemmas are needed to establish this result.

For any collection of open sets write

$$A(P) := \{i : L_{P^{(i)}}(x) > a_i \text{ for all } x > a_i\}, \quad B(P) := \{i : R_{P^{(i)}}(x) < b_i \text{ for all } x < b_i\}.$$

LEMMA 4.28. *For all sufficiently large $P \in \Lambda$, $A(P) \supseteq A(Q)$ and $B(P) \supseteq B(Q)$.*

Proof. We shall prove only the inclusion for $A(P)$. The result for the $B(P)$ will follow by dual arguments. If $i \in A(Q)$ then $L_{Q^{(i)}}(x) > a_i$ for all $x > a_i$. Thus for each $i \in A(Q)$ we can inductively construct sequences $x_k^{(i)}, y_k^{(i)}$ in (a_i, b_i) which decrease to a_i and satisfy $x_{k+1}^{(i)} < y_{k+1}^{(i)} < L_{Q^{(i)}}(x_k^{(i)})$ for all k. The intervals $I_{i,j} := (y_{k+1}^{(i)}, y_k^{(i)})$ cannot be dominated by Q, and are pairwise disjoint as i and k

run over all possible values. Thus by Lemma 4.3 there must be a P_0 in Λ which does not dominate any $I_{i,j}$. Likewise, if $P \geqslant P_0$, then P does not dominate any $I_{i,j}$. But if there were any $i \in A(Q)$ for which we could find an $x > a_i$ with $L_{P^{(i)}}(x) = a_i$, then P would dominate all $I_{i,j}$ contained in (a_i, x). It follows that when $i \in A(Q)$ then for any $x > a_i$, $L_{P^{(i)}}(x) > a_i$. Thus $i \in A(P)$. ∎

LEMMA 4.29. *Suppose we are given some $P_0 \in \Lambda$ and a collection C of indices such that, for each $i \in C$, (a_i, b_i) has a finite outer cover of size $n_i > 1$ with respect to $P_0^{(i)}$. Then there is a P_1 such that, for all $P \geqslant P_1$ and all $i \in C$, (a_i, b_i) has an outer cover of size $2n_i - 1$ with respect to $P^{(i)}$.*

Proof. Fix i and let (x_k, y_k) $(k = 1, 2, \ldots, n_i)$ be an outer cover for of (a_i, b_i) with respect to $P_0^{(i)}$. Since these intervals are a linked list, the endpoints satisfy

$$x_k < y_{k-1} < x_{k+1} < y_k \quad \text{for } 2 \leqslant k \leqslant n_i - 1 \quad \text{and} \quad x_1 = a_i,\ y_{n_i} = b_i. \tag{4.1}$$

Thus we can easily pick x'_k $(2 \leqslant k \leqslant n_i)$ and y'_k $(1 \leqslant k \leqslant n_i - 1)$ to satisfy

$$y_{k-1} < x'_k < y'_{k-1} < x_{k+1} \quad \text{for } 2 \leqslant k \leqslant n_i - 1 \tag{4.2}$$

and

$$y_{n_i-1} < x'_{n_i} < y'_{n_i-1} < b_i\,.$$

Define $x'_1 := a_i$ and $y'_{n_i} := b_i$. Then take a continuous increasing bijection that maps each x_k to x'_k and each y_k to y'_k for $1 \leqslant k \leqslant n_i$. We can construct a single function $\mathbb{R} \to \mathbb{R}$ which accomplishes the corresponding mapping on each (a_i, b_i). Let P'_0 be the image of P_0 under this transformation. Since Λ is a stable net, $P'_0 \in \Lambda$. Let P_1 be a collection in Λ that refines both P_0 and P'_0 and let P be an arbitrary collection in Λ that refines P_1. We shall show that each $P^{(i)}$ has an outer cover of size $2n_i - 1$.

If E is an interval in $P^{(i)}$ then it is contained in an interval of P_0 and in an interval of P'_0. Since the (x_j, y_j) are an outer cover for (a_i, b_i) with respect to $P_0^{(i)}$ and, correspondingly, the (x'_k, y'_k) are an outer cover for (a_i, b_i) with respect to ${P'_0}^{(i)}$, it follows that for some j and k,

$$E \subseteq (x_j, y_j) \cap (x'_k, y'_k).$$

However by (4.1) and (4.2), the only way this intersection can be non-empty is for it to equal one of

$$(x_j, y'_{j-1}) \quad \text{or} \quad (x'_j, y_j).$$

It is routine to check (again by (4.1) and (4.2)) that the collection of intervals (x'_l, y_l) $(1 \leqslant l \leqslant n_i)$ together with (x_{l+1}, y'_l) $(1 \leqslant l \leqslant n_i - 1)$ is a linked list of the correct length, and the result follows. ∎

REMARK 4.30. Note that $n_i \geqslant 2$ in the hypotheses of Lemma 4.29, and so in the conclusion $2n_i - 1 \geqslant \frac{3}{2}n_i$. In the sequel we shall need to apply Lemma 4.29

repeatedly, and this lower bound on the size of the outer cover will be easier to iterate.

LEMMA 4.31. *There is a P_0 in Λ such that for all $P \geqslant P_0$ in Λ and all i, $P^{(i)}$ is order compatible with $Q^{(i)}$.*

Proof. By Lemma 4.28, for all sufficiently large P, $A(Q) \subseteq A(P)$ and $B(Q) \subseteq B(P)$. We shall first show that for such a P, $P^{(i)}$ is order compatible with $Q^{(i)}$ for all $i \in A(P) \cup B(P)$. For if in fact $i \in A(P) \cap B(P)$ then by Corollary 4.22 there is an outer cover ordered as $\mathbb{Z}$, which clearly is order compatible with any inner cover for $Q^{(i)}$. On the other hand if $i \in A(P) \setminus B(P)$ then by Lemma 4.19 there exists an outer cover with respect to $P^{(i)}$ which is ordered as $\mathbb{Z}^-$. Correspondingly, since $i \notin B(Q)$, it follows by Lemma 4.18 that there must exist an inner cover with respect to $Q^{(i)}$ which is either finite or ordered as $\mathbb{Z}^-$. In either case $P^{(i)}$ is order compatible with $Q^{(i)}$. Finally, if $i \in B(P) \setminus A(P)$, compatibility follows by dual arguments.

