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BANACH ALGEBRAS OF OPERATOR SEQUENCES: APPROXIMATION NUMBERS

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ABSTRACT. In this paper we discuss the asymptotic behavior of the approximation numbers for operator sequences belonging to a special class of Banach algebras. Associating with every operator sequence $\{A_n\}$ from such a Banach algebra a collection $\{W^t\{A_n\}\}_{t\in T}$ of bounded linear operators on Banach spaces $\{\mathbb{E}^t\}_{t\in T}$, i.e. $W^t\{A_n\} \in \mathcal{L}(\mathbb{E}^t)$, we establish several properties of approximation numbers of A_n, among them the so-called k-splitting property, and show that the behavior of approximation numbers of A_n depends heavily on the Fredholm properties of operators $W^t\{A_n\}$.

KEYWORDS: *Approximation numbers, operator sequences, k-splitting property.*

MSC (2000): 47N40, 46H99, 47H99, 65J10, 47L80.

1. INTRODUCTION

Let $\mathbb{E}$ be a Banach space. As usual we denote by $\mathcal{L}(\mathbb{E})$ the Banach algebra of all bounded linear operators on $\mathbb{E}$. Further, let $\{\mathbb{E}_n\}$ be a sequence of finite dimensional subspaces of $\mathbb{E}$ and we assume that there is a sequence $\{L_n\}$ of projections L_n from $\mathbb{E}$ onto $\mathbb{E}_n$ converging strongly to the identity operator I on $\mathbb{E}$.

We denote by $\mathcal{F} = \mathcal{F}\{\mathbb{E}_n\}$ the collection of all bounded sequences $\{A_n\}$ of bounded linear operators $A_n \in \mathcal{L}(\mathbb{E}_n)$, i.e. of finite matrices $A_n : \mathbb{E}_n \to \mathbb{E}_n$. Provided with the operations

$$\lambda_1\{A_n\} + \lambda_2\{B_n\} := \{\lambda_1 A_n + \lambda_2 B_n\}, \quad \{A_n\}\{B_n\} := \{A_n B_n\},$$

and the supremum norm $\|\{A_n\}\| := \sup \|A_n\|$, the set $\mathcal{F}$ becomes a Banach algebra with identity $\{L_n\}$.

Note that if operators $A_n \in \mathcal{L}(\mathbb{E}_n)$ approximate an operator $A \in \mathcal{L}(\mathbb{E})$ (approximation here means that $A_n L_n$ converge strongly to A), then the Banach-Steinhaus theorem implies the boundedness of the sequence $\{A_n\}$, i.e. the sequence $\{A_n\}$ is contained in the Banach algebra $\mathcal{F}$.

It is of considerable interest to study the asymptotic behavior of the approximation numbers $s_k(A_n)$ for operator (matrix) sequences $\{A_n\} \in \mathcal{F}$ (for the definition of the approximation numbers see Section 3). We are mainly interested in the so-called k-splitting property of the approximation numbers. The approximation numbers for a sequence $\{A_n\} \in \mathcal{F}$ are said to have the k-splitting property if there is an integer $k \geqslant 0$ such that

$$\lim_{n\to\infty} s_k(A_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} s_{k+1}(A_n) > 0.$$

For instance the 0-splitting property is equivalent to the stability of a sequence $\{A_n\} \in \mathcal{F}$ (see [1]). As usual a sequence $\{A_n\} \in \mathcal{F}$ is called stable if, for $n \geqslant n_0$, all operators A_n are invertible and $\sup_{n\geqslant n_0} \|A_n^{-1}\| < \infty$.

Notice that there is a variety of concrete approximation methods for large classes of operators acting on Hilbert spaces, among them the celebrated finite section method for Toeplitz operators, for which one can show the k-splitting property (see [3], [6], [8]). For Banach spaces much less is known. The only available results are those of [1] and [9], where the finite section method for Toeplitz operators with admissible continuous symbols is treated in the spaces ℓ^p.

If $\mathbb{E}$ is a Hilbert space and L_n are the orthogonal projections onto $\mathbb{E}_n$, then $\{A_n\}^* := \{A_n^*\}$ defines an involution in $\mathcal{F}$ which makes $\mathcal{F}$ to a C^*-algebra. Note that in this case the approximation numbers of an operator $A_n \in \mathcal{L}(\mathbb{E}_n)$ are just the singular values of A_n, i.e. the eigenvalues of $(A_n^* A_n)^{1/2}$. It was proved by one of the authors and S. Roch (see [6] and [8]) that the splitting property of the singular values can be successfully studied in the context of so-called standard C^*-algebras. In short, a C^*-subalgebra $\mathcal{A}$ of the algebra $\mathcal{F}$ is called standard if there is a family $\{W^t\}_{t\in T}$ of $*$-homomorphisms of special form from the algebra $\mathcal{A}$ into algebras of bounded linear operators on Hilbert spaces $\{H^t\}_{t\in T}$, i.e. $W^t : \mathcal{A} \to \mathcal{L}(H^t)$, having the property that a sequence $\{A_n\} \in \mathcal{A}$ is stable if and only if all operators $W^t\{A_n\}$ are invertible (for detailed notion of standard algebras see e.g. [6] (Chapter 6) and compare with the definitions of the algebras $\mathcal{F}^T$ and $\mathcal{F}^T_*$ below). The main result concerning standard algebras reads as follows: For any standard algebra $\mathcal{A}$ we can form a closed two-sided ideal $\mathcal{J}^T := \mathcal{J}^T(\mathcal{A})$ such that the next theorem is in force.

THEOREM 1.1 (see [6]). *Let $\{A_n\}$ be a sequence from a standard C^*-algebra $\mathcal{A}$.*

(i) *If the coset $\{A_n\} + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{A}/\mathcal{J}^T$, then all operators $W^t\{A_n\}$ are Fredholm on H^t, the number of the non-invertible operators among the $W^t\{A_n\}$ is finite, and the singular values of A_n have the k-splitting property with*

$$k = \sum_{t\in T} \dim\ker W^t\{A_n\}.$$

(ii) *If $W^t\{A_n\}$ is not Fredholm for at least one $t \in T$, then for every integer $k \geqslant 0$*

$$s_k(A_n) \to 0 \quad \textit{as } n \to \infty.$$

It should be noted that to prove these results the singular values were thought as eigenvalues, leaving out of account that they can be alternatively viewed as approximation numbers.

The aim of this paper is a generalization of these results to the case when $\mathbb{E}$ is a Banach space. We introduce a special class of subalgebras of the Banach algebra $\mathcal{F}$, by analogy with standard algebras, for which we are able to obtain results like Theorem 1.1. Thereby, we propose a completely new proof of the k-splitting property which leads to an estimate of the convergence speed of the k-th approximation number to zero, where k is the splitting number. Note also that the results of this paper allow us to show the k-splitting property for concrete approximation methods, even in the case of Hilbert space $\mathbb{E}$, which were not explicitly treated earlier (see Section 8).

The paper is organized as follows. In Section 2 we introduce a class of subalgebras $\mathcal{F}^T$ of the Banach algebra $\mathcal{F}$ and a family $\{W^t\}_{t\in T}$ of homomorphisms from $\mathcal{F}^T$ into algebras of bounded linear operators on Banach spaces $\{\mathbb{E}^t\}$. In Sections 3 and 4 some properties of approximation numbers for operator sequences $\{A_n\} \in \mathcal{F}^T$ related with properties of the operators $W^t\{A_n\}$ are established. In Section 5, by analogy with standard algebras (see [6]), we define a class of Fredholm sequences and show that the Fredholmness of a sequence $\{A_n\} \in \mathcal{F}^T$ precisely implies the k-splitting property. Note that the new proof of the k-splitting property allows us to leave the class of standard algebras (see Remark 5.5). However, the notion of standard algebras from [6] implies that the behavior of singular values for every operator sequence $\{A_n\}$ belonging to a standard algebra is completely described by the Fredholm properties of the operators $W^t\{A_n\}$ (see Remark 5.7). Sections 6 and 7 are devoted to the proof of the k-splitting property for Fredholm sequences. Finally, in Section 8 in order to illustrate the results of this paper we analyze the behavior of approximation numbers for projections methods for operators $\alpha I + K$ and for finite sections of Toeplitz operators with $C_p + \overline{H}^\infty_p$ matrix valued symbols. A much more complicated situation will be analyzed in a forthcoming paper, namely the asymptotic behavior of approximation numbers for finite sections of Toeplitz operators with piecewise continuous symbols in the ℓ^p setting.

2. THE BANACH ALGEBRA OF OPERATOR SEQUENCES $\mathcal{F}^T$

Let $\mathbb{E}$ be an infinite dimensional Banach space and let $\{\mathbb{E}_n\}$ be a sequence of finite dimensional subspaces of $\mathbb{E}$. Moreover, we assume that there is a sequence $\{L_n\}$ of projections L_n from $\mathbb{E}$ onto $\mathbb{E}_n$ converging strongly to the identity operator I on $\mathbb{E}$.

Let further T be a (possibly infinite) index set and suppose that, for every $t \in T$, there is an infinite dimensional Banach space $\mathbb{E}^t$ with the identity operator I^t, a sequence $\{L_n^t\}$ of projections L_n^t on $\mathbb{E}^t$ converging strongly to I^t, and a sequence $\{E_n^t\}$ of invertible operators $E_n^t : \operatorname{im} L_n^t \to \operatorname{im} L_n$ such that (for brevity, we write E_n^{-t} instead of $(E_n^t)^{-1}$):

(I) $\sup_n \|E_n^t\| \|E_n^{-t}\| < \infty$.

