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CLASSIFYING THE TYPES OF PRINCIPAL GROUPOID C^*-ALGEBRAS

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ABSTRACT. Suppose G is a second countable, locally compact, Hausdorff groupoid with a fixed left Haar system. Let G^0/G denote the orbit space of G and $C^*(G)$ denote the groupoid C^*-algebra. Suppose that G is a principal groupoid. We show that $C^*(G)$ is CCR if and only if G^0/G is a T_1 topological space, and that $C^*(G)$ is GCR if and only if G^0/G is a T_0 topological space. We also show that $C^*(G)$ is a Fell Algebra if and only if G is a Cartan groupoid.

KEYWORDS: *Locally compact groupoid, C^*-algebra.*

MSC (2000): 46L05, 46L35.

1. INTRODUCTION

C^*-algebras can be classified as being continuous-trace, bounded trace, Fell Algebras, CCR (liminal), and GCR (postliminal). These are listed in order of containment. Recall that for separable C^*-algebras, an algebra is GCR if and only if it is Type I. Further, C^*-algebras that are not GCR are very poorly behaved. In the case of a transformation group C^*-algebra $C^*(H, X)$ (where H is a group that acts continuously on the space X) each of these classifications correspond to a property of the transformation group itself. For example, Phil Green was able to prove in [7] that a freely acting transformation group C^*-algebra has continuous-trace if and only if the action of the transformation group is proper. In [12] the authors have generalized Green's result to principal groupoids. In this paper we generalize three more such results.

In [6], Elliot Gootman showed the following:

THEOREM 1.1. *Suppose H and X are both second countable. Then $C^*(H, X)$ is* GCR *if and only if every stability group is* GCR *and the orbit space is T_0.*

Dana Williams considered the case for CCR transformation group C^*-algebras in [20], and proved the theorem below.

THEOREM 1.2. *Suppose that H and X are both second countable. Suppose also that at every point of discontinuity y of the map $x \mapsto S_x$, the stability group S_y is amenable, then $C^*(H,X)$ is* CCR *if and only if the stability groups are* CCR *and the orbit space is T_1.*

REMARK 1.3. Gootman has shown that the hypothesis on $x \mapsto S_x$ in Theorem 1.2 is unnecessary; however, the details have not appeared.

We also note that Thierry Fack proved versions of Theorem 1.1 and Theorem 1.2 for foliation C^*-algebras in [3].

Finally, in [8], Astrid an Huef proved :

THEOREM 1.4. *$C^*(H,X)$ is a Fell algebra if and only if (H,X) is a Cartan G-space.*

We generalize each of the above three theorems to principal groupoids. The key comes in showing that there is a continuous injection between the orbit space of the groupoid and the spectrum of the associated groupoid C^*-algebra. In fact, when the orbit space is T_0, we show that these spaces are homeomorphic.

We have also been able to further generalize the CCR and GCR results to non-principal groupoids; however, these results will appear later.

2. PRELIMINARIES

A groupoid G is a small category in which every morphism is invertible. A principal groupoid is a groupoid in which there is at most one morphism between each pair of objects. We define maps r and s from G to G by $r(x) = xx^{-1}$ and $s(x) = x^{-1}x$. These are the maps Renault calls r and d in [17]. The common image of r and s is called the unit space which we denote G^0.

We will only consider second countable, locally compact, Hausdorff groupoids G. Our main results also requires G to be principal; however, we will state this condition when it is needed. We will also assume that G has a fixed left Haar system, $\{\lambda^u\}_{u\in G^0}$.

Now consider the vector space $C_c(G)$, the space of continuous functions with compact support from G to the complex numbers, $\mathbb{C}$. We can view this space as a $*$-algebra by defining convolution and involution with the formulae:

$$f * g(x) = \int f(y)g(y^{-1}x)\, \mathrm{d}\lambda^{r(x)}(y) = \int f(xy)g(y^{-1})\, \mathrm{d}\lambda^{s(x)}(y)$$

and

$$f^*(x) = \overline{f(x^{-1})}.$$

A representation of $C_c(G)$ is a $*$-homomorphism π from $C_c(G)$ into $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ that is continuous with respect to the inductive limit topology on $C_c(G)$ and the weak operator topology on $B(\mathcal{H})$, and that is non-degenerate in the sense that the linear span of $\{\pi(f)\eta : f \in C_c(G), \eta \in \mathcal{H}\}$ is dense in $\mathcal{H}$. We define the groupoid C^*-algebra with the following theorem.

THEOREM 2.1. *For $f \in C_c(G)$, the quantity*

$$\|f\| := \sup\{\|\pi(f)\| : \pi \text{ is a representation of } C_c(G)\} \tag{2.1}$$

is finite and defines a C^*-norm on $C_c(G)$. The completion of $C_c(G)$ with respect to this norm is a C^*-algebra, denoted $C^*(G)$.

The only real issue in proving Theorem 2.1 comes in showing that $\|f\| < \infty$ for all $f \in C_c(G)$. This is a consequence of Renault's Disintegration Theorem ([18], Theorem 4.2 and [11], Theorem 3.23). The motivating example of a groupoid C^*-algebra is a transformation group C^*-algebra, $C^*(H, X)$, defined in [20] and [8].

We define the map $\pi : G \to G^0 \times G^0$ by $\pi(x) = (r(x), s(x))$. Using π, we define an equivalence relation on G^0 and endow the set of equivalence classes with the quotient topology. We call this topological space the orbit space of G, denoted G^0/G.