Thus we can restrict attention to the case $i \notin A(P) \cup B(P)$. In such a case, by Corollary 4.21, since $i \notin L(Q) \cup R(Q)$, (a_i, b_i) has a finite inner cover with respect to $Q^{(i)}$. Suppose in each such case we have picked an inner cover of least cardinality, m_i. Because there is a finite inner cover, we can find a sequence

$$a_i = L_{Q^{(i)}}(x_1) < x_1 < R_{Q^{(i)}}(x_1) = x_2 < R_{Q^{(i)}}(x_2) = x_3 < \cdots < R_{Q^{(i)}}(x_n) = b$$

and by Lemma 4.24 together with the minimality of m_i, we have $n \geqslant m_i - 2$. Thus we can choose $k_i := \lfloor \frac{m_i-2}{2} \rfloor \geqslant \frac{m_i-3}{2}$ pairwise disjoint subintervals of (a_i, b_i), each of which contains both x_{2j-1} and x_{2j} for some j. None of these intervals can be dominated by $Q^{(i)}$. Write $(a_j^{(i)}, b_j^{(i)})$ $(j = 1, 2, \ldots, k_i)$ for these intervals. Thus, by Lemma 4.3, for all sufficiently large $P \in \Lambda$, no interval of $P^{(i)}$ contains any $(a_j^{(i)}, b_j^{(i)})$. This shows, by Corollary 4.25, that there is an outer cover of (a_i, b_i) with respect to $P^{(i)}$ that has at least $n_i := \lceil \frac{k_i}{2} \rceil \geqslant \frac{m_i-3}{4}$ elements.

Let C be the set of $i \notin A(P) \cup B(P)$ for which $m_i > 15$. Then for all sufficiently large P, $P^{(i)}$ has a finite outer cover of size at least n_i for all $i \in C$, and $n_i \geqslant \frac{m_i-3}{4} > 1$. Thus we can apply Lemma 4.29 four times, and conclude that for all sufficiently large P, each $P^{(i)}$ has an outer cover of size at least

$$\left(\frac{3}{2}\right)^4 n_i > 5n_i \geqslant \frac{5m_i - 15}{4} > m_i\,.$$

It remains to deal with those i for which $m_i \leqslant 15$.

Consider the set C of all i for which $1 < m_i \leqslant 15$. By Lemma 4.23, we can find $a_i < c_i < d_i < b_i$ such that the interval (c_i, d_i) is not dominated by $Q^{(i)}$. By Lemma 4.3, for all sufficiently large P, no $P^{(i)}$ with $i \in C$ contains (c_i, d_i). Thus, again by Lemma 4.23, all (a_i, b_i) with $i \in C$ have two-element outer covers with respect to $P^{(i)}$. Applying Lemma 4.29 repeatedly five times, we conclude that for all sufficiently large P, we can find outer covers for all $P^{(i)}$ ($i \in C$) of size at least $(\frac{3}{2})^5 \times 2 > 15$.

For all remaining i, $m_i = 1$ and since every $P^{(i)}$ trivially has an outer cover of length 1, we are done. ∎

PROPOSITION 4.32. *For all sufficiently large $P \in \Lambda$, P refines Q.*

Proof. By Lemma 4.31 we can find a $P_0 \in \Lambda$ such that each $P_0^{(i)}$ is order compatible with $Q_0^{(i)}$. By Lemma 4.26 we can find order preserving bijections defined on each (a_i, b_i) which map the intervals of an outer cover of (a_i, b_i) with respect to $P_0^{(i)}$ into the intervals of an inner cover of (a_i, b_i) with respect to $Q^{(i)}$. We can patch these maps together and extend to a single map $\theta : \mathbb{R} \to \mathbb{R}$ which maps an outer cover on each (a_i, b_i) into an inner cover.

Next, transform P_0 to $P_1 := \theta(P_0)$, which belongs to Λ because of the stability property of Λ. Now pick an arbitrary interval E which is dominated by P_1, and aim to show $E \in Q$. Since by definition the (a_i, b_i) are the connected components of $\bigcup Q$, we can find i such that $E \subseteq (a_i, b_i)$. Thus $E \subseteq \theta(E')$ for some $E' \in P_0^{(i)}$. There is an outer cover of (a_i, b_i) with respect to $P_0^{(i)}$ which θ maps into an inner cover of (a_i, b_i) with respect to $Q^{(i)}$. Since E' is a subset of an interval of the outer cover, $\theta(E')$ is a subset of an interval of the inner cover, and so the same is true for E. If it so happens that $E = (x, y)$ where $a_i < x$ and $y < b_i$ then E must in fact be dominated by Q, and we are done. Thus we shall consider the case when $E = (a_i, x)$. The case of $E = (x, b_i)$ is analogous.

Let C be the set of i for which no interval (a_i, x) belongs to Q for any $x > a_i$. Applying Lemma 4.5 to the intervals (a_i, b_i) shows that we can find $P_2 \geqslant P_1$ such that, for $i \in C$, no interval (a_i, x) is dominated by P_1 for any $x > a_i$. Thus for $E = (a_i, x)$, provided at least that E is dominated by P_2, then we know $i \notin C$.

For $i \notin C$, define $c_i := \sup\{x : (a_i, x) \in Q\}$ and let $A := \{i \notin C : (a_i, c_i) \notin Q\}$ and $B := \{i \notin C : (a_i, c_i) \in Q\}$. Apply Lemma 4.3 to the intervals (a_i, c_i) for $i \in A$, and apply Lemma 4.4 to the intervals (a_i, c_i) for $i \in B$, and so conclude there is a $P_3 \geqslant P_2$ which dominates no (a, c_i) (for $i \in A$) or (a_i, x) (for $i \in B$ and $c > c_i$). Thus, provided $E = (a_i, x)$ is dominated by P_3, we know that i must be in A or B, and that in either of these cases, E is in Q.

