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GRAPH-BASED MODELS FOR KIRCHBERG ALGEBRAS

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ABSTRACT. We give a construction of Kirchberg algebras from graphs. By using product graphs in the construction we are able to provide models for general (UCT) Kirchberg algebras while maintaining the explicit generators and relations of the underlying graphs.

KEYWORDS: *Simple purely infinite C^*-algebra, Kirchberg algebra, K-theory, automorphism group, graph algebra.*

MSC (2000): Primary 46L05, 46L80; Secondary 22A22.

1. INTRODUCTION

The first examples of what are now termed Kirchberg algebras were introduced in Cuntz's 1977 paper [3]. (Following [13] we use the term *Kirchberg algebra* for a separable nuclear simple purely infinite C^*-algebra.) The brilliance of those examples led to huge advances in the field of C^*-algebras, as well as deep connections with other areas of mathematics. In a series of papers Cuntz isolated and studied the key property of *pure infiniteness* and its ramifications for K-theory (we refer to [13] for a detailed bibliography). This study was carried further by many mathematicians, notably Rørdam and Kirchberg. The culmination was the classification theorem of Kirchberg [6], also proved independently by Phillips [10]: the Kirchberg algebras satisfying the universal coefficient theorem are classified by K-theory.

Kirchberg algebras arise in many different contexts. As a result of the classification theorem, examples from different situations may be identified by computing K-theory (see [9] for an example involving dynamical systems). Alternatively, to prove a theorem about Kirchberg algebras in general, one can choose a suitable realization that lends itself to the problem at hand. One of the most useful of these has been the C^*-algebras defined by directed graphs. This idea was implicitly present in Cuntz's original paper, and for finite irreducible graphs was explicit in the papers [5], [4]. The development for arbitrary directed graphs began with the article [8], and has grown to be a mini-industry in itself (see [11] for a comprehensive survey).

There have been few applications of graph algebras to the study of Kirchberg algebras. We mention Szymański's proof [18] that any Kirchberg algebra with free K_1-group can be realized as the C^*-algebra of an irreducible row-finite graph, and the proof in [15] that Kirchberg algebras having finitely generated K-theory and free K_1-group are semiprojective (in the sense of Blackadar). Graph algebras are particularly well-suited for such arguments in that they are defined by generators and simple, highly flexible relations. Their defect is clear in the above-mentioned theorems — graph C^*-algebras have free K_1-groups, and hence cannot be used to model general Kirchberg algebras. More recently, Kumjian and Pask have introduced a notion of higher-rank graphs, or *k-graphs*, and their C^*-algebras. These have many features in common with ordinary (or 1-) graphs, and allow for more general K_1-groups. However they are much less flexible than 1-graphs, and the theory has not yet been developed as extensively.

In this paper we present examples of hybrid objects mixing elements of k-graphs of different ranks. While a general treatment seems beyond current technique (and perhaps not worth the considerable effort), the special situation developed here allows us to model arbitrary Kirchberg algebras with the same flexibility exhibited by ordinary graph algebras. In particular, we use this construction in [16] and [17] to prove interesting properties of general (UCT) Kirchberg algebras:

(i) any prime-order automorphism of the K-theory of a UCT Kirchberg algebra is induced from an automorphism of the algebra having the same order (generalizing work of [2]);

(ii) a UCT Kirchberg algebra is weakly semiprojective if and only if its K-groups are direct sums of cyclic groups.

We hope that the interest of these applications will justify the work required to establish these models.

The rest of the paper is organized as follows. We conclude the Introduction by recalling the basic notions of graph C^*-algebras in a form convenient for our purposes. In part two we construct the hybrid object underlying our algebras, and use it to define an r-discrete groupoid whose unit space is an appropriate set of paths. We remark that because the underlying object is not "row-finite," the paths we use may be finite, infinite, or semi-infinite. In part three we give generators and relations for the C^*-algebra of this groupoid, prove the gauge-invariant uniqueness theorem, and show that the C^*-algebra we have constructed is a UCT Kirchberg algebra. Finally in part four we compute the K-theory of the algebra, showing that it is equal to the direct sum of the K-theory of tensor products of the ordinary graph algebras used in the construction of the underlying object. It is here that the flexibility inherent in graph algebras may be used to construct our models of Kirchberg algebras with arbitrary K-theory.

We now briefly recall the main facts about (ordinary) graph algebras (see [14]). A *directed graph* E consists of two sets, E^0 (the *vertices*) and E^1 (the *edges*), together with two maps $o, t : E^1 \to E^0$ (*origin* and *terminus*). A *path* of length n in E is a string $e_1 e_2 \cdots e_n$ of edges with $t(e_i) = o(e_{i+1})$, $1 \leqslant i < n$. We let E^n denote the set of paths of length n, and E^* the set of all finite paths; the origin and terminus maps extend to E^* in the obvious way. For a vertex $a \in E^0$ we use the notation $E^n(a)$, respectively $E^*(a)$, for the set of paths in E of length n, respectively of arbitrary length, with origin a. We let $\mathcal{O}(E)$ denote the C^*-algebra of E. It is the universal C^*-algebra defined by generators $\{P_a | a \in E^0\}$ and $\{S_e | e \in E^1\}$ with the *Cuntz-Krieger relations*:

(i) $\{P_a | a \in E^0\}$ are pairwise orthogonal projections.

(ii) $S_e^* S_e = P_{t(e)}$, for $e \in E^1$.

(iii) $o(e) = o(f) \Longrightarrow S_e S_e^* + S_f S_f^* \leqslant P_{o(e)}$, for $e, f \in E^1$ with $e \neq f$.

(iv) $0 < \# E^1(a) < \infty \Longrightarrow P_a = \sum \{S_e S_e^* | o(e) = a\}$, for $a \in E^0$.

(These are a variant of the relations given in Theorem 2.21 of [14].)

The relationship between the C^*-algebras of a graph and a subgraph are crucial to our methods. We refer to [14]. The results are as follows. Let E be a graph and let F be a subgraph of E. We let $S = S(F)$ be the set of vertices in F^0 that do not emit more edges in E than in F. We let $\mathcal{TO}(F, S)$ denote the relative Toeplitz Cuntz-Krieger algebra of F in E. It is the universal C^*-algebra defined by generators $\{P_a | a \in F^0\}$ and $\{S_e | e \in F^1\}$ with the relations (as above) for $\mathcal{O}(F)$, modified by requiring the fourth relation only if $a \in S$. Then $\mathcal{TO}(F, S)$ is the C^*-subalgebra of $\mathcal{O}(E)$ generated by the projections and partial isometries associated to the vertices and edges of F ([14], Theorem 2.35).

2. GROUPOID MODELS FOR KIRCHBERG ALGEBRAS

We will construct models for Kirchberg algebras by using a mixture of directed 1-graphs and *k-graphs*, as studied by Kumjian and Pask in [7]. Since neither the results of [7] nor of [14] directly apply in this situation, we will carry out the necessary constructions in detail. Our models will consist of a sequence of product k-graphs connected to each other by (ordinary) 1-graphs. In fact, all details of the argument are already present in the case of two product graphs of rank 2, and it is this case that we will treat. The argument in the general case is essentially identical to the one we will give. (We remark that in the general case one may attach other product k-graphs to u_0 in the same way that $E_1 \times F_1$ is attached in what follows.)

DEFINITION 2.1. Let D be the graph with

$$D^0 = \{u_0,\ u_1,\ a_0,\ a_1\}, \quad D^1 = \{\alpha_0,\ \alpha_1\ \beta_0,\ \ldots,\ \varepsilon_1\}$$

(see Figure 1).

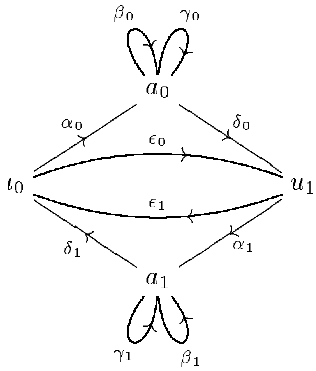

FIGURE 1

For $i = 0, 1$ let E_i and F_i be irreducible graphs. Let L_i, respectively M_i, denote the set of vertices in E_i^0, respectively F_i^0, emitting infinitely many edges. We assume that L_i and M_i are nonempty. We choose distinguished elements $v_i \in L_i$ and $w_i \in M_i$. We attach the 2-graphs $E_i \times F_i$ to the graph D by identifying $u_i \in D^0$ with $(v_i, w_i) \in E_i^0 \times F_i^0$. The entire object comprising D, $E_0 \times F_0$, and $E_1 \times F_1$ will be denoted Ω.

DEFINITION 2.2. By a *vertex* we mean an element of

$$\bigcup_i (E_i^0 \times F_i^0) \cup D^0,$$

where we identify u_i and (v_i, w_i). By an *edge* we mean an element of

$$\Big(\bigcup_i (E_i^1 \times F_i^0) \cup (E_i^0 \times F_i^1)\Big) \cup D^1.$$

DEFINITION 2.3. A *finite path element of type D* is a finite directed path in D of non-zero length. An *infinite path element of type D* is either an infinite directed path in D, or a finite path element of type D which terminates at u_0 or u_1. A *finite path element of type* (E_i, F_i) is an ordered pair $(p, q) \in E_i^* \times F_i^*$ such that p, q are not both of length zero. An *infinite path element of type* (E_i, F_i) is an ordered pair (p, q), where p, respectively q, is either an infinite path or a finite path terminating in L_i, respectively in M_i, in E_i, respectively F_i, and p, q are not both of length zero.

For a path element (p, q) of type (E_i, F_i) we define *origin* and *terminus* by

$$o(p, q) = (o(p), o(q)), \quad t(p, q) = (t(p), t(q)).$$

If (p, q) and (p', q') are path elements of type (E_i, F_i) we say that (p, q) *extends* (p', q') if p extends p' and q extends q' in the usual sense of paths in a directed graph.

A *finite path* is either a vertex, or a finite string $\mu_1 \cdots \mu_k$ of finite path elements such that:

(i) $t(\mu_i) = o(\mu_{i+1})$, and
(ii) μ_i and μ_{i+1} are of different types.

An *infinite path* is either a vertex in $L_i \times M_i$, an infinite string of finite path elements satisfying the conditions (i) and (ii) above, or a finite sequence $\mu_1 \cdots \mu_{k+1}$ such that:

(iii) $\mu_1 \cdots \mu_k$ is a finite path;
(iv) μ_{k+1} is an infinite path element;
(v) (i) and (ii) above hold.

We let X denote the set of all infinite paths. We will use the notation $\mu \preceq \nu$ to indicate that the path ν extends the path μ.

We wish to define a topology on X making it a locally compact metrizable space. First we will define the length function on paths.

DEFINITION 2.4. We define $\ell : \{\text{finite paths}\} \to \mathbb{Z}_+^2$ by

$$\ell(\mu) = \begin{cases} (0,0) & \text{if } \mu \text{ is a vertex,} \\ (\ell(p), \ell(p)) & \text{if } \mu = p \text{ is a finite path element of type } D, \\ (\ell(p), \ell(q)) & \text{if } \mu = (p,q) \text{ is a finite path element of type } (E_i, F_i). \end{cases}$$

(In the right hand side above we have used the symbol ℓ also for the usual length function on paths in a 1-graph. There should be no confusion resulting from this usage.) If $\mu = \mu_1 \cdots \mu_k$ is a finite path, we define

$$\ell(\mu) = \sum_{i=1}^{k} \ell(\mu_k).$$

We also will use the extension of ℓ to infinite paths, with values in $(\mathbb{Z}_+ \cup \{\infty\})^2$, defined in the obvious way.

LEMMA 2.5. *ℓ is additive with respect to concatenation of paths.*

Proof. The proof is left to the reader. ∎

DEFINITION 2.6. Let $\mu = \mu_1\mu_2 \cdots \mu_r$ be a finite path, decomposed as a string of finite path elements satisfying Definition 2.3(i) and (ii). We write $|\mu| = r$, the number of finite path elements in μ.

We wish to define a groupoid having X as its unit space. Since Ω is not a higher rank graph, the space of paths does not have the factorization property (Definition 1.1 of [7]). However we do have the following simple observation.