3. A MAP FROM G^0/G to $C^*(G)^\wedge$

Following [12] and [17], pages 81–82, recall that for each $u \in G^0$ there is a representation L^u induced from the point mass measure ε_u. When G is a principal groupoid, L^u acts on $L^2(G, \lambda_u)$ so that for $f \in C_c(G)$ and $\xi \in L^2(G, \lambda_u)$,

$$L^u(f)\xi(\gamma) = \int f(\gamma\alpha)\xi(\alpha^{-1})\mathrm{d}\lambda^u(\alpha).$$

The following lemma is Lemma 2.4 of [12].

LEMMA 3.1. *Suppose G is a principal groupoid. Then the representation L^u is irreducible for each $u \in G^0$. Further more, if $[u] = [v]$ then L^u is unitarily equivalent to L^v.*

We can use this construction to define a map $\psi : G^0/G \to C^*(G)^\wedge$ where $\psi([u]) = L^u$. As usual, we view L^u as its unitary equivalence class in $C^*(G)^\wedge$. Our notation is somewhat careless. We should denote the image of u under ψ by $[L^u]$ but the preceding lemma makes this carelessness less troubling.

Our goal is to show that for principal groupoids with T_0 orbit spaces, ψ is a homeomorphism. We will first show this for groupoids with T_1 orbit spaces and generalize this to T_0 orbit spaces later. Before we deal with ψ, we must first determine what the representations of $C^*(G)$ look like.

Fixing $u \in G^0$, recall from Lemma 2.13 in [17] that there is a representation M_u of $C_0(G^0)$ on $L^2(G, \lambda_u)$ defined by

$$L^u(V(\phi)f) = M_u(\phi)L^u(f). \tag{3.1}$$

PROPOSITION 3.2. *Suppose that L is an irreducible representation of $C^*(G)$, and that M is the representation of $C_0(G^0)$ defined by $M(\phi)L(f) = L(V(\phi)f)$. If $\ker M = J_F := \{\phi \in C_0(G^0) : \phi(x) = 0 \text{ for all } x \in F\}$, then there is a $u \in G^0$ such that $F = \overline{[u]}$.*

Before we can prove this proposition, we need the following two lemmas.

LEMMA 3.3. *Let U be an open subset of G^0. Then the ideal of $C^*(G)$ generated by $C_c(G|_U)$ is* $\overline{C_c(G|_{[U]})} := \mathrm{Ex}([U])$.

Proof. It suffices to see that

$$\begin{aligned} E_0 &:= C_c(G) * C_c(G|_U) * C_c(G) \\ &= \operatorname{span}\{ f * g * h : f, h \in C_c(G) \text{ and } g \in C_c(G|_U) \} \end{aligned} \tag{3.2}$$

is dense in $C_c(G|_{[U]})$ in the inductive limit topology. In view of the Stone-Weierstrass Theorem ([19], Theorem 7.33) since E_0 is self-adjoint it suffices to show E_0 separates points of $G|_{[U]}$ and vanishes at no point of $G|_{[U]}$.

Because $G|_{[U]}$ is Hausdorff, this is the same as showing that for each $\gamma \in G|_{[U]}$ and each neighborhood V of γ, there is an $F \in E_0$ with $\operatorname{supp} F \subset V$ and $F(\gamma) \neq 0$. But if $\gamma \in G|_{[U]}$, then $\gamma = \alpha\beta\delta$ with $\beta \in G|_U$, $s(\alpha) = s(\gamma)$, and $r(\delta) = r(\gamma)$.

Now notice that

$$\begin{aligned} f * g * h(\gamma) &= \int_G f * g(\gamma\eta) h(\eta^{-1}) \, \mathrm{d}\lambda^{s(\gamma)}(\eta) \\ &= \int_G \int_G f(\omega) g(\omega^{-1}\gamma\eta) h(\eta^{-1}) \, \mathrm{d}\lambda^{r(\gamma)}(\omega) \, \mathrm{d}\lambda^{s(\gamma)}(\eta). \\ &= \int_G \int_G f(\omega) g(\omega^{-1}\gamma\eta^{-1}) h(\eta) \, \mathrm{d}\lambda^{r(\gamma)}(\omega) \, \mathrm{d}\lambda_{s(\gamma)}(\eta). \end{aligned}$$

We can choose neighborhoods V_1, V_2 and V_3 of α, β and δ, respectively, such that $V_1V_2V_3 \subset V$. Notice from the integral above that if $\gamma \in \operatorname{supp}(f * g * h)$ then there exists $\omega \in \operatorname{supp} f$, $\eta \in \operatorname{supp} h$ so that $\omega^{-1}\gamma\eta^{-1} \in \operatorname{supp} g$. Since $\gamma = \omega(\omega^{-1}\gamma)\eta^{-1}\eta$, we see that $\operatorname{supp}(f * g * h) \subset (\operatorname{supp} f)(\operatorname{supp} g)(\operatorname{supp} h)$, so we have $\operatorname{supp}(f * g * h) \subset V$ provided $\operatorname{supp} f \subset V_1$, $\operatorname{supp} g \subset V_2$ and $\operatorname{supp} h \subset V_3$. Thus it suffices to take non-negative functions $f, h \in C_c(G)$ and $g \in C_c(G|_U)$ with the appropriate supports and $f(\alpha) = g(\beta) = h(\delta) = 1$ and $F = f * g * h$. ∎

LEMMA 3.4. *Suppose that L is a non-degenerate representation of $C^*(G, \lambda)$, and that M is the representation of $C_0(G^0)$ defined by $M(\phi)L(f) = L(V(\phi)f)$. Then* $\ker M = J_F$ *for a closed, G-invariant set $F \subset G^0$.*

Proof. We know $\ker M = J_F$ for closed subset F of G^0. Let $U := G^0 \setminus F$. It will suffice to see that U is G-invariant; that is, $U = [U]$.