After applying a similar argument to deal with the case $E = (x, b_i)$, we finally obtain $P_4 \geqslant P_3$ with the property that every interval dominated by P_4 must be dominated by Q, and we are done. ∎

With Proposition 4.32 we have established the final step of the proof of Theorem 4.1. However in Proposition 3.3 we saw that nets which may not be stable still give rise to ideals (though not stable ideals), and so it is natural to ask whether the conclusion of Theorem 4.1 holds without the assumption of stability. The following example shows that it does not. Thanks are due to David Pitts for suggesting that this example should be included.

EXAMPLE 4.33. Let $\mathcal{N} \setminus \{0, I\}$ be parameterized by $\mathbb{R}$ with the strongly continuous mapping $t \mapsto N_t$ and, for any $S \subseteq \mathbb{R}$, define

$$P_S := \{N_{s+1} - N_s : s \in S\}.$$

Let Ω_1 be the set of all P_S where $S \subseteq \mathbb{Q}$ and $\mathbb{Q} \setminus S$ is a G_δ set, and let Ω_2 be the singleton $\{P_{\mathbb{R}\setminus\mathbb{Q}}\}$. One readily verifies that Ω_1 and Ω_2 are nets (although not stable nets) of intervals and that Ω_1 is not cofinal in Ω_2 (or vice versa, for that matter). Nevertheless we shall show that if $\mathcal{J}_1$ and $\mathcal{J}_2$ are the ideals induced by Ω_1 and Ω_2 respectively, then $\mathcal{J}_1 \supseteq \mathcal{J}_2$.

To see this, suppose that $X \notin \mathcal{J}_1$, and aim to show $X \notin \mathcal{J}_2$. Since $X \notin \mathcal{J}_1$, there is an $\varepsilon_0 > 0$ such that for every G_δ subset L of $\mathbb{Q}$,

$$\sup\{\|(N_{s+1} - N_s)X(N_{s+1} - N_s)\|_{\mathrm{ess}} : s \in \mathbb{Q} \setminus L\} \geqslant \varepsilon_0.$$

Now consider the set T of all $x \in \mathbb{R}$ for which $\inf_{t>0} \|(N_{x+t} - N_x)X(N_{x+1} - N_{x+1-t})\|_{\mathrm{ess}} \geqslant \frac{\varepsilon_0}{3}$. If T contains any irrationals then $\|(N_{s+1} - N_s)X(N_{s+1} - N_s)\|_{\mathrm{ess}}$ must be at least $\frac{\varepsilon_0}{3}$ for some $s \in \mathbb{R} \setminus \mathbb{Q}$, and so $X \notin \mathcal{J}_2$. Thus for the remainder of this argument we can assume $T \subseteq \mathbb{Q}$.

Recall we are working over a separable Hilbert space, and so let F_j be a countable norm dense sequence in the set of compact operators. It follows that $x \in T$ if and only if

$$\|(N_{x+\frac{1}{i}} - N_x)X(N_{x+1} - N_{x+1-\frac{1}{i}}) - F_j\| > \frac{\varepsilon_0}{3} - \frac{1}{k}$$

for all $i, j, k \in \mathbb{N}$. By strong upper continuity of the norm, the set of x satisfying this last inequality is an open set, and so T is a G_δ subset of $\mathbb{Q}$. But of course, by the Baire Category Theorem, $\mathbb{Q}$ is not itself a G_δ set and so $\mathbb{Q} \setminus T$ is non-empty and we can find an $s \in \mathbb{Q} \setminus T$ such that $\|(N_{s+1} - N_s)X(N_{s+1} - N_s)\|_{\mathrm{ess}} \geqslant \frac{2\varepsilon_0}{3}$. However since $s \notin T$, for all sufficiently small $\eta > 0$,

$$\|(N_{s+\eta} - N_s)X(N_{s+1} - N_{s+1-\eta})\|_{\mathrm{ess}} < \frac{\varepsilon_0}{3}$$

and so we shall fix on a small irrational value of η for which this holds. From the last two inequalities it follows that $\|(N_{s+1} - N_s)X(N_{s+1} - N_s) - (N_{s+\eta} - N_s)X(N_{s+1} - N_{s+1-\eta})\|_{\mathrm{ess}} \geqslant \frac{\varepsilon_0}{3}$ and therefore at least one of the following must hold:

$$\|(N_{s+1-\eta} - N_{s-\eta})X(N_{s+1-\eta} - N_{s-\eta})\|_{\mathrm{ess}} \geqslant \frac{\varepsilon_0}{6},$$
$$\|(N_{s+1+\eta} - N_{s+\eta})X(N_{s+1+\eta} - N_{s+\eta})\|_{\mathrm{ess}} \geqslant \frac{\varepsilon_0}{6}.$$

However since $s \pm \eta$ is irrational this means

$$\sup\{\|EXE\|_{\mathrm{ess}} : E \in P_{\mathbb{R}\setminus\mathbb{Q}}\} \geqslant \frac{\varepsilon_0}{6}$$

and so, since Ω_2 is a singleton, the limit over Ω_2 is non-zero, and $X \notin \mathcal{J}_2$.

5. QUOTIENT NORMS

The main result of this section will be Theorem 5.8, which establishes a formula for the quotient form of $\operatorname{Alg}\mathcal{N}$ by a stable ideal. We shall first, in Corollaries 5.2 and 5.3 establish a version of the quotient norm formula for two ideals of compact character, $\mathcal{K}^+$ and $\mathcal{K}^-$. The following lemma, which is derived from a theorem of Axler, Berg, Jewell, and Shields ([1], Theorem 2) is needed for these formulas.