LEMMA 2.7. *Let μ and μ' be finite paths, and let x, x' be infinite paths, with $\mu x = \mu' x'$. Let $m = \ell(\mu) \vee \ell(\mu')$. (Here $\vee$ represents the usual lattice join in $\mathbb{Z}^2$.) Then there is a factorization $\mu x = \nu y$ with $\ell(\nu) = m$.*

Proof. The proof is accomplished by an easy induction on $\max\{|\mu|, |\mu'|\}$. ∎

DEFINITION 2.8. Let $\mu = \mu_1 \cdots \mu_k$ be a finite path decomposed as a string of finite path elements satisfying Definition 2.3(i) and (ii). Let

$$Z(\mu) = \{\sigma_1\sigma_2 \cdots \in X : \sigma_i = \mu_i \text{ for } i < k,\ \mu_k \preceq \sigma_k\}.$$

We define certain subsets of $Z(\mu)$.

$$V(\mu) = \begin{cases} Z(\mu) & \text{if } t(\mu) \notin \{u_0, u_1\}, \\ Z(\mu) \setminus Z(\mu\alpha_i) \setminus Z(\mu\varepsilon_i) & \text{if } t(\mu) = u_i. \end{cases}$$

If $t(\mu) = (y, z) \in E_i^0 \times F_i^0$, let $B \subseteq E^1(y)$ and $C \subseteq F^1(z)$ be finite subsets such that if $y \notin L_i$ then $B = \emptyset$, and if $z \notin M_i$ then $C = \emptyset$. Then we define:

$$V(\mu; B, C) = V(\mu) \setminus \bigcup_{e \in B} Z(\mu(e, z)) \setminus \bigcup_{f \in C} Z(\mu(y, f))$$

(where $\mu(e, z)$ and $\mu(y, f)$ are the concatenations of the path μ with the edges (e, z) and (y, f)).

Note that for $t(\mu) \in E_i^0 \times F_i^0$, $V(\mu) = V(\mu; \emptyset, \emptyset)$. Also note that if $t(\mu) = u_i$ for some i, then every element of $V(\mu; B, C) \setminus \{\mu\}$ extends μ farther into $E_i \times F_i$.

REMARK 2.9. We mention that the topology we intend to define on X requires that we be able to 'block' finitely many edges originating at a vertex of Ω that emits infinitely many edges. This explains our use of the notation $V(\mu; B, C)$. At a vertex emitting only finitely many edges, we can handle the situation by considering explicitly the finitely many possible extensions by an edge. Since we do not want to block all edges originating at such a vertex, we do not allow B and/or C to be nonempty at such a vertex.

REMARK 2.10. We note the following disjoint unions of sets.

(i) If $t(\mu) = u_i$ for some i, then

$$Z(\mu) = V(\mu) \cup V(\mu\alpha_i) \cup Z(\mu\varepsilon_i).$$

(ii) If $t(\mu) = (y, z) \in E_i^0 \times F_i^0$, then for B and C finite,

$$\begin{aligned} V(\mu) = V(\mu; B, C) & \\ & \cup \bigcup \{Z(\mu(e, f)) : e \in B,\ f \in C\} \\ & \cup \bigcup_{e \in B} V(\mu(e, z); \emptyset, C) \\ & \cup \bigcup \{V(\mu(e, z)\alpha_i) \cup Z(\mu(e, z)\varepsilon_i) : e \in B,\ t(e, z) = u_i\} \\ & \cup \bigcup_{f \in C} V(\mu(y, f); B, \emptyset) \end{aligned}$$

$$\cup \bigcup \{V(\mu(y,f)\alpha_i) \cup Z(\mu(y,f)\varepsilon_i) : f \in C,\ t(y,f) = u_i\},$$
$$V(\mu) = V(\mu; B, \emptyset) \cup \bigcup_{e \in B} Z(\mu(e,z)),$$
$$V(\mu) = V(\mu; \emptyset, C) \cup \bigcup_{f \in C} Z(\mu(y,f)).$$

DEFINITION 2.11. We let $\mathcal{B}$ denote the collection of all $V(\mu; B, C)$ and $Z(\mu)$. We let $\mathcal{A}$ denote the collection of all finite disjoint unions of sets in $\mathcal{B}$ (hence also $\emptyset \in \mathcal{A}$.)

LEMMA 2.12. *The collections $\mathcal{A}$ and $\mathcal{B}$ have the following properties:*

(i) $\emptyset \notin \mathcal{B}$.

(ii) *$\mathcal{B}$ is countable.*

(iii) *$\mathcal{A}$ is a ring of sets.*

(iv) *The intersection of any decreasing sequence of sets in $\mathcal{B}$ is nonempty.*

Proof. A bit of thought will likely convince the reader that this is elementary. However we have found the details slightly messy, and we warn that the following is a bit of a slog. (i) and (ii) are clear. For (iii), we must show that if A, $A' \in \mathcal{B}$ then $A \cap A'$, $A \setminus A' \in \mathcal{A}$. We will first deal with intersections. Let A, $A' \in \mathcal{B}$. First suppose that $A = Z(\mu)$ and $A' = Z(\mu')$. If $\mu \preceq \mu'$ then $A \cap A' = A'$. So suppose that μ, μ' are not comparable. We then have

$$\mu = \mu_1 \cdots \mu_{k-1}\mu_k\sigma, \quad \mu' = \mu_1 \cdots \mu_{k-1}\mu'_k\sigma', \tag{2.1}$$

decomposed as a string of finite path elements satisfying Definition 2.3(i) and (ii), where μ_k and μ'_k are not equal (and if they are comparable, then σ, respectively σ', is present, according as μ_k, respectively μ'_k is shorter). We have $Z(\mu) \cap Z(\mu') = \emptyset$, unless σ and σ' are absent, μ_k and μ'_k are of type (E, F), and

$$\mu_k = \widetilde{\mu}(e_1 \cdots e_m, z), \quad \mu'_k = \widetilde{\mu}(y, f_1 \cdots f_n), \tag{2.2}$$

with m and n both nonzero. In this case $Z(\mu) \cap Z(\mu') = Z(\mu'')$, where

$$\mu'' = \mu_1 \cdots \mu_{k-1}\widetilde{\mu}(e_1 \cdots e_m,\ f_1 \cdots f_n). \tag{2.3}$$

Next suppose that $A = V(\mu; B, C)$ and $A' = Z(\mu')$. First consider the situation where μ and μ' are comparable. If $\mu' \preceq \mu$, then we have $A \subseteq Z(\mu) \subseteq A'$. If $\mu \precneqq \mu'$, write μ and μ' as

$$\mu = \mu_1 \cdots \mu_k, \quad \mu' = \mu_1 \cdots \mu_{k-1}\mu'_k\sigma_1 \cdots \sigma_j, \quad \mu'_k = \mu_k\widetilde{\mu}. \tag{2.4}$$

There are two cases.

Case (i). Suppose that $\mu_k = \mu'_k$. Then we must have $j > 0$. If σ_1 is of type D then $V(\mu; B, C) \subseteq V(\mu)$, while $Z(\mu') \subseteq Z(\mu\alpha_i) \cup Z(\mu\varepsilon_i)$ for some i. Hence

$A \cap A' = \varnothing$ (Remark 2.10(i)). If σ_1 is of type (E, F) then we have

$$A \cap A' = \begin{cases} Z(\mu') & \text{if } \sigma_1 \text{ obeys the restrictions imposed by } B \text{ and } C, \\ \varnothing & \text{otherwise.} \end{cases}$$

Case (ii). Suppose that $\mu_k \precneqq \mu'_k$. If μ_k and μ'_k are of type D then we have

$$A \cap A' = \begin{cases} A' & \text{if } t(\mu) \in \{a_0, a_1\}, \\ \varnothing & \text{otherwise,} \end{cases}$$

for the same reason as in Case (i). If μ_k and μ'_k are of type (E, F) we assume that $\widetilde{\mu}$ obeys the restrictions imposed by B and C (since otherwise we would have $A \cap A' = \varnothing$). If $j > 0$ or if $\ell(\widetilde{\mu}) \geqslant (1, 1)$ then $A' \subseteq A$. Otherwise we have $j = 0$ and $\ell(\widetilde{\mu}) = (m, n)$ with exactly one of m, n equal to zero. For definiteness suppose $n = 0$. Then

$$A \cap A' = \begin{cases} V(\mu'; \varnothing, C) & \text{if } t(\mu') \notin \{u_0, u_1\}, \\ V(\mu'; \varnothing, C) \cup V(\mu'\alpha_i) \cup Z(\mu'\varepsilon_i) & \text{if } t(\mu') = u_i \text{ for some } i. \end{cases}$$

Now consider the situation where μ and μ' are not comparable. Write μ and μ' as in (2.1). Since $A \subseteq Z(\mu)$, we have $A \cap A' = \varnothing$ unless we are in the situation of (2.2). In this case let μ'' be as in (2.3). We have

$$\begin{aligned} A \cap A' &= V(\mu; B, C) \cap Z(\mu') = V(\mu; B, C) \cap Z(\mu'') \\ &= \begin{cases} \varnothing & \text{if } f_1 \in C, \\ V(\mu''; B, \varnothing) & \text{if } f_1 \notin C,\, t(\mu'') \notin \{u_0, u_1\}, \\ V(\mu''; B, \varnothing) \cup V(\mu''\alpha_i) \cup Z(\mu''\varepsilon_i) & \text{if } f_1 \notin C,\, t(\mu'') = u_i, \text{ some } i. \end{cases} \end{aligned}$$

Finally we let $A = V(\mu; B, C)$ and $A' = V(\mu'; B', C')$. Again, we first consider the situation where μ and μ' are comparable. If $\mu = \mu'$ then we have

$$A \cap A' = V(\mu; B \cup B', C \cup C').$$

Suppose now that $\mu \precneqq \mu'$. Write μ and μ' as in (2.4). There are two cases.

Case (iii). Suppose that $\mu_k = \mu'_k$. Then $j > 0$. If σ_1 is of type D then $A \cap A' = \varnothing$, since $A' \subseteq Z(\mu')$. If σ_1 is of type (E, F), and if $\ell(\sigma_1, \ldots, \sigma_j) \geqslant (1, 1)$, then

$$A \cap A' = \begin{cases} A' & \text{if } \sigma_1 \text{ obeys the restrictions imposed by } B \text{ and } C, \\ \varnothing & \text{otherwise.} \end{cases}$$

If $j = 1$ and $\ell(\sigma_1) = (m, n)$ where exactly one of m, n is nonzero, suppose without loss of generality that $n = 0$. Then

$$A \cap A' = \begin{cases} V(\mu'; B', C \cup C') & \text{if } \sigma_1 \text{ obeys the restrictions imposed by } B \text{ and } C, \\ \varnothing & \text{otherwise.} \end{cases}$$

Case (iv). (This case is nearly identical to the earlier Case (ii).) Suppose that $\mu_k \npreceq \mu'_k$. If μ_k and μ'_k are of type D then we have

$$A \cap A' = \begin{cases} A' & \text{if } t(\mu) \in \{a_0, a_1\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

If μ_k and μ'_k are of type (E, F) we assume that $\widetilde{\mu}$ obeys the restrictions imposed by B and C (since otherwise we would have $A \cap A' = \varnothing$). If $j > 0$ or if $\ell(\widetilde{\mu}) \geqslant (1,1)$ then $A' \subseteq A$. Otherwise we have $j = 0$ and $\ell(\widetilde{\mu}) = (m, n)$ with exactly one of m, n equal to zero. For definiteness suppose $n = 0$. Then

$$A \cap A' = V(\mu'; B', C \cup C').$$

Now consider the case where μ and μ' are not comparable. Write μ and μ' as in (2.1). As before, $A \cap A' = \varnothing$ unless we have (2.2) with σ and σ' absent. In this case we have

$$A \cap A' = V(\mu'', B, C'),$$

where μ'' is as in (2.3). This finishes the proof that $\mathcal{A}$ is closed under intersections.