We denote by $\mathcal{F}^T$ the set of all sequences $\{A_n\}$ of linear operators $A_n : \operatorname{im} L_n \to \operatorname{im} L_n$, for which there exist operators $W^t\{A_n\} \in \mathcal{L}(\mathbb{E}^t)$ such that for all $t \in T$

$$E_n^{-t} A_n E_n^t \to W^t\{A_n\}$$

holds in the sense of strong convergence for $n \to \infty$. If we define

$$\lambda_1\{A_n\} + \lambda_2\{B_n\} := \{\lambda_1 A_n + \lambda_2 B_n\}, \quad \{A_n\}\{B_n\} := \{A_n B_n\},$$

and

$$\|\{A_n\}\|_{\mathcal{F}^T} := \sup \left\{ \|A_n\|_{\mathcal{L}(\mathbb{E}_n)} = \sup_{x \in \mathbb{E}_n,\ x \neq 0} \frac{\|A_n x\|_{\mathbb{E}}}{\|x\|_{\mathbb{E}}} \ : \ n \in \mathbb{N} \right\},$$

then it is not hard to see that $\mathcal{F}^T$ becomes a Banach algebra with the unit element $\{L_n\}$ and the mappings $W^t : \mathcal{F}^T \to \mathcal{L}(\mathbb{E}^t)$, $\{A_n\} \mapsto W^t\{A_n\}$, are unital homomorphisms. Note also that the set $\mathcal{G}$ of all sequences $\{G_n\}$ with $\|G_n L_n\| \to 0$ as $n \to \infty$ forms a closed two-sided ideal of $\mathcal{F}^T$.

3. THE APPROXIMATION NUMBERS FOR $\{A_N\}$ AND THE OPERATORS $W^t\{A_n\}$

Let $\mathbb{F}$ be a finite dimensional Banach space with $\dim \mathbb{F} = m$. The k-th approximation number ($k \in \{0, 1, 2, \dots, m\}$) of an operator $A \in \mathcal{L}(\mathbb{F})$ is defined as

$$s_k(A) := \operatorname{dist}(A, \mathcal{F}_{m-k}(\mathbb{F})) := \inf\{\|A - F\|_{\mathcal{L}(\mathbb{F})} \ : \ F \in \mathcal{F}_{m-k}(\mathbb{F})\},$$

where $\mathcal{F}_{n-k}(\mathbb{F})$ denotes the collection of all operators from $\mathcal{L}(\mathbb{F})$ having the image of the dimension at most $n - k$. It is clear that

$$0 = s_0(A) \leqslant s_1(A) \leqslant s_2(A) \leqslant \cdots \leqslant s_m(A) = \|A\|_{\mathcal{L}(\mathbb{F})}.$$

Moreover one can show that (see [1])

$$s_1(A) = \begin{cases} 1/\|A^{-1}\|_{\mathcal{L}(\mathbb{F})} & \text{if } A \text{ is invertible,} \\ 0 & \text{if } A \text{ is not invertible.} \end{cases}$$

Note also that in case $\mathbb{F}$ is a Hilbert space the approximation numbers $\{s_k(A)\}_{k=1}^m$ are just the singular values of A, i.e. the eigenvalues of $(A^*A)^{1/2}$.

In this section we prove the following results which show the relation between the behavior of the approximation numbers for a sequence $\{A_n\} \in \mathcal{F}$ and the properties of the operators $W^t\{A_n\}$.

THEOREM 3.1. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}^T$ and let k be a natural number. If there is an index $t \in T$ such that $k \leqslant \dim\ker W^t\{A_n\}$, then*

$$s_k(A_n) \to 0 \quad \text{as } n \to \infty.$$

COROLLARY 3.2. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}^T$. If there is an index $t \in T$ such that the kernel of the operator $W^t\{A_n\}$ is infinite dimensional, then for each $l \in \mathbb{N}$*

$$s_l(A_n) \to 0 \quad \text{as } n \to \infty.$$

THEOREM 3.3. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}^T$. If one of the operators $W^t\{A_n\}$ is not normally solvable, then for each $l \in \mathbb{N}$*

$$s_l(A_n) \to 0 \quad \text{as } n \to \infty.$$

COROLLARY 3.4. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}^T$. If there exists a number $k \in \mathbb{N}$ such that $\liminf\limits_n s_k(A_n) > 0$, then all operators $W^t\{A_n\}$ are normally solvable and*

$$\dim\ker W^t\{A_n\} \leqslant k-1, \quad t \in T.$$

To prove Theorem 3.1, we will employ the following well-known lemma.

LEMMA 3.5. *Let $x_1, x_2, \dots, x_k$ be linearly independent vectors from a Banach space $\mathbb{F}$ and let $x_i^n \to x_i$ as $n \to \infty$, $1 \leqslant i \leqslant k$. Then there are numbers $\gamma > 0$ and $N \in \mathbb{N}$ such that*

$$\gamma \sum_{i=1}^k |\alpha_i| \leqslant \Big\| \sum_{i=1}^k \alpha_i x_i^n \Big\|_{\mathbb{F}}$$

for any scalars α_i and all $n > N$.

Proof of Theorem 3.1. Let $\{A_n\} \in \mathcal{F}^T$, let $k \in \mathbb{N}$, and let $t \in T$ be such that $k \leqslant \dim\ker W^t\{A_n\}$. We choose a system $\{x_i\}_{i=1}^k$ of linearly independent vectors belonging to the kernel of the operator $W^t\{A_n\}$ and put

$$x_i^n := L_n^t x_i, \quad n \in \mathbb{N}.$$

Further, for each $n > N$, we introduce the functionals f_i^n : span $\{x_1^n, x_2^n, \dots, x_k^n\} \to \mathbb{C}$ by the rule

$$f_i^n\Big(\sum_{j=1}^k \alpha_j x_j^n\Big) = \alpha_i, \quad 1 \leqslant i \leqslant k.$$

Lemma 3.5 gives that the functionals f_i^n are uniformly bounded:

$$\|f_i^n\| \leqslant \frac{1}{\gamma}, \quad 1 \leqslant i \leqslant k$$

for all sufficiently large n.

By the Hahn-Banach theorem, we can extend f_i^n to the whole Banach space $\mathbb{E}^t$ so that $f_i^n \in (\mathbb{E}^t)^*$ and $\|f_i^n\| \leqslant 1/\gamma$, for $1 \leqslant i \leqslant k$ and all n large enough.

Now we denote by $S_n \in \mathcal{L}(\mathbb{E}^t)$ the linear operators

$$S_n x := \sum_{i=1}^{k} f_i^n(x) x_i^n.$$

We get that the operators $R_n := L_n^t S_n L_n^t \in \mathcal{L}(\operatorname{im} L_n^t)$ are projections and $\dim \operatorname{im} R_n = k$ for all sufficiently large n. Moreover, for any $x \in \operatorname{im} L_n^t$, we have

$$\begin{aligned}
\|E_n^{-t} A_n E_n^t R_n L_n^t x\|_{\mathbb{E}^t} &= \Big\| E_n^{-t} A_n E_n^t \sum_{i=1}^{k} f_i^n(x) x_i^n \Big\|_{\mathbb{E}^t} \\
&\leqslant \sum_{i=1}^{k} |f_i^n(x)| \|E_n^{-t} A_n E_n^t L_n^t x_i\|_{\mathbb{E}^t} \leqslant \frac{\|x\|_{\mathbb{E}^t}}{\gamma} \sum_{i=1}^{k} \|E_n^{-t} A_n E_n^t L_n^t x_i\|_{\mathbb{E}^t}.
\end{aligned}$$

Since $E_n^{-t} A_n E_n^t L_n^t$ converges strongly to the operator $W^t\{A_n\}$, we obtain

$$\|E_n^{-t} A_n E_n^t L_n^t x_i\|_{\mathbb{E}^t} \to \|W^t\{A_n\} x_i\|_{\mathbb{E}^t} = 0, \quad 1 \leqslant i \leqslant k.$$

Hence $\|E_n^{-t} A_n E_n^t R_n L_n^t\|_{\mathcal{L}(\operatorname{im} L_n^t)} \to 0$ as $n \to \infty$.

Finally, we obtain (we denote $m(n) := \dim \operatorname{im} L_n$)

$$\begin{aligned}
s_k(A_n) &\leqslant \|E_n^t\| \|E_n^{-t}\| s_k(E_n^{-t} A_n E_n^t) \\
&= \|E_n^t\| \|E_n^{-t}\| \inf\{\|E_n^{-t} A_n E_n^t + F\|_{\mathcal{L}(\operatorname{im} L_n^t)} : F \in \mathcal{F}_{m(n)-k}(\operatorname{im} L_n^t)\} \\
&\leqslant \|E_n^t\| \|E_n^{-t}\| \|E_n^{-t} A_n E_n^t - E_n^{-t} A_n E_n^t (L_n^t - R_n)\|_{\mathcal{L}(\operatorname{im} L_n^t)} \\
&= \|E_n^t\| \|E_n^{-t}\| \|E_n^{-t} A_n E_n^t R_n L_n^t\|_{\mathcal{L}(\operatorname{im} L_n^t)} \to 0. \quad \blacksquare
\end{aligned}$$

To prove Theorem 3.3, we will need the following results.

THEOREM 3.6 (see [5], pp. 159–160). *Let A be a bounded linear operator on a Banach space $\mathbb{F}$.*

(i) *The operator A is normally solvable on $\mathbb{F}$ if and only if*

$$k_A := \sup_{x \in \mathbb{F}, |Ax\|_{\mathbb{F}} = 1} \inf_{x_0 \in \ker A} \|x - x_0\|_{\mathbb{F}} < \infty.$$

(ii) *If M is a closed subspace of $\mathbb{F}$ and $\dim(\mathbb{F}/M) < \infty$, then normal solvability of $A|M : M \to \mathbb{F}$ is equivalent to the normal solvability of $A : \mathbb{F} \to \mathbb{F}$.*

We prepare the proof of Theorem 3.3 by an auxiliary lemma.