If $f \in C_c(G|_U)$, then $K = \operatorname{supp} f$ is a compact subset of $G|_U$. Thus $C = r(K)$ is a compact subset of U. Therefore we can choose $\phi \in C_c(U)$ such that $\phi(u) = 1$ for all $u \in C$. Then $V(\phi)f = f$. Since ϕ vanishes on F, $M(\phi)L(f) = L(V(\phi)f) = 0$. So $f \in \ker L$, and we have shown that

$$C_c(G|_U) \subset \ker L. \tag{3.3}$$

Lemma 3.3 implies that $C_c(G|_{[U]}) \subset \ker L$. If $[U] \neq U$, then there is a $\phi \in C_c(G^0)$ such that $\operatorname{supp}\phi \subset [U]$ and ϕ is not identically zero on F. Since $V(\phi)f \in C_c(G|_{[U]})$ for all $f \in C_c(G)$, it follows that $V(\phi)f \in \ker L$. Therefore $M(\phi) = 0$, which contradicts $\ker M = J_F$. ∎

Proof of Proposition 3.2. Since G^0/G is a second countable Baire space, we know from lemma on page 222 preceding Corollary 19 in [7] that every irreducible closed set must be a point closure. Lemma 3.4 tells us that $\ker M = J_F$ where F is a closed G-invariant subset of G^0. Thus the image of F in G^0/G is closed. Suppose F is not an orbit closure. Then F is not irreducible. That is F can be written as the union $C_1 \cup C_2$ where each C_i is a closed G-invariant set such that $F \not\subset C_i$. In particular, $C_i \cap F \neq \emptyset$ for $i = 1$ or $i = 2$.

Let U_i be the G-invariant open set $G^0 \setminus C_i$. Since $\mathrm{Ex}(U_1) \cap \mathrm{Ex}(U_2) = \mathrm{Ex}(U_1)\,\mathrm{Ex}(U_2)$, it follows from Lemma 2.10 in [13] that

$$C_c(G|_{U_1})C_c(G|_{U_2})$$

is dense in $C^*(G|_{U_1}) \cap C^*(G|_{U_2})$. On the other hand

$$C_c(G|_{U_1})C_c(G|_{U_2}) \subset C_c(G|_{U_1\cap U_2}) = C_c(G|_{G^0\setminus(C_1\cup C_2)}) = C_c(G|_{G^0\setminus F}) = C_c(G|_U).$$

Thus, (3.3) implies that

$$\mathrm{Ex}(U_1) \cap \mathrm{Ex}(U_2) \subset \ker L.$$

Since L is irreducible, $\ker L$ is prime. Thus

$$\mathrm{Ex}(U_i) \subset \ker L \quad \text{for some } i = 1, 2.$$

We may as well assume that $i = 1$. Since $U_1 \cap F \neq \emptyset$ (otherwise, we would have F in C_1), we can choose $\phi \in C_c^+(G^0)$ such that $\operatorname{supp}\phi \subset U_1$ and $\phi|_F \neq 0$. If $f \in C_c(G)$, we know

$$V(\phi)f(\gamma) = \phi(r\gamma)f(\gamma),$$

thus $r(\gamma) \in U_1$ and because U_1 is invariant, $s(\gamma) \in U_1$ also. This means that $V(\phi)f$ is in $C_c(G|_{U_1})$. Thus $V(\phi)f \in \ker L$ for all $f \in C_c(G)$. It follows that $M(\phi) = 0$. But this contradicts $\phi|_F \neq 0$. Thus F must be an orbit closure as claimed. ∎

COROLLARY 3.5. *Every irreducible representation of $C^*(G)$ factors through $C^*(G|_{\overline{[u]}})$ for some $u \in G^0$.*

Proof. Suppose L is an irreducible representation and M is the associated representation satisfying (3.1). We know $\ker M = J_F$ and that $F = \overline{[u]}$ by Proposition 3.2. Let $U := G^0 \setminus F$. We must show that $\mathrm{Ex}(U) \subset \ker L$ by Lemma 2.10 of [13]. It suffices to show $C_c(G|_U) \subset \ker L$. We will do this as we did in the proof of Lemma 3.4. If $f \in C_c(G|_U)$, then $K = \operatorname{supp} f$ is a compact subset of $G|_U$. Thus $C = r(K)$ is a compact subset of U. Therefore we can choose $\phi \in C_c(U)$ such that $\phi(u) = 1$ for all $u \in C$. Then $V(\phi)f = f$. Since ϕ vanishes on F, $M(\phi)L(f) = L(V(\phi)f) = 0$. So $f \in \ker L$, and we have shown that $C_c(G|_U) \subset \ker L$. ∎

We now have all the pieces needed to show that for principal groupoids, the map ψ is a continuous open injection. Further, if the orbit space is T_1, then ψ is a homeomorphism.

PROPOSITION 3.6. *Suppose G is a principal groupoid. Then the map ψ defined above is a continuous, open, injection.*

Proof. We know that ψ is a continuous injection by Proposition 2.5 of [12].

