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LINEAR ALGEBRAIC PROPERTIES FOR JORDAN MODELS OF C_0-OPERATORS RELATIVE TO MULTIPLY CONNECTED DOMAINS

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ABSTRACT. We study C_0-operators relative to a multiply connected domain using a substitute of the characteristic function. This method allows us to prove certain relations between the Jordan model of an operator and that of its restriction to an invariant subspace.

KEYWORDS: *Hardy spaces, Hilbert spaces, functional calculus, Jordan model, quasi-equivalence, quasi-similarity.*

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INTRODUCTION

Hasumi [12], Sarason [17], and Voivhick [19] started operator theory related to function theory on multiply connected domains by providing an analogue (in the scalar case) of Beurling's theorem on invariant subspaces of the Hardy spaces of the open unit disk. Their work was continued in the work of Abrahamse–Douglas [1], [2], and of Ball [4], [5]. In particular, J.A. Ball [4] introduced the class of C_0-operators relative to a bounded finitely connected region Ω in the complex plane, whose boundary $\partial\Omega$ consists of a finite number of disjoint, analytic, simple closed curves. J. Agler [3] showed that the existence of normal boundary dilations — an analogue of Sz.-Nagy dilation theorem — still holds for annuli but it may fail for domains of connectivity greater than two (Dritschel–McCullough [11]). However it holds up to similarity (R.G. Douglas–Paulsen [10]); this allowed Zucchi [20] to provide a classification of C_0-operators relative to Ω. Since no analogue of the characteristic function of a contraction is available in that context, that study does not yield some of the results available for the unit disk. In this paper we use a substitute for the characteristic function, suggested by an analogue of Beurling's theorem provided by M.A. Abrahamse and R.G. Douglas [2]. This allows us to prove a relationship between the Jordan models of a C_0-operator relative to Ω, of its restriction to an invariant subspace, and of its compression to the orthocomplement of that subspace. In the case of the open unit disk, this result was proved by H. Bercovici and D. Voiculescu [7].

This paper is organized as follows. Section 1 contains preliminaries about bundle shifts and operators of class C_0. Here we define the notion of an operator-valued quasi-inner function and prove a useful reformulation of the description of invariant subspaces given in [2].

In Section 2, we review concepts relating quasi-equivalence and quasi- similarity, which were first introduced in [13], [14] and we prove the main result.

1. PREMIMINARIES AND NOTATIONS

In this paper, $\mathbb{C}$, $\overline{M}$, and $L(K_1, K_2)$ denote the set of complex numbers, the (norm) closure of a set M, and the set of bounded linear operators from K_1 to K_2 where K_1 and K_2 are Hilbert spaces, respectively.

1.1. HARDY SPACES. We refer to [16] for basic facts about Hardy spaces, and recall here the basic definitions.

DEFINITION 1.1. The space $H^2(\Omega)$ is defined to be the space of analytic functions f on Ω such that the subharmonic function $|f|^2$ has a harmonic majorant on Ω. For a fixed $z_0 \in \Omega$, there is a norm on $H^2(\Omega)$ defined by

$$\|f\| = \inf\{u(z_0)^{1/2} : u \text{ is a harmonic majorant of } |f|^2\}.$$

Let m be a harmonic measure for the point z_0, let $L^2(\partial\Omega)$ be the L^2-space of complex valued functions on the boundary of Ω defined with respect to m, and let $H^2(\partial\Omega)$ be the set of functions f in $L^2(\partial\Omega)$ such that $\int_{\partial\Omega} f(z)g(z)\mathrm{d}z = 0$ for every g that is analytic in a neighborhood of the closure of Ω. If f is in $H^2(\Omega)$, then there is a function f^* in $H^2(\partial\Omega)$ such that $f(z)$ approaches $f^*(\lambda_0)$ as z approaches λ_0 nontangentially, for almost every λ_0 relative to m. The map $f \to f^*$ is an isometry from $H^2(\Omega)$ onto $H^2(\partial\Omega)$. In this way, $H^2(\Omega)$ can be viewed as a closed subspace of $L^2(\partial\Omega)$.

A function f defined on Ω is in $H^\infty(\Omega)$ if it is holomorphic and bounded. $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Omega)$ and it is a Banach algebra if endowed with the supremum norm. Finally, the mapping $f \to f^*$ is an isometry of $H^\infty(\Omega)$ onto a week*-closed subalgebra of $L^\infty(\partial\Omega)$.

DEFINITION 1.2. If K is a Hilbert space, then $H^2(\Omega,K)$ is defined to be the space of analytic functions $f : \Omega \to K$ such that the subharmonic function $\|f\|^2$ is majorized by a harmonic function ν. Fix a point z_0 in Ω and define a norm on $H^2(\Omega,K)$ by

$$\|f\| = \inf\{\nu(z_0)^{1/2} : \nu \text{ is a harmonic majorant of } \|f\|^2\}.$$

As before, $H^2(\Omega,K)$ can be identified with a closed subspace of the space $L^2(\partial\Omega,K)$ of square integrable K-valued functions on $\partial\Omega$. Define $S_K : H^2(\Omega,K) \to H^2(\Omega,K)$ by $(S_K f)(z) = zf(z)$.

1.2. VECTOR BUNDLES. We present in this section and in Section 1.3 the standard definitions of analytic vector and flat unitary vector bundles. We refer to [2] for this material.

Let K be a Hilbert space. An *analytic vector bundle* over Ω with fiber K is a pair (E, p), where $p : E \to \Omega$ is a continuous surjective map such that:

(1) Each $z \in \Omega$ has a neighborhood U_z for which there is a homeomorphism $\varphi_z : U_z \times K \to p^{-1}(U_z)$ satisfying $\varphi_z(\omega, k) \in p^{-1}(\omega)$ for $\omega \in U_z$ and $k \in K$.

(2) If $z_1, z_2 \in \Omega$, there is an analytic map $\psi_{z_1,z_2} : U_{z_1} \cap U_{z_2} \to GL(K)$ satisfying $\varphi_{z_1}(\omega, k) = \varphi_{z_2}(\omega, \psi_{z_1,z_2}(\omega)k)$, where $GL(K)$ is the set of all invertible linear operators on K.

If we can choose $U_z = \Omega$ for some $z \in \Omega$, we say that (E, p) is a *trivial bundle*. If each ψ_{z_1,z_2} is a constant unitary operator for every $z_1, z_2 \in \Omega$, then (E, p) is called a *flat unitary vector bundle*.

THEOREM A. [8] *Every analytic vector bundle over Ω is analytically trivial.*

1.3. BUNDLE SHIFT. Let E be a vector bundle over Ω. A *cross section* of a vector bundle E over Ω is a continuous function f from Ω into E such that $p(f(z)) = z$ for all z in Ω. For each ω in U_z, define a map $\varphi_z^\omega : K \to p^{-1}(\omega)$ by $\varphi_z^\omega(k) = \varphi_z(\omega, k)$.