LEMMA 5.1. *Let $X \in B(\mathcal{H})$ and let P be a projection such that*

$$\|X\|_{\text{ess}} < a \quad \text{and} \quad \|XP^{\perp}\| < a.$$

Suppose $E_n = E_nP$ is a sequence of operators converging strong- to P and satisfying $\lim_n \|I - E_n\| = 1$. Given any $\varepsilon > 0$ there is an n_0 such that $\|X(I - E_n)\| \leqslant a + \varepsilon$ for all $n \geqslant n_0$.*

Proof. Rescaling as necessary, we shall assume $a = 1$ and suppose for a contradiction there is an $\varepsilon_0 > 0$ such that $\|X(I - E_n)\| > 1 + \varepsilon_0$ for infinitely many n. For each such n we pick a unit vector t_n satisfying $\|X(I - E_n)t_n\| > 1 + \varepsilon_0$ and set $x_n := (I - E_n)t_n$. Passing to a subsequence we may assume that this inequality holds for all n and, further, that x_n is weakly convergent to a limit x. Write $x_n = x + e_n$ where $\text{w-lim}\, e_n = 0$. We claim that $x = P^{\perp}x$. To see this, observe that for any fixed m and $y \in \mathcal{H}$, as $n \to \infty$

$$|\langle E_m(I - E_n)t_n, y\rangle| \leqslant \|(I - E_n^*)E_m^*y\| \longrightarrow \|(I - P)E_m^*y\| = 0$$

and thus $E_m x = \operatorname*{w-lim}_{n\to\infty} E_m x_n = 0$ for all m. Taking the limit as $m \to \infty$, our claim follows.

Since $\|X\|_{\text{ess}} < 1$, find a finite rank projection F such that $\|F^{\perp}X\| < 1$. Without loss we may assume F includes Xx in its range. Finally, pick n_0 such that

$$\|x_n\| < 1 + \frac{\varepsilon_0}{4}, \quad |\langle e_n, x\rangle| < \frac{\varepsilon_0}{8} \quad \text{and} \quad \|FXe_n\| < \frac{\varepsilon_0}{2}$$

for all $n \geqslant n_0$. It now follows that for any $n \geqslant n_0$, $\|Xx_n\| = \|FXx + FXe_n + F^{\perp}Xx + F^{\perp}Xe_n\| \leqslant \|FXx + F^{\perp}Xe_n\| + \frac{\varepsilon_0}{2}$ and

$$\begin{aligned} \|FXx + F^{\perp}Xe_n\|^2 &= \|FXx\|^2 + \|F^{\perp}Xe_n\|^2 = \|FXP^{\perp}x\|^2 + \|F^{\perp}Xe_n\|^2 \\ &< \|x\|^2 + \|e_n\|^2 \quad (\text{since } \|XP^{\perp}\| < 1 \text{ and } \|F^{\perp}X\| < 1) \\ &\leqslant \|x_n\|^2 + 2|\langle e_n, x\rangle| < 1 + \frac{\varepsilon_0}{2} < \Big(1 + \frac{\varepsilon_0}{2}\Big)^2. \end{aligned}$$

Thus, $\|Xx_n\| < 1 + \varepsilon_0$, contradicting our hypothesis. ∎

COROLLARY 5.2. *The quotient norm for* $\operatorname{Alg}\mathcal{N}/\mathcal{K}^-$ *is given by the formula*

$$\|X + \mathcal{K}^-\| = \sup\{\|NXN\|_{\text{ess}} : N \in \mathcal{N}, N < I\}.$$

Proof. Clearly for any $K \in \mathcal{K}^-$ and $N < I$ in $\mathcal{N}$, $\|X - K\| \geqslant \|N(X - K)N\|_{\mathrm{ess}} = \|NXN\|_{\mathrm{ess}}$, and thus $\|X + \mathcal{K}^-\|$ is at least as big as the supremum of the $\|NXN\|$'s. We must show the reverse inequality.

To this end, suppose that $\|NXN\|_{\mathrm{ess}} < a$ for all $N < I$, and aim to show that $\|X + \mathcal{K}^-\| \leqslant a$. By [5], $\operatorname{Alg}\mathcal{N}$ has a strongly convergent approximate identity E_k of finite rank contractions. By Lemma 4.3 of [4] we can assume that E_k converges strong-* and $\lim_n \|I - E_n\| = 1$. Choose N_n to be a sequence in $\mathcal{N}$ that increases to I, and inductively choose compact operators K_n in $\operatorname{Alg}\mathcal{N}$ as follows:

Suppose that $K_1, \ldots, K_{n-1}$ have been chosen to have the property that $K_i = K_i(N_i - N_{i-1})$ and

$$\Big\| XN_{n-1} - \sum_{i=1}^{n-1} K_i \Big\| < a$$

(declaring $N_0 := 0$ for convenience). Let $X' := XN_n - \sum_{i=1}^{n-1} K_i$, let $P := N_n - N_{n-1}$, and observe that the hypotheses of Lemma 5.1 apply to X', P, and PE_kP. The result is that we can find a k_0 for which

$$\Big\| XN_n - \sum_{i=1}^{n-1} K_i - XPE_{k_0}P \Big\| < a$$

and the induction is completed on taking $K_n := XPE_{k_0}P$.

Observe that the series $K := \sum_{i=1}^{\infty} K_i$ converges weakly, $\|X - K\| \leqslant a$, and for each n, $KN_n = \sum_{i=1}^{n} K_i$, which is compact, hence $K \in \mathcal{K}^-$. Thus $\|X + \mathcal{K}^-\| \leqslant a$. ∎

The corresponding result for $\mathcal{K}^+$ follows analogously:

COROLLARY 5.3. *The quotient norm for* $\operatorname{Alg}\mathcal{N}/\mathcal{K}^+$ *is given by the formula*

$$\|X + \mathcal{K}^+\| = \sup\{\|N^\perp X N^\perp\|_{\mathrm{ess}} : N \in \mathcal{N}, 0 < N\}.$$

We now turn our attention to estimates for the distance from stable ideals not of compact character. For these, we will need to work with the characterization of stable ideals in terms of diagonal seminorm functions (Theorem 2.4).