We will show ψ is an open map using the criteria from Proposition II.13.2 in [4]. Let $L^{u_n} \to L^u$ be a convergent net in $C^*(G)^\wedge$. Thus $M_{u_n} \to M_u$ in $C_0(G^0)^\wedge$. Each M_{u_n} corresponds to a closed subset, namely $\overline{[u_n]}$. By Lemma 2.4 of [20], we may pass to a subnet and relabel if necessary and find $v_n \in [u_n]$ so $v_n \to u$. Therefore ψ is open. ∎

REMARK 3.7. We will eventually weaken the hypothesis of Proposition 3.8 and require only that G be a principal groupoid and G^0/G be T_0.

PROPOSITION 3.8. *Suppose G is a principal groupoid in which orbits are closed. Then the map ψ defined above is a homeomorphism.*

Proof. All that is left to show is that ψ is surjective. Let L be any irreducible representation of $C^*(G)$. Since orbits are closed, we know that L is lifted from a representation on $C^*(G|_{[u]})$ from Corollary 3.5. The representation L^u is also a representation on $C^*(G|_{[u]})$. Since $C^*(G|_{[u]})$ is a transitive groupoid, and G is principal, Lemma 2.4 of [10] tells us that $C^*(G|_{[u]}) \cong K(H)$. However, the compact operators have only one irreducible representation. Therefore $L^u \cong L$. ∎

4. CCR GROUPOID C^*-ALGEBRAS

In order to prove the theorem below, a generalization of Williams' Theorem 1.2, we only use the property of Proposition 3.6 that ψ is a continuous injection.

THEOREM 4.1. *Let G be a principal groupoid. Then G is* CCR *if and only if G^0 is T_1.*

Proof. Suppose $C^*(G)$ is CCR. This implies that points of the spectrum, $C^*(G)^\wedge$, are closed. We know the map

$$\psi : G^0/G \to C^*(G)^\wedge,$$

where $\psi([u]) = L^u$, is a continuous injection by Proposition 3.6. Thus the inverse image of a point of the spectrum is one orbit which must also be closed.

Now suppose that the orbit space is T_1. Suppose L is a representation of $C^*(G)$. We know from Corollary 3.5 that L factors through $C^*(G|_{\overline{[u]}}) = C^*(G|_{[u]})$

for some $u \in G^0$. But $C^*(G|_{[u]})$ is a transitive groupoid thus

$$C^*(G|_{[u]}) \cong C^*(G_u^u) \otimes K$$

by Theorem 3.1 of [10]. This is CCR because we are assuming G is a principal groupoid. This means that L is lifted from a representation of a CCR C^*-algebra making L a representation onto the compact operators. That is, $C^*(G)$ is CCR. ∎

COROLLARY 4.2. *If G is a principal groupoid and $C^*(G)$ is* CCR *then ψ is a homeomorphism.*

Proof. This is immediate from Theorem 4.1 and Proposition 3.8. ∎

5. GCR C^*-ALGEBRAS

We can weaken the conditions in Proposition 3.8 and show that, for principal groupoids, ψ is a homeomorphism when G^0/G is a T_0 space. In doing this, we actually describe the ideal structure of the associated groupoid C^*-algebra. We will also prove a generalization of Gootman's Theorem 1.1 for principal groupoids that says $C^*(G)$ is GCR if and only if G^0/G is T_0.

We know that for principal groupoids ψ is a continuous, injective, open map from Proposition 3.6. Therefore to show ψ is a homeomorphism, we must show that ψ is onto. What we will do is show that when we require the orbit space to be T_0 rather than T_1, we can show that every irreducible representation of $C^*(G)$ is lifted from a representation of $C^*(G|_C)$ where C is a Hausdorff subset of G^0/G. This will suffice.

We will begin Proposition 5.1 below by assuming that G^0/G is T_0. We will also show that the orbit equivalence relation R on G^0 is an F_σ subset of $G^0 \times G^0$. When this is the case, Arlan Ramsay proved in Theorem 2.1 of [16] that there is a list of 14 different properties that are each equivalent to saying that G^0/G is T_0. Some of these equivalent properties include:

(1) each orbit is locally closed,
(2) G^0/G is almost Hausdorff, and
(3) G^0/G is a standard Borel space.

We will use property (2) in our proof. The idea for this proof comes from Lemma 2.3 in [21].

PROPOSITION 5.1. *Suppose G is a groupoid. If G^0/G is T_0 then there is an ordinal γ and ideals $\{I_\alpha : \alpha \leqslant \gamma\}$ such that:*

(i) *$\alpha < \beta$ implies that $I_\alpha \subset I_\beta$;*
(ii) *$I_0 = 0$ and $I_\gamma = C^*(G)$;*
(iii) *if δ is a limit ordinal, then I_δ is the ideal generated by $\{I_\alpha\}_{\alpha<\delta}$;*
(iv) *if α is not a limit ordinal, then $I_\alpha/I_{\alpha-1} \cong C^*(G|_{U_\alpha \setminus U_{\alpha-1}})$ where U_α is a saturated subset of G and each space $U_{\alpha+1} \setminus U_\alpha$ is Hausdorff;*

(v) *if L is an irreducible representation of $C^*(G)$, then L is the canonical extension of an irreducible representation of $C^*(G|_{U_\alpha \setminus U_{\alpha-1}})$.*

Also, if G is a principal groupoid, then the map ψ defined above is a homeomorphism from G^0/G into $C^*(G)^\wedge$.