LEMMA 3.7. *Let A be a bounded linear operator on a Banach space $\mathbb{F}$ and let $\{P_n\}$ be a sequence of projections P_n on $\mathbb{F}$ converging strongly to the identity operator I on $\mathbb{F}$. If $\dim \ker A < \infty$, then $\ker A(I - P_n) = \operatorname{im} P_n$ for all sufficiently large n.*

Proof. Assume the converse, i.e. that there exists a sequence of numbers $\{n_m\}_{m \in \mathbb{N}}$, $n_m \to \infty$, such that $\ker A(I - P_{n_m}) \neq \operatorname{im} P_{n_m}$ for all $m \in \mathbb{N}$. This means that for all $m \in \mathbb{N}$ there is a vector $x_{n_m} \notin \operatorname{im} P_{n_m}$ such that $x_{n_m} \in \ker A(I - P_{n_m})$.

In other words, for all $m \in \mathbb{N}$ there is a vector $y_{n_m} = (I - P_{n_m})x_{n_m} \neq 0$ such that $y_{n_m} \in \ker A$.

Now we claim that for any $k \in \mathbb{N}$ one can choose k linearly independent vectors $y_{n_{m_1}}, \dots, y_{n_{m_k}}$ with the mentioned property. This is trivial for $k = 1$. The assertion for $k > 1$ will be checked by induction. Suppose that we have already k linearly independent vectors $y_{n_{m_1}}, \dots, y_{n_{m_k}}$. Since the projections P_n converge strongly to the identity operator I, there is an $N \in \mathbb{N}$ such that for all $n > N$ the vectors $P_n y_{n_{m_1}}, \dots, P_n y_{n_{m_k}}$ are linearly independent (see Lemma 3.5). We choose a number $n_{m_{k+1}} > \max(N, n_{m_1}, \dots, n_{m_k})$ and suppose that $\sum_{i=1}^{k+1} \alpha_i y_{n_{m_i}} = 0$. Then

$$\begin{aligned} 0 &= \sum_{i=1}^{k+1} \alpha_i P_{n_{m_{k+1}}} y_{n_{m_i}} = \sum_{i=1}^{k} \alpha_i P_{n_{m_{k+1}}} y_{n_{m_i}} + \alpha_{k+1} P_{n_{m_{k+1}}} (I - P_{n_{m_{k+1}}}) x_{n_{m_{k+1}}} \\ &= \sum_{i=1}^{k} \alpha_i P_{n_{m_{k+1}}} y_{n_{m_i}}. \end{aligned}$$

Consequently, $\alpha_i = 0$ for all $1 \leqslant i \leqslant k$. Moreover,

$$\alpha_{k+1} y_{n_{m_{k+1}}} = 0 \Rightarrow \alpha_i = 0 \quad \text{for all } 1 \leqslant i \leqslant k+1.$$

Hence the vectors $y_{n_{m_1}}, y_{n_{m_2}}, \dots, y_{n_{m_{k+1}}}$ are linearly independent.

On the other hand, $\ker A$ is finite dimensional. This contradiction concludes the proof. ∎

Proof of Theorem 3.3. Let $\{A_n\} \in \mathcal{F}^T$ and let $t \in T$ be such that the operator $W^t\{A_n\}$ is not normally solvable. In view of Corollary 3.2, it is sufficient to treat only the case $\dim \ker W^t\{A_n\} < \infty$.

Contrary to the assertion of the theorem, we assume that there exist $l \in \mathbb{N}$ and $0 < d < 1$ such that $s_l(A_n) \geqslant d$ for infinitely many n.

Let $\varepsilon > 0$ be any number such that ($\sup_n \|L_n^t\| < \infty$ by the Banach-Steinhaus theorem)

$$\varepsilon < \frac{d}{4l} \Big(\sup_n \|E_n^t\| \|E_n^{-t}\| \Big)^{-1} \Big(4 \sup_n \|L_n^t\| \Big)^{-l}.$$

We claim that one can choose l vectors $z_1, \dots, z_l \in \mathbb{E}^t$ such that for all $1 \leqslant j \leqslant l$

$$\|z_j\| = 1, \quad \|W^t\{A_n\} z_j\| \leqslant \varepsilon, \tag{3.1}$$

and the following inequality takes place

$$\sum_{j=1}^{l} |\alpha_j| \leqslant \Big(4 \sup_n \|L_n^t\| \Big)^l \Big\| \sum_{j=1}^{l} \alpha_j z_j \Big\| \tag{3.2}$$

for all scalars $\alpha_1, \dots, \alpha_l$.

Let $l = 1$. By the previous lemma, there is an N_1 such that $\ker W^t\{A_n\}(I^t - L_n^t) = \operatorname{im} L_n^t$ for all $n \geqslant N_1$. Moreover, in view of Theorem 3.6, for all $n \in \mathbb{N}$

the operators $W^t\{A_n\}|\operatorname{im}(I^t - L_n^t) : \operatorname{im}(I^t - L_n^t) \to \mathbb{E}^t$ are not normally solvable. Again due to Theorem 3.6, there exists a vector $x_1 \in \operatorname{im}(I^t - L_{N_1}^t)$ such that $\|W^t\{A_n\}x_1\| = 1$ and $\|x_1\| \geqslant 1/\varepsilon$. Setting $z_1 = x_1/\|x_1\|$, we obtain the assertion for $l = 1$ ($\|L_n^t\| \geqslant 1$, since L_n^t are projections).

The assertion for $l > 1$ will be proved by induction. Suppose that we have already $l-1$ vectors $z_1, \dots, z_{l-1}$ satisfying (3.1) and (3.2). Since the operators L_n^t converge strongly to the identity operator I^t, there is a number $N_l \geqslant N_1$ such that for all $n \geqslant N_l$

$$\|(I - L_n^t)z_j\| < \varepsilon, \quad j = 1, \dots, l-1.$$

Further, due to Theorem 3.6, there exists a vector $x_l \in \operatorname{im}(I^t - L_{N_l}^t)$ such that $\|x_l\| \geqslant 1/\varepsilon$ and $\|W^t\{A_n\}x_l\| = 1$. Setting $z_l = x_l/\|x_l\|$, we get for all scalars $\alpha_1, \dots, \alpha_l$:

$$2\sup_n \|L_n^t\| \Big\| \sum_{j=1}^{l} \alpha_j z_j \Big\| \geqslant \Big\| \sum_{j=1}^{l} \alpha_j (I - L_{N_l}^t) z_j \Big\| \geqslant |\alpha_l| - \sum_{j=1}^{l-1} |\alpha_j| \varepsilon,$$

$$\sup_n \|L_n^t\| \Big\| \sum_{j=1}^{l} \alpha_j z_j \Big\| \geqslant \Big\| \sum_{i=1}^{l} \alpha_j L_{N_l}^t z_j \Big\| \geqslant \Big(\Big(4 \sup_n \|L_n^t\|\Big)^{1-l} - \varepsilon \Big) \sum_{j=1}^{l-1} |\alpha_j|.$$

Thus, $\|z_l\| = 1$, $\|W^t\{A_n\}z_l\| \leqslant \varepsilon$, and, for all scalars $\alpha_1, \dots, \alpha_l$,

$$\begin{aligned} 3\sup_n \|L_n^t\| \Big\| \sum_{j=1}^{l} \alpha_j z_j \Big\| &\geqslant \Big(\Big(4 \sup_n \|L_n^t\|\Big)^{1-l} - 2\varepsilon \Big) \sum_{j=1}^{l-1} |\alpha_j| + |\alpha_l| \\ &\geqslant \frac{3}{4} \Big(4 \sup_n \|L_n^t\|\Big)^{1-l} \sum_{j=1}^{l} |\alpha_j|. \end{aligned}$$

Now for all large n we introduce the functionals $f_m^n :$ span $\{L_n^t z_1, \dots, L_n^t z_l\} \to \mathbb{C}$ by the rule

$$f_m^n\Big(\sum_{j=1}^{l} \alpha_j L_n^t z_j \Big) = \alpha_j, \quad m = 1, 2, \dots, l.$$

From (3.2) we conclude that for all n large enough the functionals f_m^n are uniformly bounded:

$$\|f_m^n\| \leqslant 2\Big(4 \sup_n \|L_n^t\|\Big)^l, \quad m = 1, 2, \dots, l.$$

By the Hahn-Banach theorem, we can extend f_m^n to the whole Banach space $\mathbb{E}^t$ such that $f_m^n \in (\mathbb{E}^t)^*$ and $\|f_m^n\| \leqslant 2\Big(4 \sup_n \|L_n^t\|\Big)^l$ for $m = 1, \dots, l$.

