THE GEOMETRIC MEANS IN BANACH ∗-ALGEBRAS

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Dedicated to Professor Joe Diestel for his 60th birthday

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ABSTRACT. The arithmetic-geometric-harmonic inequality has played a special role in elementary mathematics. During the past twenty five years (see [1], [2] and [8] etc.) a great many mathematicians have researched on various kinds of matrix versions of the arithmetic-geometric-harmonic inequality. It is interesting to see whether the arithmetic-geometric-harmonic inequality can be extended to the context of Banach ∗-algebras. In this article we will define the geometric means of positive elements in Banach ∗-algebras and prove that the arithmetic-geometric-harmonic inequality does hold in Banach ∗-algebras.

KEYWORDS: Arithmetic mean, geometric mean, harmonic mean, Banach ∗-algebra.

MSC (2000): 47A63, 47A64.

INTRODUCTION

Let A be a Banach ∗-algebra. An element a ∈ A is called self-adjoint if a∗ = a. A is Hermitian if every self-adjoint element a of A has real spectrum: σ(a) ⊂ ℝ, where σ(a) denotes the spectrum of a. We assume in what follows that a Banach ∗-algebra A is Hermitian. Also we assume that A is unital with unit 1. Saying an element a ≥ 0 means that a = a∗ and σ(a) ⊂ [0, ∞); a > 0 means that a ≥ 0 and 0 ∉ σ(a). Thus, a > 0 implies its inverse a−1 exists. Denote the set of all invertible elements in A by Inv(A). If a, b ∈ A, then a, b ∈ Inv(A) imply ab ∈ Inv(A), and (ab)−1 = b−1a−1. Saying a ≥ b means a − b ≥ 0, and a > b means a − b > 0. The Shirali-Ford Theorem ([6] or [3], Theorem 41.5) asserted that a∗a ≥ 0 for every a ∈ A. Based on the Shirali-Ford Theorem, Okayasu [5], Tanahashi and Uchiyama [7] proved the following inequalities:

(i) If a, b ∈ A, then a ≥ 0, b ≥ 0 imply a + b ≥ 0, with a ≥ 0 implies aa ≥ 0.
(ii) If a, b ∈ A, then a > 0, b ≥ 0 imply a + b > 0.
(iii) If a, b ∈ A, then either a ≥ b > 0, or a > b ≥ 0 imply a > 0.
(iv) If a > 0, then a−1 > 0.
(v) If \( c > 0 \), then \( 0 < b < a \) if and only if \( cbc < cac \); also \( 0 < b \leq a \) if and only if \( cbc \leq cac \).

(vi) If \( 0 < a < 1 \), then \( 1 < a^{-1} \).

(vii) If \( 0 < b < a \), then \( 0 < a^{-1} < b^{-1} \); also if \( 0 < b \leq a \), then \( 0 < a^{-1} \leq b^{-1} \).

Also, Okayasu [5] showed that the following Löwner-Heinz inequality still holds in Banach \( * \)-algebras:

\[ \text{Theorem 0.1.} \]

Let \( A \) be a unital Hermitian Banach \( * \)-algebra with continuous involution. Let \( a, b \in A \) and \( p \in [0,1] \). Then \( a^p > b^p \) if \( a > b \), and \( a^p \geq b^p \) if \( a \geq b \).

It is natural to ask if there is an arithmetic-geometric-harmonic means inequality in Banach \( * \)-algebras. In this paper, we will address this problem.

1. THE LAWS OF EXPONENTS

Let \( a \in A \) and \( a > 0 \), then \( 0 \notin \sigma(a) \) and the fact of \( \sigma(a) \) being nonempty compact subset of \( \mathbb{C} \) implies that

\[ \inf \{ z : z \in \sigma(a) \} > 0 \quad \text{and} \quad \sup \{ z : z \in \sigma(a) \} < \infty. \]

Choose \( \gamma \) to be a closed rectifiable curve in \( \{ \text{Re} z > 0 \} \), the right half open plane of the complex plane, such that \( \sigma(a) \subset \text{ins} \gamma \), the inside of \( \gamma \). Let \( G \) be an open subset of \( \mathbb{C} \) with \( \sigma(a) \subset G \). If \( f : G \rightarrow \mathbb{C} \) is analytic, we define an element \( f(a) \) in \( A \) by

\[ f(a) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z-a)^{-1}dz. \]

It is known (see pp. 201–204 in [4]) that \( f(a) \) does not depend on the choice of \( \gamma \) and the Spectral Mapping Theorem:

\[ \sigma(f(a)) = f(\sigma(a)) \]

holds.