REMARK 5.2. The C^*-algebra $C^*(G|_{U_\alpha \setminus U_{\alpha-1}})$ is actually the quotient of $C^*(G|_{U_\alpha})$ by $C^*(G|_{U_{\alpha-1}})$.

Proof. First we will show that the orbit equivalence relation R on G^0 is an F_σ subset of $G^0 \times G^0$. To show that R is an F_σ set, we must show it is a countable union of closed sets of $G^0 \times G^0$. Notice that G is σ-compact and that $R = \pi(G)$ where $\pi(\gamma) = (r(\gamma), s(\gamma))$. Therefore R is an F_σ subset because π is continuous.

Now from Theorem 2.1 in [16], we know that G^0/G is almost Hausdorff. Therefore, the discussion on page 125 of [5] gives us an ordinal γ and open subsets $\{U_\alpha : \alpha \leqslant \gamma\}$ of G^0/G such that:

(a) $\alpha < \beta$ implies that $U_\alpha \subset U_\beta$;

(b) $\alpha < \gamma$ implies that $U_\alpha \setminus U_{\alpha-1}$ is a dense Hausdorff subspace in the relative topology;

(c) if δ is a limit ordinal, then $U_\delta = \bigcup_{\alpha<\delta} U_\alpha$;

(d) $U_0 = \emptyset$ and $U_\gamma = G^0/G$.

In the sequel, we will abuse notation and consider each U_α as an open invariant subset of G^0. Thus from Proposition 6.1 each U_α corresponds to an ideal $C^*(G|_{U_\alpha})$ of $C^*(G)$, which we will call I_α. Now properties (i), (ii), and (iii) follow immediately. Property (iv) follows immediately from the short exact sequence

$$0 \longrightarrow C^*(U|_{\alpha-1}) \longrightarrow C^*(G|U_\alpha) \longrightarrow C^*(G|_{U_\alpha \setminus U_{\alpha-1}}) \longrightarrow 0$$

of Lemma 2.10 in [13].

Now we must show (v). Suppose L is an irreducible representation of $C^*(G)$. Since L is an non-degenerate irreducible representation, the restriction of L to an ideal gives us an irreducible representation of the ideal. Define the set

$$S = \{\lambda : L(I_\lambda) \neq 0\}.$$

Since S is a set of ordinals, it has a smallest element. Let α be the smallest element of S. We know that α is not a limit ordinal because of property (iii). Therefore $\alpha - 1$ exists and we have

$$L(I_\alpha) \neq 0 \quad \text{and} \quad L(I_{\alpha-1}) = 0.$$

Therefore, L is the canonical extension of a representation of $I_\alpha / I_{\alpha-1}$ as needed.

Suppose G is a principal groupoid. We know that ψ is continuous, open, and injective from Proposition 3.6. Thus, to show ψ is a homeomorphism, we need only show that ψ is onto. In this proof, we need to be careful and define the following representations. Let $\mathrm{Ind}(G, u)$ be the representation L^u on $C^*(G)$ and let $\mathrm{Ind}(G_{U_\alpha}, u)$ be the representation L^u as a representation of $C^*(G|_{U_\alpha})$ for some $u \in U_\alpha$.

Now let L be any representation of $C^*(G)$. Our goal is to show that L is equivalent to $L^u = \mathrm{Ind}(G, u)$ for some $u \in G^0$. We know from part (v) that L is the canonical extension of a representation L' of $I_\alpha / I_{\alpha-1} = C^*(G|_{U_\alpha \setminus U_{\alpha-1}})$. We also know that $U_\alpha \setminus U_{\alpha-1}$ is Hausdorff which means that L' is equivalent to $\mathrm{Ind}(G|_{U_\alpha}, u)$ for some $u \in U_\alpha$. It suffices to show that the canonical extension of $\mathrm{Ind}(G|_{U_\alpha}, u)$ to $C^*(G)$ must be equal to $\mathrm{Ind}(G, u)$. Notice that the spaces each of these representations act upon are the same. The representation $\mathrm{Ind}(G|_{U_\alpha}, u)$ extends to a representation $\overline{\mathrm{Ind}}(G|_{U_\alpha}, u)$ on all of $C^*(G)$. Notice that for $f \in C_c(G)$, $g \in L^2(G_u, \lambda_u)$, $x \in G_u$ we have

$$\overline{\mathrm{Ind}}(G|_{U_\alpha}, u)(f)(\mathrm{Ind}(G|_{U_\alpha}, u)(g))\xi = \overline{\mathrm{Ind}}(G|_{U_\alpha}, u)(f * g)\xi = \mathrm{Ind}(G, u)(f * g)\xi.$$

Thus, $\mathrm{Ind}(G, u)$ is the canonical extension of $\mathrm{Ind}(G|_{U_\alpha}, u)$ as needed. ∎

We now have more than enough to prove the following theorem.

THEOREM 5.3. *Suppose G is a principal groupoid. Then $C^*(G)$ is* GCR *if and only if G^0/G is T_0.*

Proof. Suppose $C^*(G)$ is GCR. Then the spectrum of $C^*(G)$ is T_0. From Lemma 3.8, we know there is a continuous injection from the orbit space into the spectrum. Therefore, the orbit space must also be T_0.