Now we denote by $S_n \in \mathcal{L}(\mathbb{E}^t)$ the linear operators

$$S_n x := \sum_{m=1}^{l} f_m^n(x) L_n^t z_m.$$

We get, for all sufficiently large n, that the operators $R_n := L_n^t S_n L_n^t$ are projections, $R_n \in \mathcal{L}(\operatorname{im} L_n^t)$, and $\dim \operatorname{im} R_n = l$. Moreover, since $E_n^{-t} A_n E_n^t L_n^t$ converges strongly to $W^t\{A_n\}$, we obtain, for every $x \in \operatorname{im} L_n^t$ and all n large enough, (see (3.1))

$$\begin{aligned}\|E_n^{-t} A_n E_n^t R_n x\|_{\mathbb{E}^t} &= \Big\| E_n^{-t} A_n E_n^t \Big(\sum_{m=1}^{l} f_m^n(x) L_n^t z_m \Big) \Big\|_{\mathbb{E}^t} \\ &\leqslant \sum_{m=1}^{l} |f_m^n(x)| \|E_n^{-t} A_n E_n^t L_n^t z_m\|_{\mathbb{E}^t} \leqslant 4\Big(4 \sup_n \|L_n^t\|\Big)^l \varepsilon l \|x\|_{\mathbb{E}^t}.\end{aligned}$$

This implies that for all n large enough ($m(n) = \dim \operatorname{im} L_n$)

$$\begin{aligned} s_l(A_n) &\leqslant \|E_n^t\| \|E_n^{-t}\| s_l(E_n^{-t} A_n E_n^t) \\ &= \|E_n^t\| \|E_n^{-t}\| \inf\{\|E_n^{-t} A_n E_n^t - F_n\|_{\mathcal{L}(\operatorname{im} L_n^t)} \; : \; F_n \in \mathcal{F}_{m(n)-l}(\operatorname{im} L_n^t)\} \\ &\leqslant \|E_n^{-t} A_n E_n^t - E_n^{-t} A_n E_n^t (L_n^t - R_n)\|_{\mathcal{L}(\operatorname{im} L_n^t)} \sup_n \|E_n^t\| \|E_n^{-t}\| \\ &\leqslant 4l \sup_n \|E_n^t\| \|E_n^{-t}\| \Big(4 \sup_n \|L_n^t\|\Big)^l \varepsilon < d. \end{aligned}$$

This contradiction completes the proof. ∎

4. THE ALGEBRA $\mathcal{F}_*^T$

In this section we suppose, in addition, that $L_n^* \to I^* \in \mathcal{L}(\mathbb{E}^*)$ and $(L_n^t)^* \to (I^t)^* \in \mathcal{L}((\mathbb{E}^t)^*)$, $t \in T$, strongly as $n \to \infty$.

We denote by $\mathcal{F}_*^T$ the Banach algebra of all sequences $\{A_n\} \in \mathcal{F}^T$ for which one has the following strong convergence

$$\operatorname*{s-lim}_{n\to\infty} (E_n^{-t} A_n E_n^t)^* = (W^t\{A_n\})^*, \quad t \in T.$$

One can see that if a sequence $\{A_n\} \in \mathcal{F}_*^T$, then $\{A_n\}$ itself belongs to the Banach algebra $\mathcal{F}^T$ and its adjoint sequence $\{A_n^*\}$ belongs to another Banach algebra $\widetilde{\mathcal{F}}^T$ which corresponds to the Banach spaces $\mathbb{E}^*, \mathbb{E}_n^*, (\mathbb{E}^t)^*, (\mathbb{E}_n^t)^*$, the projections $L_n^*, (L_n^t)^*$, and the operators $(E_n^t)^*$.

THEOREM 4.1. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}_*^T$. If one of the operators $W^t\{A_n\}$ is not Fredholm, then for each $l \in \mathbb{N}$*

$$s_l(A_n) \to 0 \quad \textit{as } n \to \infty.$$

Proof. Let $t \in T$ be an index such that the operator $W^t\{A_n\}$ is not Fredholm. We have three possibilities for the operator $W^t\{A_n\}$ to be not Fredholm:

(i) $W^t\{A_n\}$ is not normally solvable;

(ii) $\dim \ker W^t\{A_n\} = \infty$;

(iii) $\dim \operatorname{coker} W^t\{A_n\} = \infty$.

To the case (i) and (ii) we can apply Theorem 3.3 and Corollary 3.2, respectively. Moreover, since the adjoint matrices $(E_n^t)^* A_n^* (E_n^{-t})^*$ converge strongly to the operator $(W^t(A_n))^*$ and $s_l(A_n) = s_l(A_n^*)$ for any $l \in \mathbb{N}$, the case (iii) can be also treated with the help of Corollary 3.2. ∎

COROLLARY 4.2. *Let $\{A_n\}$ be a sequence from the Banach algebra $\mathcal{F}_*^T$. If there exists a number $k \in \mathbb{N}$ such that $\liminf_n s_k(A_n) > 0$, then all operators $W^t\{A_n\}$ are Fredholm and, for all $t \in T$,*

$$\dim \ker W^t\{A_n\} \leqslant k-1 \quad \textit{and} \quad \dim \operatorname{coker} W^t\{A_n\} \leqslant k-1.$$

5. FREDHOLM SEQUENCES

In this section we suppose that, besides the condition (I), the operators E_n^t, $t \in T$, satisfy the following separation condition:

(II) $\frac{1}{\|E_n^{-\tau}\| \|E_n^t\|} E_n^{-\tau} E_n^t L_n^t \rightharpoonup 0$ weakly as $n \to \infty$ for every $\tau, t \in T$ with $\tau \neq t$.

Notice that the condition (II) coincides with the following condition (II').

(II') For every $t \in T$ and every compact operator $K^t \in \mathcal{K}(\mathbb{E}^t)$, the sequence $\{E_n^t L_n^t K^t E_n^{-t}\}$ belongs to the Banach algebra $\mathcal{F}^T$ and, for all $\tau \in T$,

$$W^\tau\{E_n^t L_n^t K^t E_n^{-t}\} = \begin{cases} K^t & \text{if } \tau = t, \\ 0 & \text{if } \tau \neq t. \end{cases}$$

LEMMA 5.1. *If the condition* (I) *is satisfied, then the conditions* (II) *and* (II') *are equivalent.*

For a proof of this lemma we recall the well-known result.

LEMMA 5.2. *Let $\mathbb{F}$ be a Banach space. If $A, A_n \in \mathcal{L}(\mathbb{F}), K \in \mathcal{K}(\mathbb{F})$, and $A_n \to A$ weakly, then $KA_n \to KA$ strongly.*

Proof of Lemma 5.1. First we show that the condition (II') follows from the condition (II). Let $t \in T$. Obviously, we have

$$W^t\{E_n^t L_n^t K^t E_n^{-t}\} = \text{s–lim}\, L_n^t K^t L_n^t = K^t.$$

Moreover, for all $\tau \neq t$, Lemma 5.2 gives

$$\begin{aligned} W^\tau\{E_n^t L_n^t K^t E_n^{-t}\} &= \text{s–lim}\, E_n^{-\tau} E_n^t L_n^t K^t E_n^{-t} E_n^\tau L_n^\tau \\ &= \text{s–lim}\, \|E_n^\tau\| \|E_n^{-t}\| E_n^{-\tau} E_n^t L_n^t K^t \frac{1}{\|E_n^\tau\| \|E_n^{-t}\|} E_n^{-t} E_n^\tau L_n^\tau = 0. \end{aligned}$$

Now we check that the condition (II′) implies the condition (II). Let $t, \tau \in T$ and $\tau \neq t$. We have to show that for any vector $x \in \mathbb{E}^t$ and any functional $f \in (\mathbb{E}^\tau)^*$

$$f\Big(\frac{1}{\|E_n^{-\tau}\|\|E_n^t\|} E_n^{-\tau} E_n^t L_n^t x\Big) \to 0 \quad \text{as } n \to \infty.$$

We take an arbitrary vector $y \in \mathbb{E}^\tau$ with $\|y\| = 1$ and define the operator K_f on the Banach space $\mathbb{E}^\tau$ by the rule

$$K_f z := f(z) y, \quad z \in \mathbb{E}^\tau.$$

Obviously, the operator $K_f \in \mathcal{K}(\mathbb{E}^\tau)$. This implies

$$\begin{aligned}
&\Big| f\Big(\frac{1}{\|E_n^{-\tau}\|\|E_n^t\|} E_n^{-\tau} E_n^t L_n^t x\Big)\Big| \\
&\qquad = \Big\| K_f \frac{1}{\|E_n^{-\tau}\|\|E_n^t\|} E_n^{-\tau} E_n^t L_n^t x \Big\| \\
&\qquad \leqslant \Big\| E_n^{-\tau} E_n^t E_n^{-t} E_n^\tau L_n^\tau K_f \frac{1}{\|E_n^{-\tau}\|\|E_n^t\|} E_n^{-\tau} E_n^t L_n^t x \Big\| \\
&\qquad\qquad + \Big| f\Big(\frac{1}{\|E_n^{-\tau}\|\|E_n^t\|} E_n^{-\tau} E_n^t L_n^t x\Big)\Big| \|(I - L_n^\tau) y\| \\
&\qquad \leqslant \|E_n^{-t} E_n^\tau L_n^\tau K_f E_n^{-\tau} E_n^t L_n^t x\| + \|f\|\|L_n^\tau\|\|x\|\|(I - L_n^\tau) y\| \to 0. \quad \blacksquare
\end{aligned}$$

Now, for each $t \in T$, we introduce the subset $\mathcal{J}^t$ of the Banach algebra $\mathcal{F}^T$

$$\mathcal{J}^t := \{\{E_n^t L_n^t K^t E_n^{-t}\} + \{G_n\} \ : K^t \in \mathcal{K}(\mathbb{E}^t), \{G_n\} \in \mathcal{G}\}.$$

Let $\mathcal{A}^T$ be the set of all sequences $\{A_n\} \in \mathcal{F}^T$ such that $\{A_n\}\mathcal{J}^t \subseteq \mathcal{J}^t$ and $\mathcal{J}^t\{A_n\} \subseteq \mathcal{J}^t$ for all $t \in T$. Obviously, $\mathcal{A}^T$ is the largest closed subalgebra of $\mathcal{F}^T$ such that, for each $t \in T$, the set $\mathcal{J}^t$ forms a closed two-sided ideal of $\mathcal{A}^T$. Moreover, it is clear that the Banach algebra $\mathcal{A}^T$ contains the unit element $\{L_n\}$.

Further, we denote by $\mathcal{J}^T$ the smallest closed two-sided ideal of $\mathcal{A}^T$ which contains all sequences $\{J_n\}$ such that $\{J_n\}$ belongs to one of the ideals $\mathcal{J}^t, t \in T$. It is easy to see that, actually, the ideal $\mathcal{J}^T$ is the closure in $\mathcal{F}^T$ of the set

$$\Big\{\{J_n\} = \sum_{i=1}^m \{J_n^{t_i}\} \ : \ m \in \mathbb{N}, \{J_n^{t_i}\} \in \mathcal{J}^{t_i}\Big\}.$$

Corresponding to the ideal $\mathcal{J}^T$ we introduce a class of Fredholm sequences by calling a sequence $\{A_n\} \in \mathcal{A}^T$ Fredholm if the coset $\{A_n\} + \mathcal{J}^T$ is invertible in the quotient algebra $\mathcal{A}^T / \mathcal{J}^T$. The following basic properties of Fredholm sequences are obvious:

(i) The set of Fredholm sequences is open in $\mathcal{A}^T$.

