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DIFFERENCES OF COMPOSITION OPERATORS ON THE BLOCH SPACES

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ABSTRACT. We study the boundedness and the compactness of the differences of two composition operators on the Bloch and the little Bloch spaces. We prove that the weak compactness of the differences on the little Bloch space is equivalent to the compactness. Moreover we will give attention to the topological structure of the space of composition operators on the Bloch space in the operator topology.

KEYWORDS: *Composition operators, Bloch spaces.*

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1. INTRODUCTION

Throughout this paper let $\mathbb{D}$ be the open unit disc and $\partial\mathbb{D}$ the unit circle. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$. We denote by $S(\mathbb{D})$ the set of analytic self-maps of $\mathbb{D}$. Every self-map φ induces the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$.

We recall that the Bloch space $\mathcal{B}$ consists of all $f \in H(\mathbb{D})$ such that

$$|||f||| = \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

Then $|||\cdot|||$ is a complete semi-norm on $\mathcal{B}$ and is Möbius invariant. Let the little Bloch space $\mathcal{B}_o$ denote the subspace of $\mathcal{B}$ consisting of functions f with

$$\lim_{|z|\to 1}(1 - |z|^2)f'(z) = 0.$$

It is well known that $\mathcal{B}$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}} = |f(0)| + |||f|||$$

and that $\mathcal{B}_o$ is a closed subspace of $\mathcal{B}$. In particular, $\mathcal{B}_o$ is the closure in $\mathcal{B}$ of the polynomials. Let $H^\infty = H^\infty(\mathbb{D})$ be the set of all bounded analytic functions on $\mathbb{D}$. Then H^∞ is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

Note that $H^\infty \subset \mathcal{B}$ and that $\|\!|f|\!\| \leqslant \|f\|_\infty$ if $f \in H^\infty$. For $\varphi \in S(\mathbb{D})$, $\|\!|\varphi|\!\| \leqslant \|\varphi\|_\infty \leqslant 1$.

Madigan and Matheson [7] shed light on the study of composition operators on $\mathcal{B}$ and $\mathcal{B}_o$ and Montes-Rodríguez [8] expressed their norms. On the other hand, MacCluer, Zhao and the second author [6] considered the topological structure of the space of composition operators on H^∞ that was originally studied by Sundberg and Shapiro [10] in the case of the Hilbert Hardy space. They gave a relationship between such a problem and the boundedness and compactness of the difference $C_\varphi - C_\psi$ of two composition operators from $\mathcal{B}$ to H^∞. Explicitly they showed that the compactness of $C_\varphi - C_\psi : H^\infty \to H^\infty$ is equivalent to the compactness of $C_\varphi - C_\psi$ acting from $\mathcal{B}$ to H^∞ and moreover that C_φ and C_ψ are in the same path component of the space of composition operators on H^∞ if and only if $C_\varphi - C_\psi : \mathcal{B} \to H^\infty$ is bounded. Also see [4] for more results. So we wonder how $C_\varphi - C_\psi$ would be acting on $\mathcal{B}$. This is our outset. In this paper we will consider the differences of composition operators on $\mathcal{B}$ and $\mathcal{B}_o$. In Section 2, we have the inequality estimating the differences of two Bloch-type derivatives which would be a useful tool to obtain our main results. In Section 3 we characterize the boundedness and the compactness of the differences of two composition operators on $\mathcal{B}$ and $\mathcal{B}_o$. Moreover we can show that the weakly compact difference $C_\varphi - C_\psi$ is compact on $\mathcal{B}_o$, using the interpolation result in the Bloch space (see [7]). Finally in Section 4 we will present some comments to study the topological structure of the space of composition operators on the Bloch space in the operator topology.

Our results involve the pseudo-hyperbolic metric. For $p \in \mathbb{D}$, let α_p be the automorphism of $\mathbb{D}$ exchanging 0 for p. Then α_p has the following form:

$$\alpha_p(z) = \frac{p-z}{1-\overline{p}z}.$$

For z, w in $\mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = |\alpha_z(w)| = \left|\frac{z-w}{1-\overline{z}w}\right|.$$

We will also use the hyperbolic metric, which is given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+\rho(z,w)}{1-\rho(z,w)}.$$

For $\varphi \in S(\mathbb{D})$, the Schwarz-Pick type derivative $\varphi^\#$ of φ is defined by

$$\varphi^\#(z) = \frac{1-|z|^2}{1-|\varphi(z)|^2}\varphi'(z). \tag{1.1}$$

By the Schwarz-Pick lemma, $|\varphi^{\#}(z)| \leqslant 1$ on $\mathbb{D}$. If φ is an automorphism of $\mathbb{D}$, the equality holds for all $z \in \mathbb{D}$ and then $\|\varphi^{\#}\|_\infty = 1$.

Let $U_{\mathcal{B}}$ be the closed unit ball of $\mathcal{B}$ with respect to the norm $\|\cdot\|_{\mathcal{B}}$ and

$$V = \{f \in \mathcal{B} : \|\!|f|\!\| \leqslant 1\}.$$

We collect here some basic properties of functions in $\mathcal{B}$ and composition operators on $\mathcal{B}$. It is known that the following hold (see [1], [9] and [11]): for $z, w \in \mathbb{D}$

$$\beta(z,w) = \sup_{f\in V} |f(z) - f(w)|, \tag{1.2}$$

$$\|\!|f|\!\| = \sup_{z\neq w} \frac{|f(z) - f(w)|}{\beta(z,w)}, \tag{1.3}$$

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\overline{z}w|^2} = 1 - \rho(z,w)^2. \tag{1.4}$$

Moreover we have

$$\sup_{f\in U_{\mathcal{B}}} (1-|w|^2)|f'(w)| = \sup_{f\in V}(1-|w|^2)|f'(w)| = 1, \tag{1.5}$$

$$\begin{aligned} \|\!|C_\varphi f|\!\| &= \sup_{z\in\mathbb{D}}(1-|z|^2)|\varphi'(z)||f'(\varphi(z))| \\ &= \sup_{z\in\mathbb{D}} |\varphi^{\#}(z)|(1-|\varphi(z)|^2)|f'(\varphi(z))|. \end{aligned} \tag{1.6}$$

2. PREREQUISITES

To investigate the behaviors of $C_\varphi - C_\psi$, we will consider the following induced distance on $\mathbb{D}$.

