

PROJECTIVE MULTIREOLUTION ANALYSES FOR DILATIONS IN HIGHER DIMENSIONS

JUDITH A. PACKER

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ABSTRACT. We continue the study of projective module wavelet frames corresponding to diagonal dilation matrices on \mathbb{R}^n with integer entries. We follow the method proposed by the author and M. Rieffel, and are able to come up with examples of non-free projective module wavelet frames which can be described via this construction. The construction is generalized to include all integral dilation matrices that are conjugates of diagonal matrices by elements in $SL(n, \mathbb{Z})$. As an application of our results, in the case $n = 3$, when the dilation matrix is a constant multiple of the identity, we embed every finitely generated projective module of $C(\mathbb{T}^3)$ as an initial module.

KEYWORDS: *Module frames, finitely generated projective modules, dilations, K-theory, wavelets, C*-algebras, Hilbert C*-module.*

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1. INTRODUCTION

Let A be an $n \times n$ integer dilation matrix. In a previous paper [18], the author and M. Rieffel studied the existence of projective multiresolution analyses corresponding to dilation by certain 2×2 matrices A in a Hilbert $C(\mathbb{T}^n)$ -module \mathcal{E} , where we recall that for $n \in \mathbb{N}, n \geq 2$, the right $C(\mathbb{T}^n)$ -module \mathcal{E} is defined as the completion of $C_c(\mathbb{R}^n)$ under the norm determined by the following $C(\mathbb{T}^n)$ -valued inner product:

$$(1.1) \quad \langle \xi, \eta \rangle_{C(\mathbb{T}^n)}(t) := \sum_{p \in \mathbb{Z}^n} (\bar{\xi} \eta)(t - p),$$

for $t \in \mathbb{R}^n$. We remark that space \mathcal{E} was first introduced by G. Zimmermann in Chapter V of his thesis [24], where the notation $L^{2,\infty}(\mathbb{Z}^n, \mathbb{T}^n)$ was used, and was studied further by J. Benedetto and G. Zimmermann in [2].

One verifies (c.f. Section 1 of [18]) that \mathcal{E} consists of all bounded continuous functions ξ on \mathbb{R}^n for which there is a constant, K , such that $\sum_{p \in \mathbb{Z}^n} |\xi(x - p)|^2 \leq K$

for each $x \in \mathbb{R}^n$. The right module action of $C(\mathbb{T}^n)$ on \mathcal{E} is given by pointwise multiplication. One also checks that

$$\|\langle \zeta, \zeta \rangle_{C(\mathbb{T}^n)}\| \geq \int_{\mathbb{R}^n} |\zeta(t)|^2 dt,$$

so that $\mathcal{E} \subseteq L^2(\mathbb{R}^n)$. In Proposition 5 of [18], sufficient decay conditions were provided on any continuous $\zeta \in L^2(\mathbb{R}^n)$ for ζ to be an element of \mathcal{E} .

In a previous paper [18], the notion of a projective multiresolution analysis for a general $n \times n$ dilation matrix was defined as follows:

DEFINITION 1.1 ([18], Definition 4). Fix $n \in \mathbb{N}$, let A be a $n \times n$ integer dilation matrix, and let \mathcal{E} be the right-rigged Hilbert $C(\mathbb{T}^n)$ -module defined above. A sequence $\{V_j\}_{j \in \mathbb{Z}}$ of subspaces of \mathcal{E} is called a *projective multiresolution analysis* for dilation by A if:

- (i) V_0 is a finitely generated projective $C(\mathbb{T}^n)$ submodule of \mathcal{E} ;
- (ii) $V_j = D^j(V_0), \forall j \in \mathbb{N}$, where D is as defined below;
- (iii) $V_j \subset V_{j+1}, \forall j \in \mathbb{Z}$;
- (iv) $\bigcup_{j=0}^{\infty} V_j$ is dense in \mathcal{E} , in the Hilbert $C(\mathbb{T}^n)$ -module topology;
- (v) $\bigcap_{-\infty}^{\infty} V_j = \{0\}$.

Here D is defined to be the Fourier transformed version of D_A ,

$$D = \mathcal{F} \circ D_A \circ \mathcal{F}^*,$$

where the dilation operator D_A is defined on $L^2(\mathbb{R}^n)$ by

$$D_A(\zeta)(x) = |\det(A)|^{1/2} |\zeta(A(x))| \quad \zeta \in L^2(\mathbb{R}^n),$$

and the Fourier transform \mathcal{F} is defined on $L^2(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(x) = \int_{\mathbb{R}^n} f(t) \bar{e}(t \cdot x) dt.$$

Here e is the exponential function defined on \mathbb{R} by $e(r) = e^{2\pi ir}$. An easy calculation shows that

$$D = D_{B^{-1}},$$

for $B = A^t$.

It turns out that condition (v) in the definitions above is implied by conditions (i) and (ii) ([18], Proposition 13).

In [18], Theorem 6, specializing to the case where $n = 2$, for any fixed diagonal dilation matrix A , and any choice of finitely generated projective $C(\mathbb{T}^2)$ -module $X(q, a)$, a projective multiresolution analysis for that dilation matrix was constructed with $V_0 \cong X(q, a)$. In this situation, the isomorphism class of each of the higher dimensional modules $V_j = D^j(V_0), j \geq 1$ was provided, and under

appropriate circumstances, we gave a constructive approach which gave module frames for \mathcal{E} analogous to dilates and translates of L^2 -wavelets.

It is the aim of this paper to continue developing the methods outlined in [18], applying the main ideas to construct $C(\mathbb{T}^n)$ -module frames for \mathcal{E} in dimensions $n \geq 3$. We discuss projective multiresolution analyses corresponding to integer-valued dilation matrices which are similar to diagonal matrices via conjugation by an element in $SL(n, \mathbb{Z})$. It is interesting to note that in [18], in all of the examples of projective multi-resolution analyses corresponding to 2×2 dilation matrices with positive determinant, where the "initial module" V_0 was not free, the wavelet module $W_0 = V_1 \ominus V_0$ was still free. Hence the module generators of W_0 in this case corresponded to multi-wavelets in the usual sense. In contrast, in this paper an example will be given of a 3×3 dilation matrix with positive determinant, and a corresponding projective multiresolution analysis where V_0, V_1 , and the wavelet module W_0 are all finitely generated *non-free* $C(\mathbb{T}^3)$ -modules. Also, in the 3×3 case, where the dilation matrix in question is an integer multiple of the identity matrix, we are able to construct a projective multiresolution analysis for that dilation matrix whose initial module is isomorphic to \mathcal{P} , where here \mathcal{P} represents an arbitrary finitely generated projective $C(\mathbb{T}^3)$ -module.

In Section 2, the class of allowable "scaling functions" which lie in \mathcal{E} is widened, in Section 3 we review K -theory and finitely generated projective modules for \mathbb{T}^n , and discuss the extent to which the two-dimensional case can be generalized to higher dimensions. In Section 4, we discuss the construction of projective multiresolution analyses for dilation matrices that are similar via an element of $SL(n, \mathbb{Z})$ to a integral diagonal dilation matrix, and in Section 5 a method is given for constructing projective module frames for \mathcal{E} from a projective multiresolution analysis whenever module frames for V_0 and W_0 are known. In particular, this result shows that W_0 is a free module, then W_i is free for all $i \in \mathbb{N}$. We finish by posing some open questions.

2. CONSTRUCTION OF SCALING FUNCTIONS WHOSE FOURIER TRANSFORMS ARE NON-COMPACTLY SUPPORTED ELEMENTS OF \mathcal{E}

In the construction of projective multiresolution analyses given in [18], continuous Meyer-type low-pass filters corresponding to dilation by an integer d with $|d| > 1$ were used, whose corresponding scaling functions had continuous Fourier transforms with compact support, hence were elements of \mathcal{E} . The tensor product construction then gave scaling functions for diagonal dilation matrices whose Fourier transforms were continuous and compactly supported, thus in \mathcal{E} . The aim of this construction was to use these filters to aid in building the projective multiresolution analyses. However in practice, one often comes across scaling functions whose Fourier transforms Φ are not compactly supported, and it is useful to have a weaker condition that guarantees they lie in \mathcal{E} . It also is of

interest to identify more precisely the $C(\mathbb{T}^n)$ -module structure of \mathfrak{E} . We do this in the following theorem.

THEOREM 2.1. *The Hilbert $C(\mathbb{T}^n)$ -module \mathfrak{E} is isomorphic to the standard $C(\mathbb{T}^n)$ -module*

$$l^2(C(\mathbb{T}^n)) = \left\{ (f_j)_{j=1}^\infty : f_j \in C(\mathbb{T}^n), \sum_{j=1}^\infty |f_j(x)|^2 \text{ converges in norm in } C(\mathbb{T}^n) \right\}.$$

In addition, if Φ is a continuous complex-valued function defined on \mathbb{R}^n , then $\Phi \in \mathfrak{E}$ if and only if the sum

$$\sum_{p \in \mathbb{Z}^n} |\Phi(x - p)|^2$$

converges uniformly on the cube $[0, 1]^n$.

Proof. Let \mathcal{H} be the Hilbert $C^*(\mathbb{Z}^n)$ -module obtained by completing the $C^*(\mathbb{Z}^n)$ -module $C_c(\mathbb{R}^n)$, where we give $C_c(\mathbb{R}^n)$ a right $C_c(\mathbb{Z}^n)$ -action defined by

$$f * a = \sum_{v \in \mathbb{Z}^n} f(t - v)a(v) \quad f \in C_c(\mathbb{R}^n), a \in C_c(\mathbb{Z}^n),$$

and a $C^*(\mathbb{Z}^n)$ -valued inner product defined by

$$\langle f_1, f_2 \rangle_{C^*(\mathbb{Z}^n)}(v) = \int_{\mathbb{R}^n} \overline{f_1(x)} f_2(x - v) dx \quad f_1, f_2 \in C_c(\mathbb{R}^n).$$

The module \mathcal{H} was studied in much greater generality for any closed subgroup H of the locally compact group G by M. Rieffel in 1974 [19], where he studied induction of group representations by algebras. In this case $H = \mathbb{Z}^n$ and $G = \mathbb{R}^n$ and \mathcal{H} serves as a strong Morita equivalence bimodule between $C^*(\mathbb{Z}^n)$ and the transformation group C^* -algebra $C_0(\mathbb{R}^n/\mathbb{Z}^n) \rtimes \mathbb{R}^n$. It then follows from the work of P. Green ([12], Theorem 2.4) that \mathcal{H} is isomorphic as a right $C^*(\mathbb{Z}^n)$ -module to the standard $C^*(\mathbb{Z}^n)$ -module $l^2(C^*(\mathbb{Z}^n))$.