(ii) The sum of a Fredholm sequence and a sequence from the ideal $\mathcal{J}^T$ is Fredholm.

(iii) The product of two Fredholm sequences is Fredholm.

THEOREM 5.3. *If a sequence* $\{A_n\} \in \mathcal{A}^T$ *is Fredholm, then all operators* $W^t\{A_n\}$ *are Fredholm on* $\mathbb{E}^t$*, and the number of the non-invertible operators among the* $W^t\{A_n\}$ *is finite.*

Proof. Let $\{A_n\} + \mathcal{J}^T$ be invertible in the quotient algebra $\mathcal{A}^T/\mathcal{J}^T$. Then there are sequences $\{B_n\} \in \mathcal{A}^T$ and $\{J_n\}, \{K_n\} \in \mathcal{J}^T$ such that

$$\{B_n\}\{A_n\} = \{L_n\} + \{J_n\} \quad \text{and} \quad \{A_n\}\{B_n\} = \{L_n\} + \{K_n\}.$$

By the definition of the ideal $\mathcal{J}^T$, there exist finite subsets $\{t_1, \ldots, t_m\}$ and $\{\tau_1, \ldots, \tau_l\}$ of T as well as sequences $\{J_n^{t_i}\} \in \mathcal{J}^{t_i}$ and $\{K_n^{\tau_i}\} \in \mathcal{J}^{\tau_i}$ such that

$$\{J_n\} = \sum_{i=1}^{m} \{J_n^{t_i}\} + \{\widehat{J}_n\} \quad \text{and} \quad \{K_n\} = \sum_{i=1}^{l} \{K_n^{\tau_i}\} + \{\widehat{K}_n\},$$

with $\{\widehat{J}_n\}, \{\widehat{K}_n\} \in \mathcal{J}^T$ and $\|\{\widehat{J}_n\}\|, \|\{\widehat{K}_n\}\| < 1/2$.

Since the sequences $\{L_n\} + \{\widehat{J}_n\}$ and $\{L_n\} + \{\widehat{K}_n\}$ are invertible in the Banach algebra $\mathcal{A}^T$, we can define the following sequences:

$$\begin{aligned}
\{\widehat{B}_n\} &:= (\{L_n\} + \{\widehat{J}_n\})^{-1}\{B_n\} \in \mathcal{A}^T, \\
\{\widehat{C}_n\} &:= \{B_n\}(\{L_n\} + \{\widehat{K}_n\})^{-1} \in \mathcal{A}^T, \\
\{\widehat{J}_n^{t_i}\} &:= (\{L_n\} + \{\widehat{J}_n\})^{-1}\{J_n^{t_i}\} \in \mathcal{J}^{t_i}, \quad i = 1, \ldots, m, \\
\{\widehat{K}_n^{\tau_i}\} &:= \{K_n^{\tau_i}\}(\{L_n\} + \{\widehat{K}_n\})^{-1} \in \mathcal{J}^{\tau_i}, \quad i = 1, \ldots, l.
\end{aligned}$$

Due to the definition of the ideals $\mathcal{J}^t$, we get

$$\{\widehat{B}_n\}\{A_n\} = \{L_n\} + \sum_{i=1}^{m} \{\widehat{J}_n^{t_i}\} = \{L_n\} + \sum_{i=1}^{m} \{E_n^{t_i} L_n^{t_i} J^{t_i} E_n^{-t_i}\} + \{G_n\}, \tag{5.1}$$

$$\{A_n\}\{\widehat{C}_n\} = \{L_n\} + \sum_{i=1}^{l} \{\widehat{K}_n^{\tau_i}\} = \{L_n\} + \sum_{i=1}^{l} \{E_n^{\tau_i} L_n^{\tau_i} K^{\tau_i} E_n^{-\tau_i}\} + \{\widehat{G}_n\}. \tag{5.2}$$

Finally, applying $W^t, t \in T$, to the equations (5.1) and (5.2) we find

$$W^t\{\widehat{B}_n\}W^t\{A_n\} = \begin{cases} I^t & t \notin \{t_1, \ldots, t_m\}, \\ I^{t_i} + J^{t_i} & t \in \{t_1, \ldots, t_m\}, \end{cases}$$

$$W^t\{A_n\}W^t\{\widehat{C}_n\} = \begin{cases} I^t & t \notin \{\tau_1, \ldots, \tau_l\}, \\ I^{\tau_i} + K^{\tau_i} & t \in \{\tau_1, \ldots, \tau_l\}. \end{cases}$$

Thus all operators $W^t\{A_n\}$ are Fredholm on $\mathbb{E}^t$, and the number of the non-invertible operators among the $W^t\{A_n\}$ is at most $m + l$. ∎

Theorem 5.3 allows us to introduce three finite numbers for a Fredholm sequence $\{A_n\}\in\mathcal{A}^T$, its nullity $\alpha(\{A_n\})$, deficiency $\beta(\{A_n\})$, and index $\operatorname{ind}(\{A_n\})$, by

$$\begin{aligned}\alpha(\{A_n\}) &= \sum_{t\in T} \dim\ker W^t\{A_n\},\\ \beta(\{A_n\}) &= \sum_{t\in T} \dim\operatorname{coker} W^t\{A_n\},\\ \operatorname{ind}(\{A_n\}) &= \alpha(\{A_n\}) - \beta(\{A_n\}).\end{aligned}$$

Applying the well-known properties of Fredholm operators, it is not hard to prove the following results:

(i) If $\{A_n\} \in \mathcal{A}^T$ is a Fredholm sequence and $\{B_n\} \in \mathcal{A}^T$ is a sufficiently small sequence, then $\alpha(\{A_n\}+\{B_n\}) \leqslant \alpha(\{A_n\}), \beta(\{A_n\}+\{B_n\}) \leqslant \beta(\{A_n\})$, and $\operatorname{ind}(\{A_n\}+\{B_n\}) = \operatorname{ind}(\{A_n\})$.

(ii) If $\{A_n\} \in \mathcal{A}^T$ is a Fredholm sequence and $\{K_n\} \in \mathcal{A}^T$ is a sequence from the ideal $\mathcal{J}^T$, then $\operatorname{ind}(\{A_n\}+\{K_n\}) = \operatorname{ind}(\{A_n\})$.

(iii) If $\{A_n\} \in \mathcal{A}^T$ and $\{B_n\} \in \mathcal{A}^T$ are Fredholm sequences, then $\operatorname{ind}(\{A_n\}\{B_n\}) = \operatorname{ind}(\{A_n\}) + \operatorname{ind}(\{B_n\})$.

The following theorem provides a relation between the nullity of a Fredholm sequence $\{A_n\} \in \mathcal{A}^T$ and the asymptotic behavior of the approximation numbers of A_n.

THEOREM 5.4. *Let $\{A_n\} \in \mathcal{A}^T$ be a Fredholm sequence. Then the approximation numbers of A_n have the k-splitting property with $k = \alpha(\{A_n\})$, i.e.*

$$\lim_{n\to\infty} s_{\alpha(\{A_n\})}(A_n) = 0 \quad \textit{and} \quad \liminf_{n\to\infty} s_{\alpha(\{A_n\})+1}(A_n) > 0.$$

This theorem will be proved in the subsequent two sections (see Lemma 6.2 and Corollary 7.5).

REMARK 5.5. To check that a sequence $\{A_n\}$ is Fredholm it is not necessary to consider the whole algebra $\mathcal{A}^T$. Indeed, let $\mathcal{A}$ be a closed subalgebra of $\mathcal{F}^T$ such that, for each $t \in T$, the set $\mathcal{J}^t$ forms a closed two-sided ideal of $\mathcal{A}$. We denote by $\mathcal{J}^T_{\mathcal{A}}$ the smallest closed two-sided ideal of $\mathcal{A}$ which contains all sequences $\{J_n\}$ which belong to one of the ideals $\mathcal{J}^t, t \in T$. Obviously, $\mathcal{A} \subset \mathcal{A}^T$ and $\mathcal{J}^T_{\mathcal{A}} = \mathcal{J}^T$. Hence, if the coset $\{A_n\} + \mathcal{J}^T_{\mathcal{A}}$ is invertible in $\mathcal{A}/\mathcal{J}^T_{\mathcal{A}}$ then $\{A_n\}$ is Fredholm. For instance, the algebra $\mathcal{F}^T_*$ provides an example for such an algebra $\mathcal{A}$ (compare with the definition of standard algebras in [6]).

Now we again suppose that $L_n^* \to I^* \in \mathcal{L}(\mathbb{E}^*)$ and $(L_n^t)^* \to (I^t)^* \in \mathcal{L}((\mathbb{E}^t)^*)$, $t \in T$, strongly as $n \to \infty$. Furthermore, we denote by $\mathcal{A}^T_F$ the set of all sequences $\{A_n\} \in \mathcal{A}^T$ except such sequences for which all operators $W^t\{A_n\}$ are Fredholm, the number of the non-invertible operators among the $W^t\{A_n\}$ is finite, but the coset $\{A_n\} + \mathcal{J}^T$ is not invertible (compare with Theorem 5.3).

Combining Theorems 4.1 and 5.4 we obtain that the behavior of the approximation numbers for a sequence $\{A_n\} \in \mathcal{F}_*^T \cap \mathcal{A}_F^T$ is completely described by the Fredholm properties of the operators $W^t\{A_n\}, t \in T$.