If E is a flat unitary vector bundle over Ω with fiber K and if f is a cross section of E, then for ω in $U_{z_1} \cap U_{z_2}$ ($z_1, z_2 \in \Omega$), the operator $(\varphi_{z_1}^\omega)^{-1}\varphi_{z_2}^\omega$ is unitary so that $\|(\varphi_{z_2}^\omega)^{-1}(f(z))\|=\|(\varphi_{z_1}^\omega)^{-1}(f(z))\|$. This means that there is a function $h_f : \Omega \to R$ defined by $h_f^E(z)=\|(\varphi_{z_2}^\omega)^{-1}(f(z))\|$, where ω is in U_{z_2}.

DEFINITION 1.3. We define $H^2(\Omega, E)$ to be the space of analytic cross sections f of E such that $(h_f^E)^2$ is majorized by a harmonic function.

We can define the *bundle shift* T_E on $H^2(\Omega,E)$ by $(T_E f)(z)=zf(z)$ for $z \in \Omega$. The operator T_E admits a functional calculus defined on the algebra $R(\Omega)$ of rational functions with poles off $\overline{\Omega}$. More precisely, if $u \in R(\Omega)$, $(u(T_E)f)(z) = u(z)f(z)$ for $z \in \Omega$ and $f \in H^2(\Omega, E)$.

1.4. QUASI-INNER FUNCTION. If E and F are flat unitary bundles over Ω that extend to an open set Ω' containing the closure of Ω, and Θ is a bounded holomorphic bundle map from E to F, then Θ can be shown to have nontangential limits a.e. relative to m on $\partial\Omega$. The limit at a point z of $\partial\Omega$ can be regarded as an operator from the fiber of E at z to the fiber of F at z.

DEFINITION 1.4. (i) A bounded holomorphic bundle map Θ is *inner* if the nontangential limits are isometric operators a.e. relative to m.

(ii) Let K and K' be Hilbert spaces and let $H^\infty(\Omega,L(K,K'))$ be the Banach space of all analytic functions $\Phi : \Omega \to L(K,K')$ with the supremum norm. For $\varphi \in H^\infty(\Omega, L(K,K'))$, we will say that φ is *quasi-inner* if there exists a constant $c > 0$ such that for every $k \in K$ and almost every $z \in \partial\Omega$ we have $\|\varphi(z)k\| \geqslant c\|k\|$.

THEOREM B. [2]. *Let T_E be a bundle shift on $H^2(\Omega,E)$. Then a closed subspace M of $H^2(\Omega,E)$ is invariant under the algebra $\{u(T_E) : u \in R(\Omega)\}$ if and only if M=$\Theta H^2(\Omega,F)$, where F is a flat unitary bundle over Ω and Θ is an inner bundle map from F to E.*

It will be convenient to reformulate Theorem B in terms of quasi-inner functions without use of vector bundles. We will say that a space M is $R(\Omega)$*-invariant* for an operator T if it is invariant under $u(T)$ for every $u \in R(\Omega)$. For a Hilbert space K, define an operator S_K on $H^2(\Omega,K)$ by $(S_K f)(z)$=$zf(z)$ for $z \in \Omega$.

The proper setting here is maps of flat unitary vector bundles, i.e., multiplicative multivalued operator-valued functions. We will convert these to usual single valued analytic functions by composing them with some bundle isomorphisms. This has been done quite often in the scalar case, see, e.g., Royden [15].

THEOREM 1.5. *Let K be a Hilbert space. Then a closed subspace M of $H^2(\Omega, K)$ is $R(\Omega)$-invariant for S_K if and only if there is a Hilbert space K' and a quasi-inner function $\varphi : \Omega \to L(K',K)$ such that $M = \varphi H^2(\Omega,K')$.*

Proof. It is clear that a subspace of the form $\varphi H^2(\Omega,K')$ with $\varphi : \Omega \to L(K',K)$ quasi-inner, is $R(\Omega)$-invariant. Conversely, consider a closed subspace $M \subset H^2(\Omega, K)$ which is $R(\Omega)$-invariant. Let $M' = \{G \in H^2(\Omega, \Omega\times K) : G(z) = (z, g(z))$ for some $g \in M\}$. Then M' is a closed subspace of $H^2(\Omega, \Omega \times K)$ which is $R(\Omega)$-invariant for $T_{\Omega\times K}$ and so, by Theorem B, there is a flat unitary bundle F over Ω with fiber K', and an inner bundle map $\Theta : F \to \Omega \times K$, such that $M' = \Theta H^2(\Omega,F)$. We know that there is a flat unitary vector bundle F' over an open set Ω' containing the closure of Ω, with fiber K', such that F is unitary equivalent to the bundle $F'|\Omega$ [2]. By Theorem A, there is an analytic isomorphism $\Lambda : \Omega' \times K' \to F'$.

Define an invertible operator $W : H^2(\Omega,K') \to H^2(\Omega, F'|\Omega)$ by $(Wf)(z) = \Lambda(z, f(z)) = \Lambda_z(f(z))$ for $f \in H^2(\Omega,K')$. Then $M' = \Theta UWH^2(\Omega, K')$ where U: $H^2(\Omega,F'|\Omega) \to H^2(\Omega,F)$ is a unitary operator. For each $z \in \Omega$, we can define a bounded operator W_z: $K' \to F_z$ by $W_z a$=$(U(Wh_a))(z)$ for $a \in K'$ where $h_a \in H^2(\Omega, K')$ defined by $h_a(z) = a$.

Let $\varphi(z)$=$\Theta_z W_z$ for $z \in \Omega$ where $\Theta_z = \Theta|F_z$. Then $\varphi \in H^\infty(\Omega, L(K', K))$ and $M = \varphi H^2(\Omega,K')$. To conclude our proof, we must verify that φ is quasi-inner.

From the fact that Λ is an analytic isomorphism, we see that the function $z \to (\Lambda_z)^{-1}$ is holomorphic on Ω', and so there is $m > 0$ such that $\|(\Lambda_z)^{-1}\| \leqslant m$ for any $z \in \Omega$. Therefore $\|W_z^{-1}\| \leqslant m$ for any $z \in \Omega$ as well, so that $\|a\|/m \leqslant \|\varphi(z)a\|$ a.e. on $\partial\Omega$ for $a \in K'$ as desired. ∎

LEMMA 1.6. *Let K_1 and K_2 be separable Hilbert spaces. If $T : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ is a bounded linear operator such that $TS_{K_1} = S_{K_2}T$, then there is a function $\psi \in H^\infty(\Omega, L(K_1, K_2))$ such that $T = M_\psi$, where $M_\psi(g)(z) = \psi(z)g(z)$ for $g \in H^2(\Omega, K_1)$, and we have $\|T\| = \|\psi\|_\infty$.*