DEFINITION 2.1. For $z, w \in \mathbb{D}$, we define

$$\flat(z,w) = \sup_{f\in V} |(1-|z|^2)f'(z) - (1-|w|^2)f'(w)|. \tag{2.1}$$

We prove here the following useful result in the next section.

PROPOSITION 2.2. *There exists a constant $C > 0$ such that*

$$\rho(z,w)^2 \leqslant \flat(z,w) \leqslant C\rho(z,w) \tag{2.2}$$

for all $z, w \in \mathbb{D}$.

Proof. For $p \in \mathbb{D}$, the automorphism α_p is an element of V and

$$\alpha_p'(z) = -\frac{1-|p|^2}{(1-\overline{p}z)^2}.$$

So, for $p,q\in\mathbb{D}$, we have that

$$\flat(p,q)\geqslant|(1-|p|^2)\alpha_p'(p)-(1-|q|^2)\alpha'(q)|\geqslant 1-\frac{(1-|p|^2)(1-|q|^2)}{|1-\overline{p}q|^2}=\rho(p,q)^2.$$

Next we will prove that for $s\in\mathbb{D}$

$$\flat(s,0)\leqslant 16\rho(s,0). \tag{2.3}$$

For $f\in V$, we have that

$$\begin{aligned}|(1-|s|^2)f'(s)-f'(0)|&=\Big|\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}\{(1-|s|^2t^2)f'(st)\}\mathrm{d}t\Big|\\&\leqslant\int_0^1 2|s|^2t|f'(st)|\mathrm{d}t+\int_0^1|s|(1-|s|^2t^2)|f''(st)|\mathrm{d}t.\end{aligned}$$

Here by 4.2.1 in [11],

$$f'(z)=2\int_{\mathbb{D}}\frac{(1-|w|^2)^2f'(w)}{(1-z\overline{w})^3}\mathrm{d}A(w)$$

where $\mathrm{d}A(w)$ is the normalized area measure on $\mathbb{D}$. Differentiating this equality with respect to a variable z,

$$\begin{aligned}\sup_{z\in\mathbb{D}}(1-|z|^2)^2|f''(z)|&\leqslant\sup_{z\in\mathbb{D}}3!\,(1-|z|^2)^2\int_{\mathbb{D}}\frac{\|\!|f|\!\|}{|1-z\overline{w}|^4}\,\mathrm{d}A(w)\\&=\sup_{z\in\mathbb{D}}3!\,(1-|z|^2)^2\,\frac{\|\!|f|\!\|}{(1-|z|^2)^2}=6\,\|\!|f|\!\|.\end{aligned}\tag{2.4}$$

Hence we have that

$$\begin{aligned}\flat(s,0)&\leqslant\sup_{f\in V}\Big(\int_0^1\frac{2|s|(1-|s|^2t^2)|f'(st)|}{1-|s|^2t^2}\mathrm{d}t+\int_0^1\frac{|s|(1-|s|^2t^2)^2|f''(st)|}{1-|s|^2t^2}\mathrm{d}t\Big)\\&\leqslant\sup_{f\in V}(2+6)\|\!|f|\!\|\int_0^1\frac{|s|\mathrm{d}t}{1-|s|^2t^2}\leqslant 8\,\beta(s,0).\end{aligned}$$

Moreover if $\rho(s,0)\leqslant 1/2$, then $\beta(s,0)\leqslant 2\rho(s,0)$. If $\rho(s,0)>1/2$, then we have that

$$\flat(s,0)\leqslant 2<4\rho(s,0).$$

Anyway we obtain the inequality (2.3),

$$\flat(s,0)\leqslant 16\rho(s,0).$$

To complete this proof, note that

$$(f\circ\alpha_q)'(z)=-\frac{1-|q|^2}{(1-\overline{q}z)^2}f'(\alpha_q(z)).$$

Then we have that by (1.4) and putting $p = \alpha_q(s)$

$$
\begin{aligned}
&|(1-|p|^2)f'(p) - (1-|q|^2)f'(q)| \\
&= |(1-|\alpha_q(s)|^2)f'(\alpha_q(s)) - (1-|q|^2)f'(\alpha_q(0))| \\
&= \Big| \frac{(1-|q|^2)(1-|s|^2)}{|1-\overline{q}s|^2} \frac{(1-\overline{q}s)^2}{1-|q|^2}(f\circ\alpha_q)'(s) - (1-|q|^2)\frac{1}{1-|q|^2}(f\circ\alpha_q)'(0)\Big| \\
&= \Big| \frac{(1-\overline{q}s)^2}{|1-\overline{q}s|^2}(1-|s|^2)(f\circ\alpha_q)'(s) - (f\circ\alpha_q)'(0)\Big| \\
&\leqslant |(1-|s|^2)(f\circ\alpha_q)'(s) - (f\circ\alpha_q)'(0)| + \Big|\frac{(1-\overline{q}s)^2}{|1-\overline{q}s|^2} - 1\Big|(1-|s|^2)|(f\circ\alpha_q)'(s)|.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\flat(p,q) &\leqslant \sup_{f\in V} |(1-|s|^2)(f\circ\alpha_q)'(s) - (f\circ\alpha_q)'(0)| \\
&\quad + \Big|\frac{(1-\overline{q}s)^2}{|1-\overline{q}s|^2} - 1\Big| \sup_{f\in V}(1-|s|^2)|(f\circ\alpha_q)'(s)| \\
&\leqslant 18\rho(s,0) + \frac{|p\overline{q}-\overline{p}q|}{|1-\overline{q}p|} \leqslant 18\rho(p,q) + 2\rho(p,q) = 20\rho(p,q). \quad \blacksquare
\end{aligned}
$$

3. DIFFERENCES OF COMPOSITION OPERATORS

In this section, we consider the behaviors of $C_\varphi - C_\psi$ on the Bloch and the little Bloch spaces.