We now show that $A \setminus A' \in \mathcal{A}$. We first suppose that $A = Z(\mu)$ and $A' = Z(\mu')$. If $\mu \preceq \mu'$ let μ and μ' be as in (2.4). We have a disjoint union:

$$A \setminus A' = (Z(\mu) \setminus Z(\mu\widetilde{\mu})) \cup \bigcup_{0 \leqslant i < j} (Z(\mu\widetilde{\mu}\sigma_1 \cdots \sigma_i) \setminus Z(\mu\widetilde{\mu}\sigma_1 \cdots \sigma_{i+1})).$$

Thus it suffices to consider $Z(\tau) \setminus Z(\tau\sigma)$. If $\sigma = e_1 \cdots e_m$ is of type D, then $Z(\tau) \setminus Z(\tau\sigma) = \bigcup_{0 \leqslant i < m} (Z(\tau e_1 \cdots e_i) \setminus Z(\tau e_1 \cdots e_{i+1}))$, and for $e \in D^1$ we have

$$Z(\tau) \setminus Z(\tau e) = \begin{cases} V(\tau) \cup Z(\tau e') & \text{if } t(\tau) = u_i, \text{ where } \{e'\} = \{\alpha_i, \varepsilon_i\} \setminus \{e\}, \\ \bigcup\{Z(\tau f) : f \in \{\beta_i, \gamma_i, \delta_i\} \setminus \{e\}\} & \text{if } t(\tau) = a_i. \end{cases}$$

If $\sigma = (e_1 \cdots e_m, f_1 \cdots f_n)$ is of type (E, F), then

$$\begin{aligned} &Z(\tau) \setminus Z(\tau\sigma) \\ &\quad = \bigcup\{V(\tau\nu\alpha_i) \cup Z(\tau\nu\varepsilon_i) : o(\sigma) \npreceq \nu \npreceq \sigma,\ t(\nu) = u_i,\ i \in \{0,1\}\} \\ &\qquad \cup \{V(\tau(e_1 \cdots e_i,\ f_1 \cdots f_j); \{e_{i+1}\}, \{f_{j+1}\}) : \\ &\qquad\qquad 0 \leqslant i \leqslant m,\ 0 \leqslant j \leqslant n,\ i + j < m + n\}, \end{aligned} \tag{2.5}$$

where in the second term we let $\{e_{m+1}\}$ and $\{f_{n+1}\}$ denote the empty set.

If μ and μ' are not comparable, an earlier part of the proof implies that $A \cap A' = \varnothing$ unless μ and μ' are as in (2.1) and (2.2), with σ and σ' absent. Then

$$A \setminus A' = \bigcup_{0 \leqslant j < n} Z(\mu(y, f_1 \cdots f_j)) \setminus Z(\mu(y, f_1 \cdots f_{j+1})),$$

and the differences in the above union are treated by (2.5).

Next suppose that $A = V(\mu; B, C)$ and $A' = Z(\mu')$. We have a disjoint union:

$$A' \setminus A = (Z(\mu') \setminus Z(\mu)) \cup (Z(\mu') \cap (Z(\mu) \setminus V(\mu; B, C))).$$

We have already proved that the first piece is in $\mathcal{A}$. We see by Remark 2.10 that $Z(\mu) \setminus V(\mu; B, C) \in \mathcal{A}$. Thus $A' \setminus A \in \mathcal{A}$ by what we have proved for intersections of sets in $\mathcal{B}$. A similar argument applies to $A \setminus A'$ in view of $A \setminus A' = (Z(\mu) \setminus Z(\mu')) \cap V(\mu; B, C)$. In the case $A = V(\mu; B, C)$ and $A' = V(\mu'; B', C')$, a similar argument together with the above case applies:

$$A \setminus A' = (Z(\mu) \setminus V(\mu'; B', C')) \cap V(\mu; B, C).$$

This completes the proof of item (iii) of the lemma.

Finally we prove (iv). Let $A_1 \supseteq A_2 \supseteq \cdots$, with $A_i = V(\mu_i; B_i, C_i)$ or $Z(\mu_i)$. From the proof of (iii) we have that $\mu_1 \preceq \mu_2 \preceq \cdots$. Then $x = \lim_i \mu_i \in \bigcap_1^\infty A_i$. ∎

LEMMA 2.13. *Let $\mathcal{E} \subseteq \mathcal{A}$ have the finite intersection property. Then $\bigcap \mathcal{E} \neq \emptyset$.*

Proof. Since $\mathcal{A}$ is countable, we may list the elements of $\mathcal{E}$ as A_1, A_2, ... Thus our assumption on $\mathcal{E}$ takes the form $\bigcap_{i=1}^p A_i \neq \emptyset$ for all p. We will construct elements $A_1', A_2', \ldots \in \mathcal{B}$ such that $A_i' \subseteq A_i$ and $A_1' \supseteq A_2' \supseteq \cdots$. Then the lemma will follow by Lemma 2.12(iv).

Let $A_1 = \bigcup_{j=1}^{k_1} A_{1j}$ be written as a disjoint union of elements of $\mathcal{B}$. We claim that there exists j_1 such that for all $p \geqslant 2$, $A_{1,j_1} \cap \bigcap_{i=2}^p A_p \neq \emptyset$. For if not, then for all $j = 1, \ldots, k_1$ there exists p_j such that $A_{1j} \cap \bigcap_{i=2}^{p_j} A_i = \emptyset$. Let $p = \max\{p_1, \ldots, p_{k_1}\}$. Then $A_{1j} \cap \bigcap_{i=2}^p A_i = \emptyset$ for $j = 1, \ldots, k_1$. Hence $A_1 \cap \bigcap_{i=2}^p A_i = \emptyset$, a contradiction. We set $A_1' = A_{1,j_1}$.

Now suppose inductively that we have found $A_1', A_2', \ldots, A_{n-1}' \in \mathcal{B}$ such that $A_i' \subseteq A_i$, $A_{i-1}' \supseteq A_i'$, and $A_{n-1}' \cap \bigcap_{i=n}^p A_i \neq \emptyset, p \geqslant n$. For $i \geqslant n$ let $A_i'' = A_i \cap A_{n-1}'$. Then $A_i'' \subseteq A_{n-1}'$ and $\bigcap_{i=n}^p A_i'' \neq \emptyset$ for all $p \geqslant n$. Let $A_n'' = \bigcup_{j=1}^{k_n} A_{nj}''$ be written as a disjoint union of elements of $\mathcal{B}$. We claim that there exists j_n such that for all $p \geqslant n+1$, $A_{n,j_n}'' \cap \bigcap_{i=n+1}^p A_i'' \neq \emptyset$. For suppose not. Then for all $j = 1, \ldots, k_n$ there exists $p_j \geqslant n+1$ such that $A_{nj}'' \cap \bigcap_{i=n+1}^{p_j} A_i'' = \emptyset$. Let $p = \max_{1 \leqslant j \leqslant k_n} \{p_j\}$. Then $A_{n,j}'' \cap \bigcap_{i=n+1}^p A_i'' = \emptyset$ for $j = 1, \ldots, k_n$. Hence $A_n'' \cap \bigcap_{i=n+1}^p A_i'' = \emptyset$, a contradiction. Therefore j_n exists as claimed, and we set $A_n' = A_{n,j_n}''$. ∎

LEMMA 2.14. *The collection, $\mathcal{B}$, of Definition* 2.11, *is a base for a locally compact metrizable topology on X.*

Proof. It follows from Lemma 2.12(ii) and (iii) that $\mathcal{B}$ is a base for a second countable topology on X. Thus metrizability will follow from local compactness and the Hausdorff property. To establish local compactness we must show that if $A \in \mathcal{B}$ is covered by $\mathcal{U} \subseteq \mathcal{B}$, then A is finitely covered by $\mathcal{U}$. Suppose not. Then for every finite subcollection $\mathcal{F} \subseteq \mathcal{U}$,

$$A \setminus \bigcup \mathcal{F} = \bigcap_{U \in \mathcal{F}} (A \setminus U) \neq \emptyset.$$

Let $\mathcal{E} = \{A \setminus U | U \in \mathcal{U}\} \subseteq \mathcal{A}$. Since $\mathcal{E}$ has the finite intersection property, Lemma 2.13 implies that $\bigcap \mathcal{E} \neq \emptyset$. But $\bigcap \mathcal{E} = A \setminus \bigcup \mathcal{U} = \emptyset$, a contradiction.

To verify the Hausdorff property, let $x \neq x'$ in X. If x and x' are comparable, say $x \preceq x'$. Then there exists a finite path element μ such that $\mu \preceq x$ and $\mu \not\preceq x'$. Then $Z(\mu)$ and $Z(o(x)) \setminus Z(\mu)$ are disjoint neighborhoods of x and x'. If x and x' are not comparable, write $x = \nu x_1$ and $x' = \nu x_1'$, where x_1 and x_1' have no common nonzero initial subpath. Let ν_1 and ν_1' be nonzero initial finite subpaths of x_1 and x_1'. Then $Z(\nu\nu_1)$ and $Z(\nu\nu_1') \setminus Z(\nu\nu_1)$ are disjoint neighborhoods of x and x'. ∎

DEFINITION 2.15. Let G be the set of triples (x, n, y) in $X \times \mathbb{Z}^2 \times X$ such that there exists $z \in X$ and decompositions $x = \mu z$, $y = \nu z$ with $\ell(\mu) - \ell(\nu) = n$.

LEMMA 2.16. *G is a groupoid with the operations*

$$(x, n, y)(y, m, z) = (x, n + m, z), \quad (x, n, y)^{-1} = (y, -n, x).$$

Proof. It suffices to show that if (x, n, y) and (y, m, z) are in G then so are $(x, n + m, z)$ and $(y, -n, x)$. It is clear that $(y, -n, x) \in G$. Let $x = \mu\zeta$, $y = \nu\zeta = \nu'\zeta'$, and $z = \sigma\zeta'$, where ζ, $\zeta' \in X$ and μ, ν, ν', σ are finite paths with $\ell(\mu) - \ell(\nu) = n$ and $\ell(\nu') - \ell(\sigma) = m$. Applying Lemma 2.7 to the equality $\nu\zeta = \nu'\zeta'$ we have a decomposition $\nu\zeta = \lambda\xi$, where $\xi \in X$ and λ is a finite path such that $\ell(\lambda) = \ell(\nu) \vee \ell(\nu')$. Then $\lambda = \nu\eta = \nu'\eta'$ for some finite paths η and η'. We obtain

$$\nu\zeta = \lambda\xi = \nu\eta\xi, \text{ so } \zeta = \eta\xi, \quad \nu'\zeta' = \lambda\xi = \nu'\eta'\xi, \text{ so } \zeta' = \eta'\xi.$$

Then $x = \mu\zeta = \mu\eta\xi$, $z = \sigma\zeta' = \sigma\eta'\xi$, and

$$\begin{aligned} \ell(\mu\eta) - \ell(\sigma\eta') &= \ell(\mu) + \ell(\eta) - \ell(\eta') - \ell(\sigma) \\ &= \ell(\mu) + \ell(\nu') - \ell(\nu) - \ell(\sigma), \quad \text{since } \nu\eta = \nu'\eta', \\ &= n + m. \quad ∎ \end{aligned}$$

DEFINITION 2.17. Consider the collection of subsets of G of the forms

$$\begin{aligned} U(\mu_1, \mu_2) &= (Z(\mu_1) \times \{\ell(\mu_1) - \ell(\mu_2)\} \times Z(\mu_2)) \cap G, \\ U_0(\mu_1, \mu_2; B, C) &= (V(\mu_1; B, C) \times \{\ell(\mu_1) - \ell(\mu_2)\} \times V(\mu_2; B, C)) \cap G, \end{aligned}$$

where μ_1, μ_2 are finite paths with $t(\mu_1) = t(\mu_2)$, and B, C are as in Definition 2.8. We will also write $U_0(\mu_1, \mu_2)$ for $U_0(\mu_1, \mu_2; \emptyset, \emptyset)$. It is easy to check that this collection is a base (of compact-open G-sets) for a locally compact, Hausdorff, totally disconnected topology making G into an r-discrete groupoid. The map $c : (x, n, y) \in G \mapsto n \in \mathbb{Z}^2$ is clearly a continuous cocycle.

LEMMA 2.18. *G is topologically free, minimal, and locally contractive.*

Proof. The proof is virtually identical to the (easy) proofs for graph algebras in [14]. ∎

COROLLARY 2.19. *$C^*_{\mathrm{r}}(G)$ is simple and purely infinite.*

Proof. This follows from [12] and [1] (see also [9]). ∎

We next define the gauge action of $\mathbb{T}^2$ on $C^*(G)$. We will use the notation ζ_m, $m \in \mathbb{Z}^2$, for the characters of $\mathbb{T}^2$: $\zeta_m(z) = z_1^{m_1} z_2^{m_2}$.

