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INTERPOLATION CLASSES AND MATRIX MONOTONE FUNCTIONS

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ABSTRACT. An interpolation function of order n is a positive function f on $(0,\infty)$ such that $\|f(A)^{1/2}Tf(A)^{-1/2}\| \leqslant \max(\|T\|, \|A^{1/2}TA^{-1/2}\|)$ for all $n \times n$ matrices T and A such that A is positive definite. By a theorem of Donoghue, the class C_n of interpolation functions of order n coincides with the class of functions f such that for each n-subset $S = \{\lambda_i\}_{i=1}^n$ of $(0,\infty)$ there exists a positive Pick function h on $(0,\infty)$ interpolating f at S. This note comprises a study of the classes C_n and their relations to matrix monotone functions of finite order. We also consider interpolation functions on general unital C^*-algebras.

KEYWORDS: *Interpolation function, matrix monotone function, Pick function.*

MSC (2000): 46B70, 46L05, 47A56.

1. INTRODUCTION

An *interpolation function h* relative to a positive operator A in a Hilbert space H is a positive continuous function defined on the spectrum of A fulfilling the condition

$$\|h(A)^{1/2}Th(A)^{-1/2}\| \leqslant \max(\|T\|, \|A^{1/2}TA^{-1/2}\|) \tag{1.1}$$

for every bounded operator T on H. By a theorem of Donoghue [6], [5] (cf. also [1], [2]), it is known that the class of interpolation functions relative to A coincides precisely with the class of restrictions to $\sigma(A)$ of positive *Pick functions*, i.e., functions of the form

$$h(\lambda) = \int_{[0,\infty]} \frac{(1+t)\lambda}{1+t\lambda}\,\mathrm{d}\varrho(t), \quad \lambda > 0, \tag{1.2}$$

where ϱ is some positive Radon measure on $[0,\infty]$. The convex cone of functions having such a representation is denoted by the symbol P'.

Now fix $n \in \mathbb{N}$, and assume that $H = \ell_2^n$ is an n-dimensional Hilbert space. We shall say that a function h defined on $\mathbb{R}_+ = (0, \infty)$ is an *interpolation function of order n* and write $h \in C_n$ if h satisfies (1.1) for *every* positive operator $A \in B(\ell_2^n)$. By the cited theorem of Donoghue, a function f belongs to C_n if and only if for every n-set $\{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$ there exists a function $h \in P'$ such that $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \ldots, n$. (Of course, the function h depends on f and the set $\{\lambda_i\}_{i=1}^n$ and is in general not unique.)

The classes C_n are related to the classes P_n' of *positive matrix monotonic functions of order n* on $\mathbb{R}_+$. This is the set of functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ having the property that for any positive definite $n \times n$-matrices A, B, the condition $A \leqslant B$ implies $h(A) \leqslant h(B)$. Indeed, it is known that $\bigcap_{n=1}^{\infty} P_n' = \bigcap_{n=1}^{\infty} C_n = P'$. The equality $\bigcap P_n' = P'$ is a well-known theorem of Löwner [13], whereas the fact that $\bigcap C_n = P'$ is essentially due to Foiaş and Lions [8].

We remark that Löwner's original proof of the fact that $\bigcap P_n' = P'$ depends on the theory of interpolation of matrix monotone functions by Pick functions. A standard source on this type of interpolation is Donoghue's book [7]. Indeed, by a result from Löwner's theory, a matrix monotone function $h \in P_n'$ can be interpolated at any subset of $\mathbb{R}_+$ consisting of $2n - 1$ points by a P'-function, but the latter condition is in general not sufficient for $h \in P_n'$ to hold. We will use this fact later in Section 3 to prove that $P_{n+1}' \subseteq C_{2n+1} \subseteq C_{2n} \subseteq P_n'$ for all n, where all inclusions are proper for appropriate values of n.