For any \( \alpha \in \mathbb{R} \), we define

\[ a^\alpha = \frac{1}{2\pi i} \int_{\gamma} z^\alpha(z-a)^{-1}dz \]

where \( z^\alpha \) is the principal \( \alpha \)-power of \( z \). Since \( A \) is a Banach \( * \)-algebra, \( a^\alpha \in A \). Since \( z^\alpha \) is analytic in \( \{ \text{Re} z > 0 \} \), by the Spectral Mapping Theorem

\[ \sigma(a^\alpha) = (\sigma(a))^\alpha = \{ z^\alpha : z \in \sigma(a) \} \subset (0,\infty). \]

Thus, we have

**Lemma 1.1.** If \( 0 < a \in A \) and \( \alpha \in \mathbb{R} \), then \( a^\alpha \in A \) with \( a^\alpha > 0 \).

Moreover, one of the laws of exponents holds in Banach \( * \)-algebras.

**Lemma 1.2.** If \( 0 < a \in A \) and \( \alpha, \beta \in \mathbb{R} \), then \( a^\alpha a^\beta = a^{\alpha+\beta} \).
Proof. Let \( \gamma \) be defined as in the discussion preceding Lemma 1.1. It is known that ([4], VII. 4.7, Riesz Functional Calculus) that the map

\[
f \mapsto f(a) = \frac{1}{2\pi i} \int_{\gamma} f(z)(z-a)^{-1}dz
\]

of \( \text{Hol}(a) \rightarrow A \) is an algebra homomorphism, where \( \text{Hol}(a) \) = all of the functions that are analytic in a neighborhood of \( \sigma(a) \). That is, \( f(a)g(a) = (fg)(a) \). Moreover, \( \sigma a z^\beta = z^{\alpha+\beta} \) holds for principal powers of \( z \) implies that

\[
a^\alpha a^\beta = \frac{1}{2\pi i} \int_{\gamma} z^\alpha z^\beta (z-a)^{-1}dz = \frac{1}{2\pi i} \int_{\gamma} z^{\alpha+\beta} (z-a)^{-1}dz = a^{\alpha+\beta}.
\]

**Lemma 1.3.** If \( 0 < a \in A \) and \( a \in \mathbb{R} \), then \( (a^a)^{-1} = (a^{-1})^a = a^{-a} \).

**Proof.** Note that \( a^0 = 1 \) ([3], Lemma 1, p. 31), and from Lemma 1.2 we have

\[
a^\alpha a^{-\alpha} = a^{\alpha+(-\alpha)} = a^0 = 1.
\]

By the uniqueness of the inverse of an element in \( A \), \( (a^a)^{-1} = a^{-a} \).

Next we want to verify that \( (a^{-1})^a = a^{-a} \). We know that \( a > 0 \) implies that

\[
\inf\{z : z \in \sigma(a)\} > 0 \quad \text{and} \quad \sup\{z : z \in \sigma(a)\} < \infty.
\]

Choose positive real numbers \( r_1 \) and \( r_2 \) such that:

\[
0 < r_1 < \inf\{z : z \in \sigma(a)\}, \quad r_2 > \sup\{z : z \in \sigma(a)\}
\]

\[
\frac{1}{r_1} > \sup\{z : z \in \sigma(a)\}, \quad 0 < \frac{1}{r_2} < \inf\{z : z \in \sigma(a)\}.
\]