DEFINITION 2.20. The gauge action $\alpha : \mathbb{T}^2 \to \mathrm{Aut}(C^*(G))$ is dual to the cocycle c. Thus for $f \in C_{\mathrm{c}}(G)$, $\alpha_z(f)(g) = \zeta_{c(g)}(z) f(g)$.

3. GENERATORS AND RELATIONS

For a finite path μ we let $s_\mu \in C_{\mathrm{c}}(G)$ denote the partial isometry

$$s_\mu = \chi_{U(\mu, t(\mu))}.$$

LEMMA 3.1. *$C^*(G)$ is generated by the set of all s_μ. Moreover, if $t(\mu_1) = o(\mu_2)$ then $s_{\mu_1\mu_2} = s_{\mu_1} s_{\mu_2}$.*

Proof. We first note that

$$U(\mu_1, \mu_2) = U(\mu_1, t(\mu_1)) \cdot U(\mu_2, t(\mu_2))^{-1},$$

so that $\chi_{U(\mu_1,\mu_2)} = s_{\mu_1} s^*_{\mu_2}$. From Remark 2.10(i) we have

$$\chi_{U_0(\mu, t(\mu))} = \begin{cases} s_\mu & \text{if } t(\mu) \notin \{u_0, u_1\}, \\ s_\mu - s_{\mu\alpha_i} - s_{\mu\varepsilon_i} & \text{if } t(\mu) = u_i. \end{cases}$$

It follows from Remark 2.10(ii) that $\chi_{U_0(\mu, t(\mu); B, C)}$ is in the span of the s_μ. Finally we note that $U_0(\mu_1, \mu_2; B, C) = U_0(\mu_1, t(\mu_1); B, C) \cdot U_0(\mu_2, t(\mu_2); B, C)^{-1}$, and hence that $\chi_{U_0(\mu_1,\mu_2;B,C)} = \chi_{U_0(\mu_1,t(\mu_1);B,C)} \cdot (\chi_{U_0(\mu_2,t(\mu_2);B,C)})^*$ is in the span of the s_μ.

The last claim follows from the fact that

$$U(\mu_1\mu_2, t(\mu_2)) = U(\mu_1, t(\mu_1)) \cdot U(\mu_2, t(\mu_2)). \quad ∎$$

In the sequel we will let A denote $C^*(G)$. We wish to give a presentation of A by generators and relations. For this we recall the hybrid graph structure of Ω given in Definition 2.2.

DEFINITION 3.2. We write

$$\Omega^{(i,j)} = \{\mu \text{ a finite path } : \ell(\mu) = (i,j)\}$$

(cf. Definition 2.3). Thus $\Omega^{(0,0)}$ is the set of vertices in Ω, while the set of edges is $\Omega^{(1,0)} \cup \Omega^{(0,1)} \cup D^1$. We write

$$\Omega^* = \bigcup_{i,j} \Omega^{(i,j)}$$

for the set of all finite paths. For $x \in \Omega^{(0,0)}$ we will write $\Omega^{(i,j)}(x) = \{\mu \in \Omega^{(i,j)} : o(\mu) = x\}$.

We remark that this notation is patterned on that of [7]. Note that the "origin" of a path in a graph corresponds to its "range" when it is thought of as a morphism in a small category.

DEFINITION 3.3. We let $\mathcal{S}$ denote the set of symbols

$$\{P_x |\ x \text{ is a vertex}\} \cup \{S_y |\ y \text{ is an edge}\}.$$

We let $\mathcal{R}$ denote the following set of relations on $\mathcal{S}$:

(i) P_x is a projection for every vertex x, S_y is a partial isometry for every edge y.

(ii) For every $a \in E_i^0$, the projections for $\{a\} \times F_i^0$ and the partial isometries for $\{a\} \times F_i^1$ satisfy the Cuntz-Krieger relations corresponding to the graph F_i (see the discussion at the end of the Section 1).

(ii′) For every $b \in F_i^0$, the projections for $E_i^0 \times \{b\}$ and the partial isometries for $E_i^1 \times \{b\}$ satisfy the Cuntz-Krieger relations corresponding to the graph E_i.

(iii) The projections for D^0 and the partial isometries for D^1 satisfy the Toeplitz-Cuntz-Krieger relations corresponding to the graph D and the vertices $\{a_0, a_1\}$.

(iv) If μ and ν are edges of types D and $E_i \times F_i$, respectively, then $S_\mu^* S_\nu = 0$.

(v) (2-graph structure of $E_i \times F_i$.) For all $e \in E_i^1$ and $f \in F_i^1$ we have

$$S_{(o(e),f)}\, S_{(e,t(f))} = S_{(e,o(f))}\, S_{(t(e),f)}, \quad S_{(t(e),f)}\, S^*_{(e,t(f))} = S^*_{(e,o(f))}\, S_{(o(e),f)}.$$

We let $\Theta = C^*\langle \mathcal{S}, \mathcal{R}\rangle$ denote the universal C^*-algebra given by these generators and relations. For a finite path written as a product of edges: $\mu = y_1 y_2 \cdots y_k$, we let S_μ denote the product $S_{y_1} S_{y_2} \cdots S_{y_k}$ (it follows from the relation (v) that this is unambiguous). It is easily seen from Definition 2.8 and Definition 2.17 that $S_\mu \mapsto s_\mu$ determines a surjective $*$-homomorphism, π, of Θ onto A. We will show below (Corollary 3.19) that π is an isomorphism. First we need to study Θ more closely.

For the next lemma recall from Definition 2.6 the notation $|\mu|$ for the number of finite path elements in μ.

LEMMA 3.4. *Let $\mu, \nu \in \Omega^*$. Suppose that $S_\mu^* S_\nu \neq 0$. Then:*

(i) $t(\mu) = t(\nu)$.

(ii) *If $|\mu| > |\nu|$ then $\nu \preceq \mu$.*

(iii) *If* $|\mu| < |\nu|$ *then* $\mu \preceq \nu$.

(iv) *If* $|\mu| = |\nu| = n$, *then* μ *and* ν *are decomposed into finite path elements satisfying Definition* 2.3(i) *and* (ii), *as*

$$\mu = \mu_1\mu_2\cdots\mu_n, \quad \nu = \mu_1\mu_2\cdots\mu_{n-1}\nu_n,$$

where μ_n *and* ν_n *are of the same type. Moreover, if* μ_n *and* ν_n *are of type* D *then one extends the other, while if they are of type* $E_i \times F_i$ *then in each coordinate one extends the other.*

Proof. (i) follows immediately from Definition 2.3 and Definition 3.3(iv). We prove (ii)–(iv) by induction on $n = \max\{|\mu|, |\nu|\}$. The lemma is easily verified if $n = 1$. Suppose it is true if $\max\{|\mu|, |\nu|\} < n$ where $n > 1$. Assume that $\max\{|\mu|, |\nu|\} = n$; say $|\mu| = n \geqslant |\nu|$. Write $\mu = \mu_1 \cdots \mu_n$ and $\nu = \nu_1 \cdots \nu_s$. We claim that $\nu_1 \preceq \mu_1$. We know that $S^*_{\mu_1} S_{\nu_1} \neq 0$. If μ_1 and ν_1 are of type D, by the inductive hypothesis we must have $\mu_1 \preceq \nu_1$ or $\nu_1 \preceq \mu_1$. But $\mu_1 \precneqq \nu_1$ implies that $S^*_{\mu_2} S^*_{\mu_1} S_{\nu_1} = 0$, a contradiction. Suppose that μ_1 and ν_1 are of type $E_i \times F_i$. If $\nu_1 \not\preceq \mu_1$ then by Definition 3.3, we must have that μ_1 and ν_1 are separately comparable in each coordinate, and that ν_1 properly extends μ_1 in at least one of the coordinates. E.g. suppose $S^*_{\mu_1} S_{\nu_1} = S_{(p,t(q)} S^*_{(t(p),q)}$ with $\ell(p) > 0$. Then $S^*_{\mu_2} S^*_{\mu_1} S_{\nu_1} = S^*_{\mu_2} S_{(p,t(q))} S^*_{(t(p),q)} = 0$, again contradicting the nonzero hypothesis. Moreover, if $|\nu| > 1$ we must have $\mu_1 = \nu_1$. For if not, $S^*_{\mu_1} S_{\nu_1} = S^*_{\mu'_1}$ where μ'_1 is of the same type as ν_1, and $\ell(\mu'_1) \neq (0,0)$. But then $S^*_{\mu'_1} S_{\nu_2} = 0$, since ν_1 and ν_2 are of different types. Therefore if $|\nu| = 1$ we have $\nu = \nu_1 \preceq \mu_1 \preceq \mu$, while if $|\nu| > 1$ we have $S^*_\mu S_\nu = S^*_{\mu_2\cdots\mu_n} S_{\nu_2\cdots\nu_s}$, and the inductive hypothesis finishes the argument. ∎

We give two corollaries that will be needed in the proof of Proposition 3.17 below.

COROLLARY 3.5. *Let* $\mu, \nu \in \Omega^*$ *with* $S^*_\mu S_\nu \neq 0$. *Suppose in addition that* $\ell(\mu) \geqslant \ell(\nu)$. *Then* $\nu \preceq \mu$.

COROLLARY 3.6. *Let* $\mu, \nu \in \Omega^*$ *with* $S^*_\mu S_\nu \neq 0$. *Suppose that* $\ell(\mu) = (j,k)$ *with* $j < k$, *and* $\ell(\nu) \leqslant (k,k)$. *Then either* $\nu \preceq \mu$, *or there are* $p \in E^*_i$ *and* $q \in F^*_i$, *with* $\ell(p) > 0$, *such that* $S^*_\mu S_\nu = S_{(p,t(q))} S^*_{(t(p),q)}$. *An analogous result holds with the roles of the two coordinates in* $E_i \times F_i$ *reversed.*

It follows from Lemma 3.4 that Θ is spanned by elements of the form $S_\mu S^*_\nu$ for which $t(\mu) = t(\nu)$. It also follows from the relations that there is an action, β, of $\mathbb{T}^2$ on Θ, defined by $\beta_z(S_\mu S^*_\nu) = \zeta_{\ell(\mu)-\ell(\nu)}(z)\, S_\mu S^*_\nu$. We note that $\pi : \Theta \to A$ is equivariant for β and α. We make some elementary computations in $\Theta \times_\beta \mathbb{T}^2$. By means of the surjection π we see that the analogous results hold in $A \times_\alpha \mathbb{T}^2$. The elements (in $C(\mathbb{T}^2, \Theta) \subseteq \Theta \times_\beta \mathbb{T}^2$) of the form $\zeta_m S_\mu S^*_\nu$, where $\mu, \nu \in \Omega^*$, make up a total subset of $\Theta \times_\beta \mathbb{T}^2$. The fixed-point algebra Θ^β sits inside $\Theta \times_\beta \mathbb{T}^2$ as the closed linear span of the constant functions $S_\mu S_\nu^*$ for which $\ell(\mu) = \ell(\nu)$. We recall the formulas for multiplication and adjoint in $C(\mathbb{T}^2, \Theta) \subseteq \Theta \times_\beta \mathbb{T}^2$:

$$f \cdot g(z) = \int f(v)\beta_v(g(v^{-1}z))\,\mathrm{d}v, \quad f^*(z) = \beta_z(f(z^{-1})^*).$$

DEFINITION 3.7. For $i = 0, 1$, let $E_{i,1} \subseteq E_{i,2} \subseteq \cdots$ and $F_{i,1} \subseteq F_{i,2} \subseteq \cdots$ be finite irreducible non-circuit subgraphs of E_i and F_i, with $\bigcup\limits_k E_{i,k} = E_i$ and $\bigcup\limits_k F_{i,k} = F_i$. Let Ω_k denote the subobject of Ω comprising D, $E_{0,k} \times F_{0,k}$, and $E_{1,k} \times F_{1,k}$. We let $\Omega_k^{(i,j)} = \Omega_k \cap \Omega^{(i,j)}$, and further let

$$X_k = \bigcup_{i,j \leqslant k} \Omega_k^{(i,j)} \quad \text{and} \quad \Theta_k^\beta = \operatorname{span}\{S_\mu S_\nu^* : \mu, \nu \in X_k,\ \ell(\mu) = \ell(\nu)\}.$$

PROPOSITION 3.8. *Θ^β is an AF-algebra, with $\{\Theta_k^\beta : k = 1, 2, \ldots\}$ as an approximating system of finite dimensional C^*-subalgebras.*