At first we study the Bloch space case. Then $C_\varphi - C_\psi$ is always bounded on $\mathcal{B}$. So we consider the compactness of $C_\varphi - C_\psi$. It is easy to prove the next lemma by adapting the proof of Proposition 3.11 in [1].

LEMMA 3.1. *Let φ and ψ be in $S(\mathbb{D})$. Then the following are equivalent:*

(i) $C_\varphi - C_\psi$ *is compact on* $\mathcal{B}$.

(ii) $\|(C_\varphi - C_\psi)f_n\|_{\mathcal{B}} \to 0$ *for any bounded sequence* $\{f_n\}$ *in* $\mathcal{B}$ *that converges to* 0 *uniformly on every compact subset of* $\mathbb{D}$.

(iii) $|||(C_\varphi - C_\psi)f_n||| \to 0$ *for any sequence* $\{f_n\}$ *as in* (ii).

To discuss the compactness on $\mathcal{B}$, we define here by $\Gamma(\varphi)$ the set of sequences $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$. Moreover we also denote by $\Gamma^\#(\varphi)$ the set of sequences $\{z_n\}$ in $\mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and $\varphi^\#(z_n) \not\to 0$. Then it is clear that $\Gamma^\#(\varphi) \subset \Gamma(\varphi)$.

It is well known that C_φ is compact on H^∞ if and only if $\|\varphi\|_\infty < 1$ if and only if $\Gamma(\varphi) = \emptyset$. In [7], it is shown that C_φ is compact on $\mathcal{B}$ if and only if $\Gamma^\#(\varphi) = \emptyset$.

Now we can characterize the compactness of $C_\varphi - C_\psi$ on $\mathcal{B}$.

THEOREM 3.2. *Let φ and ψ be in $S(\mathbb{D})$. Suppose that neither C_φ nor C_ψ is compact on $\mathcal{B}$. Then the following are equivalent:*

(i) *$C_\varphi - C_\psi$ is compact on $\mathcal{B}$.*

(ii) *Both* (a) *and* (b) *hold:*

(a) *$\Gamma^\#(\varphi) = \Gamma^\#(\psi) \neq \emptyset$. Then $\Gamma^\#(\varphi) \subset \Gamma(\varphi) \cap \Gamma(\psi)$.*

(b) *For $\{z_n\} \in \Gamma(\varphi) \cap \Gamma(\psi)$, $\lim_{n\to\infty} |\varphi^\#(z_n) - \psi^\#(z_n)| = 0$ and*

$$\lim_{n\to\infty} \varphi^\#(z_n)\rho(\varphi(z_n),\psi(z_n)) = \lim_{n\to\infty} \psi^\#(z_n)\rho(\varphi(z_n),\psi(z_n)) = 0.$$

(iii) $\lim_{|\lambda|\to 1} \|(C_\varphi - C_\psi)\alpha_\lambda\| = 0$ *and* $\lim_{|\lambda|\to 1} \|(C_\varphi - C_\psi)(\alpha_\lambda)^2\| = 0$.

Proof. The implication (i) $\Rightarrow$ (iii) can be shown by applying Lemma 3.1 instead to $f_\lambda = \alpha_\lambda - \lambda$ and $g_\lambda = \alpha_\lambda(\alpha_\lambda - \lambda)$.

Suppose that the condition (iii) holds. Since C_φ is not compact on $\mathcal{B}$, there exists a sequence $\{z_n\} \in \Gamma^\#(\varphi)$ that is satisfying $|\varphi(z_n)| \to 1$ and $\varphi^\#(z_n) \not\to 0$. For such the sequence $\{\varphi(z_n)\}$, we have that

$$\begin{aligned}
\|(C_\varphi - C_\psi)\alpha_{\varphi(z_n)}\| &\geqslant (1-|z_n|^2)|((C_\varphi - C_\psi)\alpha_{\varphi(z_n)})'(z_n)| \\
&= (1-|z_n|^2)\left|\frac{\varphi'(z_n)}{1-|\varphi(z_n)|^2} - \frac{\psi'(z_n)(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right| \\
&\geqslant \left|\varphi^\#(z_n) - \frac{(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\psi^\#(z_n)\right| \\
&\geqslant \big|\,|\varphi^\#(z_n)| - (1-\rho(\varphi(z_n),\psi(z_n))^2)|\psi^\#(z_n)|\,\big|
\end{aligned}$$

and

$$\begin{aligned}
\|(C_\varphi - C_\psi)(\alpha_{\varphi(z_n)})^2\| &\geqslant (1-|z_n|^2)|((C_\varphi - C_\psi)(\alpha_{\varphi(z_n)})^2)'(z_n)| \\
&\geqslant 2(1-\rho(\varphi(z_n),\psi(z_n))^2)|\psi^\#(z_n)|\rho(\varphi(z_n),\psi(z_n)).
\end{aligned}$$

By the condition (iii),

$$\lim_{n\to\infty} \big|\,|\varphi^\#(z_n)| - (1-\rho(\varphi(z_n),\psi(z_n))^2|\psi^\#(z_n)|\,\big| = 0, \tag{3.1}$$

$$\lim_{n\to\infty} (1-\rho(\varphi(z_n),\psi(z_n))^2)|\psi^\#(z_n)|\rho(\varphi(z_n),\psi(z_n)) = 0. \tag{3.2}$$

By the assumption that $\varphi^\#(z_n) \not\to 0$ and (3.1),

$$\lim_{n\to\infty} (1-\rho(\varphi(z_n),\psi(z_n))^2)|\psi^\#(z_n)| \neq 0.$$

Thus by (3.2), we have that

$$\lim_{n\to\infty} \rho(\varphi(z_n),\psi(z_n)) = 0 \tag{3.3}$$

for any sequence $\{z_n\}$ such that $\varphi^\#(z_n) \not\to 0$. Consequently for any sequence $\{z_n\}$ such that $|\varphi(z_n)| \to 1$,

$$\lim_{|\varphi(z_n)|\to 1} \varphi^\#(z_n)\rho(\varphi(z_n),\psi(z_n)) = 0. \tag{3.4}$$

The same is true with the role of φ and ψ interchanged, noticing that (3.3) implies $\lim_{|\varphi(z_n)|\to 1} |\varphi(z_n) - \psi(z_n)| = 0$ for any $\{z_n\} \in \Gamma^{\#}(\varphi)$.