Now suppose we know G^0/G is T_0. From Proposition 5.1, we know that every irreducible representation L of $C^*(G)$ is the canonical extension of a representation of $C^*(G_{U_\alpha \setminus U_{\alpha-1}})$ where $U_\alpha \setminus U_{\alpha-1}$ is Hausdorff. Thus $C^*(G_{U_\alpha \setminus U_{\alpha-1}})$ is CCR by Theorem 4.1. Therefore, the image of L contains the compact operators and $C^*(G)$ is GCR. ∎

6. IDEALS

We know that for an open saturated subset U of G^0, $C^*(G|_U)$ is an ideal in $C^*(G)$. When G is principal and $C^*(G)$ is GCR, all the ideals of $C^*(G)$ are of this form.

PROPOSITION 6.1. *Suppose G is a principal groupoid and $C^*(G)$ is* GCR. *Then the map $U \mapsto \mathrm{Ex}(U) \cong C^*(G|_U)$ from the collection of open saturated subsets of G^0 to the ideals of $C^*(G)$ is a bijection.*

Proof. Recall that if $C^*(G)$ is GCR, $C^*(G)^\wedge \cong \mathrm{Prim}(C^*(G))$. We also know that there is a natural correspondence between open subsets of $\mathrm{Prim}(C^*(G))$ and ideals of $C^*(G)$. Thus in order to show that Ex is a bijection, it suffices to show

$$C^*(G|_U) \cong \bigcap_{v \notin U} \ker L^v.$$

Notice that $C^*(G|_U) = \bigcap\{\ker L^v : L^v(C^*(G|_U)) = 0\}$.

It follows from the definition of L^v that if $v \in U$, $L^v(C_c(G|_U)) \neq 0$ and if $v \notin U$, $L^v(C^*(G|_U)) = 0$. Therefore

$$C^*(G|_U) = \bigcap_{v \notin U} \ker L^v$$

as needed. ∎

7. FELL ALGEBRAS

Finally, we generalize an Huef's Theorem 1.4. Many of the results involving Cartan G-spaces that an Huef used to prove (1.4) came from [14]. Thus we first must generalize some of Palais' work for Cartan G-spaces. This process leads us to some interesting results in their own right.

DEFINITION 7.1. A *subset*, N of G^0 is *wandering* if and only if the set

$$G|_N = \pi^{-1}(N, N) = \{\gamma \in G : s(g) \in N \text{ and } r(g) \in N\}$$

is relatively compact.

LEMMA 7.2. *A groupoid G is proper if and only if every compact subset of G^0 is wandering.*

Proof. Suppose G is proper so that by definition π is a proper map. That is, the inverse image of a compact set is compact. Let K be a compact subset of G^0. By assumption $\pi^{-1}(K, K)$ is compact; thus K is wandering.

Now suppose that every compact subset of G^0 is wandering. Let L be a compact subset of $G^0 \times G^0$. We must show $\pi^{-1}(L)$ is compact. Note that $L \subset W \times W$ where W is a compact subset of G^0.

Thus,

$$\pi^{-1}(L) \subset \pi^{-1}(W, W)$$

which is compact. Thus $\pi^{-1}(L)$ is a closed subset of a compact set. Therefore $\pi^{-1}(L)$ is compact. ∎

DEFINITION 7.3. We call a groupoid G a *Cartan groupoid* if and only if for every $x \in G^0$, x has a wandering neighborhood.

It is not difficult to show that a transformation group is a Cartan G-space if and only if the associated transformation group groupoid is a Cartan groupoid.

LEMMA 7.4. *If G is a Cartan groupoid, then for each $u \in G^0$, $[u]$ is closed in G^0.*

Proof. Let $u \in G^0$. Let v be a limit point of $[u]$ in G^0. Because G is a Cartan groupoid, v has a wandering neighborhood, U. We will assume that U is closed. Thus, we can find a sequence of elements $\{v_n\}$ in U that converge to v where each $v_n \in [u]$. There also exists a sequence of elements $\{\gamma_n\} \subset G$ such that for each n,

$s(\gamma_n) = v_n$ and $r(\gamma_n) = u$. Now choose one of the $\{\gamma_n\}$, call it γ_{n_0}. Notice that $r(\gamma_{n_0}^{-1}) = v_{n_0}$ and $s(\gamma_{n_0}^{-1}) = u$. Thus $\gamma_{n_0}^{-1}\gamma_n \in G|_U$ which is compact because it is relatively compact and closed . Thus we can pass to a subsequence, relabel, and assume $\{\gamma_n\}$ converges to γ. Since r and s are continuous, $r(\gamma) = u$ and $s(\gamma) = v$. Thus $v \in [u]$. ∎

Clearly, if G is proper, by Lemma 7.2 we see that G is a Cartan groupoid. We will prove a partial converse of this but first we need the following lemma.

LEMMA 7.5. *A groupoid G is proper if and only if every sequence $\{\gamma_n\} \in G$ such that $\{\pi(\gamma_n)\}$ converges has a convergent subsequence.*

Proof. Suppose that G is proper. Let $\{\gamma_n\}$ be a sequence where $\{\pi(\gamma_n)\}$ converges to (u, v). Now, let K be a compact neighborhood of (u, v). Thus $\{\pi(\gamma_n)\}$ is eventually inside of K. Since $\pi^{-1}(K)$ is compact, there is a subsequence $\{\gamma_{n_k}\}$ that converges to γ as needed.