Let \( \gamma \) be a closed rectifiable curve in \( \{\text{Re} z > 0\} \), which passes \( r_1 \) and \( r_2 \) and such that \( \sigma(a) \subset \text{ins}\gamma \). Then the curve \( 1/\gamma = \{1/z : z \in \gamma\} \) is also a closed rectifiable with \( \sigma(a) \subset \text{ins}(1/\gamma) \) and \( 1/\gamma \subset \{\text{Re} z > 0\} \). Thus,

\[
(a^{-1})^a = \frac{1}{2\pi i} \int_{1/\gamma} z^\alpha (z-a^{-1})^{-1}dz = \frac{1}{2\pi i} \int_{1/\gamma} z^\alpha (a - \frac{1}{z})^{-1}a dz
\]

\[
= a \frac{1}{2\pi i} \int_{1/\gamma} \lambda^{-a-1}(\lambda-a)^{-1}d\lambda \quad \text{(substituting : } \lambda = 1/z)\]

\[
= aa^{-a-1} = a^{-a} \quad \text{(Lemma 1.2).}
\]

**Lemma 1.4.** If \( 0 < a \in A \), \( 0 < b \in A \), \( a, \beta \in \mathbb{R} \), and \( ab = ba \), then \( a^\alpha b^\beta = b^\beta a^\alpha \).

**Proof.** Suppose that \( z \not\in \sigma(a) \), then \( ab = ba \implies (z-a)b = b(z-a) \implies b(z-a)^{-1} = (z-a)^{-1}b \). Let \( \gamma \) be defined as in the discussion preceding Lemma 1.1.
Then

\[ a^\alpha b = \left( \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z-a)^{-1} dz \right) b = \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z-a)^{-1} b dz = \frac{1}{2\pi i} \int_{\gamma} z^\alpha b(z-a)^{-1} dz = b \left( \frac{1}{2\pi i} \int_{\gamma} z^\alpha (z-a)^{-1} dz \right) = ba^\alpha. \]

Thus,

\[ ab = ba \implies a^\alpha b = ba^\alpha \implies a^\alpha b^\beta = b^\beta a^\alpha. \]

2. THE ARITHMETIC MEAN, GEOMETRIC MEAN AND HARMONIC MEAN

Naturally, for \( a, b \in A \), and \( w_1, w_2 \) are positive numbers summing to 1, their \textit{weighted arithmetic mean} can be defined as

\[ A_w(a, b) := w_1 a + w_2 b. \]

If \( a > 0, b > 0 \), their \textit{weighted harmonic mean} can be defined as

\[ H_w(a, b) := (w_1 a^{-1} + w_2 b^{-1})^{-1}. \]

From the point view of matrix analysis (see [1]), if \( a > 0, b > 0 \), and \( w_1, w_2 \) are positive numbers summing to 1, their \textit{weighted geometric mean} can be defined as

\[ G_w(a, b) := b^{1/2}(b^{-1/2}ab^{-1/2})w_1b^{1/2}. \]

Denote \( A_w(a, b), G_w(a, b) \) and \( H_w(a, b) \) by \( A(a, b), G(a, b) \) and \( H(a, b) \) respectively if \( w_1 = w_2 = 1/2 \). It is clear that \( A_w(a, b), G_w(a, b), H_w(a, b) \in A \) and \( H_w(a, b) > 0 \) and \( G_w(a, b) > 0 \) by inequalities (ii), (iv), (v) and Lemma 1.1 above. Does the following arithmetic-geometric-harmonic inequalities hold

\[ H_w(a, b) \leq G_w(a, b) \leq A_w(a, b) \]

in Banach \*-algebras?

Based on the lemmas above we can prove some properties of arithmetic mean, geometric mean and harmonic mean mentioned by Ando [1].

**THEOREM 2.1.** Suppose that \( a, b \in A \) with \( a > 0, b > 0 \), then

\[ H(a, b) = H(b, a) \quad \text{and} \quad G(a, b) = G(b, a). \]

**Proof.** \( H(a, b) = H(b, a) \) follows the definition of the harmonic mean and the fact that \( A \) is an Abelian group.