We finally remark that a third scale of classes of functions, denoted M_n, were introduced by G. Sparr [16], as a means of obtaining a new proof of Löwner's Theorem. The key observation in Sparr's proof is that the classes M_n satisfy $P_{n+1} \subseteq M_n \subseteq P_n$, where P_n is the class of all real-valued matrix monotone functions of order n on $\mathbb{R}_+$. The M_n's are moreover defined in a way which is similar to the classes C_{2n}, but there are some differences. In the sequel, we will reserve the letter M_n for the algebra of complex $n \times n$-matrices.

New proofs of Löwner's and Donoghue's Theorems can be found in [1], [2].

2. PRELIMINARIES

In this section, we begin by giving a presentation of earlier results which we shall use and discuss further later on.

Let $M_n := B(\ell_2^n)$ denote the space of complex $n \times n$ matrices, identified in the natural way with the space of bounded operators on ℓ_2^n. We shall write $A > 0$ if and only if $A \in M_n$ is a positive definite matrix. (More generally, we shall write $a > 0$ if a is a positive element of a unital C^*-algebra $\mathcal{A}$ such that $0 \notin \sigma(a)$.) The class (convex cone) P_n' of (positive) *matrix monotonic functions of order n* is by definition the set of functions $h : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$A, B \in M_n \quad \text{and} \quad 0 < A \leqslant B \quad \text{imply} \quad h(A) \leqslant h(B).$$

(Here $h(A)$ and $h(B)$ denote the usual functional calculus in the C^*-algebra M_n.) In this notation, the well-known theorem of Löwner [13] becomes

$$\bigcap_{n=1}^{\infty} P'_n = P', \tag{2.1}$$

where P' is the class of functions representable in the form (1.2) with some positive Radon measure ϱ on $[0, \infty]$. We shall occasionally need to use the class of (not necessarily positive) *Pick functions* on $\mathbb{R}_+$, which we denote by P or sometimes $P(\mathbb{R}_+)$. This is the class of functions $h : \mathbb{R}_+ \to \mathbb{R}$ which are real-analytic on $\mathbb{R}_+$ and admit of analytic continuation to the upper half-plane in $\mathbb{C}$ and have non-negative imaginary parts there. It can be shown [7] that

$$P' = \{f \in P : f > 0 \text{ on } \mathbb{R}_+\}. \tag{2.2}$$

In [12] it was shown that all the classes P'_n are different, i.e.

$$P'_{n+1} \subsetneqq P'_n, \quad n \in \mathbb{N}. \tag{2.3}$$

(As noted in [12], (2.3) was previously asserted by Donoghue ([7], p. 83), but without a detailed proof.)

In 1961, Foiaş and Lions [8] introduced the class of "interpolation functions" and established their basic properties. For $A \in M_n$ such that $A > 0$, we define the A-norm on M_n by $\|T\|_A = \|A^{1/2}TA^{-1/2}\|$. We note that for $c \geqslant 0$, the statement $\|T\|_A \leqslant c$ is equivalent to $A^{-1/2}T^*ATA^{-1/2} \leqslant c^2$, i.e., $T^*AT \leqslant c^2A$. We shall say that a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is an *interpolation function of order n*, and that it *belongs to the class C_n* if and only if

$$\|T\|_{h(A)} \leqslant \max(\|T\|, \|T\|_A), \quad \forall\ T, A \in M_n : A > 0,$$

or, equivalently,

$$(\forall\ T, A \in M_n) :\ A > 0, \quad T^*T \leqslant 1, \quad T^*AT \leqslant A \quad \text{imply} \quad T^*h(A)T \leqslant h(A).$$

Evidently $C_{n+1} \subseteq C_n$ for all n. In [8], Foiaş and Lions proved an equivalent of the following statement:

$$\bigcap_{n=1}^{\infty} C_n = P'. \tag{2.4}$$

See Remark 3.2 (cf. also [1] and [2], Section 4).