Proof. It is clear that $\{S_\mu S_\nu^* : \mu, \nu \in X_k,\ \ell(\mu) = \ell(\nu)\}$ is a finite set, and it follows from Lemma 3.4 that Θ_k^β is a finite dimensional C^*-algebra. Since $\bigcup\limits_k \Theta_k^\beta$ is dense in Θ^β, Θ^β is an AF-algebra. ∎

COROLLARY 3.9. *$A^\alpha = C^*(c^{-1}(0))$ is an AF-algebra.*

Proof. Since $\pi : \Theta \to A$ is equivariant, and Θ^β is AF, then so is A^α. For any r-discrete groupoid G with continuous cocycle c taking values in a discrete abelian group, and α the induced action of the dual group on $C^*(G)$, it is a fact that $C^*(G)^\alpha = C^*(c^{-1}(0))$. ∎

A short computation shows that Θ^β is a hereditary subalgebra of $\Theta \times_\beta \mathbb{T}^2$. The definition of $\widehat{\beta}$ is: $\widehat{\beta}_n(f)(z) = \zeta_n(z) f(z)$, for $f \in C(\mathbb{T}^2, \Theta)$, from which we find

$$\widehat{\beta}_n(\zeta_m S_\mu S_\nu^*) = \zeta_{m+n} S_\mu S_\nu^*.$$

LEMMA 3.10. *Let σ, σ', μ, and ν be finite paths with $t(\mu) = t(\nu)$, $t(\sigma) = o(\mu)$, and $t(\sigma') = o(\nu)$, and such that $\ell(\sigma\mu) = \ell(\sigma'\nu)$. Then $\zeta_{\ell(\sigma')} S_\mu S_\nu^* = S_\sigma^* \cdot S_{\sigma\mu} S_{\sigma'\nu}^* \cdot \zeta_{\ell(\sigma')} S_{\sigma'}$, where $\cdot$ represents multiplication in $\Theta \times_\beta \mathbb{T}^2$.*

Proof. Let $m = \ell(\sigma')$. We compute:

$$\begin{aligned} S_\sigma^* \cdot S_{\sigma\mu} S_{\sigma'\nu}^* \cdot \zeta_m S_{\sigma'}(z) &= \int\!\!\int S_\sigma^* \beta_v(S_{\sigma\mu} S_{\sigma'\nu}^* \beta_w((\zeta_m S_{\sigma'})(w^{-1}v^{-1}z)))\,\mathrm{d}w\,\mathrm{d}v \\ &= \int\!\!\int S_\sigma^* S_{\sigma\mu} S_{\sigma'\nu}^* \beta_{vw}(\zeta_m(w^{-1}v^{-1}z) S_{\sigma'})\,\mathrm{d}w\,\mathrm{d}v \end{aligned}$$

$$= S_\mu S^*_{\sigma'\nu} \int\int \zeta_m(w^{-1}v^{-1}z)\zeta_m(vw)S_{\sigma'}\,\mathrm{d}w\,\mathrm{d}v$$
$$= \zeta_{\ell(\sigma')}(z)S_\mu S^*_\nu. \quad \blacksquare$$

Let I_0 denote the ideal in $\Theta \times_\beta \mathbb{T}^2$ generated by Θ^β. Note that it follows from Lemma 3.10 that $\zeta_{\ell(\sigma')}S_\mu S^*_\nu \in I_0$, where μ, ν, and σ' are as in the statement.

COROLLARY 3.11. $\widehat{\beta}_{(1,1)}(I_0) \subseteq I_0$.

Proof. It suffices to show that $\widehat{\beta}_{(1,1)}(\Theta^\beta) \subseteq I_0$. So let μ and ν be finite paths with $t(\mu) = t(\nu)$ and $\ell(\mu) = \ell(\nu)$. Let σ and σ' be finite paths with $\ell(\sigma) = \ell(\sigma') = (1,1)$, $t(\sigma) = o(\mu)$, and $t(\sigma') = o(\nu)$ (σ and σ' exist since edges in D have length $(1,1)$). Then we have

$$\begin{aligned}\widehat{\beta}_{(1,1)}(S_\mu S^*_\nu) &= \zeta_{(1,1)}S_\mu S^*_\nu \\ &= S^*_\sigma \cdot S_{\sigma\mu}S^*_{\sigma'\nu} \cdot \zeta_{(1,1)}S_{\sigma'}, \quad \text{by Lemma 3.10,} \\ &\in I_0. \quad \blacksquare\end{aligned}$$

We will let $I_n = (\widehat{\beta}_{(1,1)})^n(I_0)$ for $n \in \mathbb{Z}$. We have a composition series:

$$\cdots \lhd I_1 \lhd I_0 \lhd I_{-1} \lhd \cdots \lhd \Theta \times_\beta \mathbb{T}^2.$$

LEMMA 3.12. $\Theta \times_\beta \mathbb{T}^2 = \overline{\bigcup_{n\in\mathbb{Z}} I_n}$.

Proof. Let $m \in \mathbb{Z}^2$ and finite paths μ, ν with $t(\mu) = t(\nu)$ be given. We will show that there exists $k \in \mathbb{Z}$ such that $\zeta_m S_\mu S^*_\nu \in I_{-k}$. Choose $k \in \mathbb{Z}$ with

$$m + (k,k) \geqslant (1,1), \text{ and} \tag{3.1}$$
$$m + (k,k) + \ell(\nu) \geqslant \ell(\mu) + (1,1). \tag{3.2}$$

It follows from (3.1) that there exists a finite path σ' with $t(\sigma') = o(\nu)$ and $\ell(\sigma') = m + (k,k)$ (this is because every vertex is the terminus of a path of length $(1,1)$ with origin in $E_i \times F_i$ for some i). Then it follows from (3.2) that there exists a finite path σ with $t(\sigma) = o(\mu)$ and $\ell(\sigma) = \ell(\sigma') - \ell(\mu) + \ell(\nu)$. We have

$$\begin{aligned}(\widehat{\beta}_{(1,1)})^k(\zeta_m S_\mu S^*_\nu) &= \zeta_{m+(k,k)}S_\mu S^*_\nu \\ &= S^*_\sigma \cdot S_{\sigma\mu}S^*_{\sigma'\nu} \cdot \zeta_{m+(1,1)}S_{\sigma'}, \quad \text{by Lemma 3.10,} \\ &\in I_0.\end{aligned}$$

It follows that $\zeta_m S_\mu S^*_\nu \in I_{-k}$. $\blacksquare$

We now wish to show that $\pi : \Theta \to A$ is an isomorphism. To accomplish this we need a detailed study of the finite dimensional approximating subalgebras of Θ^β. (This analysis will be needed again in the proof of Lemma 4.2.) We begin with several definitions. We remark that in general, the structure of the finite dimensional subalgebras of the AF core of a Toeplitz graph algebra is made complicated by the fact that the Cuntz-Krieger relations are not satisfied at all vertices, and hence that there will be nonzero *defect* projections at some vertices. In Lemma 4.3 below we treat this situation, which is analogous to, but much easier than, the situation of Θ^β. The case of Θ^β is further complicated by the hybrid graph structure of Ω.

DEFINITION 3.13. For $x \in \Omega_k^{(0,0)} \setminus \{a_0, a_1\}$ we define projections $\lambda_k(x), \rho_k(x)$, and $\omega_k(x) \in \Theta_k^\beta$ by

$$\lambda_k(x) = P_x - \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(1,0)}(x)\}$$
$$\rho_k(x) = P_x - \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(0,1)}(x)\}$$
$$\omega_k(x) = \begin{cases} \lambda_k(x)\rho_k(x) & \text{if } x \notin \{u_0, u_1\}, \\ \lambda_k(x)\rho_k(x) - S_{\alpha_i}S_{\alpha_i}^* - S_{\varepsilon_i}S_{\varepsilon_i}^* & \text{if } x = u_i. \end{cases}$$

Moreover, we let $\lambda_k(a_i) = \rho_k(a_i) = \omega_k(a_i) = 0$.

REMARK 3.14. Note that $\lambda_k(u_i)$ is a projection by relation (iv) of Definition 3.3. We list some easy consequences of the definitions. Recall (from the end of Section 1) that $S(E_{i,k})$ is the set of vertices of $E_{i,k}$ that do not emit edges in E_i other than those emitted in $E_{i,k}$:

$$\lambda_k(x) \neq 0 \text{ if and only if } x = (y,z) \text{ with } y \notin S(E_{i,k});$$
$$\rho_k(x) \neq 0 \text{ if and only if } x = (y,z) \text{ with } z \notin S(F_{i,k});$$
$$\omega_k(x) \neq 0 \text{ if and only if } x = (y,z) \text{ with } y \notin S(E_{i,k}) \text{ and } z \notin S(F_{i,k});$$
$$\lambda_k(x)S_\mu = 0 \text{ whenever } \mu = \mu_1\mu_2 \text{ with } \ell(\mu_1) = (1,0),\ \mu \in \Omega_k^*;$$
$$\rho_k(x)S_\mu = 0 \text{ whenever } \mu = \mu_1\mu_2 \text{ with } \ell(\mu_1) = (0,1),\ \mu \in \Omega_k^*;$$
$$\omega_k(x)S_\mu = 0 \text{ whenever } \ell(\mu) \neq (0,0),\ \mu \in \Omega_k^*;$$
$$\lambda_k(o(\mu))S_\mu = S_\mu\lambda_k(t(\mu)) \text{ if } \ell(\mu) = (0,1),\ \mu \in \Omega_k^*;$$
$$\rho_k(o(\mu))S_\mu = S_\mu\rho_k(t(\mu)) \text{ if } \ell(\mu) = (1,0),\ \mu \in \Omega_k^*.$$

DEFINITION 3.15. $\mu \in X_k$ is *maximal* if whenever $\mu \preceq \mu'$ with $\mu' \in X_k$; then $\mu' = \mu$.

We remark that $\mu \in X_k$ is maximal if and only if one of the following occurs:
(i) $\ell(\mu) = (k,k)$.
(ii) $\ell(\mu) = (k-1,k)$ or $(k,k-1)$, and $t(\mu) \in \{a_0, a_1\}$.

DEFINITION 3.16. We define certain non-zero projections in Θ_k^β. They are of four kinds:
(i) $S_\mu S_\mu^*$, where $\mu \in X_k$ is maximal.
(ii) $S_\mu\lambda_k(t(\mu))S_\mu^*$, where $\ell(\mu) = (j,k)$ with $j < k$, and $t(\mu) = (y,z)$ with $y \notin S(E_{i,k})$.
(iii) $S_\mu\rho_k(t(\mu))S_\mu^*$, where $\ell(\mu) = (k,j)$ with $j < k$, and $t(\mu) = (y,z)$ with $z \notin S(F_{i,k})$.

(iv) $S_\mu \omega_k(t(\mu)) S_\mu^*$, where $\ell(\mu) \leqslant (k-1,k-1)$, and $t(\mu) = (y,z)$ with $y \notin S(E_{i,k})$ and $z \notin S(F_{i,k})$.

We let $\theta_i(\mu)$ denote the projection in Definition 3.16 above constructed from the path μ, where the arabic numeral i corresponds to the roman numeral of the relevant item n the definition. (The symbol $\theta_i(\mu)$ ought to include an indication of the integer k implicit in its definition. We have omitted this indication in order to reduce a bit the bristliness of our notation.)

PROPOSITION 3.17. *Let k be fixed.*

(i) *$\theta_i(\mu)$ is a minimal projection in Θ_k^β.*

(ii) *$\theta_i(\mu)$ and $\theta_{i'}(\nu)$ are equivalent in Θ_k^β if and only if $i = i'$, $\ell(\mu) = \ell(\nu)$, and $t(\mu) = t(\nu)$.*

(iii) $\sum_{i,\mu} \theta_i(\mu) = 1_{\Theta_k^\beta}$.

Proof. First note that if $\ell(\mu) = \ell(\nu)$ and $t(\mu) = t(\nu)$ then $(S_\nu S_\mu^*)\theta_i(\mu)(S_\nu S_\mu^*)^* = \theta_i(\nu)$, proving the reverse direction of (ii). We now claim that if σ, $\tau \in X_k$ with $\ell(\sigma) = \ell(\tau)$ are such that $\theta_i(\mu) S_\sigma S_\tau^* \theta_{i'}(\nu) \neq 0$, then $i = i'$, $t(\mu) = t(\nu)$, $\ell(\mu) = \ell(\nu)$, and moreover, $\sigma \preceq \mu$, $\tau \preceq \nu$. This will conclude the proof of (ii). (i) will also follow, since then $\theta_i(\mu) S_\sigma S_\tau^* \theta_i(\mu) \neq 0$ implies that σ, $\tau \preceq \mu$. Then $\ell(\sigma) = \ell(\tau)$ implies that $\sigma = \tau$, and hence that $\theta_i(\mu) S_\sigma S_\tau^* \theta_i(\mu) = \theta_i(\mu)$.