LEMMA 5.4. *Let $\|\cdot\|_N$ be a diagonal seminorm function which takes the values $\|\cdot\|_N = j_N$ for all $0 < N < I$. Let $X \in \operatorname{Alg}\mathcal{N}$ satisfy $\|X\|_N < a$ for all N. Then there is a $T \in \operatorname{Alg}\mathcal{N}$ with $\|X - T\| < a$ and $\|T\|_N = 0$ for all N.*

Proof. A routine compactness argument shows that there is a strictly increasing sequence N_i ($i \in \mathbb{Z}$) with $\lim_{n\to-\infty} N_i = 0$, $\lim_{n\to\infty} N_i = I$, and

$$\|(N_{i+1} - N_{i-1})X(N_{i+1} - N_{i-1})\| < a \quad \text{for all } i. \tag{5.1}$$

Now consider the possible values of $\|\cdot\|_N$ at $N = 0$. These are 0, i_0^+, and e_0^+. We shall modify the initial tail of the sequence N_i according to which of these values occurs. In the first case, make no changes. In the second case (i.e. $\|\cdot\|_0 = i_0^+$) we can renumber the sequence so that $N_0 = 0$ and dispense with the negative-index terms, while stipulating that estimate (5.1) still holds. In the third case (i.e. $\|\cdot\|_0 = e_0^+$) we can again dispense with the negative-index terms, as long as we accept that now a weaker estimate holds for the first term:

$$\|(N_2 - N)X(N_2 - N)\|_{\mathrm{ess}} < a \quad \text{for all } 0 < N < N_2 \tag{5.2}$$

(and the original norm estimates still hold for the remaining terms $i \geqslant 2$). However, in this case we can use Corollary 5.3 to find $K_0 = N_2 K_0 N_2 \in \mathcal{K}^+$ such that $\|XN_2 - K_0\| < a$. The estimate (5.1) now holds with X replaced by $X - K_0$ for $i = 1$ and for $i \geqslant 3$. We can recover (5.1) for $i = 2$ by changing N_1 to be a new value between N_0 and N_2 chosen so that $\|(N_2 - N_1)K_0(N_2 - N_1)\|$ is sufficiently small. Making this change does not affect any of the other norm estimates.

In the same way we can modify the sequence based on the values of $\|\cdot\|_N$ at $N = I$, possibly dispensing with the tail and terminating the sequence at a finite point, $N_{n_0} = I$. In the case that the analogue of estimate (5.2) applies to $N_{n_0} - N_{n_0-2}$, we will find a $K_1 = N_{n_0-2}^\perp K_0 N_{n_0-2}^\perp \in \mathcal{K}^-$ such that $\|N_{n_0-2}^\perp X - K_1\| < a$ and adjust N_{n_0-1} so that estimate (5.1) still holds for all i for $X - K_1$.

Thus, after adjusting our sequence N_i appropriately, estimate (5.1) will hold for either X itself, or else the operator X adjusted by subtracting possibly one or other of K_0 and K_1. Since (5.1) implies that $\|N_{i-1}^\perp X N_{i+1}\| < a$ for all i, it follows by a version of Arveson's Distance Formula due to Power [12], that there is an operator T satisfying $N_{i-1}^\perp T N_{i+1} = 0$ for all i, such that $\|X - T\| < a$.

Clearly $\|T\|_N = j_N(T) = 0$ for all $0 < N < I$. Next focus on $N = 0$. If $\|\cdot\|_0 = 0$ then there is nothing to prove. If $\|\cdot\|_0 = i_0^+$ then since in this case the sequence of N_i's terminates at $N_0 = 0$ and so $TN_2 = 0$, therefore $\|T\|_0 = 0$. Lastly, if $\|\cdot\|_0 = e_0^+$, since TN_2 is again 0 and we adjusted X in the previous paragraph by the operator $K_0 \in \mathcal{K}^+$ for which $e_0^+(K_0) = 0$, then we can restore X to its original value and replace T with $T + K_0$, observing that $\|T + K_0\|_0$ is also zero. A similar argument applies for $N = I$ and we are done. ∎

Recall from Definition 3.14 in [11] that a diagonal seminorm $\|\cdot\|_N$ function is called a greatest diagonal seminorm function if there is an $X \in \operatorname{Alg}\mathcal{N}$ and an $a > 0$, so that $\|\cdot\|_N$ is the largest diagonal seminorm function for which $\|X\|_N < a$ for all $N \in \mathcal{N}$. Since the collection of all diagonal seminorm functions is a complete lattice, given any X and $a > 0$, there is a greatest diagonal seminorm function for X and a. It is advantageous to work with greatest diagonal seminorm functions because of a certain regularity they exhibit, as shown by the following lemma, quoted from Lemma 3.16 in [11]:

LEMMA 5.5. *A diagonal seminorm function, $\|\cdot\|_N$, is a greatest diagonal seminorm function if and only if it has the following lower semicontinuity property: $\|\cdot\|_N^- = 0$ whenever there is a sequence N_n increasing to N with $\|\cdot\|_{N_n} \neq j_{N_n}$ for all n, and $\|\cdot\|_N^+ = 0$ whenever there is a sequence N_n decreasing to N with $\|\cdot\|_{N_n} \neq j_{N_n}$ for all n.*

PROPOSITION 5.6. *Let $\|\cdot\|_N$ be a greatest diagonal seminorm function and let $X \in \operatorname{Alg}\mathcal{N}$ satisfy $\|X\|_N < a$ for all N. Then there is a $T \in \operatorname{Alg}\mathcal{N}$ with $\|X - T\| < a$ and $\|T\|_N = 0$ for all N.*