Proof. Define $Y \in (S_{K_1 \oplus K_2})'$ by $Y = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}$. Then by the Proposition 1.9 in [2], $Y = M_\omega$ where $\omega \in H^\infty(\Omega, L(K_1 \oplus K_2))$. Let $\omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$. Take $\psi = \omega_{21}$, then $T = M_\psi$ and one can check easily that $\|T\| = \|\psi\|_\infty$. ∎

COROLLARY 1.7. *Let $\varphi_1 : \Omega \to L(K_1, K)$ and $\varphi_2 : \Omega \to L(K_2, K)$ be quasi-inner functions.*

Then the subspaces $\varphi_1 H^2(\Omega, K_1)$ and $\varphi_2 H^2(\Omega, K_2)$ of $H^2(\Omega, K)$ are equal if and only if there exist functions $\varphi \in H^\infty(\Omega, L(K_1, K_2))$ and $\psi \in H^\infty(\Omega, L(K_2, K_1))$ such that $\varphi\psi = I_{K_2}$, $\psi\varphi = I_{K_1}$ and $\varphi_1(z) = \varphi_2(z)\varphi(z)$ for any $z \in \Omega$. In particular, K_1 and K_2 have the same dimension.

Proof. The condition $\varphi_1(z) = \varphi_2(z)\varphi(z)$ with φ invertible clearly implies $\varphi_1 H^2(\Omega, K_1) = \varphi_2 H^2(\Omega, K_2)$. Conversely, assume that $\varphi_1 H^2(\Omega, K_1) = \varphi_2 H^2(\Omega, K_2)$. Define an operator $T : H^2(\Omega, K_1) \to H^2(\Omega, K_2)$ as follows. For $f \in H^2(\Omega, K_1)$, $Tf = g$ such that $\varphi_1 f = \varphi_2 g$. Since φ_i $(i = 1, 2)$ is a quasi-inner function, T is well-defined and invertible. Since $S_{K_2}T = TS_{K_1}$, by the previous lemma $T = M_\varphi$ for a function $\varphi \in H^\infty(\Omega, L(K_1, K_2))$. Note that the invertibility of T is equivalent to the invertibility of φ. It follows that $\varphi_1 f = \varphi_2 \varphi f$ for any $f \in H^2(\Omega, K_1)$ and so $\varphi_1 = \varphi_2\varphi$. Since $\varphi(z)$ is invertible for any $z \in \Omega$, K_1 and K_2 have the same dimension. ∎

1.5. THE CLASS C_0.

The theory of Jordan models for contractions of class C_0 was developed by Sz.-Nagy–Foias, Moore–Nordgren, and Bercovici–Voiculescu.

We will present in this section the definition of C_0-operators relative to Ω. Reference for this material is Zucchi [20].

Let H be a Hilbert space and K_1 be a compact subset of the complex plane. If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, for $r = p/q$ a rational function with poles off K_1, we can define an operator $r(T)$ by $q(T)^{-1}p(T)$.

DEFINITION 1.8. If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, we say that K_1 is a *spectral set* for the operator T if $\|r(T)\| \leqslant \max\{|r(z)| : z \in K_1\}$, whenever r is a rational function with poles off K_1.

If $T \in L(H)$ is an operator with $\overline{\Omega}$ as a spectral set and with no normal summand with spectrum in $\partial\Omega$, i.e., T has no reducing subspace $M \subseteq H$ such that $T|M$ is normal and $\sigma(T|M) \subseteq \partial\Omega$, then we say that T satisfies *Hypothesis (h)*.

THEOREM 1.9 ([20], Theorem 3.1.4). *Let* $T \in L(H)$ *be an operator satisfying Hypothesis* (h). *Then there is a unique norm continuous representation* Ψ_T *of* $H^\infty(\Omega)$ *into* $L(H)$ *such that :*

(i) $\Psi_T(1)=I_H$, *where* $I_H \in L(H)$ *is the identity operator;*

(ii) $\Psi_T(g)=T$, *where* $g(z)=z$ *for all* $z \in \Omega$;

(iii) Ψ_T *is continuous when* $H^\infty(\Omega)$ *and* $L(H)$ *are given the weak***-topology. Moreover* Ψ_T *is contractive, i.e.,* $\|\Psi_T(f)\| \leqslant \|f\|$ *for all* $f \in H^\infty(\Omega)$.

From now on we will indicate $\Psi_T(f)$ by $f(T)$ for all $f \in H^\infty(\Omega)$.

DEFINITION 1.10. An operator T satisfying hypothesis (h) is said to be of *class* C_0 *relative to* Ω if there exists $u \in H^\infty(\Omega)\backslash\{0\}$ such that $u(T)=0$.

By Theorem 1 in [15], if T is of class C_0 relative to Ω, then there is a quasi-inner function $m_T \in H^\infty(\Omega)$ such that $\ker(\Psi_T) = m_T H^\infty(\Omega)$ and m_T is said to be a *minimal function* of T.

1.6. JORDAN MODEL.

DEFINITION 1.11. Let H and H' be Hilbert spaces. An operator $T \in L(H)$ is called a *quasiaffine transform* of an operator $T' \in L(H')$ if there exists an injective operator $X \in L(H, H')$ with dense range in H' such that $T'X=XT$. We write $T \prec T'$ if T is a quasiaffine transform of T'. The operators T and T' are *quasisimilar* $(T \sim T')$ if $T \prec T'$ and $T' \prec T$.

Let θ and θ' be two functions in $H^\infty(\Omega)$. We say that θ *divides* θ' (or $\theta|\theta'$) if θ' can be written as $\theta'=\theta\cdot\phi$ for some $\phi \in H^\infty(\Omega)$. We will use the notation $\theta \equiv \theta'$ if $\theta|\theta'$ and $\theta'|\theta$.

DEFINITION 1.12. (i) Given a quasi-inner function $\theta \in H^\infty(\Omega)$, the *Jordan block* $S(\theta)$ is the operator acting on the space $H(\theta)=H^2(\Omega) \ominus \theta H^2(\Omega)$ as follows:

$$S(\theta) = P_{H(\theta)} S | H(\theta) \tag{1.1}$$

where $S \in L(H^2(\Omega))$ is defined by $(Sf)(z)=zf(z)$.

(ii) Let $\Theta=\{\theta_i \in H^\infty(\Omega)\colon i = 1,2,3,\ldots\}$ be a family of quasi-inner functions. Then Θ is called a *model function* if $\theta_i|\theta_j$ whenever $j \leqslant i$. The *Jordan operator* $S(\Theta)$ determined by the model function Θ is the C_0-operator defined as $S(\Theta)=\bigoplus_{i<\gamma'} S(\theta_i)$, $\gamma'=\min\{k\colon \theta_k \equiv 1\}$.

We will call $S(\Theta)$ the *Jordan model* of the operator T if $S(\Theta) \sim T$. From [20], we can get following results:

THEOREM C. *For every operator* T *of class* C_0 *relative to* Ω *acting on a separable space* H, *there is a unique Jordan model for* T.