To prove the claim, assume that $\theta_i(\mu) S_\sigma S_\tau^* \theta_{i'}(\nu) \neq 0$. We first consider the situation $S_\mu^* S_\sigma \neq 0$. Suppose first that μ is maximal. If $\ell(\mu) = (k,k)$ then $\sigma \preceq \mu$ by Corollary 3.5. If $\ell(\mu) \neq (k,k)$ then $t(\mu) \in \{a_0, a_1\}$. By Lemma 3.4, $t(\sigma) = t(\mu)$. Since μ is maximal, $\mu \precneqq \sigma$ is impossible. Then by Lemma 3.4 we have $\sigma \preceq \mu$. Next suppose that $\ell(\mu) = (j,k)$ with $j < k$ and $t(\mu) = (y,z)$ with $y \notin S(E_{i,k})$. If $\sigma \not\preceq \mu$, Corollary 3.6 implies that $S_\mu^* S_\sigma = S_{(p,t(q))} S_{(t(p),q)}^*$ with $\ell(p) > 0$. But then $\lambda(t(\mu)) S_\mu^* S_\sigma = \lambda(t(\mu)) S_{(p,t(q))} S_{(t(p),q)}^* = 0$. Hence we must have $\sigma \preceq \mu$. An analogous argument handles the case $\ell(\mu) = (k,j)$ with $j < k$ and $t(\mu) = (y,z)$ with $z \notin S(F_{i,k})$. Finally suppose $\ell(\mu) \leqslant (k-1,k-1)$ and $t(\mu) = (y,z)$ with $y \notin S(E_{i,k})$ and $z \notin S(F_{i,k})$. If $\mu \precneqq \sigma$ then $\sigma = \mu\sigma'$ with $\ell(\sigma') \neq (0,0)$. Then

$$\omega_k(y,z) S_\mu^* S_\sigma = \omega_k(y,z) S_{\sigma'} = 0,$$

a contradiction. If $\sigma \not\preceq \mu$, then Lemma 3.4 implies that $S_\mu^* S_\sigma = S_{(p,t(q))} S_{(t(p),q)}^*$ with $\ell(p) \neq 0$ and $\ell(q) \neq 0$. But then

$$\omega_k(y,z) S_\mu^* S_\sigma = \omega_k(y,z) S_{(p,t(q))} S_{(t(p),q)}^* = 0,$$

a contradiction. Thus in all cases we have $\sigma \preceq \mu$.

Now write $\mu = \sigma\mu'$. We have $S_\mu^* S_\sigma S_\tau^* = S_{\mu'}^* S_\tau^* = S_{\tau\mu'}^*$. Note that $t(\mu) = t(\tau\mu')$ and $\ell(\mu) = \ell(\tau\mu')$. We then have

$$\theta_i(\mu) S_\sigma S_\tau^* = \theta_i(\mu) S_\mu S_\mu^* S_\sigma S_\tau^* = \theta_i(\mu) S_\mu S_{\tau\mu'}^* = S_\mu S_{\tau\mu'}^* \theta_i(\tau\mu'),$$
$$\theta_i(\mu) S_\sigma S_\tau^* \theta_{i'}(\nu) = S_\mu S_{\tau\mu'}^* \theta_i(\tau\mu') \theta_{i'}(\nu).$$

Thus $\theta_i(\tau\mu')\theta_{i'}(\nu) \neq 0$, and hence $\theta_i(\tau\mu')S_\gamma S_\gamma^*\theta_{i'}(\nu) \neq 0$ for $\gamma = \tau\mu'$ and for $\gamma = \nu$. It follows from the above that $\tau\mu' \preceq \nu$ and $\nu \preceq \tau\mu'$, and hence that $\tau\mu' = \nu$. Thus $\ell(\mu) = \ell(\nu)$ and $t(\mu) = t(\nu)$. It follows from Remark 3.14 that $i = i'$, finishing the proof of (i) and (ii).

Before proving (iii) we make some preliminary observations. Let $x \in \Omega_k^{(0,0)} \setminus \{a_0, a_1\}$. Using Definition 3.13 and Remark 3.14 we find that

$$P_x = \lambda_k(x) + \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(1,0)}(x)\}, \tag{3.3}$$

$$P_x = \rho_k(x) + \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(0,1)}(x)\}, \tag{3.4}$$

$$\begin{aligned}
P_x &= \Big(\lambda_k(x) + \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(1,0)}(x)\}\Big)\Big(\rho_k(x) + \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(0,1)}(x)\}\Big) \\
&= \lambda_k(x)\rho_k(x) + \sum\{\lambda_k(x)S_\mu S_\mu^* : \mu \in \Omega_k^{(0,1)}(x)\} \\
&\quad + \sum\{\rho_k(x)S_\nu S_\nu^* : \nu \in \Omega_k^{(1,0)}(x)\} \\
&\quad + \sum\{S_{\mu\nu}S_{\mu\nu}^* : \mu \in \Omega_k^{(0,1)}(x), \nu \in \Omega_k^{(1,0)}(x)\} \\
&= \omega_k(x) + \sum\{S_\mu \lambda_k(t(\mu))S_\mu^* : \mu \in \Omega_k^{(0,1)}(x)\} \\
&\quad + \sum\{S_\nu \rho_k(t(\nu))S_\nu^* : \nu \in \Omega_k^{(1,0)}(x)\} + \sum\{S_\mu S_\mu^* : \mu \in \Omega_k^{(1,1)}(x)\}.
\end{aligned} \tag{3.5}$$

Note that (3.5) still holds if $x \in \{a_0, a_1\}$. Next, let $\mu_1 \in \Omega_k^{(0,1)}$ and $\mu_2 \in \Omega_k^{(1,0)}$. If $t(\mu_1) = o(\mu_2)$ then

$$\lambda_k(o(\mu_1))S_{\mu_1}\rho_k(o(\mu_2))S_{\mu_2} = S_{\mu_1}\lambda_k(t(\mu_1))S_{\mu_2}\rho_k(t(\mu_2)) = 0,$$

and if $t(\mu_2) = o(\mu_1)$ then

$$\rho_k(o(\mu_2))S_{\mu_2}\lambda_k(o(\mu_1))S_{\mu_1} = S_{\mu_2}\rho_k(t(\mu_2))S_{\mu_1}\lambda_k(t(\mu_1)) = 0,$$

by Remark 3.14. Finally, if $t(\mu_1) = u_i$ and $\mu_3 \in \{\alpha_i, \varepsilon_i\}$ then

$$\lambda_k(o(\mu_1))S_{\mu_1}S_{\mu_3} = S_{\mu_1}\lambda_k(u_i)S_{\mu_3} = S_{\mu_1}S_{\mu_3},$$

by Definition 3.13 and Definition 3.3(iv). Hence for $\mu_3 \in \Omega_k^{(1,1)}(u_i)$,

$$\lambda_k(o(\mu_1))S_{\mu_1}S_{\mu_3} = \begin{cases} S_{\mu_1\mu_3} & \text{if } \mu_3 \in \{\alpha_i, \varepsilon_i\}, \\ 0 & \text{otherwise.} \end{cases}$$

An analogous result holds with μ_2 replacing μ_1 and ρ replacing λ.

Now we prove (iii). Let $x \in \Omega_k^{(0,0)}$, and consider the expression in (3.5). In each term of the form $S_\mu S_\mu^* = S_\mu P_{t(\mu)} S_\mu^*$, substitute for $P_{t(\mu)}$ with formula (3.5) if $\ell(\mu) \leqslant (k-1, k-1)$, with formula (3.3) if $(0,k) \leqslant \ell(\mu) \leqslant (k-1,k)$, and with formula (3.4) if $(k,0) \leqslant \ell(\mu) \leqslant (k,k-1)$. Similarly, if $\ell(\mu) \leqslant (k-1,k-1)$, use (3.5) to substitute for $P_{t(\mu)}$ in terms of the forms $S_\mu\lambda_k(t(\mu))S_\mu^* = S_\mu P_{t(\mu)}\lambda_k(t(\mu))S_\mu^*$ and $S_\mu\rho_k(t(\mu))S_\mu^* = S_\mu P_{t(\mu)}\rho_k(t(\mu))S_\mu^*$. Use the above observations to eliminate zero terms and to simplify, then repeat. This process must stop in a finite number of steps, giving

$$\begin{aligned} P_x &= \sum\{S_\mu \omega_k(t(\mu))S_\mu^* : o(\mu) = x,\ \ell(\mu) \leqslant (k-1,k-1)\} \\ &\quad + \sum_{j=0}^{k-1} \sum\{S_\mu \lambda_k(t(\mu))S_\mu^* : o(\mu) = x,\ \ell(\mu) = (j,k)\} \\ &\quad + \sum_{j=0}^{k-1} \sum\{S_\mu \rho_k(t(\mu))S_\mu^* : o(\mu) = x,\ \ell(\mu) = (k,j)\} \\ &\quad + \sum\{S_\mu S_\mu^* : o(\mu) = x,\ \mu \in X_k \text{ maximal}\} \\ &= \sum_i \sum\{\theta_i(\mu) : o(\mu) = x\}. \end{aligned}$$

Now (iii) follows by summing over $x \in \Omega_k^{(0,0)}$. ∎

THEOREM 3.18. *(Gauge-invariant uniqueness theorem) Let*

$$\phi : (\Theta, \mathbb{T}^2, \beta) \to (C, \mathbb{T}^2, \gamma)$$

be an equivariant $*$-homomorphism between C^*-dynamical systems. If $\phi\big|_{\Theta^\beta}$ is injective then ϕ is injective.

Proof. We have that $\widetilde{\phi} : \Theta \times_\beta \mathbb{T}^2 \to C \times_\gamma \mathbb{T}^2$ is injective, since it is so on the ideals I_n. Therefore $\widetilde{\widetilde{\phi}} : \Theta \times_\beta \mathbb{T}^2 \times_{\widehat{\beta}} \mathbb{Z}^2 \to C \times_\gamma \mathbb{T}^2 \times_{\widehat{\gamma}} \mathbb{Z}^2$ is injective. The result now follows from Takesaki-Takai duality. ∎

COROLLARY 3.19. *$A = C^*(G)$ is isomorphic to Θ, and is simple, purely infinite, nuclear and classifiable, i.e. a* UCT *Kirchberg algebra.*

Proof. Let $\lambda : C^*(G) \to C^*_{\mathrm{r}}(G)$ be the left regular representation. Then $\lambda \circ \pi : \Theta \to C^*_{\mathrm{r}}(G)$. The action of $\mathbb{T}^2$ on $C^*(G)$ clearly descends equivariantly to $C^*_{\mathrm{r}}(G)$, so that $\lambda \circ \pi$ is equivariant. We note that $\lambda \circ \pi\big|_{\Theta^\beta}$ is injective. This follows from the facts that Θ^β is AF, and that the minimal projections $\theta_i(\mu)$ (Proposition 3.17) in the subalgebras Θ_k^β have nonzero image in $C_c(G)^\alpha \subseteq C^*_{\mathrm{r}}(G)^\alpha$. Then $\lambda \circ \pi$ is injective by Theorem 3.18, and hence π is injective, and hence an isomorphism. It follows also that λ is injective (this could also be deduced from nuclearity of $C^*(G)$, proved below, and [1]). We have that $C^*(G) \times_\alpha \mathbb{T}^2 = \widetilde{\pi}(\Theta \times_\beta \mathbb{T}^2)$ is AF, and hence A, which is strongly Morita equivalent to $A \times_\alpha \mathbb{T}^2 \times_{\widehat{\alpha}} \mathbb{Z}^2$, is nuclear and classifiable. Simplicity and pure infiniteness follow from Corollary 2.19. ∎

The following proposition is necessary for our application of the results of this paper in [16]. The proof is immediate from the description of Θ by generators and relations. (We remark that in that application, the graphs F_i will be chosen to represent Kirchberg algebras having K-theory of the form $(\mathbb{Z}, 0)$ or $(0, \mathbb{Z})$, while the graphs E_i will be chosen to represent Kirchberg algebras with preassigned K_0 and trivial K_1.)