Observe that \( G(a, b) = G(b, a) \) is equivalent to

\[ a^{-1/2}b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}a^{-1/2} = (a^{-1/2}ba^{-1/2})^{1/2}. \]
Since positive elements are equal if and only if their squares are equal (see Lemma 6 of [7]), using Lemma 1.2 this is in turn equivalent to
\[
 a^{-1/2}b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}[b^{1/2}a^{-1}b^{1/2}](b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}a^{-1/2} = a^{-1/2}ba^{-1/2}.
\]

Since the term in square brackets is just \((b^{-1/2}ab^{-1/2})^{-1}\) by Lemma 1.3, the left hand side of the expression above does indeed reduce to the right hand side when we use Lemma 1.2 again.

**Theorem 2.2.** Suppose that \(a, b, c \in A\) with \(a > 0, b > 0\) and \(c \in \text{Inv}(A)\), then
\[
c^* H(a, b)c = H(c^*ac, c^*bc) \quad \text{and} \quad c^* G(a, b)c = G(c^*ac, c^*bc).
\]

**Proof.** Since \(c \in \text{Inv}(A)\), \(c^{-1}\) exists. Hence
\[
c^* H(a, b)c = c^* \left( \frac{1}{2}a^{-1} + \frac{1}{2}b^{-1} \right)^{-1} = \left( \frac{1}{2}c^{-1}a^{-1}(c^*c)^{-1} + \frac{1}{2}c^{-1}b^{-1}(c^*c)^{-1} \right)^{-1} = \frac{1}{2}(c^*ac)^{-1} + \frac{1}{2}(c^*bc)^{-1}
\]

It is analogous with the proof of Theorem 2.1, we now verify the second equality:
\[
c^* G(a, b)c = G(c^*ac, c^*bc)
\]
\[
\iff c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c
\]
\[
= (c^*bc)^{1/2}((c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2})^{1/2}(c^*bc)^{1/2}
\]
\[
\iff (c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}
\]
\[
= ((c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2})^{1/2}
\]
\[
\iff ((c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2})^2
\]
\[
= (c^*bc)^{-1/2}(c^*ac)(c^*bc)^{-1/2}.
\]

The last equality is true, since by Lemma 1.2
\[
((c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2})^2
\]
\[
= ((c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2})
\]
\[
(c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}
\]
\[
= (c^*bc)^{-1/2}c^* b^{1/2}(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}c^* b^{1/2}
\]
\[
(b^{-1/2}ab^{-1/2})^{1/2}b^{1/2}c(c^*bc)^{-1/2}.
\]
and Lemma 1.3, we have

\[ 1, \text{ numbers summing to } 248 \]

\[ Q \]

**Theorem 2.3.** Suppose that \( a, b \in A \) with \( a > 0, b > 0 \). Then

\[ H_w(a, b)^{-1} = A_w(a^{-1}, b^{-1}) \quad \text{and} \quad G_w(a^{-1}, b^{-1}) = G_w(a, b)^{-1}. \]

**Proof.** The first equality is obvious from its definitions. Using Lemma 1.2 and Lemma 1.3, we have

\[ G_w(a^{-1}, b^{-1}) = (b^{-1})^{1/2}((b^{-1})^{-1/2}a^{-1}(b^{-1})^{-1/2}w_1(b^{-1})^{1/2} \]

\[ = (b^{1/2})^{-1}((b^{-1/2}ab^{-1/2})^{-1}w_1(b^{1/2})^{-1} \]

\[ = (b^{1/2}(b^{-1/2}ab^{-1/2})w_1b^{1/2})^{-1} = G_w(a, b)^{-1}. \]

**Theorem 2.4.** Suppose that \( a, b \in A \) with \( a > 0, b > 0 \), and \( w_1, w_2 \) are positive numbers summing to 1, then

\[ H_w(a, b) \leq G_w(a, b) \leq A_w(a, b). \]