PROPOSITION 1.13. *Let T be of class C_0 relative to Ω acting on a separable space H and H' be $R(\Omega)$-invariant for T. If $T \sim \bigoplus_{\alpha<\gamma} S(\theta_\alpha)$ and $T|H' \sim \bigoplus_{\alpha<\gamma'} S(\theta'_\alpha)$, then $\theta'_\alpha|\theta_\alpha$ for every $\alpha \leqslant \min\{\gamma,\gamma'\}$.*

1.7. SCALAR MULTIPLES. Let K and K' be Hilbert spaces and $\varphi\in H^\infty(\Omega,L(K,K'))$ be a quasi-inner function. We set $H(\varphi) = H^2(\Omega,K') \ominus \varphi H^2(\Omega,K)$ and denote by $S(\varphi)$ the compression of $S_{K'}$ to $H(\varphi)$, i.e., $S(\varphi) = P_{H(\varphi)}\, S_{K'}|H(\varphi)$, where $P_{H(\varphi)}$ denotes the orthogonal projection onto $H(\varphi)$.

DEFINITION 1.14. The function $\varphi\in H^\infty(\Omega,L(K,K'))$ is said to have a *scalar multiple* $u \in H^\infty(\Omega)$, $u\neq 0$, if there exists $\psi\in H^\infty(\Omega,L(K',K))$ satisfying the relation $\varphi(z)\psi(z) = u(z)I_{K'}$ for $z \in \Omega$.

THEOREM 1.15. *Suppose that $\varphi\in H^\infty(\Omega,L(K,K'))$ is a quasi-inner function and $u\in H^\infty(\Omega)$. Then the following assertions are equivalent :*

(i) *u is a scalar multiple of φ.*

(ii) *$u(S(\varphi)) = 0$.*

(iii) *$uH^2(\Omega,K') \subset \varphi H^2(\Omega,K)$.*

Proof. Assume (i), and let $\psi\in H^\infty(\Omega,L(K',K))$ such that $\varphi(z)\psi(z)=u(z)\cdot I_{K'}$ for $z\in\Omega$. Then $u(S(\varphi))H(\varphi)=P_{H(\varphi)}u(S_{K'})H(\varphi) \subset P_{H(\varphi)}uH^2(\Omega,K') \subset P_{H(\varphi)}\varphi H^2(\Omega,K)$. Thus $u(S(\varphi))=0$. Thus (i)→(ii).

Next, assume (ii). Then $u(S_{K'})H(\varphi)=uH(\varphi)\subset\varphi H^2(\Omega,K)$. It follows that $uH^2(\Omega, K') = uH(\varphi)+u\varphi H^2(\Omega,K)\subset\varphi H^2(\Omega,K)$. Thus (ii) → (iii).

To prove (iii)→(i), let $M=\{f\in H^2(\Omega,K):ug=\varphi f$ for some $g\in H^2(\Omega,K')\}$. Then $\overline{M}$ is $R(\Omega)$-invariant for S_K. By Theorem 1.5, there is a Hilbert space K_1 and a quasi-inner function $\varphi_1\in H^\infty(\Omega,L(K_1,K))$ such that $\overline{M}=\varphi_1 H^2(\Omega,K_1)$. From Theorem 2.2.4 in [20], $u=\theta F$ where θ is a function such that $|\theta|$ is constant almost everywhere on each component of $\partial\Omega$ and F is an outer function in $H^\infty(\Omega)$. By the definition of M, $\theta H^2(\Omega,K')=\overline{\theta F H^2(\Omega,K')}=\overline{uH^2(\Omega,K')}=\overline{\varphi M}=\varphi\overline{M}=\varphi\varphi_1 H^2(\Omega,K_1)$. Since θ is quasi-inner, $\theta I_{K'}\in H^\infty(\Omega,L(K'))$ is also quasi-inner. (Note that $(\theta I_{K'})(z) = \theta(z)I_{K'}$). Then by Corollary 1.7, there exist $\varphi_2\in H^\infty(\Omega,L(K',K_1))$ such that $\theta I_{K'} = \varphi\varphi_1\varphi_2$. Then $uI_{K'} = \varphi(F\varphi_1\varphi_2)$, i.e. $u(z)I_{K'} = \varphi(z)(F(z)\varphi_1(z)\varphi_2(z))$. Since $F\varphi_1\varphi_2 \in H^\infty(\Omega, L(K',K))$, u is a scalar multiple of φ. ∎

In the next statement, $\operatorname{adj}\varphi : \Omega \to L(\mathbb{C}^n)$ is defined by $(\operatorname{adj}\varphi)(z)=\operatorname{adj}(\varphi(z))$ which is the algebraic adjoint of $\varphi(z)$ (i.e., $A\cdot\operatorname{adj}(A)=\operatorname{adj}(A)\cdot A=\det(A)I_{\mathbb{C}^n}$ for $A \in L(\mathbb{C}^n)$).

PROPOSITION 1.16. *Let K and K' be Hilbert spaces with* $\dim K = \dim K' = n$ $(<\infty)$.

(i) *If $\varphi \in H^\infty(\Omega,L(\mathbb{C}^n))$ is a quasi-inner function, then θ, defined by $\theta(z) = \det(\varphi(z))$, is quasi-inner.*

(ii) *If $\varphi\in H^\infty(\Omega,L(\mathbb{C}^n))$ is a quasi-inner function, then* $\operatorname{adj}\varphi$ *is quasi-inner.*

(iii) *If $\varphi\in H^\infty(\Omega,L(K,K'))$ is a quasi-inner function, then $S(\varphi)$ is of class C_0.*

Proof. (i) and (ii): Since φ is quasi-inner, there exists $m > 0$ such that for $h\in\mathbb{C}^n$, $m\|h\| \leqslant \|\varphi(z)h\|$ a.e. $z \in \partial\Omega$. Then

$$m^n \leqslant |\det(\varphi(z))| \quad \text{and} \quad m^{n-1} \leqslant |\operatorname{adj}(\varphi(z))| \quad \text{a.e.} \quad z \in \partial\Omega.$$

From those facts one can conclude that (i) and (ii) are true.

(iii): By Theorem 1.15, it is enough to prove that φ has a scalar multiple $u\in H^\infty(\Omega)$. Let $\psi(z)=\operatorname{adj}(\varphi(z))$ and $u(z)=\det(\varphi(z))$. Then by (ii), $\psi\in H^\infty(\Omega,L(K', K))$ and by (i), $u(\neq 0)\in H^\infty(\Omega)$. Since $\varphi(z)[\operatorname{adj}(\varphi(z))]=[\det(\varphi(z))]I_{K'}$ for $z \in \Omega$, it is proven. ∎

Let θ and θ' be two quasi-inner functions in $H^\infty(\Omega)$. If $\theta \equiv \theta'$ i.e., θ and θ' belong to the same equivalence class under the equivalence relation $\equiv$ between $H^\infty(\Omega)$ functions introduced after Definition 1.13, then it is convenient to regard them as the same element in $H^\infty(\Omega)$, and introduce the following definition.