Considering again the estimation of $|||(C_\varphi - C_\psi)\alpha_{\varphi(z_n)}|||$ for any sequence $\{z_n\}$ such that $\varphi^{\#}(z_n) \not\to 0$,

$$
\begin{aligned}
|||(C_\varphi - C_\psi)\alpha_{\varphi(z_n)}||| &\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| - |\psi^{\#}(z_n)| \left| 1 - \frac{(1-|\varphi(z_n)|^2)(1-|\psi(z_n)|^2)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2} \right| \\
&\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| - |\psi^{\#}(z_n)| \, |(1-|\varphi(z_n)|^2)\alpha'_{\varphi(z_n)}(\varphi(z_n)) \\
&\qquad - (1-|\psi(z_n)|^2)\alpha'_{\varphi(z_n)}(\psi(z_n))| \\
&\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| - |\psi^{\#}(z_n)| \flat(\varphi(z_n), \psi(z_n)) \\
&\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| - C|\psi^{\#}(z_n)| \rho(\varphi(z_n), \psi(z_n)).
\end{aligned}
$$

The last inequality follows from Proposition 2.2. Hence by (3.3), we get

$$
\lim_{n\to\infty} |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| = 0 \tag{3.5}
$$

for any sequence $\{z_n\}$ such that $\varphi^{\#}(z_n) \not\to 0$.

For any sequence $\{z_n\}$ such that $|\varphi(z_n)| \to 1$, $|\psi(z_n)| \to 1$ and $\varphi^{\#}(z_n) \to 0$, we will use

$$
\lim_{|\psi(z_n)|\to 1} \psi^{\#}(z_n)\rho(\varphi(z_n), \psi(z_n)) = 0
$$

and so obtain

$$
\lim_{n\to\infty} \varphi^{\#}(z_n) = \lim_{n\to\infty} \psi^{\#}(z_n) = 0.
$$

Consequently we get (b).

Furthermore, from (3.3) and (3.5), we have that if $\{z_n\}$ is in $\Gamma^{\#}(\varphi)$, then $|\psi(z_n)| \to 1$ and $\psi^{\#}(z_n) \not\to 0$. This means that $\Gamma^{\#}(\varphi) \subset \Gamma^{\#}(\psi)$. Similarly we can get $\Gamma^{\#}(\psi) \subset \Gamma^{\#}(\varphi)$ and this implies that $\Gamma^{\#}(\varphi) = \Gamma^{\#}(\psi)$. Recall that $\Gamma^{\#}(\varphi) \subset \Gamma(\varphi)$ and $\Gamma^{\#}(\psi) \subset \Gamma(\psi)$. Then we have that $\Gamma^{\#}(\varphi) \subset \Gamma(\varphi) \cap \Gamma(\psi)$.

Finally assume that (ii) holds. Let $\{f_n\}$ be a sequence in $\mathcal{B}$ such that $|||f_n||| \leqslant 1$ and f_n converges to 0 uniformly on every compact subset of $\mathbb{D}$. To prove that $|||(C_\varphi - C_\psi)f_n||| \to 0$, suppose not. We may assume that for some $\varepsilon > 0$, $|||(C_\varphi - C_\psi)f_n||| > \varepsilon$ for all n. Then for each n, there exists a sequence $\{z_n\} \subset \mathbb{D}$ such that

$$
|\varphi^{\#}(z_n)(1-|\varphi(z_n)|^2)f_n'(\varphi(z_n)) - \psi^{\#}(z_n)(1-|\psi(z_n)|^2)f_n'(\psi(z_n))| > \varepsilon. \tag{3.6}
$$

This implies that either $|\varphi(z_n)|$ or $|\psi(z_n)|$ tends to 1. Suppose that $|\varphi(z_n)| \to 1$ and $\psi(z_n) \to \omega$. If $|\omega| < 1$, then $\{z_n\}$ is not in $\Gamma(\varphi) \cap \Gamma(\psi)$. By the condition (a), we have $\varphi^{\#}(z_n) \to 0$. On the other hand, $|\omega| < 1$ implies that $f_n'(\psi(z_n)) \to 0$. This contradicts (3.6). Hence we obtain $|\omega| = 1$. This means that $|\varphi(z_n)|, |\psi(z_n)| \to 1$. Then by the conditions of (b), we have that

$$
\begin{aligned}
&|\varphi^{\#}(z_n)(1-|\varphi(z_n)|^2)f_n'(\varphi(z_n)) - \psi^{\#}(z_n)(1-|\psi(z_n)|^2)f_n'(\psi(z_n))| \\
&\qquad \leqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| |||f_n||| + C|\psi^{\#}(z_n)|\rho(\varphi(z_n), \psi(z_n)) \to 0.
\end{aligned}
$$

This contradicts (3.6). Thus we finish the proof. ∎

EXAMPLE 3.3. We present analytic self-maps of $\mathbb{D}$ such that neither C_φ nor C_ψ is compact on $\mathcal{B}$ but $C_\varphi - C_\psi$ is compact on $\mathcal{B}$.

(i) Let $\varphi(z) = (z+1)/2$ and $\psi(z) = \varphi(z) + t(z-1)^3$ for small t.

(ii) Let $\sigma(z) = (1+z)/(1-z)$ and $\varphi(z) = (\sigma(z)^{1/2} - 1)/(\sigma(z)^{1/2} + 1)$ be a lens map. And let $\psi(z) = 1 - \sqrt{2(1-z)}$.