THEOREM 5.6. *Let (A_n) be a sequence from the set $\mathcal{F}_*^T \cap \mathcal{A}_F^T$.*

(i) *If all operators $W^t\{A_n\}$ are Fredholm on $\mathbb{E}^t$ and the number of the non-invertible operators among the $W^t\{A_n\}$ is finite, then the approximation numbers of A_n have the k-splitting property with $k = \sum_{t\in T} \dim\ker W^t\{A_n\}$.*

(ii) *Otherwise, $s_l(A_n) \to 0$ as $n \to \infty$ for each $l \in \mathbb{N}$.*

REMARK 5.7. Taking into account Theorem 5.41 from [6] one can show that any standard algebra is contained in the set $\mathcal{F}_*^T \cap \mathcal{A}_F^T$.

Finally, we would like to mention that in Hilbert space setting there is a general notion of Fredholm sequences which does not depend on the underlying algebra $\mathcal{F}^T$ (see [6], [7]). Thereby, in case of sequences in a standard algebra this general notion reduces to the notion of Fredholm sequences considered in this section (which, at least formally, depends on the underlying algebra $\mathcal{F}^T$).

6. THE $\alpha(\{A_n\})$-TH APPROXIMATION NUMBER FOR A FREDHOLM SEQUENCE $\{A_n\} \in \mathcal{A}^T$

Let $\{A_n\} \in \mathcal{A}^T$ be a Fredholm sequence. In this section we show that the first $\alpha(\{A_n\})$ approximation numbers of A_n converge to zero.

Due to (5.1), there exist sequences $\{B_n\} \in \mathcal{A}^T, \{G_n\} \in \mathcal{G}$, a finite subset $\{t_1, \dots, t_m\}$ of T, and compact operators $K^{t_i} \in \mathcal{K}(\mathbb{E}^{t_i})$ such that

$$\{B_n\}\{A_n\} = \{L_n\} + \sum_{i=1}^{m} \{E_n^{t_i} L_n^{t_i} K^{t_i} E_n^{-t_i}\} + \{G_n\}.$$

Hence, the operators $W^t\{A_n\}$ have a trivial kernel for every $t \in T \setminus \{t_1, \dots, t_m\}$, and

$$\alpha(\{A_n\}) = \sum_{i=1}^{m} \dim\ker W^{t_i}\{A_n\}.$$

For each i let k_i refer to $\dim\ker W^{t_i}\{A_n\}$ and let $\{x_{i,l}\}_{l=1}^{k_i}$ with $\|x_{i,l}\| = 1$ be a basis of $\ker W^{t_i}\{A_n\}$. We define vectors $x_{i,l}^n \in \operatorname{im} L_n$ by

$$x_{i,l}^n := \frac{1}{\|E^{t_i}\|} E_n^{t_i} L_n^{t_i} x_{i,l}, \quad i = 1, \dots, m,\ l = 1, \dots, k_i.$$

LEMMA 6.1. *There exist numbers $N \in \mathbb{N}$ and $\gamma > 0$ such that*

$$\gamma \sum_{i=1}^{m} \sum_{l=1}^{k_i} |\alpha_{i,l}| \leqslant \Big\| \sum_{i=1}^{m} \sum_{l=1}^{k_i} \alpha_{i,l} x_{i,l}^n \Big\|$$

for all $n > N$ and all scalars $\alpha_{i,l}$.

Proof. Assume the converse, i.e. that there is an infinite set $\mathbb{N}_1 \subset \mathbb{N}$, a sequence of numbers $\{\gamma_n\}_{n\in\mathbb{N}_1}$ with $\gamma_n \to 0$, and sequences of scalars $\{\alpha_{i,l}^n\}_{n\in\mathbb{N}_1}$ such that

$$\gamma_n \sum_{i=1}^{m}\sum_{l=1}^{k_i} |\alpha_{i,l}^n| > \Big\| \sum_{i=1}^{m}\sum_{l=1}^{k_i} \alpha_{i,l}^n x_{i,l}^n \Big\|, \quad n \in \mathbb{N}_1.$$

For each $n \in \mathbb{N}_1$ we find $1 \leqslant i_n \leqslant m$ and $1 \leqslant l_n \leqslant k_{i_n}$ such that $|\alpha_{i_n,l_n}^n| = \max_{i,l} |\alpha_{i,l}^n|$ and put $\beta_{i,l}^n := \alpha_{i,l}^n / \alpha_{i_n,l_n}^n$. We get $|\beta_{i,l}^n| \leqslant 1, \beta_{i_n,l_n}^n = 1$, and

$$\Big\| \sum_{i=1}^{m}\sum_{l=1}^{k_i} \beta_{i,l}^n x_{i,l}^n \Big\| < \gamma_n \sum_{i=1}^{m}\sum_{l=1}^{k_i} |\beta_{i,l}^n| \leqslant \gamma_n \sum_{i=1}^{m} k_i = \gamma_n \alpha(\{A_n\}) \to 0.$$

Since for any fixed i and l the sequence $\{\beta_{i,l}^n\}_{n\in\mathbb{N}_1}$ is bounded, one can choose an infinite subset $\mathbb{N}_2$ of $\mathbb{N}_1$ and scalars $\beta_{i,l}$ such that $|\beta_{i,l}| \leqslant 1$, at least one of $\beta_{i,l}$ is equal to one, and $\beta_{i,l}^n \to \beta_{i,l},\ n \in \mathbb{N}_2$. Without lose of generality we suppose that $\beta_{1,1} = 1$. Taking into account that $\|x_{i,l}^n\| \leqslant$ const, we get

$$\sum_{i=1}^{m}\sum_{l=1}^{k_i} \beta_{i,l} x_{i,l}^n = z^n \quad \text{with } \|z^n\| \to 0,\ n \in \mathbb{N}_2.$$

By the definition of the vectors $x_{i,l}^n$, we obtain

$$\begin{aligned} &\sum_{l=1}^{k_1} \beta_{1,l} L_n^{t_1} x_{1,l} \\ &= \|E_n^{t_1}\| \|E_n^{-t_1}\| \Big(\sum_{i=2}^{m}\sum_{l=1}^{k_i} \beta_{i,l} \frac{1}{\|E_n^{-t_1}\| \|E_n^{t_i}\|} E_n^{-t_1} E_n^{t_i} L_n^{t_i} x_{i,l} + \frac{1}{\|E_n^{-t_1}\|} E_n^{-t_1} z^n \Big). \end{aligned} \tag{6.1}$$

Since the vectors $\{x_{1,l}\}_1^{k_1}$ are linearly independent, we can take a functional $f \in (\mathbb{E}^{t_1})^*$ such that

$$f(x_{1,1}) = 1 \quad \text{and} \quad f(x_{1,l}) = 0 \quad \text{for all } l = 2, \dots, k_1.$$

Applying the functional f to the left hand side of (6.1), we get

$$f(\sum_{l=1}^{k_1} \beta_{1,l} L_n^{t_1} x_{1,l}) \to \sum_{l=1}^{k_1} \beta_{1,l} f(x_{1,l}) = \beta_{1,1} = 1.$$

On the other hand, the separation condition (II) implies that

$$f(\|E_n^{t_1}\| \|E_n^{-t_1}\| (\sum_{i=2}^{m}\sum_{l=1}^{k_i} \beta_{i,l} \frac{1}{\|E_n^{-t_1}\| \|E_n^{t_i}\|} E_n^{-t_1} E_n^{t_i} L_n^{t_i} x_{i,l} + \frac{1}{\|E_n^{-t_1}\|} E_n^{-t_1} z^n))$$

converge to zero. This contradiction completes the proof. ∎

Now, for each $n>N$, we introduce the functionals $f^n_{i,l} : \operatorname{span}\{x^n_{1,1}, \ldots, x^n_{m,k_m}\} \to \mathbb{C}$ by the rule

$$f^n_{i,l}\Big(\sum_{j=1}^{m}\sum_{v=1}^{k_i} \alpha_{j,v}x^n_{j,v}\Big) = \alpha_{i,l}, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant l \leqslant k_i.$$

By the Hahn-Banach theorem, we can extend $f^n_{i,l}$ to the whole Banach space $\mathbb{E}$ so that $f^n_{i,l} \in \mathbb{E}^*$ and $\|f^n_{i,l}\| \leqslant 1/\gamma$ for all n large enough (see Lemma 6.1).

Further, we denote by $S_n \in \mathcal{L}(\mathbb{E})$ the linear operators

$$S_n x := \sum_{i=1}^{m}\sum_{l=1}^{k_i} f^n_{i,l}(x)x^n_{i,l}.$$

We get that the operators $R_n := L_n S_n L_n$ are projections, $R_n \in \mathcal{L}(\mathbb{E}_n)$, and $\dim \operatorname{im} R_n = \sum\limits_{i=1}^{m} k_i = \alpha(\{A_n\})$ for all sufficiently large n. Moreover, for any $x \in \mathbb{E}_n$, we have

$$\|A_n R_n L_n x\|_{\mathbb{E}} = \Big\|A_n \sum_{i=1}^{m}\sum_{l=1}^{k_i} f^n_{i,l}(x)x^n_{i,l}\Big\|_{\mathbb{E}} \leqslant \sum_{i=1}^{m}\sum_{l=1}^{k_i} |f^n_{i,l}(x)|\Big\|A_n \frac{1}{\|E^{t_i}\|}E^{t_i}_n L^{t_i}_n x_{i,l}\Big\|_{\mathbb{E}}$$

$$\leqslant \ \text{const}\ \|x\|_{\mathbb{E}} \sum_{i=1}^{m}\sum_{l=1}^{k_i} \|E^{-t_i}_n A_n E^{t_i}_n L^{t_i}_n x_{i,l}\|_{\mathbb{E}^{t_i}}. \tag{6.2}$$

Since, for each i, $E^{-t_i}_n A_n E^{t_i}_n L^{t_i}_n$ converges strongly to the operator $W^{t_i}\{A_n\}$, it follows that

$$\|E^{-t_i}_n A_n E^{t_i}_n L^{t_i}_n x_{i,l}\|_{\mathbb{E}^{t_i}} \to \|W^{t_i}\{A_n\}x_{i,l}\|_{\mathbb{E}^{t_i}} = 0, \quad 1 \leqslant i \leqslant m,\ 1 \leqslant l \leqslant k_i.$$

Hence, we conclude that

$$\begin{aligned} s_{\alpha(\{A_n\})}(A_n) &= \inf\{\|A_n + F\|_{\mathcal{L}(\mathbb{E}_n)} : F \in \mathcal{F}_{m(n)-\alpha(\{A_n\})}(\mathbb{E}_n)\} \\ &\leqslant \|A_n - A_n(L_n - R_n)\|_{\mathcal{L}(\mathbb{E}_n)} = \|A_n R_n L_n\|_{\mathcal{L}(\mathbb{E}_n)} \to 0. \end{aligned}$$

Thus, we have proved the following result.