DEFINITION 1.17. Let F be a family of functions in $H^\infty(\Omega)$. A quasi-inner function $\theta\in H^\infty(\Omega)$ is called *the greatest common quasi-inner divisor* of F if θ divides every element in F and if θ is a multiple of any other common quasi-inner divisor of F. *The greatest common quasi-inner divisor* of F is denoted by $\bigwedge F$ (or $\bigwedge_{i\in I} f_i$ if $F=\{f_i : i\in I\}$, or $f_1\wedge f_2$ if $F=\{f_1, f_2\}$).

2. QUASI-EQUIVALENCE AND QUASI-SIMILARITY

2.1. NORMAL FORM.

DEFINITION 2.1. A *quasi-unit* $\mathbf{X}$ of order n is a collection of $n \times n$ matrices over $H^\infty(\Omega)$ such that the family $\det(\mathbf{X})=\{\det(X) : X\in\mathbf{X}\}$ is relatively prime, i.e. $\bigwedge \det(\mathbf{X}) \equiv 1$.

DEFINITION 2.2. If A and B are $m \times n$ matrices over $H^\infty(\Omega)$, then A is said to be *quasi-equivalent* to B if there exist quasi-units $\mathbf{X}$ and $\mathbf{Y}$ of order m and n respectively such that $\mathbf{X}A=B\mathbf{Y}$ where $\mathbf{X}A=\{XA : X \in \mathbf{X}\}$ and $B\mathbf{Y}=\{BY : Y \in \mathbf{Y}\}$.

A matrix E over $H^\infty(\Omega)$ is in *normal form* (or simply *normal*) provided

$$E = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \tag{2.1}$$

where D is a diagonal matrix of nonzero quasi-inner functions and each one except the first divides its predecessor.

DEFINITION 2.3. Let $D_k(A)$ be the greatest common quasi-inner divisor of all minors of rank k of A (k is no larger than $\min\{m,n\}$) and D_0=1. Then the *invariant factors* for a $m \times n$ matrix A over $H^\infty(\Omega)$ are defined by

$$\xi_k(A) = \frac{D_k(A)}{D_{k-1}(A)} \quad \text{for } k \geqslant 1$$

such that some minors of rank k are not 0.

The following result is proved as Theorem 3.1 in [14].

PROPOSITION 2.4. *Every $n\times n$ matrix over $H^\infty(\Omega)$ is quasi-equivalent to a normal matrix. In fact, for any $n\times n$ matrix A over $H^\infty(\Omega)$, A is quasi-equivalent to the normal matrix formed by the invariant factors of A.*

The following result is proved as in the case of the open unit disk [6].

COROLLARY 2.5. *Let φ be a quasi-inner function in $H^\infty(\Omega,L(\mathbb{C}^n))$. If φ is quasi-equivalent to a normal matrix N whose diagonal entries are $\theta_0,\dots,\theta_{n-1}$, then* $\det\varphi\equiv\theta_0\cdots\theta_{n-1}$.

Let f_1 and f_2 be in $H^\infty(\Omega)$. If M is the w^*-closure of $f_1H^\infty(\Omega)+f_2H^\infty(\Omega)$, then by the same way as Theorem 1 in [15], we can get $M=(f_1\wedge f_2)H^\infty(\Omega)$.

PROPOSITION 2.6. *Let φ_1, $\varphi_2\in H^\infty(\Omega)$ be functions such that $\varphi_1\wedge\varphi_2\equiv 1$. If $f\in L^2(\partial\Omega,\mathbb{C}^n)$ and $\varphi_1 f$, $\varphi_2 f\in H^2(\partial\Omega,\mathbb{C}^n)$, then $f\in H^2(\partial\Omega,\mathbb{C}^n)$.*

Proof. Since $\varphi_1\wedge\varphi_2\equiv1$, the w^*-closure of $\varphi_1H^\infty(\partial\Omega)+\varphi_2H^\infty(\partial\Omega)$ is $H^\infty(\partial\Omega)$. Thus there are nets $\{f_\alpha\}$ and $\{g_\alpha\}$ in $H^\infty(\partial\Omega)$ such that $h_\alpha=\varphi_1 f_\alpha+\varphi_2 g_\alpha$ converges to 1, i.e.

$$\int_{\partial\Omega}(h_\alpha-1)h\,\mathrm{d}m\to 0 \tag{2.2}$$

for any $h\in L^1(\partial\Omega)$. We will prove that $h_\alpha f\to f$ weakly in $L^2(\partial\Omega,\,\mathbb{C}^n)$, i.e. $((h_\alpha f-f),g)\to0$ for any $g\in L^2(\partial\Omega,\mathbb{C}^n)$. Indeed, if $f=(f_1,\dots,f_n)$ and $g=(g_1,\dots,g_n)$, then $(h_\alpha f-f,g)=\int_{\partial\Omega}(h_\alpha-1)h\,\mathrm{d}m$ where $h=\sum_i f_i\overline{g}_i\in L^1(\partial\Omega)$. From (2.2), we have

$$h_\alpha f\to f \text{ weakly in } L^2(\partial\Omega).$$

Since a subspace of a Banach space is norm closed if and only if it is weakly closed [9], $H^2(\partial\Omega,\mathbb{C}^n)$ is weakly closed. Since $\varphi_1 f$, $\varphi_2 f\in H^2(\partial\Omega,\mathbb{C}^n)$, $h_\alpha f\in H^2(\partial\Omega,\mathbb{C}^n)$. If follows that $f\in H^2(\partial\Omega,\mathbb{C}^n)$. ∎

The following results are proved as in the case of the open unit disk [6].

PROPOSITION 2.7. *Let φ_1 and φ_2 be quasi-inner functions in $H^\infty(\Omega,L(\mathbb{C}^n))$. If φ_1 and φ_2 are quasi-equivalent, then $S(\varphi_1)$ and $S(\varphi_2)$ are quasisimilar.*

COROLLARY 2.8. *Let φ be a quasi-inner function in $H^\infty(\Omega,L(\mathbb{C}^n))$. If φ is quasi-equivalent to a normal matrix N whose diagonal entries are $\theta_0,\dots,\theta_{n-1}$ ($\theta_{i+1}|\theta_i$ for i=$0,1,\dots,n-1$), then $S(\varphi)\sim\bigoplus_{i=0}^{n-1}S(\theta_i)$.*

Proof. Since $S(N)\sim\bigoplus_{i=0}^{n-1}S(\theta_i)$, by Proposition 2.7, $S(\varphi)\sim\bigoplus_{i=0}^{n-1}S(\theta_i)$, because "$\sim$" is an equivalence relation. ∎

COROLLARY 2.9. *Let φ_1 and φ_2 be quasi-inner functions in $H^\infty(\Omega,L(\mathbb{C}^n))$. If $S(\varphi_1)$ is a quasi-affine transform of $S(\varphi_2)$, then φ_1 and φ_2 are quasi-equivalent.*

COROLLARY 2.10. *Let φ be a quasi-inner function in $H^\infty(\Omega,L(\mathbb{C}^n))$. If $S(\varphi) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, then* $\det\varphi \equiv \theta_0 \cdots \theta_{n-1}$.