These functions satisfy the conditions of Theorem 3.2.

Now we study the little Bloch space case. At first we characterize the boundedness of $C_\varphi - C_\psi$ on $\mathcal{B}_o$.

THEOREM 3.4. *Let φ and ψ be in $S(\mathbb{D})$. Then the following are equivalent:*

(i) $C_\varphi - C_\psi$ *is bounded on* $\mathcal{B}_o$.

(ii) *Both* (a) *and (b) hold:*

(a) $\varphi - \psi \in \mathcal{B}_o$.

(b) $\lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\varphi'(z)| = 0$ *and* $\lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\psi'(z)|=0$.

(iii) $\varphi - \psi \in \mathcal{B}_o$ *and* $\varphi^2 - \psi^2 \in \mathcal{B}_o$.

Proof. Suppose that $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$, that is, $(C_\varphi - C_\psi)f \in \mathcal{B}_o$ for any $f \in \mathcal{B}_o$. Taking f as the identity function and $f(z) = z^2$, we have $\varphi - \psi \in \mathcal{B}_o$ and $\varphi^2 - \psi^2 \in \mathcal{B}_o$. Then the implication (i) ⇒ (iii) holds.

Next suppose the condition (iii) is true. As $\varphi^2 - \psi^2 = (\varphi - \psi)(\varphi + \psi) \in \mathcal{B}_o$, we have

$$\lim_{|z|\to 1}(1-|z|^2)|(\varphi'(z)-\psi'(z))(\varphi(z)+\psi(z)) + (\varphi(z)-\psi(z))(\varphi'(z)+\psi'(z))| = 0.$$

Since $\varphi - \psi \in \mathcal{B}_o$, we obtain

$$\lim_{|z|\to 1}(1-|z|^2)|(\varphi(z)-\psi(z))(\varphi'(z)+\psi'(z))| = 0.$$

Thus

$$\begin{aligned}
&(1-|z|^2)|\varphi(z)-\psi(z)||\varphi'(z)| \\
&\qquad = (1-|z|^2)|\varphi(z)-\psi(z)|\left|\frac{\varphi'(z)+\psi'(z)}{2} + \frac{\varphi'(z)-\psi'(z)}{2}\right| \\
&\qquad \leqslant 2^{-1}\{(1-|z|^2)|(\varphi(z)-\psi(z))(\varphi'(z)+\psi'(z))| \\
&\qquad\qquad + (1-|z|^2)|(\varphi(z)-\psi(z))(\varphi'(z)-\psi'(z))|\}.
\end{aligned}$$

So by the above estimate we have

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\psi'(z)| = 0.$$

Similarly we obtain

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\psi'(z)| = 0.$$

So the condition (ii) holds.

Finally we will show the implication (ii) $\Rightarrow$ (i). Let $f_n(z) = z^n$. Then $f_n \in \mathcal{B}_o$ and

$$(C_\varphi - C_\psi)f_n(z) = (\varphi(z) - \psi(z))(\varphi^{n-1}(z) + \cdots + \psi^{n-1}(z)).$$

So

$$\begin{aligned}
&(1-|z|^2)|((C_\varphi - C_\psi)f_n)'(z)| \\
&\leqslant (1-|z|^2)|\varphi'(z) - \psi'(z)||\varphi^{n-1}(z) + \cdots + \psi^{n-1}(z)| \\
&\quad + (1-|z|^2)|\varphi(z) - \psi(z)||(n-1)\varphi^{n-2}(z)\varphi'(z) + \cdots + (n-1)\psi^{n-2}(z)\psi'(z)| \\
&\leqslant n(1-|z|^2)|\varphi'(z) - \psi'(z)| + \frac{n(n-1)}{2}(1-|z|^2)|\varphi(z) - \psi(z)|(|\varphi'(z)| + |\psi'(z)|).
\end{aligned}$$

Thus by the condition (ii), we have that $(C_\varphi - C_\psi)f_n \in \mathcal{B}_o$. Hence $(C_\varphi - C_\psi)p \in \mathcal{B}_o$ for any polynomial p. As the set of polynomials is dense in $\mathcal{B}_o$ and $C_\varphi - C_\psi$ is bounded on $\mathcal{B}$, we can prove that $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$. ∎

We present here analytic self-maps φ and ψ of $\mathbb{D}$ such that neither C_φ nor C_φ is bounded on $\mathcal{B}_o$ but $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$.

EXAMPLE 3.5. Let $S(z) = \exp(-(1+z)/(1-z))$ be a singular inner function. It is known that S is a function in H^∞ that is not in $\mathcal{B}_o$. Let $p(z) = (z-1)/2$. And we put $\varphi = (S+p)/2$ and $\psi = (S-p)/2$. Then we can check these functions have the condition (ii) of Theorem 3.4.

Next we characterize the compactness of $C_\varphi - C_\psi$ on $\mathcal{B}_o$.

THEOREM 3.6. *Let φ and ψ be in $S(\mathbb{D})$. Then the following are equivalent:*

(i) *$C_\varphi - C_\psi$ is compact on $\mathcal{B}_o$.*

(ii) *Both* (a) *and* (b) *hold:*

(a) $\lim_{|z|\to 1} \varphi^\#(z)\rho(\varphi(z),\psi(z)) = \lim_{|z|\to 1} \psi^\#(z)\rho(\varphi(z),\psi(z)) = 0.$

(b) $\lim_{|z|\to 1} \left|\varphi^\#(z) - \psi^\#(z)\right| = 0.$

Proof. Suppose that the conditions (a) and (b) hold. Then we can check that $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$. Indeed, we have

$$\begin{aligned}
&(1-|z|^2)|\varphi'(z) - \psi'(z)| \\
&\qquad = (1-|z|^2)\left|\frac{1-|\varphi(z)|^2}{1-|\varphi(z)|^2}\varphi'(z) - \frac{1-|\psi(z)|^2}{1-|\psi(z)|^2}\psi'(z)\right| \\
&\qquad \leqslant |\varphi^\#(z) - \psi^\#(z)| + |\psi(z)|^2|\varphi^\#(z) - \psi^\#(z)| + |\,|\varphi(z)|^2 - |\psi(z)|^2||\varphi^\#(z)| \\
&\qquad \leqslant 2|\varphi^\#(z) - \psi^\#(z)| + 2\rho(\varphi(z),\psi(z))|\varphi^\#(z)|.
\end{aligned}$$

So the conditions (a) and (b) imply that $\varphi - \psi \in \mathcal{B}_o$.