Proof. Let $S = \{N : \|\cdot\|_N = j_N\}$. By Lemma 5.5, this is an open set and so decomposes as the disjoint union of open intervals (M_n, N_n). For $N \notin S$ write $\|\cdot\|_N = a_N^- \vee a_N^+$ where $a_N^\pm$ is one of 0, $e_N^\pm$, or $i_N^\pm$. Note that also by Lemma 5.5, $a_{M_n}^-$ is zero unless it happens that M_n is the upper endpoint of another component of S (i.e. $M_n = N_k$ for some k). Thus it will suffice to construct $T_n = (N_n - M_n)T_n(N_n - M_n)$ with the property that $\|T_n\|_N = 0$ for all N and $\|(N_n - M_n)(X - T_n)(N_n - M_n)\| < a$, for then

$$T := X - \sum_{n=1}^{\infty}(N_n - M_n)(X - T_n)(N_n - M_n)$$

will have the desired properties. However if we restrict to the nest $(N_n - M_n)\mathcal{N}$ and apply Lemma 5.4 to $(N_n - M_n)X(N_n - M_n)$ in the algebra of this nest, we obtain the desired T_n. ∎

COROLLARY 5.7. *Let $\mathfrak{I}$ be a stable ideal not of compact character and let $\mathcal{F}$ be a set of diagonal seminorm functions that specify $\mathfrak{I}$ as in Theorem 2.4. Then the quotient norm is given by the formula*

$$\|X + \mathfrak{I}\| = \inf\Big\{\sup_{N\in\mathcal{N}} \|X\|_N : \|\cdot\|_N \in \mathcal{F}\Big\}.$$

Proof. If $T \in \mathfrak{I}$ and $\varepsilon > 0$, there is a $\|\cdot\|_N$ in $\mathcal{F}$ such that $\|T\|_N < \varepsilon$ for all N. Thus

$$\|X - T\| \geqslant \|X - T\|_N \geqslant \|X\|_N - \varepsilon$$

for all N. This proves $\|X + \mathfrak{I}\| \geqslant \inf\Big\{\sup_{N\in\mathcal{N}} \|X\|_N : \|\cdot\|_N \in \mathcal{F}\Big\}$ and it remains to establish the reverse inequality.

Suppose that $\inf\Big\{\sup_{N\in\mathcal{N}} \|X\|_N : \|\cdot\|_N \in \mathcal{F}\Big\} < a$ and aim to show that $\|X + \mathfrak{I}\| < a$. It follows there is a $\|\cdot\|_N \in \mathcal{F}$ such that $\|X\|_N < a$ for all N. Although $\|\cdot\|_N$ need not be a *greatest* diagonal seminorm, we can take $\|\cdot\|'_N$ to be the greatest diagonal seminorm for X and a, so that

$$\|X\|_N \leqslant \|X\|'_N < a$$

for all N. Applying Proposition 5.6 to $\|\cdot\|'_N$, there is a $T \in \operatorname{Alg}\mathcal{N}$ with $\|X - T\| < a$ and $\|T\|_N \leqslant \|T\|'_N = 0$. If follows that $T \in \mathfrak{I}$ and so $\|X + \mathfrak{I}\| < a$. ∎

THEOREM 5.8. *Let $\mathfrak{I}$ be a stable ideal of $\operatorname{Alg}\mathcal{N}$ and let Ω be a stable net that specifies it in the sense of Theorem 3.18. Then the quotient norm is given by the formula*

$$\|X + \mathfrak{I}\| = \lim_{P\in\Omega} \sup_{E\in\mathcal{P}} \|EXE\|_{\mathrm{ess}}.$$

Proof. If $\varepsilon > 0$ we can find a $T \in \mathcal{J}$ such that $\|X - T\| < \|X + \mathcal{J}\| + \varepsilon$ and so,

$$\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} = \lim_{P\in\Omega} \sup_{E\in P} \|E(X-T)E\|_{\mathrm{ess}} \leqslant \|X - T\| < \|X + \mathcal{J}\| + \varepsilon .$$

Thus $\lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} \leqslant \|X + \mathcal{J}\|$ and it remains to prove the reverse inequality.

Let Ω' be the set of all collections $P_{T,a}$ as T ranges over $\mathcal{J}$ and $a > 0$ (see Remark 3.14). In Theorem 3.18 we saw that Ω' specifies $\mathcal{J}$. Since Ω also specifies $\mathcal{J}$, it follows by Theorem 4.1 that the two nets Ω and Ω' are mutually cofinal. Thus for any $X \in \operatorname{Alg}\mathcal{N}$

$$\lim_{P\in\Omega} \sup_{E\in\mathcal{P}} \|EXE\|_{\mathrm{ess}} = \lim_{P\in\Omega'} \sup_{E\in\mathcal{P}} \|EXE\|_{\mathrm{ess}}.$$

Let $\mathcal{F}$ be a set of diagonal seminorms that specify $\mathcal{J}$ as in Theorem 2.4. Suppose that $\lim_{P\in\Omega'} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} < a$, and find a $P \in \Omega'$ such that $\sup_{E\in P} \|EXE\|_{\mathrm{ess}} < a$. Since $P \in \Omega'$, there is a $T \in \mathcal{J}$ and a $c > 0$ such that $P = P_{T,c}$. By rescaling T, we can assume $c = a$. Next, since $T \in \mathcal{J}$, we can find a diagonal seminorm function $\|\cdot\|_N$ in $\mathcal{F}$ such that $\|T\|_N < a$ for all $N \in \mathcal{N}$. Thus by Lemma 3.17 $\|\cdot\|_N$ is compatible with $P_{T,a}$ and so also with $P_{X,a}$, because this contains $P_{T,a}$. Again, by Lemma 3.17, we conclude that $\|X\|_N < a$ for all N. Thus by Corollary 5.7, $\|X + \mathcal{J}\| \leqslant a$. The conclusion is thus that

$$\|X + \mathcal{J}\| \leqslant \lim_{P\in\Omega'} \sup_{E\in P} \|EXE\|_{\mathrm{ess}} = \lim_{P\in\Omega} \sup_{E\in P} \|EXE\|_{\mathrm{ess}}$$

and the result follows. ∎

6. SUMS OF IDEALS

In this section we shall study the algebraic sum of two stable ideals, and show that this is also stable. In Proposition 6.3 we present a natural description of the sum of two stable ideals in terms of the stable nets giving rise to the summands.