Proof. Let N be a normal matrix whose diagonal entries are $\theta_0, \ldots, \theta_{n-1}$. Since $S(N) \sim \bigoplus_{i=0}^{n-1} S(\theta_i)$, $S(\varphi) \sim S(N)$. By Corollary 2.9, φ and N are quasi-equivalent. Then by Corollary 2.5, $\det\varphi \equiv \theta_0 \cdots \theta_{n-1}$. ∎

2.2. MAIN RESULTS. In this section, first of all we show how to use Theorem 1.5 and Corollary 1.7.

THEOREM 2.11. *Let F and F' be two separable Hilbert spaces and φ be a quasi-inner function in $H^\infty(\Omega,L(F,F'))$.*

(i) *If $M \subset H(\varphi)$ is $R(\Omega)$-invariant for $S(\varphi)$, then there is a Hilbert space K and there are quasi-inner functions $\varphi_1 \in H^\infty(\Omega,L(F,K))$ and $\varphi_2 \in H^\infty(\Omega,L(K,F'))$ such that $\varphi(z)=\varphi_2(z)\varphi_1(z)$ for $z\in\Omega$ and*

$$M = \varphi_2 H^2(\Omega,K) \ominus \varphi H^2(\Omega,F). \tag{2.3}$$

(ii) *Conversely, if K, φ_1 and φ_2 are as above , then* (2.3) *defines a $R(\Omega)$-invariant subspace of $H(\varphi)$. Moreover, if $S(\varphi)=\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is the triangularization of $S(\varphi)$ with respect to the decomposition $H(\varphi)=M \oplus (H(\varphi) \ominus M)$, then $T_2=S(\varphi_2)$ and $S(\varphi_1)$ is similar to T_1.*

Proof. (i) Since M is $R(\Omega)$-invariant, the space $M \oplus \varphi H^2(\Omega,F)$ is also $R(\Omega)$ -invariant subspace of $H^2(\Omega,F')$ and so Theorem 1.5 implies the existence of a Hilbert space K and of a quasi-inner function $\varphi_2\in H^\infty(\Omega,L(K,F'))$ such that (2.3) holds.

The inclusion $\varphi H^2(\Omega,F) \subset \varphi_2 H^2(\Omega,K)$ implies that for any $f\in H^2(\Omega,F)$ there is $\phi_f\in H^2(\Omega,K)$ such that $\varphi f=\varphi_2\phi_f$. Let $M'=\{\phi_f\in H^2(\Omega,K) : \varphi f=\varphi_2\phi_f$ for some $f\in H^2(\Omega,F)\}$. Since $\varphi(rf)=\varphi_2(r\phi_f)$ for any $r\in R(\Omega)$ and $f\in H^2(\Omega,F)$, M' is also a $R(\Omega)$-invariant subspace of $H^2(\Omega,K)$, and so $M'=\varphi_3 H^2(\Omega,K')$ for some Hilbert space K' and a quasi-inner function $\varphi_3\in H^\infty(\Omega,L(K',K))$ by Theorem 1.5. It follows that $\varphi H^2(\Omega,F)=\varphi_2\varphi_3 H^2(\Omega,K')$ by the definition of M'. By Corollary 1.7, there is a function $\varphi_4\in H^\infty(\Omega,L(F,K'))$ such that $\varphi=\varphi_2\varphi_3\varphi_4$.

Let $\varphi_1=\varphi_3\varphi_4\in H^\infty(\Omega,L(F,K))$. Since φ and φ_2 are quasi-inner functions, so is φ_1. Thus φ_1 is a quasi-inner function satisfying $\varphi=\varphi_2\varphi_1$.

(ii) The $R(\Omega)$-invariance of the subspace M described by (2.3) is obvious.

Since $H(\varphi) \ominus M=H^2(\Omega,F') \ominus \varphi_2 H^2(\Omega,K)=H(\varphi_2)$, we have

$$T_2{}^* = S(\varphi)^*|H(\varphi) \ominus M = S^*_{F'}|H(\varphi_2) = S(\varphi_2)^*.$$

Thus $T_2 = S(\varphi_2)$. It remains to prove similarity of T_1 and $S(\varphi_1)$. Define $Y : H^2(\Omega, K) \to \varphi_2 H^2(\Omega, K)$ by $Yf = \varphi_2 f$. Clearly Y is onto. Since φ_2 is a quasi-inner function, Y is one-to-one. Since $Y(\varphi_1 H^2(\Omega, F)) = \varphi_2\varphi_1 H^2(\Omega, F) = \varphi H^2(\Omega, F)$, $\varphi_2 H^2(\Omega, K) = M \oplus \varphi H^2(\Omega, F)$ and $H^2(\Omega, K)$=$H(\varphi_1) \oplus \varphi_1 H^2(\Omega, F)$, we have $P_M Y(H(\varphi_1))$=M. Thus we can define a bounded linear operator $F : H(\varphi_1) \to M$ by $Fg = P_M \varphi_2 g$ for $g \in H(\varphi_1)$, and F is onto. Since φ_2 is a quasi-inner function, $\ker F = \{g \in H(\varphi_1)\colon \varphi_2 g \in \varphi H^2(\Omega, F)\}$ =$\{g \in H(\varphi_1) : g \in \varphi_1 H^2(\Omega, F)\} = \{0\}$. It follows that $F \in L(H(\varphi_1), M)$ is bijective. By the Open Mapping Theorem, F is invertible and clearly $T_1 F = FS(\varphi_1)$. ∎

Fix $n \geqslant 1$, and consider the mapping $\Gamma_n : L(F) \to L(\bigotimes^n F)$ given by $\Gamma_n(T) = T \otimes T \otimes \cdots \otimes T$, where F is a Hilbert space and $T \in L(F)$.

Define a unitary representation $\pi_n : S_n \to L(\bigotimes^n F)$ where S_n denotes the group of permutations of $\{1, 2, \ldots, n\}$, defined by

$$\pi_n(\sigma)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes \cdots \otimes x_{\sigma^{-1}(n)}, \tag{2.4}$$

$\sigma \in S_n$, $x_j \in F$, $1 \leqslant j \leqslant n$.