One easily checks that $\mathfrak{E} = \mathcal{F}(\mathcal{H})$ and that

$$\mathcal{F}(f * a) = \mathcal{F}(f) \cdot \mathcal{F}(a) \quad f \in \mathcal{H}, a \in C^*(\mathbb{Z}^n)$$

and

$$\mathcal{F}(\langle f_1, f_2 \rangle_{C^*(\mathbb{Z}^n)}) = \langle \mathcal{F}(f_1), \mathcal{F}(f_2) \rangle_{C(\mathbb{T}^n)},$$

where here, by abuse of notation, we also denote the Fourier transform from $C^*(\mathbb{Z}^n)$ to $C(\mathbb{T}^n)$ by \mathcal{F} . Hence \mathfrak{E} is isomorphic to \mathcal{H} . But we have seen that \mathcal{H} is isomorphic to $l^2(C^*(\mathbb{Z}^n))$ as a $C^*(\mathbb{Z}^n)$ -module. Thus \mathfrak{E} is isomorphic to the standard $C(\mathbb{T}^n)$ -module $l^2(C(\mathbb{T}^n))$ as a $C(\mathbb{T}^n)$ -module.

As for the second statement, suppose that Φ is continuous on \mathbb{R}^n and that

$$\sum_{p \in \mathbb{Z}^n} |\Phi(x - p)|^2$$

converges uniformly the cube $[0, 1]^n$. Since each term $|\Phi(x - p)|^2$ is continuous on $[0, 1]^n$, by the theory of uniform convergence, the sum $\sum_{p \in \mathbb{Z}^n} |\Phi(x - p)|^2$ is continuous on the compact cube $[0, 1]^n$, hence is bounded from above by some positive constant K . But then for any $x \in \mathbb{R}^n$, there exists $m \in \mathbb{Z}^n$ with $x' = x + m \in [0, 1]^n$, so that

$$\sum_{p \in \mathbb{Z}^n} |\Phi(x - p)|^2 = \sum_{p \in \mathbb{Z}^n} |\Phi(x' - m - p)|^2 \leq K,$$

and hence $\Phi \in \mathfrak{E}$.

The converse follows directly from Proposition 1 of [18]. ■

REMARK 2.2. We note that the construction of the isomorphism between \mathcal{H} and $l^2(C^*(\mathbb{Z}^n))$ from [12] depends on the choice of a Borel section $\alpha : \mathbb{R}^n / \mathbb{Z}^n \rightarrow \mathbb{R}^n$. Thus there is no canonical choice of isomorphism between \mathfrak{E} and $l^2(C(\mathbb{T}^n))$.

We give an example of the use of the above theorem to show that the Fourier transform of the Haar scaling function in dimension 1 for dilation by an integer $d > 1$ is an element of \mathfrak{E} . Some ideas behind this construction can be found in Section 4 of the reference [11]. Fix $d \in \mathbb{N}, d > 1$, and find an orthonormal basis for \mathbb{R}^d , with the initial vector fixed as $\vec{r}_0 = (\frac{1}{\sqrt{d}}, \frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}})$, and the other vectors as $\vec{r}_1, \dots, \vec{r}_{d-1}$. For $0 \leq l \leq d - 1$, write

$$\vec{r}_l = (a_{l,0}, a_{l,1}, \dots, a_{l,d-1}).$$

Let $\vec{v}(x)$ denote the row vector whose entries consist of the following d -periodic functions defined on $\mathbb{R} : (1, e(\frac{x}{d}), e(\frac{2x}{d}), \dots, e(\frac{(d-1)x}{d}))$. Now define

$$\mu_l(x) = \vec{r}_l \cdot \vec{v}(x) \quad 0 \leq l \leq d - 1,$$

where the “ \cdot ” denotes dot product, so that

$$\mu_l(x) = \sum_{j=0}^{d-1} a_{l,j} e(\frac{jx}{d}) \quad 0 \leq l \leq d - 1.$$

The functions $\mu_i(x), 0 \leq i \leq d - 1$ are continuous and periodic modulo d , and

$$\mu_0(x) = \sum_{j=0}^{d-1} e(\frac{jx}{d}),$$

so that $\mu_0(0) = 1$. As in [11], one easily calculates by the law of characters on $\mathbb{Z}/d\mathbb{Z}$ that:

$$\langle \mu_l, \mu_k \rangle_{C(\mathbb{R}/\mathbb{Z})} = \sum_{n=0}^{d-1} da_{l,n} a_{k,n} = d\delta_{l,k},$$

since the vectors $\vec{r}_0, \vec{r}_1, \dots, \vec{r}_{d-1}$ form an orthonormal basis for \mathbb{R}^d . (Here the use of the inner product notation refers to the definition given in equation (1.1).)

Defining $m_i(x) = \mu_i(dx), 0 \leq i \leq d - 1$, it is clear that the functions $\{m_i : 0 \leq i \leq d - 1\}$ will be continuous and periodic modulo 1. They will also satisfy the two conditions

$$m_0(0) = \sqrt{d} \quad \text{and} \quad \langle m_l, m_k \rangle_{C(\mathbb{R}/\frac{1}{d}\mathbb{Z})}(x) = d\delta_{lk}(x).$$

The functions $\{m_j : 0 \leq j \leq d - 1\}$ are all trigonometric polynomials, and m_0 satisfies Cohen's condition as stated in [23]. In the usual way one uses the telescoping property to prove that when m_0 is the trigonometric low-pass filter defined above, the corresponding scaling function viewed in the frequency domain will take on the closed form

$$(2.1) \quad \Phi(x) = \prod_{j=1}^{\infty} \left[\frac{1}{\sqrt{d}} m_0\left(\frac{x}{d^j}\right) \right] = \frac{e(x) - 1}{2\pi i x},$$

which is familiar from ordinary wavelet theory as being the Fourier transform of the Haar scaling function. By an application of Theorem 2.1, we immediately see that Φ is an element of the Hilbert $C(\mathbb{T})$ -module \mathcal{E} . In the same way one can show that the Fourier transform for the Haar scaling function corresponding to dilation by $-d$ for $d \in \mathbb{N}, d > 1$, is also in \mathcal{E} .

3. PROJECTIVE MULTIREOLUTION ANALYSES IN HIGHER DIMENSIONS: DIAGONAL MATRICES WITH INTEGER ENTRIES

In this section, we first discuss finitely generated projective modules over $C(\mathbb{T}^n), n \geq 3$, and then go on to construct projective multi-resolution analyses in higher dimensions. We also discuss the relationship of these projective modules to the K -groups of $C(\mathbb{T}^n)$. The modules we consider are adapted from $C(\mathbb{T}^2)$ -modules first constructed in Section 3 of [21], and were also reviewed in Section 3 of [18]. For completeness, we briefly describe these $C(\mathbb{T}^2)$ -modules here. Throughout this section, unless otherwise specified, we view functions on \mathbb{T}^n as functions defined on \mathbb{R}^n which are periodic modulo \mathbb{Z}^n .

THEOREM 3.1. *For $q \in \mathbb{N}$ and $a \in \mathbb{Z}$, let $X(q, a)$ denote the right $C(\mathbb{T}^2)$ -module consisting of the space of continuous complex-valued functions f on $\mathbb{T} \times \mathbb{R}$ which satisfy*

$$f(s, t - q) = e(as)h(s, t),$$

with module action given by

$$f \cdot F(s, t) = f(s, t)F(s, t),$$

for $f \in X(q, a)$ and $F \in C(\mathbb{T}^2)$. Then $X(q, a)$ is a finitely generated, projective $C(\mathbb{T}^2)$ -module; following the results from [21], we say that $X(q, a)$ has dimension q and twist $-a$. The set $\{X(q, a) : q \in \mathbb{N}, a \in \mathbb{Z}\}$ parameterizes the isomorphism classes of finitely

generated projective $C(\mathbb{T}^2)$ -modules, in the sense that if X is a finitely generated projective $C(\mathbb{T}^2)$ -module, there exist unique values of q and a such that $X \cong X(q, a)$. Moreover, if $q_1, q_2 \in \mathbb{N}$ and $a_1, a_2 \in \mathbb{Z}$ are fixed, then

$$X(q_1, a_1) \oplus X(q_2, a_2) \cong X(q_1 + q_2, a_1 + a_2)$$

as finitely generated projective $C(\mathbb{T}^2)$ -modules. Furthermore, cancellation holds for all finitely generated $C(\mathbb{T}^2)$ -modules; that is, if

$$X(q_1, a_1) \oplus X(q_2, a_2) \cong X(q_1, a_1) \oplus X(q_3, a_3),$$

then $X(q_2, a_2) \cong X(q_3, a_3)$, so that $q_2 = q_3$ and $a_2 = a_3$.

Proof. For the proof of this result, refer to Theorem 3.9 of [21]. ■

It follows that for any positive integer q , $X(q, 0)$ is isomorphic to the free $C(\mathbb{T}^2)$ -module on q generators; moreover, we obtain the well-known result that $K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2 = \{[q, a] : q, a \in \mathbb{Z}\}$ [21].