Now suppose for every $\{\gamma_n\} \in G$ such that $\pi(\gamma_n)$ converges to (u, v), $\{\gamma_n\}$ has a convergent subsequence $\{\gamma_{n_k}\}$ where $\{\gamma_{n_k}\}$ converges to γ. Let K be a compact subset of $G^0 \times G^0$. We must show $\pi^{-1}(K)$ is compact. Let $\{\gamma_n\} \subset \pi^{-1}(K)$. It suffices to show $\{\gamma_n\}$ has a convergent subsequence. Since $\{\pi(\gamma_n)\} \subset K$, $\{\pi(\gamma_n)\}$ has a convergent subsequence in K, call it $\{\pi(\gamma_{n_k})\}$ where $\{\pi(\gamma_{n_k})\} \to (u, v)$. So, by assumption, we can find a subsequence and relabel so that $\{\gamma_{n_k}\}$ converges to $\gamma \in \pi^{-1}(K)$. ∎

LEMMA 7.6. *A groupoid G is proper if and only if G is Cartan and G^0/G is Hausdorff.*

Proof. Suppose G is Cartan and G^0/G is Hausdorff. Let $\{\gamma_n\}$ be a sequence in G such that $\{\pi(\gamma_n)\}$ converges to (u, v). By Lemma 7.5, we must show that there exists a convergent subsequence of $\{\gamma_n\}$ that converges to γ.

Because the quotient map is continuous,

$$[r(\gamma_n)] \to [u] \quad \text{and} \quad [s(\gamma_n)] \to [v]$$

in G^0/G. Since the orbit space is Hausdorff, and for each n

$$[r(\gamma_n)] = [s(\gamma_n)],$$

we must have $[u] = [v]$. Thus there exist $\gamma \in G$ so that $r(\gamma) = u$ and $s(\gamma) = v$. Which also means that

$$r(\gamma_n) \to r(\gamma) \quad \text{and} \quad s(\gamma_n) \to s(\gamma).$$

That is,

$$\pi(\gamma_n) \to \pi(\gamma) = (u, v).$$

Since r is open, we can pass to a subsequence, relabel, and find $\eta_n \to \gamma$ with $r(\eta_n) = r(\gamma_n)$. Then $\eta_n^{-1}\gamma_n$ makes sense and $\pi(\eta_n^{-1}\gamma_n) \to (v, v)$. By taking a wandering neighborhood U of v, we can pass to a subsequence, relabel, and assume that $\eta_n^{-1}\gamma_n \to \beta$ with $\beta \in G|_{\{v\}}$. But then $\gamma_n \to \gamma\beta$ as needed.

Now suppose G is proper. Since G is locally compact, Lemma 7.2 tells us that G is Cartan. We must show that G^0/G is Hausdorff. It suffices to show that limits of convergent nets are unique.

Suppose $\{x_n\} \in G^0$ and

$$[x_n] \to [u] \quad \text{and} \quad [x_n] \to [v].$$

Notice that the quotient map

$$q : G^0 \to G^0/G$$

is open. This is true because $q(U) = s(r^{-1}(U))$ for any open set $U \in G^0$ and r and s are continuous and open. Thus using Proposition 2.13.2 of [4], we can pass to a subnet, relabel, and assume that x_n converges to x in G^0 and that there are $\{v_n\} \subset G^0$ such that $[v_n] = [x_n]$ with v_n converging to some v. Similarly, we can find $\{u_n\} \subset G^0$ such that $[u_n] = [x_n] = [v_n]$.

Let $\gamma_n \in G$ be such that $r(\gamma_n) = u_n$ and $s(\gamma_n) = v_n$. If K is a compact neighborhood of u and v, then $\{\gamma_n\}$ is eventually in the compact set $\pi^{-1}(K, K)$. Thus we can pass to a subnet, relabel, and assume that γ_n converges to γ in G. But then $(\gamma) = u$ and $s(\gamma) = v$. That is $[u] = [v]$. ∎

Because of the correspondence between open saturated subsets and ideals, saturated sets give us a key to the structure of $C^*(G)$. For Cartan groupoids, we can take the saturation of wandering neighborhoods and see that in addition to getting a saturated set, some of the useful properties of wandering neighborhoods are preserved.

LEMMA 7.7. *Suppose G is a principal Cartan groupoid and U is an open wandering neighborhood. Let $V := [U]$ be the saturation of U. Then $V/G|_V$ and $U/G|_U$ are homeomorphic.*

Proof. Suppose that

$$q_U : U \to U/G|_U \quad \text{and} \quad q_V : V \to V/G|_V$$

are the corresponding quotient maps for the orbit spaces for $G|_U$ and $G|_V$. Now consider the map

$$f : U/G|_U \to V/G|_V \quad \text{so that } f(q_U(x)) = q_V(x)$$

for $x \in U$. We will show f is a homeomorphism. Clearly, f is well defined.

Suppose

$$q_V(x_1) = q_V(x_2) \quad \text{where } x_1, x_2 \in U.$$

This means there exist $\gamma \in G|_V$ so that $r(\gamma) = x_1$ and $s(\gamma) = x_2$. Since we know x_1 and x_2 are in U, $\gamma \in G|_U$. Therefore

$$q_U(x_1) = q_U(x_2)$$

and f is injective.

Now let $q_V(y) \in V/G|_V$. Since $y \in V$ and $V = [U]$, y is in the orbit of x for some $x \in U$. This means that $q_V(y) = q_V(x) = f(q_U(x))$ and f is surjective.