LEMMA 6.2. *If a sequence $\{A_n\} \in \mathcal{A}^T$ is Fredholm, then the $\alpha(\{A_n\})$-th approximation number of A_n tends to zero as $n \to \infty$.*

Note that the inequality (6.2) gives us an estimate for the convergence speed of the $\alpha(\{A_n\})$-th approximation number of A_n to zero.

7. THE $(\alpha(\{A_n\})+1)$-ST APPROXIMATION NUMBER FOR A FREDHOLM SEQUENCE $\{A_n\} \in \mathcal{A}^T$

Let $\{A_n\} \in \mathcal{A}^T$ be a Fredholm sequence. In this section we show that the $(\alpha(\{A_n\}) + 1)$-st approximation number of A_n is bounded away from zero.

First we recall the concept of the generalized invertibility. Let $\mathbb{F}$ be a Banach space. An operator $A^{(-1)} \in \mathcal{L}(\mathbb{F})$ is said to be a generalized inverse of $A \in \mathcal{L}(\mathbb{F})$

if the equality $AA^{(-1)}A = A$ holds. One can show that if $A^{(-1)}$ is a generalized inverse of A then the following relations take place

$$A^{(-1)}A = I - P_{\ker A} \quad \text{and} \quad AA^{(-1)} = P_{\operatorname{im} A}, \tag{7.1}$$

where $P_{\ker A}$ and $P_{\operatorname{im} A}$ are projections onto the kernel and the image of A, respectively. Notice that any Fredholm operator is generalized invertible. Moreover, it is not hard to prove the next lemma.

LEMMA 7.1. *Let A be a Fredholm operator on a Banach space $\mathbb{F}$. For given a generalized inverse $A^{(-1)}$ and a right regularizer B of A $(I - AB \in \mathcal{K}(\mathbb{F}))$, there is a compact operator $X \in \mathcal{K}(\mathbb{F})$ such that $A^{(-1)} = B + X$.*

LEMMA 7.2. *If a sequence $\{A_n\} \in \mathcal{A}^T$ is Fredholm, then*

$$\liminf_n s_{\beta(\{A_n\})+1}(A_n) > 0.$$

For a proof of this lemma we recall the well-known result.

LEMMA 7.3. *Let $\mathbb{F}$ be a Banach space. If $A, A_n \in \mathcal{L}(\mathbb{F}), K \in \mathcal{K}(\mathbb{F})$, and $A_n \to A$ strongly, then $\|A_nK - AK\| \to 0$ as $n \to \infty$.*

Proof of Lemma 7.2. Due to (5.2), there exist sequences $\{B_n\} \in \mathcal{A}^T, \{G_n\} \in \mathcal{G}$, a finite subset $\{\tau_1, \dots, \tau_l\}$ of T, and compact operators $K^{\tau_i} \in \mathcal{K}(\mathbb{E}^{\tau_i})$ such that

$$\{A_n\}\{B_n\} = \{L_n\} + \sum_{i=1}^{l} \{E_n^{\tau_i} L_n^{\tau_i} K^{\tau_i} E_n^{-\tau_i}\} + \{G_n\}.$$

Hence, for each $1 \leqslant i \leqslant l$ the operator $W^{\tau_i}\{B_n\}$ is a right regularizer of $W^{\tau_i}\{A_n\}$ and

$$\beta(\{A_n\}) = \sum_{i=1}^{l} \dim \operatorname{coker} W^{\tau_i}\{A_n\}.$$

Further, let $(W^{\tau_i}\{A_n\})^{(-1)}$ be generalized inverses of $W^{\tau_i}\{A_n\})$. By Lemma 7.1, there are compact operators $X^{\tau_i} \in \mathcal{K}(\mathbb{E}^{\tau_i})$ such that

$$(W^{\tau_i}\{A_n\})^{(-1)} = W^{\tau_i}\{B_n\} + X^{\tau_i}, \quad i = 1, \dots, l.$$

In view of (7.1), we obtain for any $i = 1, \dots, l$

$$\begin{aligned} P_{\operatorname{im} W^{\tau_i}\{A_n\}} &= W^{\tau_i}\{A_n\}(W^{\tau_i}\{A_n\})^{(-1)} = W^{\tau_i}\{A_n\}(W^{\tau_i}\{B_n\} + X^{\tau_i}) \\ &= I + K^{\tau_i} + W^{\tau_i}\{A_n\}X^{\tau_i}. \end{aligned} \tag{7.2}$$

Now we introduce the sequence $\{C_n\} \in \mathcal{A}^T$ by

$$\{C_n\} := \{B_n\} + \sum_{i=1}^{l} \{E_n^{\tau_i} L_n^{\tau_i} X^{\tau_i} E_n^{-\tau_i}\}.$$

Taking into account Lemma 7.3 and (7.2), we get

$$
\begin{aligned}
\{A_n\}\{C_n\} &= \{L_n\} + \sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}K^{\tau_i}E_n^{-\tau_i}\} + \{G_n\} + \{A_n\}\sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}X^{\tau_i}E_n^{-\tau_i}\} \\
&= \{L_n\} + \sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}K^{\tau_i}E_n^{-\tau_i}\} + \{G_n\} + \sum_{i=1}^{l}\{E_n^{\tau_i}E_n^{-\tau_i}A_nE_n^{\tau_i}L_n^{\tau_i}X^{\tau_i}E_n^{-\tau_i}\} \\
&= \{L_n\} + \sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}(K^{\tau_i} + W^{\tau_i}\{A_n\}X^{\tau_i})E_n^{-\tau_i}\} + \{\widehat{G}_n\} \\
&= \{L_n\} - \sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}(I - P_{\operatorname{im} W^{\tau_i}\{A_n\}})E_n^{-\tau_i}\} + \{\widehat{G}_n\},
\end{aligned}
$$

with a sequence $\{\widehat{G}_n\} \in \mathcal{G}$. Since $\|\widehat{G}_n\| \to 0$, we conclude that for all sufficiently large n the matrix $L_n + \widehat{G}_n$ is invertible and $\|(L_n + \widehat{G}_n)^{-1}\| \leqslant 2$. Thus, for all n large enough

$$
\begin{aligned}
0 < \frac{1}{2} &\leqslant \|(L_n + \widehat{G}_n)^{-1}\|^{-1} = s_1(L_n + \widehat{G}_n) \\
&= \inf\{\|L_n + \widehat{G}_n + F\| : F \in \mathcal{F}_{m(n)-1}(\mathbb{E}_n)\} \\
&\leqslant \inf\Big\{\Big\|L_n + \widehat{G}_n + F - \sum_{i=1}^{l}\{E_n^{\tau_i}L_n^{\tau_i}(I - P_{\operatorname{im} W^{\tau_i}\{A_n\}})E_n^{-\tau_i}\}\Big\| : \\
&\qquad\qquad\qquad F \in \mathcal{F}_{m(n)-\beta(\{A_n\})-1}(\mathbb{E}_n)\Big\} \\
&= \inf\{\|A_nC_n + F\| : F \in \mathcal{F}_{m(n)-\beta(\{A_n\})-1}(\mathbb{E}_n)\} \\
&\leqslant \inf\{\|A_nC_n + FC_n\| : F \in \mathcal{F}_{m(n)-\beta(\{A_n\})-1}(\mathbb{E}_n)\} \\
&\leqslant \sup_n \|C_n\| \inf\{\|A_n + F\| : F \in \mathcal{F}_{m(n)-\beta(\{A_n\})-1}(\mathbb{E}_n)\} \\
&\leqslant \text{ const } s_{\beta(\{A_n\})+1}(A_n). \quad \blacksquare
\end{aligned}
$$

We complete this section by the following result which shows that, in the case at hand, the nullity and the deficiency of a Fredholm sequence coincide.

THEOREM 7.4. *The index of a Fredholm sequence is equal to zero.*

Proof. Let $\{A_n\} \in \mathcal{A}^T$ be a Fredholm sequence. Comparing Lemma 6.2 with Lemma 7.2, we conclude that $\beta(\{A_n\}) \geqslant \alpha(\{A_n\})$. In other words, $\operatorname{ind}(\{A_n\}) \leqslant 0$. Further, let $\{B_n\} \in \mathcal{A}^T$ be such that

$$\{B_n\}\{A_n\} = \{L_n\} + \{J_n\} \quad \text{and} \quad \{A_n\}\{B_n\} = \{L_n\} + \{K_n\},$$

where $\{J_n\}, \{K_n\} \in \mathcal{J}^T$. Hence, the sequence $\{B_n\}$ is also Fredholm and $\operatorname{ind}(\{B_n\}) \leqslant 0$. On the other hand, we have

$$\operatorname{ind}(\{B_n\}) + \operatorname{ind}(\{A_n\}) = \operatorname{ind}(\{B_n\}\{A_n\}) = \operatorname{ind}(\{L_n\} + \{J_n\}) = \operatorname{ind}(\{L_n\}) = 0.$$

Thus, $\operatorname{ind}(\{A_n\}) = -\operatorname{ind}(\{B_n\}) = 0. \quad \blacksquare$

COROLLARY 7.5. *If a sequence $\{A_n\} \in \mathcal{A}^T$ is Fredholm, then*

$$\liminf_{n\to\infty} s_{\alpha(\{A_n\})+1}(A_n) > 0.$$

8. SOME EXAMPLES

In this section we illustrate the results of this paper by a few examples.