The above analysis for finitely generated projective modules over $C(\mathbb{T}^2)$ extends to finitely generated projective modules over $C(\mathbb{T}^n)$. By definition, the stable isomorphism classes of finitely generated projective modules over $C(\mathbb{T}^n)$ give the positive cone of the abelian group $K_0(C(\mathbb{T}^n)) \cong K^0(\mathbb{T}^n)$, hence generate the entire group $K^0(\mathbb{T}^n)$, where $K^0(\mathbb{T}^n)$ represents the Grothendieck group from stable isomorphism class of complex vector bundles over \mathbb{T}^n . It is well-known that the ring $K^*(\mathbb{T}^n) \cong K_*(C(\mathbb{T}^n))$ is isomorphic to the exterior algebra over \mathbb{Z} on n generators, $\bigwedge_{\mathbb{Z}}^* \{e_1, \dots, e_n\}$ [8]. Here the Chern character $ch : K_*(C(\mathbb{T}^n)) \rightarrow \check{H}^*(\mathbb{T}^n, \mathbb{Q})$ is integral and gives an isomorphism $ch_0 : K_0(C(\mathbb{T}^n)) \rightarrow \check{H}^{\text{even}}(\mathbb{T}^n, \mathbb{Z})$ and $ch_1 : K_1(C(\mathbb{T}^n)) \rightarrow \check{H}^{\text{odd}}(\mathbb{T}^n, \mathbb{Z})$ (see [13] for a proof), and $\check{H}^*(\mathbb{T}^n, \mathbb{Z})$ under cup product is known to be isomorphic to the exterior algebra $\bigwedge_{\mathbb{Z}}^* \{e_1, \dots, e_n\}$ with $\check{H}^k(\mathbb{T}^n, \mathbb{Z}) \cong \bigwedge_{\mathbb{Z}}^k \{e_1, \dots, e_n\}$. For future reference we discuss some calculations involving the Chern character on $K_*(C(\mathbb{T}^n))$ under the identification with the exterior algebra on n generators discussed above; see the reference [15] for more details.

Recall from [18] that the Chern character on $K_0(C(\mathbb{T}^2))^+$ maps $[X(q, a)]$ to $q - ae_1 \wedge e_2 \in \bigwedge_{\mathbb{Z}}^{\text{even}} \{e_1, e_2\}$. Also recall that if $A = \begin{pmatrix} b & c \\ d & f \end{pmatrix} \in GL(2, \mathbb{Z})$ is viewed an automorphism of \mathbb{T}^2 by the formula

$$A(z, w) = (z^b w^c, z^d w^f) \quad (z, w) \in \mathbb{T}^2,$$

then the corresponding map in K -theory, $A^* : K_0(C(\mathbb{T}^2)) \rightarrow K_0(C(\mathbb{T}^2))$ is given by

$$(3.1) \quad A^*(q - ae_1 \wedge e_2) = q - \det(A)ae_1 \wedge e_2.$$

More generally, viewing $C(\mathbb{T}^n)$ as the C^* -algebra of all continuous functions on the n real variables $s_1, s_2, \dots, s_{n-1}, t$, and fixing $q \in \mathbb{N}$ and $a_1, a_2, \dots, a_{n-1} \in \mathbb{Z}$, let $X(q, a_1, a_2, \dots, a_{n-1})$ be the $C(\mathbb{T}^n)$ -module consist of all continuous functions

$f : \mathbb{R}^n \rightarrow \mathbb{C}$ satisfying the identity

$$f(s_1, s_2, \dots, s_{n-1}, t - q) = e(a_1s_1 + \dots + a_{n-1}s_{n-1})f(s_1, s_2, \dots, s_{n-1}, t).$$

It can be shown using methods from Section 3 of [1] and [15] that the $C(\mathbb{T}^n)$ -module $X(q, a_1, a_2, \dots, a_{n-1})$ is a finitely generated projective $C(\mathbb{T}^n)$ -module whose first Chern class corresponds to the element $-\sum_{1 \leq k \leq n-1} a_k e_k \wedge e_n \in \Lambda_{\mathbb{Z}}^2\{e_1, \dots, e_n\} \cong H^2(\mathbb{T}^n, \mathbb{Z})$. For general n , this method does not produce all finitely generated projective modules of $C(\mathbb{T}^n)$, as in general the positive cone of $K_0(C(\mathbb{T}^n))$ seems to be unknown; see Exercise 6.10.2 of [3]. Also, as indicated in [3] and demonstrated in Section 4 of [17], cancellation of projective modules over $C(\mathbb{T}^n)$ fails for $n \geq 5$, although cancellation holds for $n \leq 4$. However, the failure of cancellation to hold does not affect calculations at the level of the K -groups, and combined with the natural automorphisms of \mathbb{T}^n provided by elements in $GL(n, \mathbb{Z})$ one can write down specific formulas for a large class of examples of elements from $K_0(C(\mathbb{T}^n))^+$. In the special cases when $n = 2$ and $n = 3$ we can write down all of them. We can also write down analogues of Equation (3.1) which tell us how certain elements of $GL(n, \mathbb{Z})$ affect elements in $K_0(C(\mathbb{T}^n))^+$, by using the exterior algebra methods outlined in the proof of Equation (3.1). We will use these facts in what follows when we construct our projective multi-resolution analyses for certain higher dimensional dilation matrices.

We now discuss the construction of projective multi-resolution analyses where the initial space V is a finitely generated projective module over $C(\mathbb{T}^n)$, for $n \geq 3$. For simplicity, we will concentrate on the case where the module V in question is of the form $X(q, 0, \dots, a_j, 0, \dots, 0)$ as defined above, for $q \in \mathbb{N}, 1 \leq j \leq n - 1$ and $a_1, a_2, \dots, a_{n-1} \in \mathbb{Z}$. In this case $X(q, 0, \dots, a_j, 0, \dots, 0)$ is the $C(\mathbb{T}^n)$ -module consisting of all continuous complex-valued functions from \mathbb{R}^n to \mathbb{C} which are periodic modulo \mathbb{Z} in the first $n - 1$ variables and which satisfy the identity

$$f(s_1, s_2, \dots, s_{n-1}, t - kq) = e(ka_j s_j) f(s_1, s_2, \dots, s_{n-1}, t) \quad s_1, s_2, \dots, s_{n-1}, t \in \mathbb{R}, k \in \mathbb{Z},$$

and where the action of $C(\mathbb{T}^n)$ on $X(q, 0, 0, \dots, a_j, 0, \dots, 0)$ is by pointwise multiplication. Here, as previously, j represents a fixed integer between 1 and $n - 1$. By our discussion in Section 2, the module $X(q, 0, \dots, a_j, 0, \dots, 0)$ is exactly the pullback of the $C(\mathbb{T}^2)$ -module $X(q, a_j)$ obtained from the projection of \mathbb{T}^n onto \mathbb{T}^2 given by projection in the j th and n th coordinates, and therefore, upon identifying $K_0(C(\mathbb{T}^n))$ with $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_n\}$ as in section two,

$$[X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0)]_{K_0(C(\mathbb{T}^n))}$$

can be identified with the element $q - a_j e_j \wedge e_n \in \Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_n\}$. The $C(\mathbb{T}^n)$ -valued inner product on $X(q, 0, \dots, 0, a_j, \dots, 0)$ is given by

$$\langle h_1, h_2 \rangle_{C(\mathbb{T}^n)} = \sum_{k=0}^{q-1} \overline{h_1(s_1, s_2, \dots, s_{n-1}, t - k)} \cdot h_2(s_1, s_2, \dots, s_{n-1}, t - k).$$

We now remark that it is possible to generalize Theorem 4 of [18] to this n -dimensional setting, and embed $X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0)$ as a finitely generated projective module in the Hilbert $C(\mathbb{T}^n)$ -module $\Xi \subseteq L^2(\mathbb{R}^n)$ described in Section 1, as follows:

THEOREM 3.2. *Fix $n \in \mathbb{N}$, and let A be the $n \times n$ diagonal matrix*

$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

where $d_1, d_2, \dots, d_n \in \mathbb{N}$ and $|d_j| > 1, 1 \leq j \leq n$. Fix $q \in \mathbb{N}$ and $a_j \in \mathbb{Z}$, and let $X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0)$ be the $C(\mathbb{T}^n)$ -module defined in Section 2. Then there exists a $C(\mathbb{T}^n)$ -module monomorphism

$$\mathcal{R} : X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0) \rightarrow \Xi \subseteq L^2(\mathbb{R}^n)$$

which satisfies:

- (i) $\langle \mathcal{R}(h_1), \mathcal{R}(h_2) \rangle_{C(\mathbb{T}^n)} = \langle h_1, h_2 \rangle_{C(\mathbb{T}^n)}$, for all $h_1, h_2 \in (q, 0, \dots, a_j, \dots, 0)$;
- (ii) $\mathcal{R}(X(q, 0, \dots, a_j, \dots, 0)) \subseteq D\mathcal{R}(X(q, 0, \dots, a_j, \dots, 0))$, where D is the conjugate of the dilation operator with respect to the dilation matrix A by the Fourier transform.

Proof. Without loss of generality we let $j = n - 1$ since the proof for the other cases is the same, up to a permutation of the variables. We recall that the inner product on Ξ is defined by

$$\langle \lambda, \mu \rangle_{\Xi}(s_1, s_2, \dots, s_{n-1}, t) := \sum_{v \in \mathbb{Z}^n} (\overline{\lambda \mu})((s_1, s_2, \dots, s_{n-1}, t) - v).$$

As in the proof of Theorem 4 of [18], we want to find some appropriate function $\gamma \in \Xi$ and then define

$$\mathcal{R}(h) = h \cdot \gamma \quad h \in X(q, 0, 0, \dots, 0, a_{n-1}).$$

As in the proof of that theorem, one checks that in order for equation (i) in the statement of the theorem to hold, it is necessary to have

$$\sum_{(m_1, \dots, m_n) \in \mathbb{Z}^n} |\gamma((s_1 - m_1, \dots, s_{n-1} - m_{n-1}, t - m_n q))|^2 = 1.$$

Then, using the fact that $\widehat{D}_A = D_{A^{-1}}$, one checks that if one wants (ii) to hold, it is necessary that

$$D_A(\mathcal{R}(X(q, 0, 0, \dots, 0, a_{n-1}))) \subseteq \mathcal{R}(X(q, 0, 0, \dots, 0, a_{n-1})),$$

which as in the proof of Theorem 4 of [18] will imply the following condition on the function γ :

$$\left\{ \sqrt{\prod_{m=1}^n |d_m|} \gamma(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n t) f(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n t) : f \in X(q, a) \right\} \\ \subseteq \{ \gamma(s_1, \dots, s_{n-1}, t) f(s_1, \dots, s_{n-1}, t) : f \in X(q, 0, \dots, 0, a_{n-1}) \}.$$