Moreover we have

$$(1-|z|^2)|\varphi(z)-\psi(z)||\varphi'(z)| \leqslant 2\rho(\varphi(z),\psi(z))|\varphi^{\#}(z)|(1-|\varphi(z)|^2)$$
$$(1-|z|^2)|\varphi(z)-\psi(z)||\psi'(z)| \leqslant 2\rho(\varphi(z),\psi(z))|\psi^{\#}(z)|(1-|\psi(z)|^2).$$

By the condition (a),

$$\lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\varphi'(z)| = \lim_{|z|\to 1}(1-|z|^2)|\varphi(z)-\psi(z)||\psi'(z)| = 0.$$

Thus, by Theorem 3.4, $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$.

Here we recall that $C_\varphi - C_\psi$ is compact on $\mathcal{B}_o$ if and only if

$$\lim_{|z|\to 1}\sup\{(1-|z|^2)|((C_\varphi - C_\psi)f)'(z)| : f \in \mathcal{B}_o, \|f\|_{\mathcal{B}} \leqslant 1\} = 0 \tag{3.7}$$

by Lemma 1 of [7].

For $f \in \mathcal{B}$ with $\|f\|_{\mathcal{B}} \leqslant 1$, we have

$$\begin{aligned}
&(1-|z|^2)|((C_\varphi - C_\psi)f)'(z)| \\
&\qquad \leqslant |\varphi^{\#}(z)(1-|\varphi(z)|^2)f'(\varphi(z)) - \psi^{\#}(z)(1-|\psi(z)|^2)f'(\psi(z))| \\
&\qquad \leqslant |\varphi^{\#}(z)-\psi^{\#}(z)|(1-|\psi(z)|^2)|f'(\psi(z))| \\
&\qquad\qquad + |\varphi^{\#}(z)||(1-|\varphi(z)|^2)f'(\varphi(z)) - (1-|\psi(z)|^2)f'(\psi(z))| \\
&\qquad \leqslant |\varphi^{\#}(z)-\psi^{\#}(z)| + |\varphi^{\#}(z)|\flat(\varphi(z),\psi(z)) \\
&\qquad \leqslant |\varphi^{\#}(z)-\psi^{\#}(z)| + C|\varphi^{\#}(z)|\rho(\varphi(z),\psi(z)).
\end{aligned}$$

Hence by the conditions (a) and (b), we obtain (3.7) and so $C_\varphi - C_\psi$ is compact on $\mathcal{B}_o$.

Conversely suppose that $C_\varphi - C_\psi$ is compact on $\mathcal{B}_o$. For a sequence $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1$, take

$$f_n(z) = 2^{-1}\alpha_{\varphi(z_n)}(z) = 2^{-1}\frac{\varphi(z_n)-z}{1-\overline{\varphi(z_n)}z}.$$

Then $f_n \in \mathcal{B}_o$ and $\|f_n\|_{\mathcal{B}} = 1$. So

$$\begin{aligned}
(1-|z_n|^2)|((C_\varphi - C_\psi)f_n)'(z_n)| &\geqslant 2^{-1}(1-|z_n|^2)\left|\frac{\varphi'(z_n)}{1-|\varphi(z_n)|^2} - \frac{\psi'(z_n)(1-|\varphi(z_n)|^2)}{(1-\overline{\varphi(z_n)}\psi(z_n))^2}\right| \\
&\geqslant 2^{-1}|\,|\varphi^{\#}(z_n)| - |\psi^{\#}(z_n)|(1-\rho(\varphi(z_n),\psi(z_n))^2)|.
\end{aligned}$$

Thus

$$\lim_{|z_n|\to 1}|\,|\varphi^{\#}(z_n)| - |\psi^{\#}(z_n)|(1-\rho(\varphi(z_n),\psi(z_n))^2)| = 0. \tag{3.8}$$

Next for the same sequence $\{z_n\} \subset \mathbb{D}$ as above, put

$$g_n(z) = 4^{-1}(\alpha_{\varphi(z_n)}(z))^2.$$

Then $g_n \in \mathcal{B}_o$ and $\|g_n\|_{\mathcal{B}} = 1$. So

$$(1-|z_n|^2)|((C_\varphi - C_\psi)g_n)'(z_n)| \geqslant 4^{-1}|\psi^{\#}(z_n)|(1-\rho(\varphi(z_n),\psi(z_n))^2)\rho(\varphi(z_n),\psi(z_n)).$$

Thus we have

$$\lim_{|z_n|\to 1} |\psi^{\#}(z_n)|\rho(\varphi(z_n),\psi(z_n))(1-\rho(\varphi(z_n),\psi(z_n))^2) = 0, \tag{3.9}$$

and the same is of course true with the roles of φ and ψ interchanged. Now if $\rho(\varphi(z_n),\psi(z_n))|\varphi^{\#}(z_n)|$ does not approach 0 as $|z_n| \to 1$, then $1-\rho(\varphi(z_n),\psi(z_n))^2$ approaches 0 by (3.9) with φ playing the role of ψ, so by (3.8) we would have $\varphi^{\#}(z_n) \to 0$. This is a contradiction. Hence $\rho(\varphi(z_n),\psi(z_n))|\varphi^{\#}(z_n)| \to 0$ as $|z_n| \to 1$, and by symmetry $\rho(\varphi(z_n),\psi(z_n))|\psi^{\#}(z_n)| \to 0$. And so the condition (a) is true.