The homomorphism π_n can be extended to a $*$-homomorphism, still denoted π_n, from the C^*-algebra consisting of all formal sums $\sum\limits_{\sigma \in S_n} \alpha_\sigma \sigma$ $(\alpha_\sigma \in \mathbb{C})$ to $L(\otimes^n F)$. We will use the alternating projection a_n defined by

$$a_n = \frac{1}{n!} \sum_{\sigma \in S_n} \varepsilon(\sigma)\sigma, \tag{2.5}$$

where $\varepsilon(\sigma)$ is the sign of σ, i.e. $\varepsilon(\sigma) = +1$ or -1 according to whether σ is an even or odd permutation. Let $n \geqslant 1$ be a natural number. We use the notation $\bigwedge^n F$ for $\pi_n(a_n)$ $(\otimes^n F)$. The space $\bigwedge^n F$ is called the nth *exterior power* of F. If $B \in L(F)$, we denote by $\bigwedge^n B$ the operator $\Gamma_n(B) | \bigwedge^n F$.

PROPOSITION 2.12. *If A and B are quasi-equivalent quasi-inner functions in $H^\infty(\Omega, L(\mathbb{C}^n))$, then $\bigwedge^k A$ and $\bigwedge^k B$ are quasi-equivalent, for $1 \leqslant k \leqslant n$.*

Proof. This is same as Proposition 6.5.17 in [6]. ∎

PROPOSITION 2.13. *If* $A = \begin{pmatrix} \theta_0 & 0 & \ldots & 0 \\ 0 & \theta_1 & \ldots & 0 \\ & & & \\ 0 & 0 & \ldots & \theta_{n-1} \end{pmatrix}_{n \times n}$ *is normal, then $S(\bigwedge^k A)$ has minimal function $\theta_0\theta_1 \cdots \theta_{k-1}$ for $k = 1, \ldots, n$.*

Proof. Since $\bigwedge^k A$ is also a diagonal quasi-inner function with diagonal entries $\theta_{i_1}\theta_{i_2} \cdots \theta_{i_k}$ where $i_p \neq i_q$ for $p \neq q$ ([6]), the minimal function of $S(\bigwedge^k A)$ is $\theta_0\theta_1 \cdots \theta_{k-1}$. ∎

If $\{M_i\}_{i\in I}$ is a family of subsets of the Hilbert space H, we denote by $\bigvee_{i\in I} M_i$ the closed linear span generated by $\bigcup_{i\in I} M_i$.

DEFINITION 2.14. Let $T \in L(H)$ be an operator with spectrum in $\overline{\Omega}$. A subset $G \subseteq H$ with the property that $\bigvee\{r(T)m \,;\, r \in R(\Omega),\, m \in G\} = H$, is called an $R(\Omega)$-*cyclic set* for T. The *multiplicity* μ_T of T is the smallest cardinality of an $R(\Omega)$-cyclic set for T. The operator T is said to be *multiplicity-free* if $\mu_T = 1$. If $\mu_T = 1$, any vector $x \in H$ such that $\bigvee\{r(T)x \,;\, r \in R(\Omega)\} = H$ is said to be $R(\Omega)$-*cyclic* for T.

Recall that if $\mu_T \leqslant n$, then Jordan model of T is $\bigoplus_{j=0}^{n-1} S(\theta_j)$ [20].

PROPOSITION 2.15. *Assume that $T \in L(H)$ is an operator of class C_0 relative to Ω such that $\mu_T = n < \infty$, H' is a $R(\Omega)$-invariant subspace for T, and $T = \begin{pmatrix} T' & Y \\ 0 & T'' \end{pmatrix}$ is the triangularization of T with respect to the decomposition $H = H' \oplus (H \ominus H')$. If $\bigoplus_{j<n} S(\theta_j)$, $\bigoplus_{j<n} S(\theta'_j)$, and $\bigoplus_{j<n} S(\theta''_j)$ are the Jordan models of T, T', T'', respectively, then*

$$\theta_0 \cdots \theta_{k-1} | \theta'_0 \cdots \theta'_{k-1} \theta''_0 \cdots \theta''_{k-1}$$

for every k such that $1 \leqslant k < n$, and

$$\theta_0 \cdots \theta_{n-1} \equiv \theta'_0 \cdots \theta'_{n-1} \theta''_0 \cdots \theta''_{n-1}.$$

Proof. Let $f \in H^\infty(\Omega, L(\mathbb{C}^n))$ be a quasi-inner function such that f is a normal matrix whose diagonal entries are $\theta_0, \ldots, \theta_{n-1}$. By Corollary 2.8, $S(f) = \bigoplus_{j=0}^{n-1} S(\theta_j) \sim T$. Thus there is an injective operator $X \in L(H, H(f))$ with dense range such that $S(f)X = XT$.

Let M be the closure of XH'. Since H' is a $R(\Omega)$-invariant subspace for T, so is M for $S(f)$. Then by Theorem 2.11, there are quasi-inner functions $f_1 \in H^\infty(\Omega, L(\mathbb{C}^n))$ and $f_2 \in H^\infty(\Omega, L(\mathbb{C}^n))$ such that

$$f = f_2 f_1 \quad \text{and} \quad M = f_2 H^2(\Omega, \mathbb{C}^n) \ominus f H^2(\Omega, \mathbb{C}^n).$$

If $S(f) = \begin{pmatrix} T_1 & Z \\ 0 & T_2 \end{pmatrix}$ is the triangularization of $S(f)$ with respect to the decomposition $H(f) = M \oplus (H(f) \ominus M)$, then by Theorem 2.11, T_1 is similar to $S(f_1)$ and $T_2 = S(f_2)$.

Let $X' = X|H'$. Then $T_1 X' = S(f)X|H' = XT|H' = X'T'$ and so $T_1 \sim T' \sim \bigoplus_{j=0}^{n-1} S(\theta'_j)$. Since T_1 is similar to $S(f_1)$, $S(f_1) \sim \bigoplus_{j=0}^{n-1} S(\theta'_j)$. Define $X'' : H(f) \ominus M \to H \ominus H'$ by $X'' = X^*|H(f) \ominus M$. Then X'' is injective with dense range in $H \ominus H'$ and $X'' T_2{}^* = X^* S(f)^* | H(f) \ominus M = T^* X^* | H(f) \ominus M = (T'')^* X''$. Thus