For $f \in X(q, 0, \dots, 0, a_{n-1})$, f must satisfy the equalities:

$$f(d_1(s_1 - 1), \dots, d_{n-1}(s_{n-1} - 1), d_n t) = f(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n t), \\ f(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n(t - q)) = e^{(d_{n-1} d_n q s)} f(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n(t - q)).$$

Once more we check that in order for

$$\left(\prod_{l=1}^n |d_l| \right)^{1/2} \gamma(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n t) f(d_1 s_1, \dots, d_{n-1} s_{n-1}, d_n t)$$

to be an element of $\mathcal{R}(X(q, 0, \dots, 0, a_{n-1}))$ for every $f \in X(q, 0, \dots, 0, a_{n-1})$, it is sufficient that there exist $\check{m} \in X(q, 0, \dots, 0, -(d_{n-1} d_n - 1) a_{n-1})$ satisfying

$$\gamma(d_1 s_1, d_2 s_2, \dots, d_{n-1} s_{n-1}, d_n t) = \frac{\check{m}(s_1, \dots, s_{n-1}, t)}{\sqrt{d}} \gamma(s_1, \dots, s_{n-1}, t),$$

for $d = |d_1 d_2 \cdots d_n|$. But now we just use the tensor product construction to do this, that is, we find using the methods described in classical wavelet theory and summarized in [17] a continuous function m' of the $n - 2$ real variables s_1, s_2, \dots, s_{n-2} , which is periodic modulo \mathbb{Z}^{n-2} and such that m' is continuously differentiable and non-zero in a large enough neighborhood of the origin:

$$m'(0, \dots, 0) = \sqrt{|d_1 d_2 \cdots d_{n-2}|}, \\ \sum_{l=1}^{n-2} \sum_{i_j=0}^{|d_l|-1} \left| m'(s_1 - \frac{i_1}{|d_1|}, s_2 - \frac{i_2}{|d_2|}, \dots, s_{n-2} - \frac{i_{n-2}}{|d_{n-2}|}) \right|^2 = |d_1 d_2 \cdots d_{n-2}|$$

(we can choose m' to be a tensor product of $n - 2$ standard Haar filter functions each with dilation factor d_i constructed in Section 2), and define

$$\tilde{m} \in X(q, -(d_{n-1} d_n - 1) a_{n-1}), \tilde{m} : \mathbb{R}^2 \rightarrow \mathbb{C}$$

as in the proof of Theorem 7 of [18], with d_1, d_2 , and a in that theorem replaced by d_{n-1}, d_n , and a_{n-1} respectively. Finally, let $\check{m} : \mathbb{R}^n \rightarrow \mathbb{C}$ be defined by

$$\check{m}(s_1, \dots, s_{n-1}, t) = m'(s_1, \dots, s_{n-2}) \tilde{m}(s_{n-1}, t).$$

Then if we let

$$\gamma(s_1, \dots, s_{n-1}, t) = \prod_{i=1}^{\infty} \frac{\check{m}(\frac{s_1}{(d_1)^i}, \dots, \frac{s_{n-1}}{(d_{n-1})^i}, \frac{t}{(d_n)^i})}{\sqrt{d}},$$

it is clear from the tensor product construction that γ satisfies the desired conditions, by separation of variable techniques. ■

Just as in the 2-dimensional case, by modifying the proof of Theorem 6 of [18] we can define a nested sequence of finitely generated projective $C(\mathbb{T}^n)$ -modules

$$\{V_i\}_{i=0}^\infty$$

by setting

$$V_i = D^i(V_0).$$

Using the exact same method as in Theorem 6 of [18] for the two dimensional case, we associate an ordinary Hilbert space multiresolution analysis to V_0 and we obtain the decomposition

$$\Xi = V_0 \bigoplus_{i \geq 0} W_i,$$

where for each i , W_i is the finitely generated projective $C(\mathbb{T}^n)$ -module defined by

$$W_i = V_{i+1} \ominus V_i.$$

We now want to calculate the image of $V_1 = D(V_0)$ in $K_0(C(\mathbb{T}^n))$, where V_0 is isomorphic to $X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0)$ as in Theorem 3.2. Again without loss of generality, by permuting the variables, we can take V_0 isomorphic to $X(q, 0, 0, \dots, 0, a_{n-1})$. Somewhat surprising generalizations of Theorem 7 of [18] now arise:

THEOREM 3.3. *Let A be the $n \times n$ diagonal matrix*

$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

where $d_1, d_2, \dots, d_n \in \mathbb{N}$ and $|d_j| > 1, 1 \leq j \leq n$, and let V_0 be the image of $X(q, 0, \dots, 0, a_{n-1})$ in $\Xi \subseteq L^2(\mathbb{R}^n)$ defined in Theorem 3.2, so that $V_1 = D(V_0) \supseteq V_0$. Then V_1 is isomorphic to

$$X(|\det(A)|q, 0, 0, \dots, 0, \prod_{j=1}^{n-2} |d_j| \cdot \text{sign}(d_{n-1}d_n)a_{n-1})$$

as a finitely generated projective $C(\mathbb{T}^n)$ -module.

Proof. Let us view $C(\mathbb{T}^n)$ as a tensor product C^* -algebra,

$$C(\mathbb{T}^n) \cong C(\mathbb{T}^{n-2}) \otimes_{\mathbb{C}} C(\mathbb{T}^2),$$

and we view $X(q, 0, 0, \dots, 0, a_{n-1})$ as a tensor product of modules with respect to this decomposition:

$$X(q, 0, 0, \dots, 0, a_{n-1}) \cong C(\mathbb{T}^{n-2}) \otimes_{\mathbb{C}} X(q, a_{n-1}),$$

by recalling that $C(\mathbb{T}^{n-2})$ is a free module over itself. With respect to this notation, the module V_0 can also be viewed as a tensor product:

$$V_0 \cong V'_0 \otimes_{\mathbb{C}} V''_0.$$

Here by construction, V'_0 is a singly generated free module over $C(\mathbb{T}^{n-2})$ and can alternatively be viewed as the intersection of $\Xi' \subseteq L^2(\mathbb{R}^{n-2})$ with an ordinary multi-resolution analysis level zero space V'_0 corresponding to translation by \mathbb{Z}^{n-2} and dilation by the $(n-2) \times (n-2)$ diagonal matrix

$$A' = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_{n-2} \end{pmatrix}.$$

The module V''_0 corresponds to the image of $X(q, a_{n-1})$ in $\Xi \subseteq L^2(\mathbb{R}^2)$ constructed in Theorem 4 of [18]. With respect to this decomposition, the unitary operator D can also be written as a tensor product:

$$D = D' \otimes D'',$$

where D' is the dilation operator in the frequency domain corresponding to the matrix A' above, and D'' is the dilation operator in the frequency domain corresponding to the 2×2 diagonal matrix $A'' = \begin{pmatrix} d_{n-1} & 0 \\ 0 & d_n \end{pmatrix}$. Thus we can view $D(V_0)$ as being isomorphic to the tensor product of modules

$$D'(V'_0) \otimes_{\mathbb{C}} D''(V''_0).$$

Now tensor product techniques and standard results from multiresolution analysis show that

$$D'(V'_0) \cong \bigoplus_{k=1}^{|\det(A')|} [C(\mathbb{T}^{n-2})]_k$$

as a $C(\mathbb{T}^{n-2})$ -module, and the results of Theorem 7 of [18] show that

$$D''(V''_0) \cong X(|d_{n-1}d_n|q, \text{sign}(d_{n-1}d_n)a_{n-1})$$

as a $C(\mathbb{T}^2)$ -module. Thus $D(V_0)$ is isomorphic to the tensor product of modules

$$\bigoplus_{k=1}^{|\det(A')|} [C(\mathbb{T}^{n-2})]_k \otimes_{\mathbb{C}} X(|d_{n-1}d_n|q, \text{sign}(d_{n-1}d_n)a_{n-1}).$$

But for each $k, 1 \leq k \leq n-2$,

$$\begin{aligned} [C(\mathbb{T}^{n-2})] \otimes X(|d_{n-1}d_n|q, \text{sign}(d_{n-1}d_n)a_{n-1}) \\ \cong X(|d_{n-1}d_n|q, 0, \dots, 0, \text{sign}(d_{n-1}d_n)a_{n-1}) \end{aligned}$$

as a $C(\mathbb{T}^n)$ -module. So by the distributive properties for tensor products and direct sums of $C(\mathbb{T}^n)$ -modules,

$$\begin{aligned} & \left(\bigoplus_{k=1}^{|\det(A')|} [C(\mathbb{T}^{n-2})]_k \right) \otimes_{\mathbb{C}} X(|d_{n-1}d_n|q, \text{sign}(d_{n-1}d_n)a_{n-1}) \\ & \cong \bigoplus_{k=1}^{|\det(A')|} [X(|d_{n-1}d_n|q, 0, 0, \dots, 0, \text{sign}(d_{n-1}d_n)a_{n-1})]_k \\ & \cong X(|\det(A)|q, 0, 0, \dots, 0, |\det(A')| \cdot \text{sign}(d_{n-1}d_n)a_{n-1}), \end{aligned}$$

by an easy generalization of Theorem 6 of [18] to this situation. Hence

$$V_1 \cong X(|\det(A)|q, 0, 0, \dots, 0, |\det(A')| \cdot \text{sign}(d_{n-1}d_n)a_{n-1})$$

as finitely generated $C(\mathbb{T}^n)$ -modules, as desired. ■

For an example, we consider the case $n = 3$. In this situation, given a non-free $C(\mathbb{T}^3)$ -module V_0 , a non-free module $W_0 = V_1 \oplus V_0$ can easily arise even when all of the dilation factors have positive signs.

EXAMPLE 3.4. Let $n = 3$, and $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. For $a \in \mathbb{Z} \setminus \{0\}$, embed

$X(1, 0, a)$ as a finitely generated projective submodule V_0 of $\mathcal{E} \subseteq L^2(\mathbb{R}^3)$ by means of Theorem 3.2. Then Theorem 3.3 shows that $V_1 = D(V_0)$ is isomorphic to $X(8, 0, 2a)$, so that

$$W_0 = V_1 \oplus V_0 \cong X(7, 0, a),$$

a finitely generated projective $C(\mathbb{T}^3)$ -module which is not free. In the last calculation we used the fact that finitely generated projective modules over $C(\mathbb{T}^n)$ satisfy the cancellation property for $n \leq 4$.