Next we take the functions

$$h_n(z) = 2^{-1}\alpha_{\psi(z_n)}(z) = 2^{-1}\frac{\psi(z_n)-z}{1-\overline{\psi(z_n)}z}$$

for a sequence $\{z_n\} \subset \mathbb{D}$ such that $|z_n| \to 1$ and consider the similar estimation as in the proof of Theorem 3.2. So we can obtain

$$(1-|z_n|^2)|((C_\varphi - C_\psi)h_n)'(z_n)| \geqslant 2^{-1}|\varphi^{\#}(z_n) - \psi^{\#}(z_n)| - C|\varphi^{\#}(z_n)|\rho(\varphi(z_n),\psi(z_n)).$$

By the condition (a), the second term of the right-side in the above inequality tends to 0 as $|z_n| \to 1$. Consequently

$$\lim_{|z_n|\to 1} |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| = 0$$

and the condition (b) also holds. ∎

REMARK 3.7. As we check the conditions that characterize the compactness, we can obtain the following. Suppose that $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$. Then the compactness of $C_\varphi - C_\psi$ on $\mathcal{B}$ is equivalent to the compactness on $\mathcal{B}_o$.

Next we consider weakly compactness of differences on $\mathcal{B}_o$ and can show the following using the interpolation result in the Bloch space (see [7]).

THEOREM 3.8. *Let $\varphi,\psi \in S(\mathbb{D})$ and suppose that $C_\varphi - C_\psi$ is bounded on $\mathcal{B}_o$. Then if $C_\varphi - C_\psi$ is weakly compact on $\mathcal{B}_o$, it is compact on $\mathcal{B}_o$.*

Proof. By the Gantmacher's theorem (see [2]), $C_\varphi - C_\psi$ is weakly compact on $\mathcal{B}_o$ if and only if $(C_\varphi - C_\psi)f \in \mathcal{B}_o$ for every $f \in \mathcal{B}$.

We suppose that $C_\varphi - C_\psi$ is not compact on $\mathcal{B}_o$, that is, there is a sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and a constant $\delta \in (0,1)$ such that:

(a) $\lim_{n\to\infty} |\varphi^{\#}(z_n)|\rho(\varphi(z_n),\psi(z_n)) \geqslant \delta$,

or

(b) $\lim_{n\to\infty} |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| \geqslant \delta$.

At first we discuss the case (a). Then we have $|\varphi(z_n)| \to 1$. Indeed, assume that $\varphi(z_n) \to w \in \mathbb{D}$. Then by the boundedness of $C_\varphi - C_\psi$ on $\mathcal{B}_o$ and the condition (a), $\lim_{n\to\infty}(1-|z_n|^2)|\varphi'(z_n)| = 0$. And so $\varphi^{\#}(z_n) \to 0$. This contradicts the hypothesis.

On the other hand, if $|\psi(z_n)| \not\to 1$, then similarly we have $\psi^{\#}(z_n) \to 0$. Since $|\varphi(z_n)| \to 1$, we can choose an R-separated subsequence (which we denote by the same) of $\{\varphi(z_n)\}$ and so by Proposition 1 of [7], there exists a function $f \in \mathcal{B}$ such that $(1-|\varphi(z_n)|^2)f'(\varphi(z_n)) = 1$ for all n. Then

$$(1-|z_n|^2)|((C_\varphi - C_\psi)f)'(z_n)| \geqslant |\varphi^{\#}(z_n)| - |\psi^{\#}(z_n)(1-|\psi(z_n)|^2)f'(\psi(z_n))|.$$

This contradicts the hypothesis (a).

So we obtain $|\varphi(z_n)| \to 1$ and $|\psi(z_n)| \to 1$. We can use the argument of an R-separated sequence again. That is, we can choose an R-separated subsequence (which we denote by the same) of $\{\varphi(z_n)\} \cup \{\psi(z_n)\}$, a function $g \in \mathcal{B}$ such that $(1-|\varphi(z_n)|^2)g'(\varphi(z_n)) = 1$ and $(1-|\psi(z_n)|^2)g'(\psi(z_n)) = 0$. Thus

$$(1-|z_n|^2)|((C_\varphi - C_\psi)g)'(z_n)| = |\varphi^{\#}(z_n)| \geqslant \delta.$$

This is a contradiction.

Next we consider the case (b). That is, we assume that for some sequence $\{z_n\}$ in $\mathbb{D}$ such that $|z_n| \to 1$ and a constant $\delta \in (0,1)$, $\varphi^{\#}(z_n)\rho(\varphi(z_n),\psi(z_n)) \to 0$ and $\lim_{|z_n|\to 1} |\varphi^{\#}(z_n) - \psi^{\#}(z_n)| \geqslant \delta$. Here we have

$$\begin{aligned}(1-|z_n|^2)&|\varphi'(z_n) - \psi'(z_n)| \\ &\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)|(1-|\psi(z_n)|^2) - 2|\varphi^{\#}(z_n)|\rho(\varphi(z_n),\psi(z_n)).\end{aligned}$$

By the fact that $\varphi - \psi \in \mathcal{B}_o$ and the assumption above, then $|\psi(z_n)| \to 1$. Again, by passing to a subsequence it be assumed that $\{\psi(z_n)\}$ is R-separated. So there exists a function $f \in \mathcal{B}$ such that

$$(1-|\psi(z_n)|^2)|f'(\psi(z_n))| = 1$$

for all n. Then

$$\begin{aligned}(1-&|z_n|^2)|((C_\varphi - C_\psi)f)'(z_n)| \\ &\geqslant |\varphi^{\#}(z_n) - \psi^{\#}(z_n)|(1-|\psi(z_n)|^2)|f'(\psi(z_n))| - C|\varphi^{\#}(z_n)|\rho(\varphi(z_n),\psi(z_n)).\end{aligned}$$

This implies that

$$\lim_{n\to\infty}(1-|z_n|^2)|((C_\varphi - C_\psi)f)'(z_n)| \geqslant \delta.$$

This also is a contradiction. ∎

4. CONNECTED COMPONENTS OF $\mathcal{C}(\mathcal{B})$

For a Banach space X of analytic functions on $\mathbb{D}$, let $\mathcal{C}(X)$ be the space of composition operators on X with the operator norm topology. We write $C_\varphi \sim_X C_\psi$ if C_φ and C_ψ are in the same path component of $\mathcal{C}(X)$.