PROPOSITION 3.20. *Let Γ_i be a subgroup of* Aut (E_i) *fixing the vertex v_i. There is a homomorphism $\Gamma_0 \times \Gamma_1 \to$* Aut (Θ) *defined on generators by letting Γ_0 act on E_0, Γ_1 on E_1, and the trivial action on D, F_0 and F_1. Moreover, if $x_0 \in E_0^0$ is fixed by Γ_0, then the corner of Θ defined by the projection $P_{(x_0,w_0)}$ is invariant for the action of $\Gamma_0 \times \Gamma_1$.*

4. THE K-THEORY OF A

We may now omit the use of the $*$-isomorphism π, and identify Θ with A. We let B denote the subalgebra of A generated by the edges of Ω in $\bigcup_i E_i \times F_i$. Thus B is isomorphic to the direct sum of the algebras $\mathcal{O}(E_i) \otimes \mathcal{O}(F_i)$. This isomorphism can be described explicitly by means of generators and relations as follows

$$S_{(e,z)} \mapsto S_e \otimes P_z, \quad S_{(y,f)} \mapsto P_y \otimes S_f, \quad P_{(y,z)} \mapsto P_y \otimes P_z.$$

We note that even though P_{u_i} contains the range projections of S_{α_i} and S_{ε_i}, this does not affect the above isomorphism. It is here that the infinite valence of the graphs E_i and F_i at the vertices v_i and w_i is crucial.

Note that the K-theory of B is well-understood and easily computed (see, e.g., [16]). In Theorem 4.7 we will prove that the inclusion of B into A induces an isomorphism in K-theory. This fact is exploited in [16] and [17]. Let us briefly describe how UCT Kirchberg algebras with prescribed K-theory may be constructed in this way (see [16] for details). Let G_0 and G_1 be the desired K-groups. Let E_0 and E_1 be directed graphs defining Kirchberg algebras with $K_*\mathcal{O}(E_i) = (G_i, 0)$, and satisfying the hypothesis of Definition 2.1 (such graphs are described in [16]). Let F_0 and F_1 be directed graphs (also satisfying the hypothesis of Definition 2.1) defining the C^*-algebras stable-$\mathcal{O}_\infty$ and -$\mathcal{P}_\infty$ (i.e. having K-theory $(\mathbb{Z}, 0)$ and $(0, \mathbb{Z})$ — see [16]). By the Künneth theorem we have $K_*(B) = (G_0, G_1)$.

We now return to the study of the AF-algebra $A \times_\alpha \mathbb{T}^2$. It follows from the fact that the ideals I_n are AF that the inclusion of I_n into $A \times_\alpha \mathbb{T}^2$ induces an injection in K_0. We let ϕ denote the automorphism $\widehat{\alpha}_{(1,1)\,*}$ of $K_0(A \times_\alpha \mathbb{T}^2)$. We have

$$K_0(A \times_\alpha \mathbb{T}^2) = \bigcup_{n \in \mathbb{Z}} \phi^n(K_0(A^\alpha)).$$

We let $W = K_0(A \times_\alpha \mathbb{T}^2)$ and $W_n = K_0(I_n)$. Thus $W_n \supseteq W_{n+1}$ and $W = \bigcup_{n \in \mathbb{Z}} W_n$. Since $W_0 \cong K_0(A^\alpha)$, W_0 is generated by elements of the form $[S_\mu S_\mu^*]$, where μ is a finite path. If $t(\mu) = t(\nu)$ and $\ell(\mu) = \ell(\nu)$ then $S_\mu S_\nu^*$ is a partial isometry in A^α implementing an equivalence between $S_\mu S_\mu^*$ and $S_\nu S_\nu^*$, so that $[S_\mu S_\mu^*] = [S_\nu S_\nu^*]$.

LEMMA 4.1. *Let μ and σ be finite paths with $t(\sigma) = o(\mu)$. Then $\widehat{\alpha}_{\ell(\sigma)*}([S_\mu S_\mu^*]) = [S_{\sigma\mu} S_{\sigma\mu}^*]$.*

Proof. Let $f \in C(\mathbb{T}^2, A)$ be given by $f = \zeta_{\ell(\sigma)} S_{\sigma\mu} S_\mu^*$. Routine computations give

$$f^* f = \zeta_{\ell(\sigma)} S_\mu S_\mu^* = \widehat{\alpha}_{\ell(\sigma)}(S_\mu S_\mu^*), \quad \text{and} \quad f f^* = S_{\sigma\mu} S_{\sigma\mu}^*. \quad ∎$$

We note that B is invariant under α. The crossed product $B \times_\alpha \mathbb{T}^2$ is an AF-subalgebra of $A \times_\alpha \mathbb{T}^2$, and is isomorphic to a direct sum of tensor products of AF-algebras:

$$B \times_\alpha \mathbb{T}^2 \cong \bigoplus_i (\mathcal{O}(E_i) \times \mathbb{T}) \otimes (\mathcal{O}(F_i) \times \mathbb{T}).$$

LEMMA 4.2. *Let $i : B \times_\alpha \mathbb{T}^2 \to A \times_\alpha \mathbb{T}^2$ be the inclusion map. Then i_* is injective in K_0.*

Before giving the proof, we give a preliminary lemma describing the finite dimensional approximants to the AF core of the relative Toeplitz algebra of an ordinary graph. We remark that this is an easier version of Proposition 3.17.

LEMMA 4.3. *Let E be a finite directed graph. Let $S \subseteq E^0$ not contain any sink of E. For $k \geqslant 1$ let $C_k(E,S)$ be the finite dimensional C^*-subalgebra of $\mathcal{TO}(E,S)$ given by*

$$C_k(E,S) = \operatorname{span}\Big\{ S_p S_q^* : p,\, q \in \bigcup_{j \leqslant k} E^j,\ \ell(p) = \ell(q) \Big\}.$$

For $y \in E^0 \setminus S$ let $\xi_y \in C_k(E,S)$ be given by

$$\xi_y = P_y - \sum_{e \in E^1(y)} S_e S_e^*.$$

For $0 \leqslant j < k$ and $y \in E^0 \setminus S$ let

$$N_j^{(k)}(y) = \{ S_p \xi_y S_p^* : p \in E^j,\ t(p) = y \}.$$

For $y \in E^0$ let

$$N_k^{(k)}(y) = \{ S_p S_p^* : p \in E^k,\ t(p) = y \}.$$

Set

$$N^{(k)} = \bigcup \{ N_j^{(k)}(y) : 0 \leqslant j < k,\ y \in E^0 \setminus S \} \cup \bigcup \{ N_k^{(k)}(y) : y \in E^0 \}.$$

Then

(i) *$N^{(k)}$ is a maximal family of pairwise orthogonal minimal projections in $C_k(E,S)$.*

(ii) *Let $a \in N_i^{(k)}(y)$ and $b \in N_j^{(k)}(z)$, where $0 \leqslant i,j \leqslant k$ and $y, z \in E^0$. Then a and b are equivalent in $C_k(E,S)$ if and only if $i = j$ and $y = z$.*

Proof. We first note that

(4.1) $\xi_y S_e = 0$ for $y \in E^0 \setminus S,\ e \in E^1$,

(4.2) $S_p^* S_q = 0$ if and only if p and q are not comparable, for $p,\ q \in E^*$.

It follows easily that

$$S_p \xi_{t(p)} S_p^* S_q \xi_{t(q)} S_q^* \neq 0 \text{ if and only if } p = q \text{ and } t(p),\ t(q) \in E^0 \setminus S;$$
$$S_p S_p^* S_q S_q^* = 0 \text{ if } \ell(p) = \ell(q) = k \text{ and } p \neq q;$$
$$S_p \xi_{t(p)} S_p^* S_q S_q^* = 0 \text{ if } \ell(p) < \ell(q) \text{ and } t(p) \in E^0 \setminus S.$$

Therefore the projections of $N^{(k)}$ are pairwise orthogonal. We now consider the minimality and equivalence of projections in $N^{(k)}$ together. For this, fix paths r, $s \in \bigcup_{j \leqslant k} E^j$ with $\ell(r) = \ell(s)$. Let $t(p), t(q) \in E^0 \setminus S$ with $\ell(p), \ell(q) < k$, and suppose that

$$S_p \xi_{t(p)} S_p^* S_r S_s^* S_q \xi_{t(q)} S_q^* \neq 0. \tag{4.3}$$

Then (4.1) and (4.2) imply that $r \preceq p$, $s \preceq q$, and $S_p^* S_r = S_q^* S_s$. Hence $t(p) = t(q)$ and $\ell(p) = \ell(q)$, as required by (ii). Moreover if $p = q$ then the product in (4.3) equals $S_p \xi_{t(p)} S_p^*$, proving that $S_p \xi_{t(p)} S_p^*$ is minimal. If $p, q \in E^k$ and if

$$S_p S_p^* S_r S_s^* S_q S_q^* \neq 0, \tag{4.4}$$

then $r \preceq p$, $s \preceq q$, and $S_r^* S_p = S_s^* S_q$, so $t(p) = t(q)$ as required by (ii). Moreover if $p = q$ then the product in (4.4) equals $S_p S_p^*$, proving that $S_p S_p^*$ is minimal. Finally, let $t(p) \in E^0 \setminus S$ with $\ell(p) < k$, and $\ell(q) = k$, and consider $S_p \xi_{t(p)} S_p^* S_r S_s^* S_q S_q^* \neq 0$. It follows from (4.1) and (4.2) that $r \preceq p$ and $s \preceq q$. Then $S_r S_s^* S_q = S_{q'}$ where $\ell(q') = k$. Since $\ell(p) < k$, it follows from the same considerations that $\xi_{t(p)} S_p^* S_{q'} = 0$. Thus $S_p \xi_{t(p)} S_p^*$ and $S_q S_q^*$ are inequivalent.

For the reverse implication in (ii), note that if $\ell(p) = \ell(q) < k$ and $y = t(p) = t(q) \in E^0 \setminus S$ then

$$(S_q \xi_y S_p^*)^* (S_q \xi_y S_p^*) = S_p \xi_y S_p^*, \quad (S_q \xi_y S_p^*)(S_q \xi_y S_p^*)^* = S_q \xi_y S_q^*,$$

and if $\ell(p) = \ell(q) = k$ and $t(p) = t(q)$, then

$$(S_q S_p^*)^* (S_q S_p^*) = S_p S_p^*, \quad (S_q S_p^*)(S_q S_p^*)^* = S_q S_q^*.$$

Finally, we show that $\sum N^{(k)} = 1$. For convenience we will let $\xi_y = 0$ for $y \in S$. Then for any $y \in E^0$ we have:

$$P_y = \xi_y + \sum_{e_1 \in E^1(y)} S_{e_1} S_{e_1}^* = \xi_y + \sum_{e_1 \in E^1(y)} S_{e_1} \Big(\xi_{t(e_1)} + \sum_{e_2 \in E^1(t(e_1))} S_{e_2} S_{e_2}^* \Big) S_{e_1}^*$$

$$= \xi_y + \sum_{p\in E^1(y)} S_p\xi_{t(p)}S_p^* + \sum_{p\in E^2(y)} S_pS_p^* = \cdots =$$
$$= \sum_{i=0}^{k-1} \sum_{p\in E^i(y)} S_p\xi_{t(p)}S_p^* + \sum_{p\in E^k(y)} S_pS_p^*.$$

The result follows by summing over $y \in E^0$. ∎

Proof of Lemma 4.2. By repeated application of ϕ it suffices to show that $i_* : K_0(B^\alpha) \to K_0(A^\alpha)$ is injective. Letting $A_k^\alpha = \Theta_k^\beta$ (via the isomorphism π), we have $A^\alpha = \overline{\bigcup_k A_k^\alpha}$. We define finite dimensional approximating subalgebras to B^α in a manner similar to the Θ_k^β. Namely, let

$$X_k^{(0)} = \bigcup_{i=0}^{1} \bigcup_{j,j'\leqslant k} E_{i,k}^j \times F_{i,k}^{j'}.$$

(Thus $X_k^{(0)}$ is the set of paths μ in X_k that do not contain edges from D.) Now let $B_k^\alpha = \text{span}\,\{S_\mu S_\nu^* : \mu,\ \nu \in X_k^{(0)},\ \ell(\mu) = \ell(\nu)\}$. It is clear that $B^\alpha = \overline{\bigcup_k B_k^\alpha}$ and that $B_k^\alpha \subseteq A_k^\alpha$. We will show that the inclusion $B_k^\alpha \subseteq A_k^\alpha$ induces an injection in K_0. This will prove the lemma.