We can now easily identify the isomorphism class of the V_i as $C(\mathbb{T}^n)$ -modules, for $n \geq 2$. The proof is left to the reader as an exercise:

COROLLARY 3.5. Let $q \in \mathbb{N}, a_{n-1} \in \mathbb{Z}$, integers d_1, d_2, \dots, d_n ,

$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

and V_0 be as in Theorem 3.3, so that V_0 is a submodule of $\mathcal{E} \subseteq L^2(\mathbb{R}^n)$ isomorphic to $X(q, 0, 0, \dots, 0, a_{n-1})$. For each $i \in \mathbb{N}$ define the finitely generated projective $C(\mathbb{T}^n)$ -module V_i by

$$V_i = D^i(V_0).$$

Then

$$V_i \cong X\left(\left[\prod_{j=1}^n |d_j|\right]^i q, 0, 0, \dots, 0, \left[\prod_{j=1}^{n-2} |d_j|\right]^i [\text{sign}(d_{n-1}d_n)]^i a_{n-1}\right),$$

so that

$$W_i = V_{i+1} \ominus V_i$$

is stably isomorphic to $X\left(\left[\left(\prod_{j=1}^n |d_j|\right)^{i+1} - \left(\prod_{j=1}^{n-2} |d_j|\right)^{i-1}\right] q, \mathbb{C}\right)$, where

$$\mathbb{C} = \left(\left[\prod_{j=1}^{n-2} |d_j|\right]^{i+1} [\text{sign}(d_{n-1}d_n)]^{i+1} - \left[\prod_{j=1}^{n-2} |d_j|\right]^i [\text{sign}(d_{n-1}d_n)]^i\right) a.$$

We note that for $n \geq 5$, in general we can only calculate the *stable* isomorphism type of $W_i = V_{i+1} \ominus V_i$, because finitely generated projective $\mathbb{C}(\mathbb{T}^n)$ -modules do not have the cancellation property in general for $n \geq 5$. We note that this particular exercise would combine the methods of Theorem 3.3 and Theorem 8 of [18] for the calculation.

4. PROJECTIVE MULTIREOLUTION ANALYSES FOR CONJUGATES OF DIAGONAL DILATION MATRICES

Without much extra effort, we can obtain results for dilation matrices that are conjugates of diagonal dilation matrices by elements in $SL(n, \mathbb{Z})$. That is, given any $S \in SL(n, \mathbb{Z})$, we can define the $\mathbb{C}(\mathbb{T}^n)$ -module

$$X_S(q, 0, 0, \dots, a_j, 0, \dots, 0) = D_S(X(q, 0, 0, \dots, a_j, 0, \dots, 0))$$

and inject it as a submodule of \mathcal{E} . We are indebted to Dana Williams for the suggestion that this last fact should be true, and for showing us how to prove a major part of it. We first prove the result for $n = 2$, and then move on to the case for general n .

PROPOSITION 4.1. *Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ be fixed, let and $S \in SL(2, \mathbb{Z})$, and set $M = S^{-1}AS$, where*

$$A = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

and d_1, d_2 are integers both having absolute value strictly greater than 1. Write $D = \mathcal{F}D_A\mathcal{F}^{-1}$ and $D' = \mathcal{F}D_M\mathcal{F}^{-1}$. Then there exists a finitely generated projective $\mathbb{C}(\mathbb{T}^2)$ submodule of \mathcal{E} isomorphic to $X(q, a)$ denoted by V'_0 , such that $V'_1 = D'(V'_0) \supseteq V'_0$, and more generally, $V'_i = D'(V'_{i-1}) \supseteq V'_{i-1}$, for all $i \in \mathbb{N}$, with

$$\overline{V'_0 \cup \bigcup_{i \in \mathbb{N}} V'_i} = \mathcal{E}.$$

Furthermore,

$$V'_i \cong X(|\det(M)|^i q, [\text{sign}(\det(M))]^i a),$$

so that

$$W_0 = V_1 \ominus V_0 \cong X((|\det(M)| - 1)q, (\text{sign}(\det(M)) - 1)a).$$

Proof. Let the map $\mathcal{R} : X(q, a) \rightarrow V_0$ and the nested sequence of finitely generated projective $C(\mathbb{T}^2)$ -modules $\{V_i = D^i(V_0)\}_{i=0}^\infty$ be as defined in Theorem 6 of [18], so that

$$\overline{V_0 \cup \bigcup_{i \in \mathbb{N}} V_i} = \Xi.$$

Note that

$$D' = D_{S^t} \circ D \circ D_{(S^t)^{-1}},$$

since $\mathcal{F}D_R\mathcal{F}^{-1} = D_{(R^t)^{-1}}$ for all $R \in GL(n, \mathbb{R})$. Define $V'_0 = D_{S^t}(V_0)$. Then

$$D_{S^t}(V_0) = \{(\gamma \circ S^t) \cdot (h \circ S^t), h \in X(q, a)\},$$

where γ is the function constructed in Theorem 6 of [18], so that $(\gamma \circ S^t) \cdot (h \circ S^t) \in \Xi$ for all $h \in X(q, a)$. Hence $V'_0 \subseteq \Xi$. Furthermore, if $(\gamma \circ S^t) \cdot (h \circ S^t) \in V'_0$, and $g \in C(\mathbb{T}^2)$, then

$$[(\gamma \circ S^t) \cdot (h \circ S^t)]g(s, t) = [(\gamma \circ S^t)(h) \cdot (g \circ (S^t)^{-1}) \circ S^t](s, t).$$

Since $g \circ (S^t)^{-1} \in C(\mathbb{T}^2)$, $(h) \cdot (g \circ (S^t)^{-1}) \in X(q, a)$, so that

$$[(\gamma \circ S^t) \cdot (h \circ S^t)] \cdot g \in V'_0,$$

and V'_0 is a $C(\mathbb{T}^2)$ -submodule of Ξ .

Now observe that

$$D'(V'_0) = D_{S^t} \circ D(V_0) \supseteq D_{S^t}(V_0) = V'_0.$$

We finally remark that $V'_0 \cong X(q, a)$. Indeed for any $R \in SL(2, \mathbb{Z})$, $q \in \mathbb{N}$, and $a \in \mathbb{Z}$, consider the following $C(\mathbb{T}^2)$ module:

$$X_R(q, a) = \{h \circ R : h \in X(q, a)\}.$$

The right action of $C(\mathbb{T}^2)$ on $X_R(q, a)$ is defined as usual by pointwise multiplication. One easily checks that $f \cdot g \in X_R(q, a)$ whenever $f \in X_R(q, a)$ and $g \in C(\mathbb{T}^2)$, and another calculation shows that the map on $X_R(q, a) \times X_R(q, a)$ defined by

$$\langle f_1, f_2 \rangle_{C(\mathbb{T}^2)}(s, t) = \sum_{j=0}^{q-1} \overline{f_1(s, t + j)} f_2(s, t + j)$$

gives a $C(\mathbb{T}^2)$ -valued inner product on $X_R(q, a)$, with

$$\langle f_1, f_2 \cdot g \rangle_{C(\mathbb{T}^2)} = \langle f_1, f_2 \rangle_{C(\mathbb{T}^2)} \cdot g \quad \text{for } f_1, f_2 \in X_R(q, a)$$

and $g \in C(\mathbb{T}^2)$. It is evident from the construction that in terms of $K_0(C(\mathbb{T}^2))$,

$$[X_R(q, a)]_{K_0(C(\mathbb{T}^2))} = (R^{-1})^*[X(q, a)]_{K_0(C(\mathbb{T}^2))} = [X(q, a)]_{K_0(C(\mathbb{T}^2))},$$

by equation (3.1). We now define a map $\mathcal{R}' : X_{S^t}(q, a) \rightarrow V'_0$ by

$$\mathcal{R}'(f)(s, t) = \gamma \circ S^t(s, t) \cdot f(s, t),$$

where again γ is the function constructed in Theorem 6 of [18]. It is clear that \mathcal{R}' preserves the module structure, and we can check that \mathcal{R}' preserves the $C(\mathbb{T}^2)$ -valued inner product as well. Let $f_i = h_i \circ S^t$, where $h_i \in X(q, a), i = 1, 2$. Then

$$\langle f_1, f_2 \rangle_{C(\mathbb{T}^2)} = \langle h_1, h_2 \rangle_{C(\mathbb{T}^2)} \circ S^t,$$

and a similar calculation shows that

$$\mathcal{R}'(f_i) = \mathcal{R}(h_i) \circ S^t \quad i = 1, 2.$$

Hence, for $f_1, f_2 \in X_{S^t}(q, a)$ and $h_1, h_2 \in X(q, a)$ as above,

$$\begin{aligned} \langle f_1, f_2 \rangle_{C(\mathbb{T}^2)}(s, t) &= \langle h_1, h_2 \rangle_{C(\mathbb{T}^2)} \circ S^t(s, t) = \langle \mathcal{R}(h_1), \mathcal{R}(h_2) \rangle_{C(\mathbb{T}^2)} \circ S^t(s, t) \\ &= \sum_{(m,n) \in \mathbb{Z}^2} \overline{[(\gamma \circ S^t) \cdot (f_1)]} \cdot [(\gamma \circ S^t) \cdot (f_2)]((s, t) - (m, n)) \\ &= \langle \mathcal{R}'(f_1), \mathcal{R}'(f_2) \rangle_{C(\mathbb{T}^2)}, \end{aligned}$$

so that \mathcal{R}' preserves the $C(\mathbb{T}^2)$ -valued inner products, as desired. Since V'_0 is isomorphic to $X_{S^t}(q, a)$ and $X_{S^t}(q, a)$ is isomorphic to $X(q, a)$, it follows that V'_0 is isomorphic to $X(q, a)$ as a $C(\mathbb{T}^2)$ -module. Finally, we will show that

$$V'_i \cong X(|\det(M)|^i q, [\text{sign}(\det(M))^i a].$$

Note that $\det(M) = \det(A)$, as M and A are similar, so it is enough to show that

$$V'_i \cong X(|\det(A)|^i q, [\text{sign}(\det(A))^i a].$$

We do this for the case $i = 1$, and note that higher dimensional cases are similar. It is enough to show that

$$D_{S^t} \circ D'(V_0) \cong X(|\det(A)|^q, [\text{sign}(\det(A)) a].$$

For ease of notation, we now write $d_1 = b_1|d_1|, d_2 = b_2|d_2|$, where $b_1, b_2, \in \{-1, 1\}$. Then $\text{sign}[\det(A)] = b_1 b_2$. Let $(d_1, a) = w$, with $w > 0$, so that we can write $a = r \cdot w$, and $d_1 = y \cdot w$, where $(r, y) = 1$, i.e. r and y are relatively prime. Then $|d_1| = |y \cdot w| = |y| \cdot w = y b_1 \cdot w$, so that $|y| = y \cdot b_1$. Arguments similar to those used in the proof of Theorem 7 of [18] show that

$$D(V_0) \cong \bigoplus_{j=0}^{w-1} [X(|y d_2| q, b_1 b_2 r)]_j, \text{ so that}$$

$$D_{S^t} \circ D(V_0) \cong \bigoplus_{k=0}^{w-1} [[X_{S^t}(|y d_2| q, b_1 b_2 r)]_j].$$