8.1. EXAMPLE 1 : PROJECTION METHODS FOR OPERATORS $\alpha I + K$. Let $\mathbb{E}$ be an infinite dimensional Banach space, and let $\{L_n\}$ and $\{R_n\}$ be two sequences of finite dimensional projections on $\mathbb{E}$ with $\operatorname{im} R_n = \operatorname{im} L_n$ and $L_n, R_n \to I$ strongly as $n \to \infty$.

We denote by T_1 the index set $\{1\}$ and by $\mathcal{F}^{T_1}$ the Banach algebra of all operator sequences $\{A_n\}, A_n \in \mathcal{L}(\operatorname{im} L_n)$, for which there exists

$$\text{s–lim}\, A_n =: W^1\{A_n\}.$$

Further we introduce the subset $\mathcal{J}^1 = \mathcal{J}^{T_1}$ of the Banach algebra $\mathcal{F}^{T_1}$

$$\mathcal{J}^1 = \{\{L_n K L_n\} + \{G_n\} \quad : K \in \mathcal{K}(\mathbb{E}), \|G_n\| \to 0\}.$$

Now let $\mathcal{A}_1$ stand for the set of all sequences $\{A_n\} \in \mathcal{F}^{T_1}$ having the form

$$A_n = \alpha L_n + R_n K L_n + G_n$$

with a compact operator $K \in \mathcal{K}(\mathbb{E}), \alpha \in \mathbb{C}$, and $\|G_n\| \to 0$. One can check that $\mathcal{A}_1$ is a subalgebra of $\mathcal{F}^{T_1}$ and that $\mathcal{J}^1 = \mathcal{J}^{T_1}$ forms a closed two-sided ideal of $\mathcal{A}_1$.

THEOREM 8.1. *Let $\{A_n\} \in \mathcal{A}_1$, i.e. $\{A_n\} = \{\alpha L_n + R_n K L_n + G_n\}$.*

(i) *If $\alpha \neq 0$, then the approximation numbers of A_n have the k-splitting property with $k = \dim\ker(\alpha I + K)$.*

(ii) *If $\alpha = 0$, then $s_l(A_n) \to 0$ for each $l \in \mathbb{N}$.*

Proof. If $\alpha \neq 0$, then the coset $\{A_n\} + \mathcal{J}^{T_1}$ is invertible in the quotient algebra $\mathcal{A}_1/\mathcal{J}^{T_1}$ (see Lemma 7.3). Thus, the first assertion follows immediately from Theorem 5.4.

If $\alpha = 0$, then $W^1\{A_n\} = K$ is a compact operator. This implies that either $W^1\{A\}$ is not normally solvable or $\dim\operatorname{im} W^1\{A_n\} < \infty$. Hence, in this case we can apply Theorem 3.3 or Corollary 3.2. ∎

8.2. EXAMPLE 2 : FINITE SECTIONS OF TOEPLITZ OPERATORS WITH $C_p + \overline{H}_p^\infty$ SYMBOLS. For $1 < p < \infty$, let ℓ^p_N denote the Banach space of all sequences $x : \mathbb{Z}_+ \to \mathbb{C}^N, \mathbb{Z}_+ = \{i \in \mathbb{Z} : i \geqslant 0\}$, such that

$$\|x\|_{\ell^p_N} := \Big(\sum_{i=0}^{\infty} \|x_i\|_p^p\Big)^{1/p} < \infty,$$

where $\|x_i\|_p^p = \|(x_i^1, x_i^2, \dots, x_i^N)\|_p^p = |x_i^1|^p + |x_i^2|^p + \cdots + |x_i^N|^p$.

Further, let $\mathbb{T}$ be the complex unit circle and let M^p stand for the set of all functions $a \in L^\infty(\mathbb{T})$ for which the Toeplitz operator $T(a)$ is bounded on $\ell^p = \ell^p_1$. We denote by C^p the closure in M^p of the trigonometric polynomials and by $\overline{H}^\infty$ the subset of $L^\infty(\mathbb{T})$ which consists of all functions with vanishing positive Fourier coefficients. Furthermore, let $\overline{H}^\infty_p := \overline{H}^\infty \cap M^p$, $C_p + \overline{H}^\infty_p := \{f + g : f \in C^p, g \in \overline{H}^\infty_p\}$, and $(C_p + \overline{H}^\infty_p)_{N\times N}$ be the collection of $N \times N$-matrices with entries from $C_p + \overline{H}^\infty_p$.

It is clear that if $a \in (C_p + \overline{H}^\infty_p)_{N\times N}$, then the Hankel operator $H(a)$ is compact on ℓ^p_N. Moreover we have

THEOREM 8.2 (see Theorem 2.94 of [2]). *The Toeplitz operator $T(a)$ with $a \in (C_p + \overline{H}^\infty_p)_{N\times N}$ is Fredholm on ℓ^p_n if and only if the function a is invertible in $(C_p + \overline{H}^\infty_p)_{N\times N}$.*

Now, for $n \in \mathbb{N}$, we introduce the projections P_n and operators W_n on the space ℓ^p_N acting by the rule

$$P_n x = P_n(x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots) := (x_0, x_1, \dots, x_{n-1}, 0, 0, \dots),$$
$$W_n x = W_n(x_0, x_1, \dots, x_{n-1}, x_n, x_{n+1}, \dots) := (x_{n-1}, x_{n-2}, \dots, x_0, 0, 0, \dots).$$

We denote by T_2 the index set $\{1, 2\}$ and by $\mathcal{F}^{T_2}$ the Banach algebra of all operator sequences $\{A_n\}$, $A_n \in \mathcal{L}(\operatorname{im} P_n)$, for which there exist

$$\text{s–lim}\, A_n =: W^1\{A_n\} \quad \text{and} \quad \text{s–lim}\, W_n A_n W_n =: W^2\{A_n\}.$$

Further we introduce the subsets $\mathcal{J}^1$ and $\mathcal{J}^2$ of the Banach algebra $\mathcal{F}^{T_2}$

$$\mathcal{J}^1 = \{\{P_n K P_n\} + \{G_n\} \ : K \in \mathcal{K}(\ell^p_N), \|G_n\| \to 0\},$$
$$\mathcal{J}^2 = \{\{W_n K W_n\} + \{G_n\} \ : K \in \mathcal{K}(\ell^p_N), \|G_n\| \to 0\}.$$

Now let $\mathcal{A}_2$ stand for the set of all sequences $\{A_n\} \in \mathcal{F}^{T_2}$ having the form

$$A_n = P_n T(a) P_n + P_n K P_n + W_n L W_n + G_n,$$

where $a \in (C_p + \overline{H}^\infty_p)_{N\times N}$, K and L are compact operators on ℓ^p_N, and $\|G_n\| \to 0$. One can check that $\mathcal{A}_2$ is a subalgebra of $\mathcal{F}^{T_2}_*$ and that each of the sets $\mathcal{J}^1, \mathcal{J}^2$ forms a closed two-sided ideal of $\mathcal{A}_2$.

THEOREM 8.3. *Let $\{A_n\} \in \mathcal{A}_2$, i.e. $\{A_n\} = \{P_n T(a) P_n + P_n K P_n + W_n L W_n + G_n\}$.*

(i) *If the function a is invertible in $(C_p + \overline{H}^\infty_p)_{N\times N}$, then the operators $W^1\{A_n\} = T(a) + K$ and $W^2\{A_n\} = T(\widetilde{a}) + L$ $(\widetilde{a}(t) = a(1/t))$ are Fredholm on ℓ^p_N and the approximation numbers of A_n have the k-splitting property with $k = \dim\ker W^1\{A_n\} + \dim\ker W^2\{A_n\}$.*

(ii) *Otherwise, $s_l(A_n) \to 0$ for each $l \in \mathbb{N}$.*

Proof. By Theorem 8.2 the operator $W^1\{A_n\}$ is Fredholm on ℓ^p_N if and only if the function a is invertible in $(C_p + \overline{H}^\infty_p)_{N\times N}$. From this and the formula (see e.g. 7.7(2) of [2])

$$\{P_nT(a)P_n\}\{P_nT(b)P_n\} = \{P_nT(ab)P_n\} - \{P_nH(a)H(\widetilde{b})P_n\} - \{W_nH(\widetilde{a})H(b)W_n\}$$

we deduce that if the operator $W^1\{A_n\}$ is Fredholm then the coset $\{A_n\} + \mathcal{J}^{T_2}$ is invertible in $\mathcal{A}_2/\mathcal{J}^{T_2}$ (its inverse is the coset $\{P_nT(a^{-1})P_n\} + \mathcal{J}^{T_2}$). Hence $\mathcal{A}_2 \subset \mathcal{A}_F^{T_2} \cap \mathcal{F}_*^{T_2}$ and the assertions of the theorem follow immediately from Theorem 5.6. ∎

REMARK 8.4. Using the same Banach algebra $\mathcal{F}^{T_2}$ one can show that if a function $a \in M^2 = L^\infty(\mathbb{T})$ is locally sectorial over QC (for the definition see e.g. Section 2.84 of [2]) then the singular values for the finite sections $\{P_nT(a)P_n\}$ have the k-splitting property with $k = \dim\ker T(a) + \dim\ker T(\widetilde{a})$.

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