We note that $B_k^\alpha \cong \bigoplus_i C_k(E_{i,k}, S(E_{i,k})) \otimes C_k(F_{i,k}, S(F_{i,k}))$. Thus every minimal projection in B_k^α corresponds to a tensor product of minimal projections from $C_k(E_{i,k}, S(E_{i,k}))$ and $C_k(F_{i,k}, S(F_{i,k}))$, and two such are equivalent in B_k^α if and only if they are separately equivalent in each factor. Note that, while $\lambda_k(x)$ and $\rho_k(x)$ (Definition 3.13) belong to B_k^α, $\omega_k(x) \notin B_k^\alpha$ if $x \in \{u_0, u_1\}$. We define $\omega_k^{(0)}(x) \in B_k^\alpha$ by

$$\omega_k^{(0)}(x) = \lambda_k(x)\rho_k(x).$$

Then $\omega_k^{(0)}(u_i)$ is a minimal projection in B_k^α (although it is not minimal in Θ_k^β). Note that if $x = (y,z) \in \Omega_k^{(0,0)}$ then $\lambda_k(x)$ corresponds to $\xi_y \otimes P_z$, $\rho_k(x)$ corresonds to $P_y \otimes \xi_z$, and $\omega_k^{(0)}(x)$ corresponds to $\xi_y \otimes \xi_z$, while if $\mu = (p,q) \in X_k^{(0)}$, then $S_\mu S_\mu^*$ corresponds to $S_pS_p^* \otimes S_qS_q^*$. For $\mu \in X_k^{(0)}$ with $\ell(\mu) \leqslant (k-1, k-1)$ and $t(\mu) = (y,z)$ with $y \notin S(E_{i,k})$ and $z \notin S(F_{i,k})$, we define

$$\theta_4^{(0)}(\mu) = S_\mu\omega_k^{(0)}(t(\mu))S_\mu^* \in B_k^\alpha,$$

analogously to Definition 3.16(iv). Then the projections in B_k^α of the form $\theta_i(\mu)$, $1 \leqslant i \leqslant 3$, and $\theta_4^{(0)}(\mu)$, form a complete family of pairwise orthogonal minimal projections. By Lemma 4.3 we see that they satisfy the conditions for equivalence given in Proposition 3.17(ii). From Definition 3.13 we see that if $\theta_4^{(0)}(\mu) \neq \theta_4(\mu)$, then

$$\theta_4^{(0)}(\mu) = \theta_4(\mu) + \theta_{i_1}(\tau_1) + \theta_{i_2}(\tau_2) + \cdots,$$

where $\ell(\tau_1), \ell(\tau_2), \ldots > \ell(\mu)$. This observation has the following consequence. Choose bases for $K_0(A_k^\alpha)$ and $K_0(B_k^\alpha)$ consisting of classes of minimal projections as above. If the bases are ordered by increasing length of the underlying paths, then the matrix of the map $K_0(B_k^\alpha) \to K_0(A_k^\alpha)$ induced from inclusion is lower triangular, with 1's on the diagonal. Thus the map is injective. ∎

We let Y denote $K_0(B \times_\alpha \mathbb{T}^2)$. By Lemma 4.2 we may identify Y with $i_*(Y) \subseteq W$. We now give a key lemma, that is based on the fact that the (ordinary) graph D connecting the 2-graphs $E_i \times F_i$ defines the Cuntz algebra $\mathcal{O}_2$ (with trivial K-theory).

LEMMA 4.4. *$(2\phi - \mathrm{id})W \subseteq Y$.*

Proof. Let $x \in W$. For $n \in \mathbb{Z}$ large enough we have $\phi^n(x) \in W_0$. If $(2\phi - \mathrm{id})\phi^n(x) \in Y$, then since Y is ϕ-invariant we get $(2\phi - \mathrm{id})(x) \in \phi^{-n}Y = Y$. So we may assume that $x \in W_0$. Since $W_0 = K_0(A^\alpha)$ is generated by elements of the form $[S_\mu S_\mu^*]$ for finite paths μ, we may assume that $x = [S_\mu S_\mu^*]$.

Case (i). Suppose $t(\mu) \in E_i^0 \times F_i^0$ for some i. Since the K_0-class of the projection is unchanged if the path is replaced by a new path with the same length and terminus, we may assume that $\mu \in (E_i \times F_i)^*$, and so that $x \in Y$.

Case (ii). Suppose $t(\mu) \in \{a_0, a_1\}$. For definiteness we suppose $t(\mu) = a_0$. For the rest of this argument we will omit the subscript on a_0, β_0, γ_0 and δ_0. Then we may assume that $\mu = \nu\beta^m$ for some $m \geqslant 0$ and some path ν with $t(\nu) = a$. We note that

$$S_{\beta^m} S_{\beta^m}^* = S_{\beta^m}(S_\beta S_\beta^* + S_\gamma S_\gamma^* + S_\delta S_\delta^*) S_{\beta^m}^*, \quad \text{while } \phi[S_{\beta^m} S_{\beta^m}^*] = [S_{\beta^{m+1}} S_{\beta^{m+1}}^*].$$

Hence $[S_{\beta^m} S_{\beta^m}^*] = 2[S_{\beta^{m+1}} S_{\beta^{m+1}}^*] + [S_{\beta^m\delta} S_{\beta^m\delta}^*] \in 2\phi[S_{\beta^m} S_{\beta^m}^*] + Y$, since $t(\beta^m\delta) = u_i$. Therefore $(2\phi - \mathrm{id})[S_{\beta^m} S_{\beta^m}^*] \in Y$. Thus

$$\begin{aligned}(2\phi - \mathrm{id})[S_\mu S_\mu^*] &= (2\phi - \mathrm{id})\phi[S_{\nu\beta^m} S_{\nu\beta^m}^*] \\ &= \widehat{\alpha}_{\ell(\nu)*} \circ (2\phi - \mathrm{id})[S_{\beta^m} S_{\beta^m}^*], \quad \text{by Lemma 4.1,} \\ &\in \widehat{\alpha}_{\ell(\nu)*}(Y) \subseteq Y. \quad ∎\end{aligned}$$

LEMMA 4.5. *$\ker(\mathrm{id} - \phi) \subseteq Y$.*

Proof. Let $x \in \ker(\mathrm{id} - \phi)$. Then

$$x = \phi(x) = \phi(x) - (\mathrm{id} - \phi)(x) = (2\phi - \mathrm{id})(x) \in Y. \quad ∎$$

The preceding and following lemmas will allow us to show that the K-theory of $A \times_\alpha \mathbb{T}^2 \times_\phi \mathbb{Z}$ is given by the subalgebra B. We let $\psi = \widehat{\alpha}_{(1,0)*}$, so that ϕ and ψ generate the action of $\mathbb{Z}^2$ on W. We note that since B is invariant for α, Y is invariant for ψ as well as for ϕ.

LEMMA 4.6. *$W/(\mathrm{id} - \phi)W \cong Y/(\mathrm{id} - \phi)Y$, and the isomorphism is equivariant for ψ.*

Proof. First we show that $W = (\mathrm{id} - \phi)W + Y$. Let $x \in W$. By Lemma 4.4 we have $\phi(x) = (\mathrm{id} - \phi)(x) + (2\phi - \mathrm{id})(x) \in (\mathrm{id} - \phi)W + Y$. Applying ϕ^{-1} we see that $x \in (\mathrm{id} - \phi)W + Y$. Now we have

$$\frac{W}{(\mathrm{id} - \phi)W} = \frac{(\mathrm{id} - \phi)W + Y}{(\mathrm{id} - \phi)W} \cong \frac{Y}{Y \cap ((\mathrm{id} - \phi)W)}.$$

We will show that $Y \cap ((\mathrm{id} - \phi)W) = (\mathrm{id} - \phi)Y$, which will conclude the proof. The containment "$\supseteq$" is clear. For the containment "$\subseteq$", let $y \in Y$ with $y = (\mathrm{id} - \phi)(x)$ for some $x \in W$. Then $\phi(x) = y + (2\phi - \mathrm{id})(x) \in Y$, by Lemma 4.4. It follows that $x \in Y$, so that $y \in (\mathrm{id} - \phi)Y$. ∎

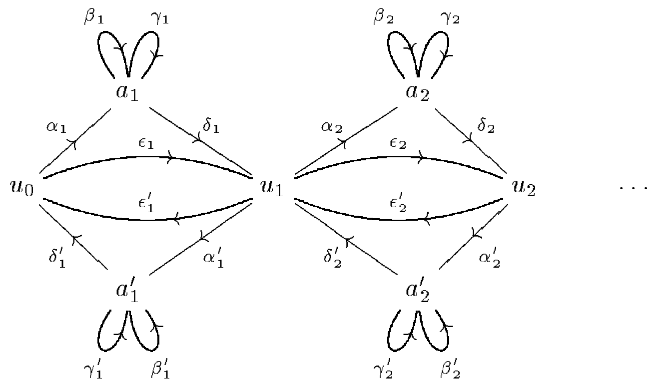

FIGURE 2

THEOREM 4.7. $K_*(A) \cong K_*(B)$.

Proof. Lemmas 4.5 and 4.6 and the Pimsner-Voiculescu exact sequence show that $K_*(A \times_\alpha \mathbb{T}^2 \times_\phi \mathbb{Z})$ and $K_*(B \times_\alpha \mathbb{T}^2 \times_\phi \mathbb{Z})$ are isomorphic, equivariantly for ψ. Another application of Pimsner-Voiculescu, together with Takai-Takesaki duality, gives a commuting diagram of long exact sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & \frac{W}{(\mathrm{id}-\phi)W} \arrow[r, "\mathrm{id}-\psi"] & \frac{W}{(\mathrm{id}-\phi)W} \arrow[r] & K_0(A) \arrow[r] & {} \\
\cdots \arrow[r] & \frac{Y}{(\mathrm{id}-\phi)Y} \arrow[r, "\mathrm{id}-\psi"] \arrow[u, "\cong"'] & \frac{Y}{(\mathrm{id}-\phi)Y} \arrow[r] \arrow[u, "\cong"'] & K_0(B) \arrow[r] \arrow[u, "i_*"'] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r] & \ker(\mathrm{id}-\phi) \arrow[r, "\mathrm{id}-\psi"] & \ker(\mathrm{id}-\phi) \arrow[r] & K_1(A) \arrow[r] & \cdots \\
{} \arrow[r] & \ker(\mathrm{id}-\phi) \arrow[r, "\mathrm{id}-\psi"] \arrow[u, equal] & \ker(\mathrm{id}-\phi) \arrow[r] \arrow[u, equal] & K_1(B) \arrow[r] \arrow[u, "i_*"'] & \cdots
\end{tikzcd}$$

It follows from the five lemma that $K_*(A) \cong K_*(B)$. ∎

THEOREM 4.8. *Let $k \geqslant 1$ be given. For $0 \leqslant i$ and $1 \leqslant j \leqslant k$ let $E_{i,j}$ be an irreducible directed graph with distinguished vertex $v_{i,j}$ emitting infinitely many edges. For $i \geqslant 0$ let D_i be a copy of the graph D in Definition* 2.1 *(with vertices u_{i-1}, u_i, a_i, a'_i — see figure* 2*). Let Ω be the object obtained from the 1-graphs $\{D_i\}$ and the product k-graphs $\{E_{i_1} \times \cdots \times E_{i,k}\}$ by identifying the vertex u_i with $(v_{i,1}, \ldots, v_{i,k})$ as in Definition* 2.1. *Let A be the C^*-algebra defined by the generators $\mathcal{S}$ and relations $\mathcal{R}$ as in Definition* 3.3 *(modified in the obvious way). Then A is the unique stable* UCT *Kirchberg algebra with K-theory equal to*

$$\bigoplus_{i=0}^{\infty} K_*\Big(\bigotimes_{j=1}^{k} \mathcal{O}(E_{i,j})\Big).$$

Proof. This follows from Corollary 3.19 and Theorem 4.7. (The uniqueness is a result of Zhang, [19].) ∎

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