Again, the discussion given in the first part of the proof shows that

$$[X_{S^t}(|y d_2| q, b_1 b_2 r)]_j \cong [X(|y d_2| q, b_1 b_2 r)]_j,$$

so that

$$D_{S^t} \circ D(V_0) \cong \bigoplus_{j=0}^{w-1} [[X(|y d_2| q, b_1 b_2 r)]_j],$$

and since

$$\bigoplus_{j=0}^{w-1} [X(|yd_2|q, b_1b_2r)]_j \cong X(|d_1d_2|q, \text{sign}(\det(A))a),$$

the desired result has been proved. ■

THEOREM 4.2. *Let $q \in \mathbb{N}$ and $a \in \mathbb{Z}$ be fixed, and let $M \in M(n, \mathbb{Z})$ and $S \in SL(2, \mathbb{Z})$ be such that*

$$SMS^{-1} = A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

where $d_1, d_2, \dots, d_n \in \mathbb{N}$ and $|d_j| > 1, 1 \leq j \leq n$. Then there exists a finitely generated projective $C(\mathbb{T}^n)$ submodule of \mathfrak{E} isomorphic to $X(q, 0, \dots, a_j, 0, \dots, 0)$, denoted by V'_0 , such that $V'_1 = D'(V'_0) \supseteq V'_0$ and

$$\overline{V'_0 \cup \bigcup_{i \in \mathbb{N}} V'_i} = \mathfrak{E}.$$

Proof. Let the $C(\mathbb{T}^n)$ -module V_0 which is isomorphic to $X(q, 0, \dots, a_j, \dots, 0)$ be as in the proof of Theorem 3.2, and let $D' = \mathcal{F} \circ D_M \circ \mathcal{F}^{-1}$. As in the proof of Proposition 4.1 we define the $C(\mathbb{T}^n)$ -submodule V'_0 which is isomorphic to $X_{S^t}(q, 0, \dots, a_j, \dots, 0)$ and which satisfies the condition

$$V'_0 \subseteq D'(V'_0),$$

where as before $V'_0 = D_{S^t}(V_0)$. The proof is then very similar to that given in the proof of Proposition 4.1, and we leave details to the reader. ■

REMARK 4.3. Let $S \in SL(2, \mathbb{Z}), n \geq 3$, and put $B = S^t$. The main difference in passing to the case $n \geq 3$ is that $X_B(q, 0, \dots, a_j, 0, \dots, 0)$ will not be isomorphic to $X(q, 0, 0, \dots, a_j, 0, \dots, 0)$ in general, unlike the two-dimensional case. Recall that $[X(q, 0, 0, \dots, 0, a_j, 0, \dots, 0)]_{K_0(C(\mathbb{T}^n))}$ can be identified with the element $q - a_j e_j \wedge e_n \in \bigwedge_{\mathbb{Z}}^{\text{even}} \{e_1, \dots, e_n\}$. On the other hand, suppose that we write

$$B = (b_{jk})_{1 \leq j, k \leq n}.$$

Upon modifying the methods of Proposition 3.4 of [18], we see that

$$[X_S(q, 0, 0, \dots, a_j, 0, \dots, 0)]_{K_0(C(\mathbb{T}^n))}$$

is identified with the element $q - a_j \left(\sum_{l=1}^n b_{jl} e_l \right) \wedge \left(\sum_{k=1}^n b_{nk} e_k \right) \in \bigwedge_{\mathbb{Z}}^{\text{even}} \{e_1, \dots, e_n\}$, which of course need not be equal to $q - a_j e_j \wedge e_n$ for $n \geq 3$.

Suppose further that $d_1 = d_2 = \cdots = d_{n-1} = d_n$. Then A commutes with any matrix $B \in SL(n, \mathbb{Z})$, i.e. $B^{-1}AB = A$. In this way we can obtain a large variety of $C(\mathbb{T}^n)$ submodules V'_0 of \mathcal{E} such that

$$V'_0 \subseteq D(V'_0).$$

We develop this idea for the case $n = 3$.

EXAMPLE 4.4. Let $n = 3$ and let our dilation matrix be the standard dilation by d in all variables, i.e. $A = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$. Let $B = (b_{jk})_{1 \leq j, k \leq 3}$ be an arbitrary element of $SL(3, \mathbb{Z})$. Then the above argument shows that we can take

$$V'_0 \cong X_B(q, 0, 1)$$

for any $q \in \mathbb{N}$, and build a projective multiresolution analysis for dilation by A . One calculates from the formulas given above that

$$\begin{aligned} [X_B(q, 0, a)]_{K_0(C(\mathbb{T}^3))} &= [q + a[b_{22}b_{31} - b_{21}b_{32}]e_1 \wedge e_2 + a[b_{23}b_{31} - b_{21}b_{33}]e_1 \wedge e_3 \\ &\quad + a[b_{23}b_{32} - b_{22}b_{33}]e_2 \wedge e_3]_{K_0(C(\mathbb{T}^3))}. \end{aligned}$$

With this example in hand, whenever the 3×3 dilation matrix is a constant integral multiple of the identity, we can easily embed projective modules isomorphic to any nonzero representative of the positive cone of $[K_0(C(\mathbb{T}^3))]$ as the initial space of one of our projective multiresolution analyses, as long as we keep in mind that every nonzero member of this positive cone can be represented as $[q + c_1e_1 \wedge e_2 + c_2e_1 \wedge e_3 + c_3e_2 \wedge e_3]$ for $q \in \mathbb{N}$ and $c_1, c_2, c_3 \in \mathbb{Z}$ via its exterior algebra representation in $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, e_2, e_3\}$.

THEOREM 4.5. Let $n = 3$ and let our dilation matrix be the standard dilation by d in all variables, i.e. $A = \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$, where d is an integer whose absolute value is greater than 1. Then, identifying $K_0(C(\mathbb{T}^3))$ with $\Lambda_{\mathbb{Z}}^{\text{even}}\{e_1, e_2, e_3\} = \Lambda_{\mathbb{Z}}^0\{e_1, e_2, e_3\} \oplus \Lambda_{\mathbb{Z}}^2\{e_1, e_2, e_3\}$, given any $[q + c_1e_1 \wedge e_2 + c_2e_1 \wedge e_3 + c_3e_2 \wedge e_3] \in [K_0(C(\mathbb{T}^3))]^+$, there exists a finitely generated projective $C(\mathbb{T}^3)$ -module $V'_0 \subseteq \mathcal{E}$ with

$$V'_0 \cong [q + c_1e_1 \wedge e_2 + c_2e_1 \wedge e_3 + c_3e_2 \wedge e_3]$$

such that

$$D(V'_0) \supseteq V'_0.$$

Hence, setting

$$V'_n = D^n(V'_0)$$

for $n \geq 0$, the nested sequence $\{V'_n\}$ gives a projective multi-resolution analysis corresponding to the dilation matrix A .

Proof. Let the greatest common divisor of c_1, c_2 and c_3 be $a \in \mathbb{N}$, and write $c_1 = ax, c_2 = ay$, and $c_3 = az$, where x, y , and z are relatively prime. By Example 4.4, in order to prove the theorem, it is sufficient to find $B = (b_{jk})_{1 \leq j, k \leq 3} \in SL(3, \mathbb{Z})$, such that $[b_{22}b_{31} - b_{21}b_{32}] = x, [b_{23}b_{31} - b_{21}b_{33}] = y$, and $[b_{23}b_{32} - b_{22}b_{33}] = z$. The three terms involving the b_{ij} are nothing more than -1 times the cofactors of B obtained from its last two rows, so the question then becomes one of whether it is possible to find $B \in SL(3, \mathbb{Z})$ with specified relatively prime cofactors along the last two rows. Of course this is possible; first choose b_{11}, b_{12} , and $b_{13} \in \mathbb{Z}$ such that

$$-b_{11}z + b_{12}y - b_{13}x = 1.$$

Now let $(x, z) = \nu$, so that $(\nu, y) = 1$ as x, y , and z are relatively prime. Write $x = M\alpha$ and $z = \nu\beta$ where $(\alpha, \beta) = 1$. Since α and β are relatively prime, we can find $\sigma, \tau \in \mathbb{Z}$ such that $\alpha\tau + \beta\sigma = y$. Now we set

$$b_{21} = -\alpha, b_{22} = 0, b_{23} = \beta, \quad b_{31} = \sigma, b_{32} = \nu, b_{33} = \tau.$$