**Proof.** Firstly we verify the arithmetic-geometric means inequality: \( G_w(a, b) \leq A_w(a, b) \). With the help of inequality (v),

\[ G_w(a, b) \leq A_w(a, b) \]

\[ \iff b^{1/2}(b^{-1/2}ab^{-1/2})w_1b^{1/2} \leq w_1a + w_2b \]

\[ \iff b^{1/2}(b^{-1/2}ab^{-1/2})w_1b^{1/2} \leq b^{1/2}(w_1b^{-1/2}ab^{-1/2} + w_2)b^{1/2} \]

\[ \iff (b^{-1/2}ab^{-1/2})w_1 \leq w_1b^{-1/2}ab^{-1/2} + w_2 \]

\[ \iff w_1n + w_2 - n^{w_1} \geq 0, \]

where \( n := b^{-1/2}ab^{-1/2} \). Lemma 1.1 and inequality (v) imply \( n > 0 \), and hence \( \sigma(n) \subset (0, \infty) \).

Let \( f(z) = w_1z + w_2 - z^{w_1} \), where \( z^{w_1} \) is the principal of the power function. Then \( f(z) \) is analytic in the right half open plane \( \{ \text{Re} z > 0 \} \) of the complex plane. Next we claim that \( f(z) \geq 0 \) on the positive real line. In fact, let \( x = z - 1 \) in the Bernoulli inequality:

\[ (1 + x)^{w_1} \leq 1 + w_1x, \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad -1 < x. \]

We have

\[ z^{w_1} \leq w_1z + (1 - w_1), \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad 0 < z, \]

that is,

\[ f(z) \geq 0, \quad \text{if } 0 < w_1 < 1 \quad \text{and} \quad 0 < z. \]

The Spectral Mapping Theorem implies

\[ \sigma(f(n)) = f(\sigma(n)) \subset [0, \infty). \]
So
\[ f(n) = w_1 n + w_2 - n^{w_1} \geq 0. \]
Hence
\[ G_w(a, b) \leq A_w(a, b). \]
Replacing \( a \) and \( b \) by \( a^{-1} \) and \( b^{-1} \) respectively in the arithmetic-geometric means inequality, Theorem 2.3 and inequality (vii) guarantee that
\[ H_w(a, b) \leq G_w(a, b). \]

In general, for \( a_1, a_2, \ldots, a_n \in A \), and an \( n \)-tuple of positive numbers \( w_1, w_2, \ldots, w_n \) summing to 1, their weighted arithmetic mean in \( A \) can be defined as
\[ A_w(a_1, a_2, \ldots, a_n) := w_1 a_1 + w_2 a_2 + \cdots + w_n a_n. \]
If \( a_i > 0, 1 \leq i \leq n \), their weighted harmonic mean in \( A \) can be defined as
\[ H_w(a_1, a_2, \ldots, a_n) := \left( w_1 a_1^{-1} + w_2 a_2^{-1} + \cdots + w_n a_n^{-1} \right)^{-1}. \]
From the point of view of matrix analysis (see [8]), if \( a_i > 0, 1 \leq i \leq n \), and \( w_1, \ldots, w_n \) are positive numbers summing to 1, their weighted geometric mean in \( A \) can be defined as
\[
G_w(a_1, a_2, \ldots, a_n) := a_1^{1/2} (a_n^{-1/2} a_{n-1}^{1/2} \cdots (a_3^{-1/2} a_2^{1/2} a_2^{-1/2} a_1^{-1/2})^{a_1} \cdots a_2^{1/2} a_3^{-1/2} a_2^{1/2} a_n^{-1/2} a_{n-1} a_n^{-1/2},
\]
where \( a_i = 1 - \left( w_{i+1} / \sum_{j=1}^{i+1} w_j \right) \) for \( i = 1, \ldots, n - 1 \). Note that this geometric mean is just the inductive generalization of \( n = 2 \) case, which was discussed in Theorem 2.3 and 2.4.

Based on Theorem 2.4 with the same inductive proof as in [8], we have

**THEOREM 2.5.** Suppose that \( a_i \in A, 1 \leq i \leq n \), with \( a_i > 0, 1 \leq i \leq n \), and \( w_1, \ldots, w_n \) are positive numbers summing to 1, then
\[ H_w(a_1, \ldots, a_n) \leq G_w(a_1, \ldots, a_n) \leq A_w(a_1, \ldots, a_n). \]

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