LEMMA 6.1. *Let $\mathcal{J}_1$ and $\mathcal{J}_2$ be two stable ideals of* $\operatorname{Alg}\mathcal{N}$. *Then the algebraic sum, $\mathcal{J}_1 + \mathcal{J}_2$, is a stable ideal.*

Proof. Clearly $\mathcal{J}_1 + \mathcal{J}_2$ is an automorphism invariant ideal, and we need only show that it is norm closed. By a standard argument,

$$\frac{\operatorname{Alg}\mathcal{N}}{\mathcal{J}_1} \supseteq \frac{\mathcal{J}_1 + \mathcal{J}_2}{\mathcal{J}_1} \cong \frac{\mathcal{J}_2}{\mathcal{J}_1 \cap \mathcal{J}_2}.$$

Since $\frac{\mathcal{J}_2}{\mathcal{J}_1\cap\mathcal{J}_2}$ is complete as a normed space, it suffices to show that the above algebra isomorphism is isometric, for then $\frac{\mathcal{J}_1+\mathcal{J}_2}{\mathcal{J}_1}$ is closed in $\frac{\operatorname{Alg}\mathcal{N}}{\mathcal{J}_1}$, and $\mathcal{J}_1 + \mathcal{J}_2$ is its preimage under the quotient map. To this end, choose $X \in \mathcal{J}_2$ and let the stable nets Ω_1 and Ω_2 respectively induce $\mathcal{J}_1$ and $\mathcal{J}_2$. Clearly $\|X + \mathcal{J}_1\| \leqslant \|X + (\mathcal{J}_1 + \mathcal{J}_2)\|$.

On the other hand, by Theorem 5.8, given $\varepsilon > 0$, we can find $P_1 \in \Omega_1$ such that $\sup_{E \in P_1} \|EXE\|_{\text{ess}} < \|X + \mathcal{J}_1\| + \varepsilon$ and, since $X \in \mathcal{J}_2$, we can certainly find $P_2 \in \Omega_2$ such that $\sup_{E \in P_2} \|EXE\|_{\text{ess}} < \|X + \mathcal{J}_1\| + \varepsilon$. But then $P_1 \cup P_2 \in \Omega_1 + \Omega_2$ and, by Proposition 3.12 and Theorem 5.8,

$$\|X + (\mathcal{J}_1 \cap \mathcal{J}_2)\| = \lim_{P \in \Omega_1 + \Omega_2} \sup_{E \in P} \|EXE\|_{\text{ess}} \leqslant \max_{i=1,2} \sup_{E \in P_i} \|EXE\|_{\text{ess}} < \|X + \mathcal{J}_1\| + \varepsilon.$$

Since ε was arbitrary, the result follows. ∎

LEMMA 6.2. *Let $\|\cdot\|_N^{(1)}$ and $\|\cdot\|_N^{(2)}$ be two diagonal seminorm functions and write $\|\cdot\|_N^{(1)} \wedge \|\cdot\|_N^{(2)}$ for their meet in the lattice of diagonal seminorm functions. Suppose that $\|\cdot\|_N^{(1)}$ is in fact a greatest diagonal seminorm function and that for some $Y \in \operatorname{Alg}\mathcal{N}$ and $\varepsilon > 0$*

$$\|Y\|_N^{(1)} \wedge \|Y\|_N^{(2)} < \varepsilon \quad \textit{for all } N \in \mathcal{N}.$$

Then $Y = Y_1 + Y_2$ for some $Y_i \in \operatorname{Alg}\mathcal{N}$ where $\|Y_1\|_N^{(1)} < \varepsilon$ and $\|Y_2\|_N^{(2)} < \varepsilon$ for all $N \in \mathcal{N}$.

Proof. By Lemma 5.5, $S := \{N : \|\cdot\|_N^{(1)} = j_N\}$ is an open set, which decomposes as the disjoint union of intervals (M_n, N_n). Let $Y_1 := Y - \sum_n (N_n - M_n)Y(N_n - M_n)$. Clearly $\|Y_1\|_N^{(1)} = 0$ for all $N \in S$ and observe that for $N \notin S$, $\|\cdot\|_N = a_N^- \vee a_N^+$ where $a_N^\pm$ is one of 0, $e_N^\pm$, or $i_N^\pm$. Also by Lemma 5.5, $a_{M_n}^-$ (respectively, $a_{N_n}^+$) is zero unless it happens that M_n is the upper endpoint (respectively, M_n is the lower endpoint) of another component of S, and so $\|Y_1\|_N^{(1)} = 0$ for all $N \in \mathcal{N}$.

Next, consider $Y' := Y - Y_1 = \sum_n (N_n - M_n)Y(N_n - M_n)$. If $N \in S$ then $\|\cdot\|_N^{(1)} = j_N \geqslant \|\cdot\|_N^{(2)}$ and so, since $\|Y\|_N^{(i)} \geqslant \|Y'\|_N^{(i)}$, we cannot have $\|Y'\|_N^{(2)} \geqslant \varepsilon$. Thus $\|Y'\|_N^{(2)} < \varepsilon$ for all $N \in S$.

For $N \notin S$ write $\|\cdot\|_N^{(i)-}$ and $\|\cdot\|_N^{(i)+}$ for the left and right respective parts of $\|\cdot\|_N^{(i)}$. In other words, $\|\cdot\|_N^{(i)} = \|\cdot\|_N^{(i)-} \vee \|\cdot\|_N^{(i)+}$ and each $\|\cdot\|_N^{(i)\pm}$ is one of 0, $e_N^\pm$, $i_N^\pm$. Let $A^- := \{n : \|Y'\|_{N_n}^{(2)-} \geqslant \varepsilon\}$ and $A^+ := \{n : \|Y'\|_{M_n}^{(2)+} \geqslant \varepsilon\}$. If $n \in A^-$ then

$$\|Y'\|_{N_n}^{(1)-} < \varepsilon \leqslant \|Y'\|_{N_n}^{(2)-}.$$

Conversely, any $N \in (M_n, N_n)$ that satisfies $\|Y'\|_N^{(1)} \geqslant \varepsilon$, must have $\|Y'\|_N^{(2)} < \varepsilon$, and so $\|\cdot\|_N^{(2)} \neq j_N$ (since j_N dominates all other diagonal seminorms). It follows by Lemma 5.5 that since $\|Y'\|_{N_n}^{(2)-} \neq 0$, N_n cannot be the limit from below of such N. Thus there is an $L_n \in (M_n, N_n)$ such that $\|Y'\|_N^{(1)} < \varepsilon$ for all $N \in [L_n, N_n)$.