Then one checks that $[b_{22}b_{31} - b_{21}b_{32}] = x, [b_{23}b_{31} - b_{21}b_{33}] = y, [b_{23}b_{32} - b_{22}b_{33}] = z$, and that the matrix B with entries as defined above satisfies

$$\det(B) = 1,$$

so is an element of $SL(3, \mathbb{Z})$, as desired. Finally, setting $V'_0 = X_B(q, 0, a)$, we have

$$D(V'_0) \supseteq V'_0$$

and

$$\begin{aligned} [V'_0]_{K_0(C(\mathbb{T}^3))} &= [q + a[b_{22}b_{31} - b_{21}b_{32}]e_1 \wedge e_2 + a[b_{22}b_{31} - b_{21}b_{33}]e_1 \wedge e_3 \\ &\quad + a[b_{23}b_{32} - b_{22}b_{33}]e_2 \wedge e_3]_{K_0(C(\mathbb{T}^3))} \\ &= [q + c_1e_1 \wedge e_2 + c_2e_1 \wedge e_3 + c_3e_2 \wedge e_3]_{K_0(C(\mathbb{T}^3))}, \end{aligned}$$

as desired. We leave it to the reader as an exercise to calculate the isomorphism class of each V'_n in $K_0(C(\mathbb{T}^3))$ by using the combined methods of Proposition 4.2 and Corollary 3.5. ■

By rearranging the variables, the above discussion generalizes slightly to allow us to deal with diagonal dilation matrices and finitely generated projective modules whose Chern characters take on the values $q_j - ae_{i_j} \wedge e_{k_j}$ for $i_j \neq k_j$ in the exterior algebra formulation of $K_0(C(\mathbb{T}^n))$. Using separation of variable techniques, we can generalize even further as follows: suppose that $[P] \in K_0(C(\mathbb{T}^n))^+$ has its representation in $\bigwedge_{\mathbb{Z}}^{\text{even}}\{e_1, \dots, e_n\}$ given by

$$[P] = \bigwedge_{j=1}^l (q_j - a_1e_{i_j} \wedge e_{k_j}),$$

where $q_j \in \mathbb{N}, 2l \leq n$, and $\{i_1, k_1, i_2, k_2, \dots, i_l, k_l\}$ are distinct elements $\{1, 2, \dots, n\}$. Without loss of generality by rearranging the variables we assume that $i_1 < k_1 <$

$i_2 < k_2 < \dots < i_l < k_l$. Then by repeated use of Theorem 3.3 and tensor products, it is possible to find a finitely generated projective $C(\mathbb{T}^n)$ -submodule $V_0 \subseteq \Xi$ such that

$$[V_0] = [P] \in K_0(C(\mathbb{T}^n)) \quad \text{and} \quad V_0 \subseteq D(V_0),$$

where D is dilation in the frequency domain by an $n \times n$ diagonal matrix A with integer entries all having absolute value greater than 1. We leave details to the reader.

5. BUILDING MODULE FRAMES FROM PROJECTIVE MULTIREOLUTION ANALYSES

In this section, we present a useful algorithm for constructing a module frame for Ξ from a projective multiresolution analysis corresponding to a dilation matrix A .

Before beginning, we fix a bit of notation: if $v \in \mathbb{Z}^n$, recall that the translation operator T_v on $L^2(\mathbb{R}^n)$ is defined by

$$T_v(f)(x) = f(x - v) \quad f \in L^2(\mathbb{R}^n).$$

For any $v \in \mathbb{Z}^n$, let ε_v denote the Fourier transformed version of T_v acting on $L^2(\mathbb{R}^n)$ defined by $\varepsilon_v = \mathcal{F} \circ T_v \circ \mathcal{F}^*$. A standard calculation shows that ε_v is the multiplication operator on $L^2(\mathbb{R}^n)$ defined by

$$\varepsilon_v(\zeta)(t) = e(-v \cdot t)\zeta(t) \quad \zeta \in L^2(\mathbb{R}^n),$$

where recall e is the exponential function defined on \mathbb{R} by $e(r) = e^{2\pi ir}$.

We now present the theorem on the construction of module frames.

THEOREM 5.1. *Let $A \in GL(n, \mathbb{Q})$ be a $n \times n$ dilation matrix with integer entries, let $|\det(A)| = d$, and suppose that $\{V_j\}_{j=0}^\infty$ is a projective multiresolution analysis for Ξ corresponding to dilation by A . Suppose that $\{\Phi_1, \Phi_2, \dots, \Phi_s\}$ is a module frame for V_0 , and that $\{\Psi_1, \Psi_2, \dots, \Psi_r\}$ is a module frame for the finitely generated projective $C(\mathbb{T}^n)$ -module $W_0 = V_1 \ominus V_0$. Then it is possible to enumerate a countably infinite subset $\{v_l : l \geq 0\} \subseteq \mathbb{Z}^n$ such that for each $i \in \mathbb{N} \cup \{0\}$, $\{v_l : 0 \leq l \leq d^i - 1\}$ is a set of coset representatives for $\mathbb{Z}^n / A^i(\mathbb{Z}^n)$. Furthermore, letting D be dilation by A in the frequency domain, for each $i \in \mathbb{N}$,*

$$\{D^i \varepsilon_{v_{k_1}}(\Psi_1), D^i \varepsilon_{v_{k_2}}(\Psi_2), \dots, D^i \varepsilon_{v_{k_r}}(\Psi_r) : 0 \leq k_l \leq d^i - 1, 1 \leq l \leq r\}$$

forms a module frame for the finitely generated projective $C(\mathbb{T}^n)$ -module $W_i = V_{i+1} \ominus V_i$, so that a module frame for all of Ξ is given by the set

$$\{\Phi_1, \Phi_2, \dots, \Phi_s\} \cup \bigcup_{j=0}^\infty \bigcup_{k=1}^r \{D^j \varepsilon_{v_l}(\Psi_k) : 0 \leq l \leq d^j - 1\}.$$

Proof. We start the proof by showing how to construct the appropriate sets of coset representatives, enumerated in a consistent fashion. Let $\{\beta_i : 0 \leq i \leq d - 1\}$ be a set of coset representatives for $\mathbb{Z}^n / A(\mathbb{Z}^n)$ with $\beta_0 = 0$. Fix $b \in \mathbb{N}$. Then it is easy to check that $\left\{ \sum_{i=0}^{b-1} A^i(\beta_{b_i}) : 0 \leq b_i \leq d - 1 \right\}$ gives a family of coset representatives for $\mathbb{Z}^n / A^b(\mathbb{Z}^n)$. The set $\{v_l : 0 \leq l \leq d^i - 1\}$ is given by giving l its d -adic expansion in terms of powers of the matrix A and the coset representatives β_i , i.e. if we write $l = \sum_{i=0}^{b-1} (b_i)d^i$, where $b_i \in \{0, 1, \dots, d - 1\}$, we then set $v_l = \sum_{i=0}^{b-1} A^i(\beta_{b_i})$. One can check that as one passes from the enumeration of coset representatives of $\mathbb{Z}^n / A^b(\mathbb{Z}^n)$ to the enumeration of representatives for $\mathbb{Z}^n / A^{b+1}(\mathbb{Z}^n)$, this definition of v_l is consistent, i.e. it does not depend on b .

We will establish the first claim of the theorem by induction. The statement is true for $i = 0$, by hypothesis. We prove it is true for $i = 1$, as the proof of this fact will serve as a model for the general case. We note that $V_2 = V_1 \oplus W_1$. But also $V_2 = D(V_1) = D(V_0 \oplus W_0)$. One easily checks that

$$D(V_0 \oplus W_0) = D(V_0) \oplus D(W_0) = V_1 \oplus D(W_0).$$

We note that

$$W_1 = D(W_0),$$

since the orthogonal complement of V_1 in V_2 with respect to the Hilbert module inner product must be unique. We know that $\{\Psi_k : 1 \leq k \leq r\}$ is a module frame for W_0 and we want to show that

$$\{D\varepsilon_{\beta_{k_1}}(\Psi_1), D\varepsilon_{v_{k_2}}(\Psi_2), \dots, D\varepsilon_{\beta_{k_r}}(\Psi_r) : 0 \leq k_l \leq d - 1, 1 \leq l \leq r\}$$

is a module frame for V_1 . Fix $\zeta \in W_1 = D(W_0)$. We must show that

$$\zeta(x) = \sum_{k=1}^r \sum_{l=d^j}^{d^{j+1}-1} D\varepsilon_{v_l}(\Psi_k)(x) \langle D\varepsilon_{\beta_l}(\Psi_k), \zeta \rangle_{C(\mathbb{T}^n)}(x),$$

for $\{\beta_l : 0 \leq l \leq d^{j+1} - 1\}$ a set of coset representatives for $\mathbb{Z}^n / A^{j+1}(\mathbb{Z}^n)$, where without loss of generality we choose $\beta_0 = 0$. Since $\zeta \in D(W_0)$, we can write $\zeta = D(\tilde{\zeta})$ for some $\tilde{\zeta} \in W_0$. Using the definition of module frame for W_0 , we get

$$\tilde{\zeta}(x) = \sum_{k=1}^r \Psi_k(x) \langle \Psi_k, \tilde{\zeta} \rangle_{C(\mathbb{T}^n)}(x).$$

Since $\zeta(x) = \frac{1}{\sqrt{d}}\xi([A^t]^{-1}(x))$, to obtain the desired result it is enough to show that

$$\begin{aligned} & \sum_{k=1}^r \sum_{l=0}^{d-1} D\varepsilon_{v_l}(\Psi_k)(x) \langle D\varepsilon_{\beta_l}(\Psi_k), \zeta \rangle_{C(\mathbb{T}^n)}(x) \\ &= \frac{1}{\sqrt{d}} \sum_{k=1}^r \Psi_k([A^t]^{-1}(x)) \langle \Psi_k, \xi \rangle_{C(\mathbb{T}^n)}([A^t]^{-1}(x)). \end{aligned}$$