In [6], MacCluer, Zhao and the second author characterized connected components and isolated points in $\mathcal{C}(H^\infty)$ by introducing a topology on $S(\mathbb{D})$.

Similarly we can define $d_\beta(\varphi, \psi)$ in the analogous way:

$$d_\beta(\varphi, \psi) = \sup_{z \in \mathbb{D}} \beta(\varphi(z), \psi(z)).$$

PROPOSITION 4.1. *Let φ and ψ be in $S(\mathbb{D})$. Then we have*

$$|||C_\varphi - C_\psi||| \leqslant d_\beta(\varphi, \psi).$$

Proof. Let $f \in V$. By (1.3),

$$1 \geqslant |||f||| \geqslant \frac{|f(\varphi(z)) - f(\psi(z))|}{\beta(\varphi(z), \psi(z))}.$$

Hence we have $|f(\varphi(z)) - f(\psi(z))| \leqslant \beta(\varphi(z), \psi(z))$. Suppose that $d_\beta(\varphi, \psi) < \infty$. Then $\|f \circ \varphi - f \circ \psi\|_\infty \leqslant d_\beta(\varphi, \psi)$. This means that $C_\varphi - C_\psi$ is bounded from $\mathcal{B}$ to H^∞. Hence we have $|||(C_\varphi - C_\psi)f||| \leqslant \|(C_\varphi - C_\psi)f\|_\infty \leqslant d_\beta(\varphi, \psi)$. ∎

PROPOSITION 4.2. *Let φ and ψ be in $S(\mathbb{D})$. Then we have*

$$\|C_\varphi - C_\psi\|_{\mathcal{B}} \leqslant \beta(\varphi(0), \psi(0)) + d_\beta(\varphi, \psi) \leqslant 2d_\beta(\varphi, \psi).$$

Proof. For $f \in U_{\mathcal{B}}$, then

$$\begin{aligned} \|(C_\varphi - C_\psi)f\|_{\mathcal{B}} &= |f(\varphi(0)) - f(\psi(0))| + |||(C_\varphi - C_\psi)f||| \\ &\leqslant |f(\varphi(0)) - f(\psi(0))| + d_\beta(\varphi, \psi). \end{aligned}$$

Hence we get

$$\begin{aligned} \|C_\varphi - C_\psi\|_{\mathcal{B}} &\leqslant \sup_{f \in U_{\mathcal{B}}} |f(\varphi(0)) - f(\psi(0))| + d_\beta(\varphi, \psi) \\ &\leqslant \beta(\varphi(0), \psi(0)) + d_\beta(\varphi, \psi) \quad \leqslant 2d_\beta(\varphi, \psi). \end{aligned}$$ ∎

COROLLARY 4.3. *Let φ and ψ be in $S(\mathbb{D})$. If $C_\varphi \sim_{H^\infty} C_\psi$, then $C_\varphi \sim_{\mathcal{B}} C_\psi$.*

Proof. Let φ, ψ be in $S(\mathbb{D})$ such that $C_\varphi \sim_{H^\infty} C_\psi$. By [6], it is shown that $d_\rho(\varphi, \psi) < 1$ and there is a curve γ from φ to ψ in the topological space $S(\mathbb{D}, d_\rho)$. Since the $x \mapsto (1/2)\log((1+x)/(1-x))$ is continuous on $(0, 1)$, Proposition 4.2 implies that γ induces a curve from C_φ to C_ψ in $\mathcal{C}(\mathcal{B})$. ∎

Finally we present a problem concerning the component of $\mathcal{C}(\mathcal{B})$.

PROBLEM. *Is the set of compact composition operators on $\mathcal{B}$ a (path) component in $\mathcal{C}(\mathcal{B})$?*

ADDENDUM (Addendum). The authors have continued a research concerning the problem above and recently published a paper [5].

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REFERENCES

[1] C.C. COWEN, B.D. MACCLUER, *Composition Operators on Spaces of Analytic Functions*, CRC Press, Boca Raton 1995.

[2] N. DUNFORD, J.T. SCHWARTZ, *Linear Operators. Part I. General Theory*, Wiley, New York 1988.

[3] J.B. GARNETT, *Bounded Analytic Functions*, Academic Press, New York 1981.

[4] T. HOSOKAWA, K. IZUCHI, D. ZHENG, Isolated points and essential components of composition operators on H^∞, *Proc. Amer. Math. Soc.* **130**(2002), 1765–1773.

[5] T. HOSOKAWA, S. OHNO, Topological structures of the sets of composition operators on the Bloch spaces, *J. Math. Anal. Appl.* **314**(2006), 736–748.

[6] B.D. MACCLUER, S. OHNO, R. ZHAO, Topological structure of the space of composition operators on H^∞, *Integral Equations Operator Theory* **40**(2001), 481–494.

[7] K. MADIGAN, A. MATHESON, Compact composition operators on the Bloch space, *Trans. Amer. Math. Soc.* **347**(1995), 2679–2687.

[8] A. MONTES-RODRÍGUEZ, The Pick-Schwarz lemma and composition operators on Bloch spaces, *Rend. Circ. Mat. Palermo (2) Suppl.* **56**(1998), 167–170.

[9] J.H. SHAPIRO, *Composition Operators and Classical Function Theory*, Springer-Verlag, New York 1993.

[10] J.H. SHAPIRO, C. SUNDBERG, Isolation amongst the composition operators, *Pacific J. Math.* **145**(1990), 117–152.

[11] K. ZHU, *Operator Theory in Function Spaces*, Marcel Dekker, New York 1990.

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