Suppose that $\{q_U(x_n)\}$ converges to $q_U(x)$. We must show that $\{q_V(x_n)\}$ converges to $q_V(x)$. Suppose the contrary. Thus we can find a neighborhood, W, of $q_V(x)$ for which there is a subsequence which we relabel and assume $\{q_V(x_n)\} \notin W$ for all n. Because $\{q_U(x_n)\}$ converges to $q_U(x)$, and q_U is an open map, it follows from Proposition 2.13.2 in [4] that we can find a sequence $\{y_n\}$ and a subsequence of $\{x_n\}$ and relabel so that $y_n \to x$ and $[y_n] = [x_n]$ in U. Therefore $q_V(y_n) = q_V(x_n)$ for all n and, since q_V is continuous, $\{q_V(x_n)\}$ converges to $q_V(x)$. This is a contradiction; thus f is continuous.

Suppose $q_V(u_n) \to q_V(u)$ where we can suppose that each u_n as well as each u belong to U. Since q_V is open, we can pass to a subsequence, relabel, and assume that there are v_n in V such that $q_V(v_n) = q_V(u_n)$ and $v_n \to u$. Since U is open, we eventually have each $v_n \in U$. Since q_U is continuous, for large n, $q_U(v_n) \to q_U(u)$. It follows from Proposition II.13.2 in [4] that f is open. ∎

LEMMA 7.8. *Suppose V is the saturation of an open wandering set, then $G|_V$ is proper.*

Proof. Because G is a Cartan groupoid, $G|_V$ is also a Cartan groupoid. Thus, to show that $G|_V$ is proper, it suffices to show that the orbit space, $V/G|_V$, is Hausdorff. From Lemma 7.2, we know that $G|_U$ is proper, thus by Lemma 7.6, $U/G|_U$ is Hausdorff. But Lemma 7.7 tells us that $U/G|_U \cong V/G|_V$. Therefore $V/G|_V$ is also Hausdorff. ∎

With this newly defined structure of a Cartan groupoid, we have the machinery to generalize Theorem 1.4.

THEOREM 7.9. *Suppose G is a principal groupoid. Then G is a Cartan Groupoid if and only if $A = C^*(G)$ is a Fell algebra.*

Proof. Suppose G is a Cartan groupoid. We must show that for every irreducible representation, π of A, π is a Fell point of $\widehat{A}$. Let $x \in G^0$ and U be an open wandering neighborhood of x. Let V be the saturation of U which is also open.

Since G is a Cartan groupoid, the orbits of G are closed by Lemma 7.4. Therefore $G^0/G \cong \widehat{A}$ by Proposition 3.8. Let π be the representation of A that corresponds to $[x]$.

Since V is a saturated open subset of G, Lemma 2.10 of [13] tells us $C^*(G|_V)$ is an ideal in A. Thus π is an irreducible representation of $C^*(G|_V)$. Also, from Lemma 7.8, we know that $G|_V$ is a principal proper groupoid; thus Theorem 2.3 of [12] tells us that the ideal $C^*(G|_V)$ has continuous-trace. We know continuous-trace C^*-algebras are Fell algebras, thus π is a Fell point of the open subset $C^*(G|_V)^\wedge$ of $\widehat{A}$ which means π is a Fell point of $\widehat{A}$ also.

Now suppose A is a Fell algebra. Let $x \in G^0$. We must show x has a wandering neighborhood.

Since A is CCR, $G^0/G \cong \widehat{A}$ by Corollary 4.2.

Let π_x be the representation corresponding to $[x]$. Since π_x is a Fell point, from Corollary 3.4 in [1] we know it has an open Hausdorff neighborhood in $\widehat{A}$. This neighborhood is of the form $\widehat{J}$ where J is an ideal of A. We also know from Lemma 6.1 that

$$J \cong C^*(G|_V)$$

for some open, saturated subset V of G^0. Notice that $x \in V$.

Since J has Hausdorff spectrum and is a Fell algebra, J has continuous-trace. Therefore by Theorem 2.3 of [12], $G|_V$ is proper. Thus, we know from Lemma 7.2 that every compact subset of V is wandering.

Let N be a compact neighborhood of x in V. Therefore N is a wandering neighborhood of x in G^0. ∎

The proof of the following corollary is trivial in the transformation group case; however it requires much of the machinery established thus far to prove it in the groupoid case.

COROLLARY 7.10. *Suppose G is a principal groupoid. If $x \in G^0$ has a wandering neighborhood and $y \in [x]$, then y has a wandering neighborhood.*

Proof. Let U be an open wandering neighborhood of x. We know that $G|_{[U]}$ is proper. Therefore $C^*(G|_{[U]})$ has continuous-trace which means it is a Fell algebra. Thus by Theorem 7.9, $G|_U$ is a Cartan groupoid. So we know every element of $[U]$ has a wandering neighborhood in $[U]$; therefore, every element has a wandering neighborhood in G^0. ∎

COROLLARY 7.11. *Let G be a principal groupoid so that $C^*(G)$ is* GCR*. The largest Fell ideal of $C^*(G)$ is $C^*(G|_Y)$ where*

$$Y = \{x \in G^0 : \textit{ there exists a wandering neighborhood of } x\}.$$

Proof. Since G is principal and $C^*(G)$ is GCR, by Lemma 6.1 we know every closed ideal is of the form $C^*(G|_Y)$ for some open G-invariant subset $Y \in G^0$. From Corollary 7.10 we see that the Y defined above is G-invariant. Also notice that Y is open. Now apply Theorem 7.9 and we see that $C^*(G|_Y)$ is a Fell algebra and that any ideal that is also a Fell algebra, must be contained in $C^*(G|_Y)$. ∎

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