We now calculate that

$$\begin{aligned} & \sum_{k=1}^r \sum_{l=0}^{d-1} D\varepsilon_{\beta_l}(\Psi_k) \langle D\varepsilon_{\beta_l}(\Psi_k), \zeta \rangle_{C(\mathbb{T}^n)}(x) \\ &= \sum_{k=1}^r \sum_{l=0}^{d-1} \varepsilon_{A^{-1}(\beta_l)} D(\Psi_k)(x) \left\langle \varepsilon_{A^{-1}(\beta_l)} D^1(\Psi_k), \frac{1}{\sqrt{d}} \xi \circ [A^t]^{-1} \right\rangle_{C(\mathbb{T}^n)}(x) \\ &= \frac{1}{\sqrt{d^2}} \sum_{k=1}^r \sum_{l=0}^{d-1} e(-A^{-1}(\beta_l) \cdot x) (\Psi_k)([A^t]^{-1}(x)) \langle \varepsilon_{A^{-1}(\beta_l)} D^{j+1}(\Psi_k), \xi \circ [A^t]^{-1} \rangle \\ &= \frac{1}{d\sqrt{d}} \sum_{k=1}^r \sum_{l=0}^{d-1} e(-A^{-1}(\beta_l) \cdot x) \Psi_k \circ [A^t]^{-1}(x) \\ & \quad \cdot \sum_{v \in \mathbb{Z}^n} \overline{e(-A^{-1}(\beta_l) \cdot (x+v)) \Psi_k \circ [A^t]^{-1}(x+v)} \cdot \xi([A^t]^{-1}(x+v)) \\ &= \frac{1}{d\sqrt{d}} \sum_{k=1}^r \sum_{l=0}^{d-1} \sum_{v \in \mathbb{Z}^n} \sum_{m=0}^{d-1} e(A^{-1}(\beta_l) \cdot [A^t](v) + \gamma_m) \Psi_k \circ [A^t]^{-1}(x) \\ & \quad \cdot \overline{(\Psi_k \circ [A^t]^{-1}(x + [A^t](v) + \gamma_m))} \cdot \xi([A^t]^{-1}(x + [A^t](v) + \gamma_m)), \end{aligned}$$

(where $\{\gamma_m : 0 \leq m \leq d-1\}$ is a set of coset representatives for $\mathbb{Z}^n/[A^t](\mathbb{Z}^n)$ with $\gamma_0 = 0$)

$$\begin{aligned} &= \frac{1}{d\sqrt{d}} \sum_{k=1}^r \sum_{m=0}^{d-1} \sum_{l=0}^{d-1} e(A^{-1}(\beta_l) \cdot \gamma_m) \\ & \quad \cdot \sum_{v \in \mathbb{Z}^n} \Psi_k([A^t]^{-1}(x)) \overline{(\Psi_k \circ [A^t]^{-1}(x + \gamma_m + v))} \cdot \xi([A^t]^{-1}(x + \gamma_m + v)). \end{aligned}$$

By the law of characters on the finite abelian group $\mathbb{Z}^n/[A^t](\mathbb{Z}^n)$,

$$\sum_{l=0}^{d-1} e((A^{-1}\beta_l) \cdot \gamma_m) = 0$$

unless $\gamma_m = 0$, that is, unless $m = 0$, and equals d when $m = 0$, in which case $\beta_0 = \gamma_0 = \vec{0} \in \mathbb{Z}^n$. Our sum therefore becomes

$$\begin{aligned} &= \frac{1}{d\sqrt{d}} \cdot d \sum_{k=1}^r \sum_{v \in \mathbb{Z}^n} \Psi_k([A^t]^{-1}(x)) \overline{\Psi_k([A^t]^{-1}(x) + v)} \cdot \zeta([A^t]^{-1}(x) + v) \\ &= \frac{1}{\sqrt{d}} \sum_{k=1}^r \Psi_k([A^t]^{-1}(x)) \langle \Psi_k, \zeta \rangle_{C(\mathbb{T}^n)}([A^t]^{-1}(x)) \\ &= \frac{1}{\sqrt{d}} \zeta([A^t]^{-1}(x)) = D(\zeta)(x) = \zeta(x), \end{aligned}$$

as desired.

We now suppose that $\{\Psi_{j,1}, \Psi_{j,2}, \dots, \Psi_{j,r_j}\}$ is a module frame for W_j , the orthogonal complement of V_j in V_{j+1} with respect to the Hilbert module $C(\mathbb{T}^n)$ -valued inner product. The same argument used to show that $W_1 = D(W_0)$ shows that $W_{j+1} = D(W_j)$. It is easy now to see that

$$\{D\varepsilon_{\beta_{k_1}}(\Psi_{j,1}), D\varepsilon_{\beta_{k_2}}(\Psi_{j,2}), \dots, D\varepsilon_{\beta_{k_r}}(\Psi_{j,r_j}) : 0 \leq k_l \leq d-1, 1 \leq l \leq r_j\}$$

is a module frame for V_{j+1} . This is because the same argument as for the case $j = 0$ shows that if $\zeta \in W_{j+1} = D(W_j)$, then $\zeta = D(\xi)$ for some $\xi \in W_j$, and

$$\begin{aligned} &\sum_{k=1}^r \sum_{l=0}^{d-1} D\varepsilon_{\beta_l}(\Psi_{j,k})(x) \langle D\varepsilon_{\beta_l}(\Psi_{j,k}), \zeta \rangle_{C(\mathbb{T}^n)}(x) \\ &= \frac{1}{\sqrt{d}} \sum_{k=1}^{r_j} \Psi_{j,k}([A^t]^{-1}(x)) \langle \Psi_{j,k}, \zeta \rangle_{C(\mathbb{T}^n)}([A^t]^{-1}(x)), \end{aligned}$$

from which follows the desired statement about the module frame for W_{j+1} .

We now prove by induction that $\{D^i \varepsilon_{v_l}(\Psi_k) : 0 \leq l \leq d^i - 1, 1 \leq k \leq r\}$ forms a module frame for the finitely generated projective $C(\mathbb{T}^n)$ -module W_i , where here $\{v_l : 0 \leq l \leq d^i - 1\}$ is a set of coset representatives for $\mathbb{Z}^n / A^i(\mathbb{Z}^n)$ as defined in the first paragraph of the proof of the theorem. The statement is true for $i = 0$, by hypothesis, and we now assume it is true for $i = j$, so that $\{D^j \varepsilon_{v_l}(\Psi_k) : 0 \leq l \leq d^j - 1, 1 \leq k \leq r\}$ forms a module frame for the finitely generated projective $C(\mathbb{T}^n)$ -module W_j , where here $\{v_l : 0 \leq l \leq d^j - 1\}$ is a set of coset representatives for $\mathbb{Z}^n / A^j(\mathbb{Z}^n)$. From our remarks above, we know that a module frame for W_{j+1} is given by the set

$$\{D\varepsilon_{\beta_m}(D^j \varepsilon_{v_l}(\Psi_k)) : 0 \leq m \leq d-1, 0 \leq l \leq d^j - 1, 1 \leq k \leq r\}.$$

We now show that this takes on the desired form. We first note the relations

$$\varepsilon_{\beta_m} D^j = D^j \varepsilon_{A^j(\beta_m)} \quad \text{and} \quad \varepsilon_{A^j(\beta_m)} \varepsilon_{v_l} = \varepsilon_{[A^j(\beta_m) + v_l]}.$$

Thus we see that

$$\{D^{j+1} \varepsilon_{[A^j(\beta_m) + v_l]}(\Psi_k) : 0 \leq m \leq d-1, 0 \leq l \leq d^j - 1, 1 \leq k \leq r\}$$

forms a module frame for W_{j+1} . We finally note that as m runs through all the values between 0 and $d-1$, and l goes from 0 to d^j-1 , the set $\{A^j(\beta_m) + v_l\}$ runs through all the coset representatives of $\mathbb{Z}^n / A^{j+1}(\mathbb{Z}^n)$, as explained in our consistent enumeration of coset representatives given in the first paragraph of the proof. Thus

$$\{A^j(\beta_m) + v_l : 0 \leq m \leq d-1, 0 \leq l \leq d^j-1\} = \{v_l : 0 \leq l \leq d^{j+1}-1\},$$

so that

$$\{D^{j+1}\varepsilon_{v_l}(\Psi_k) : 0 \leq l \leq d^{j+1}-1, 1 \leq k \leq r\}$$

forms a module frame for W_{j+1} . The induction step of the first statement of the theorem is done, so we have proved the first statement constructing module frames for W_i for each $i \in \mathbb{N} \cup \{0\}$. The rest of the theorem follows from the fact that we have a projective multiresolution analysis for \mathcal{E} , so that once we know module frames for V_0 and each W_i , the union of these module frames gives a standard module frame for all of \mathcal{E} .

Finally, we remark that if W_0 is a free $C(\mathbb{T}^n)$ -module of rank r , then W_i will be free of rank rd^i . This follows by choosing the module frame for W_0 in such a way that $\langle \Psi_i, \Psi_j \rangle_{C(\mathbb{T}^2)} = \delta_{i,j} 1_{C(\mathbb{T}^2)}$, performing a calculation involving the law of characters to show that

$$\langle D\varepsilon_{v_l}(\Psi_i), D\varepsilon_{v_m}(\Psi_j) \rangle_{C(\mathbb{T}^2)} = \delta_{i,j} \delta_{l,m} 1_{C(\mathbb{T}^n)},$$

so that W_1 is free of rank rd . Using induction exactly as before, we see that W_i is free of rank rd^i for all $i \in \mathbb{N}$. ■

6. CONCLUSION

The methods we have used so far may seem fairly specialized. The question then arises as to how to construct examples beyond the special cases of diagonal matrices with integer entries, and their conjugates by elements of $SL(n, \mathbb{Z})$, and various special ad-hoc dilation matrices, such as $Q = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$, the matrix associated to the quincunx lattice. In the quincunx lattice case, we have been able to construct an initial module V_0 in \mathcal{E} isomorphic to the projective $C(\mathbb{T}^2)$ -module $X(1,1)$ and this leads to a wavelet module W_0 that is a free $C(\mathbb{T}^2)$ -module of rank one. Letting ϕ_1 and ϕ_2 be the module frame for V_0 as constructed following the proof of Proposition 17 in [18], and letting $\{\psi\}$ be a module frame for the singly-generated module W_0 , Theorem 5.1 gives a way to construct a module frame of \mathcal{E} that corresponds to the projective multiresolution analysis.

By using recent constructions of smooth scaling functions in the ordinary multiresolution analysis setting by M. Bownik [4], and M. Bownik and D. Speegle [5], it seems likely that the construction of projection multi-resolution analyses can be generalized to a wider class of dilation matrices of arbitrary dimension.

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JUDITH A. PACKER, DEPARTMENT OF MATHEMATICS, CB 395, UNIVERSITY OF COLORADO, BOULDER, CO, 80309-0395, USA
E-mail address: packer@euclid.colorado.edu

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