By the same token, if $n \in A^+$ we can find $G_n \in (M_n, N_n)$ so that $\|Y'\|_N^{(1)} < \varepsilon$ for all $N \in (M_n, G_n]$. Without loss, insist that $M_n < G_n < L_n < N_n$ for $n \in A^- \cap A^+$.

Let $Y_1' := \sum_{i \in A^-} (N_i - L_i) Y (N_i - L_i) + \sum_{j \in A^+} (G_j - M_j) Y (G_j - M_j)$. Then $\|Y_1'\|_N^{(1)} < \varepsilon$ for all N, by a similar argument to the first paragraph, and clearly $\|Y' - Y_1'\|_N^{(2)} < \varepsilon$ for all N. Thus $Y = (Y_1 + Y_1') + (Y' - Y_1')$, which has the desired properties. ∎

PROPOSITION 6.3. *Let $\mathfrak{I}_1$ and $\mathfrak{I}_2$ be stable ideals specified respectively by the stable nets Ω_1 and Ω_2. For $P_1 \in \Omega_1$ and $P_2 \in \Omega_2$ define*

$$P_1 \cdot P_2 := \{E_1 E_2 : E_1 \in P_1, E_2 \in P_2\},$$
$$\Omega_1 \cdot \Omega_2 := \{P_1 \cdot P_2 : P_1 \in \Omega_1, P_2 \in O_2\}.$$

Then $\Omega := \Omega_1 \cdot \Omega_2$ is a stable net, and $\mathfrak{I}_1 + \mathfrak{I}_2 = \Big\{ X \in \operatorname{Alg} \mathcal{N} : \lim_{P \in \Omega} \sup_{E \in P} \|EXE\|_{\mathrm{ess}} = 0 \Big\}$.

Proof. It is routine to verify that $\Omega := \Omega_1 \cdot \Omega_2$ is a stable net and whenever $Y = Y_1 + Y_2 \in \mathfrak{I}_1 + \mathfrak{I}_2$ then $\lim_{P \in \Omega} \sup_{E \in P} \|EYE\|_{\mathrm{ess}} = 0$. It remains to suppose that $\lim_{P \in \Omega} \sup_{E \in P} \|EYE\|_{\mathrm{ess}} = 0$ for some $Y \in \operatorname{Alg} \mathcal{N}$ and show that $Y \in \mathfrak{I}_1 + \mathfrak{I}_2$.

Since the ideals $\mathfrak{I}_i$ $(i = 1, 2)$ are stable ideals, there are families of diagonal seminorm functions $\mathcal{F}_i$ which determine them. Let Ω_i' be the set of $P_{T,a}$ as T varies over $\mathfrak{I}_i$ and $a > 0$. As we saw in Theorem 3.18, each Ω_i' determines $\mathfrak{I}_i$ and so, by Theorem 4.1, each Ω_i' is mutually cofinal with each Ω_i. Thus it is easily seen that $\Omega' := \Omega_1' \cdot \Omega_2'$ and Ω are mutually cofinal.

Given $\varepsilon > 0$, since $\lim_{P \in \Omega'} \sup_{E \in P} \|EYE\|_{\mathrm{ess}} = \lim_{P \in \Omega} \sup_{E \in P} \|EYE\|_{\mathrm{ess}} = 0$, we can find a $P \in \Omega'$ such that $\|EYE\|_{\mathrm{ess}} < \varepsilon$ for all $E \in P$. However $P = P_1 \cdot P_2$ where each $P_i = P_{T_i, \varepsilon}$ for a suitably scaled $T_i \in \mathfrak{I}_i$. Thus we can find $\| \cdot \|_N^{(i)} \in \mathcal{F}_i$ such that $\|T_i\|_N^{(i)} < \varepsilon$ for all N. Since the proof of Lemma 6.2 requires $\| \cdot \|_N^{(1)}$ to be a greatest diagonal seminorm function, let us replace $\| \cdot \|_N^{(1)}$ with the greatest diagonal seminorm function for which $\|T_1\|_N^{(1)} < \varepsilon$ for all N.

From the definition, each P_i is compatible with $\| \cdot \|_N^{(i)}$ in the sense of Definition 3.16. It is straightforward to check that therefore $P = P_1 \cdot P_2$ is compatible with $\| \cdot \|_N^{(1)} \wedge \| \cdot \|_N^{(2)}$. Therefore also $P_{Y,\varepsilon} \supseteq P$ is compatible with $\| \cdot \|_N^{(1)} \wedge \| \cdot \|_N^{(2)}$ and so, by Lemma 3.17, $\|Y\|_N^{(1)} \wedge \|Y\|_N^{(2)} < \varepsilon$.

It follows from Lemma 6.2 that $Y = Y_1 + Y_2$ where $\|Y_i\|_N^{(i)} < \varepsilon$ for all N and $i = 1, 2$. Although for technical reasons we replaced the original $\| \cdot \|_N^{(1)}$ from $\mathcal{F}_1$ with the greatest diagonal seminorm function for T_1 and ε, nevertheless since that seminorm function dominated the original one, the current estimate for Y_1 holds with the original $\| \cdot \|_N^{(1)}$ from $\mathcal{F}_1$ too. But thus, by Theorem 5.8, Y is at most distance 2ε from $\mathcal{I}_1 + \mathcal{I}_2$. Finally, since ε was arbitrary, Lemma 6.1 shows that $Y \in \mathcal{I}_1 + \mathcal{I}_2$. ∎

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