$T_2 \sim T'' \sim \bigoplus_{j=0}^{n-1} S(\theta''_j)$. It follows that $S(f_2) \sim \bigoplus_{j=0}^{n-1} S(\theta''_j)$. Fix k such that $1 \leqslant k < n$ and note that $\bigwedge^k f = \bigwedge^k f_2 \wedge \bigwedge^k f_1$. By Proposition 2.13, the minimal function of $S(\bigwedge^k f)$ is $\theta_0\theta_1 \cdots \theta_{k-1}$. By Corollary 2.9, there are normal matrices N_1 and N_2 which are quasi-equivalent to f_1 and f_2, respectively and diagonal entries of N_1 (N_2) are $\theta'_0, \theta'_1, \ldots, \theta'_{n-1}$ ($\theta''_0, \theta''_1, \ldots, \theta''_{n-1}$, respectively). By Proposition 2.12, $\bigwedge^k f_1$ and $\bigwedge^k N_1$ are quasi-equivalent. By Proposition 2.7, $S(\bigwedge^k f_1)$ and $S(\bigwedge^k N_1)$ are quasisimilar. Thus the minimal functions of $S(\bigwedge^k f_1)$ is $\theta'_0\theta'_1 \cdots \theta'_{k-1}$. Similarly, the minimal function of $S(\bigwedge^k f_2)$ is $\theta''_0\theta''_1 \cdots \theta''_{k-1}$. By Theorem 1.15, there are functions g', $g'' \in H^\infty(\Omega, L(\bigwedge^k \mathbb{C}^n))$ such that $g'(\bigwedge^k f_1) = \theta'_0\theta'_1 \cdots \theta'_{k-1} I$ and $g''(\bigwedge^k f_2) = \theta''_0\theta''_1 \cdots \theta''_{k-1} I$. Combining these relations we get $g'g''(\bigwedge^k f) = g'g''(\bigwedge^k f_2 \bigwedge^k f_1) = \theta'_0\theta'_1 \cdots \theta'_{k-1}\theta''_0\theta''_1 \cdots \theta''_{k-1} I$ and this corollary follows because $\theta_0\theta_1 \cdots \theta_{k-1}$ is the least scalar multiple of $\bigwedge^k f$ by Theorem 1.15.

Next, for $k = n$, since $S(f) \sim \bigoplus_{j=0}^{n-1} S(\theta_j)$, $S(f_1) \sim \bigoplus_{j=0}^{n-1} S(\theta'_j)$, and $S(f_2) \sim \bigoplus_{j=0}^{n-1} S(\theta''_j)$, by Corollary 2.10, $\det(f) \equiv \theta_0\theta_1 \cdots \theta_{n-1}$, $\det(f_1) \equiv \theta'_0\theta'_1 \cdots \theta'_{n-1}$, and $\det(f_2) \equiv \theta''_0\theta''_1 \cdots \theta''_{n-1}$. From the fact $f = f_2 f_1$, we can get $\det(f) = (\det(f_2))(\det(f_1))$ which proves the case $k = n$. ∎

When $T \in L(H)$ is an operator of class C_0 relative to Ω and $K = \bigvee\{r(T)h : r \in R(\Omega)\}$, let m_h denote the minimal function of $T|K$. We have the following proposition from Theorem 4.3.10 in [20].

PROPOSITION 2.16. *Let $T \in L(H)$ be an operator of class C_0 relative to Ω. If $\bigoplus_{j<\omega} S(\theta_j)$ is the Jordan model of T, then for any $k = 1, 2, 3, \ldots$, there are $R(\Omega)$-invariant subspaces $M_{-1}, M_0, \ldots, M_{k-2}$ and $h_0, h_1, \ldots, h_{k-1}$ in H such that*

$$h_i \in M_{i-1} \quad \text{and} \quad m_{h_i} = m_{T|M_{i-1}} \tag{2.6}$$

for $i = 0, 1, \ldots, k-1$, and

$$K_i \vee M_i = M_{i-1} \quad \text{and} \quad K_i \cap M_i = \{0\} \tag{2.7}$$

for $i = 0, 1, \ldots, k-1$, where $M_{-1} = H$ and $K_i = \bigvee\{r(T)h_i : r \in R(\Omega)\}$.

THEOREM 2.17. *Assume that $T \in L(H)$ is an operator of class C_0 relative to Ω, H' is a $R(\Omega)$-invariant subspace for T, and $T = \begin{pmatrix} T' & Y \\ 0 & T'' \end{pmatrix}$ is the triangularization of T with respect to the decomposition $H = H' \oplus (H \ominus H')$. If $\bigoplus_{j<\gamma} S(\theta_j)$, $\bigoplus_{j<\gamma} S(\theta'_j)$, and $\bigoplus_{j<\gamma} S(\theta''_j)$ are the Jordan models of T, T', T'', respectively, then*

$$\theta_0 \cdots \theta_{k-1} | \theta'_0 \cdots \theta'_{k-1}\theta''_0 \cdots \theta''_{k-1}$$

for every $k = 1, 2, 3, \ldots$.

Proof. Fix $k \geqslant 1$. Since $T \sim \bigoplus_{j<\omega} S(\theta_j)$, by Proposition 2.16 and proof of Theorem 4.3.10 in [20], there is a $R(\Omega)$-invariant subspace M for T such that $T_1 = T|M \sim \bigoplus_{j<k} S(\theta_j)$. Clearly $H_1{}' = M \cap H'$ is $R(\Omega)$-invariant for T_1.

Let $T_1 = \begin{pmatrix} T_1' & Y_1 \\ 0 & T_1'' \end{pmatrix}$ be the triangularization of T_1 with respect to the decomposition $M = H_1' \oplus (M \ominus H_1')$. If $\bigoplus_j S(\phi_j')$ and $\bigoplus_j S(\phi_j'')$ are Jordan models of T_1' and T_1'', respectively, then

$$\theta_0 \cdots \theta_{k-1} \equiv \phi_0' \cdots \phi_{k-1}' \phi_0'' \cdots \phi_{k-1}'' \tag{2.8}$$

by Proposition 2.15. (Note $\mu_{T_1} \leqslant k$.) Since $T'|H_1{}' = T_1{}'$, by Proposition 1.13, $\phi_i{}'|\theta_i{}'$ for $i = 0, \ldots, k-1$.

Next, let $H_1'' = M \ominus H_1'$, $H'' = H \ominus H'$, and $X : H_1{}'' \to H''$ be orthogonal projection. If $a \in \ker X$, then $a \in H' \cap (M \ominus H_1') \subset H' \cap M = H_1{}'$. Since $a \in H_1''(= M \ominus H_1')$, $a = 0$. Thus X is one-to-one. Moreover, H' is invariant for T, and H'' is invariant for T^*. Thus $T^* P_{H''} = P_{H''} T^* P_{H''}$ and so $P_{H''} T = (T^* P_{H''})^* = (P_{H''} T^* P_{H''})^* = P_{H''} T P_{H''} = T'' P_{H''}$. Since $P_{H''} T_1'' = P_{H''} P_{M \ominus H_1'} T|M \ominus H_1' = P_{H''} T|M \ominus H_1'$, $T'' X = X T_1''$. Since X is one-to-one, T_1'' is quasi-similar to a restriction of T'' to an invariant subspace and so we can get $\phi_i{}''|\theta_i{}''$ for $i = 0, \ldots, k-1$. Thus from (2.8), we can conclude that $\theta_0 \cdots \theta_{k-1} | \theta_0' \cdots \theta_{k-1}' \theta_0'' \cdots \theta_{k-1}''$. ∎

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