In 1967, Donoghue [6], [5] proved a stronger version of the Foiaş–Lions Theorem. In order to formulate Donoghue's Theorem in its full generality, let H be a Hilbert space, and A, B fixed positive, injective (possibly unbounded) operators in H such that there exists a positive number r such that, in the sense of quadratic forms,

$$\frac{1}{r}A(1+A)^{-1} \leqslant B \leqslant r(1+A). \tag{2.5}$$

Consider the condition

$$\|T\|_B \leqslant \max(\|T\|, \|T\|_A), \quad \forall\, T \in B(H). \tag{2.6}$$

This condition is equivalent to the following statement: for all $T \in B(H)$ such that $T^*T \leqslant 1$ and $T^*AT \leqslant A$ holds: $T^*BT \leqslant B$. In particular, if we take $T = E$ to be an orthogonal projection, this implication says: $EAE \leqslant E$ implies $EBE \leqslant B$. But for orthogonal projections, the condition $EAE \leqslant A$ is equivalent to that A and E commute. Thus B commutes with every orthogonal projection which commutes with A, that is, B is affiliated with A. It now follows from von Neumann's Bicommutator Theorem that $B = f(A)$ for some Borel measurable positive function f on $\sigma(A)$. With somewhat more effort, it is possible to prove that f may be taken to be continuous.

FACT 2.1. *Suppose that* (2.5) *and* (2.6) *holds. Then there exists a (unique) continuous positive function h on $\sigma(A)$ such that $B = h(A)$.*

For a proof of Fact 2.1, we refer to Lemma 2 of [6], or Lemma 1.1 of [2]. We remark that, in our applications of Fact 2.1 in this paper, the operators A and B will be bounded above and below, whence the condition (2.5) will be trivially satisfied.

DEFINITION 2.2. Let $P'|\sigma(A)$ be the convex cone of restrictions to $\sigma(A)$ of functions in P' (of the form (1.2)). Let C_A be the class of continuous functions $h : \sigma(A) \to \mathbb{R}_+$ such that the corresponding operator $B = h(A)$ fulfills (2.6). We refer to C_A as the class of *interpolation functions* with respect to A.

THEOREM 2.3. *The class of interpolation functions with respect to A coincides precisely with the set of restrictions to $\sigma(A)$ of P'-functions. In other words,*

$$C_A = P'|\sigma(A). \tag{2.7}$$

The original formulation of this theorem ([6], Theorem 1) is in the guise of interpolation theory. A proof of this theorem in the present form is given in Theorem 7.1 of [1] (the finite-dimensional case) and [2] (the infinite-dimensional case).

The following corollary is immediate from Theorem 2.3.

COROLLARY 2.4. *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to C_n if and only if for every n-set $S = \{\lambda_i\}_{i=1}^n$, there exists a P'-function h interpolating f at S, i.e. $f(\lambda_i) = h(\lambda_i)$ for $i = 1, \ldots, n$.*

3. A STUDY OF THE CLASSES C_n AND P'_n

We shall now consider the problem of finding the precise relations between the classes of monotone functions and interpolation functions of finite order. In [1], it was observed that $P'_{n+1} \subseteq C_{2n} \subseteq P'_n$. We shall now see that this observation can quite easily be improved, by using two theorems from the Löwner theory, as stated in Donoghue's book [7], Chapter XIV.

We have the following theorem.

THEOREM 3.1. *For all $n \in \mathbb{N}$ holds:*

$$P'_{n+1} \subseteq C_{2n+1} \subseteq C_{2n} \subseteq P'_n. \tag{3.1}$$

Moreover, P'_n and C_n are different classes for all n,

$$P'_n \subsetneq C_n. \tag{3.2}$$

Proof. "$P'_{n+1} \subseteq C_{2n+1}$": Let $f \in P'_{n+1}$ and let $S = \{\lambda_i\}_{i=1}^{2n+1} \subseteq \mathbb{R}_+$ be a subset consisting of $2n+1$ points, where $0 < \lambda_1 < \cdots < \lambda_{2n+1}$. Then by Theorem I, p. 128 of [7], there exists a function $h \in P$, rational of degree at most n, such that $h(\lambda_i) = f(\lambda_i)$, $i = 1, \ldots, 2n+1$. Following Donoghue [7], we associate to the set S the polynomial

$$S(\lambda) = \prod_{i=1}^{2n+1} (\lambda - \lambda_i).$$

By Theorem III, p. 131 in [7] we have

$$(f(\lambda) - h(\lambda))S(\lambda) \geqslant 0, \quad \lambda > 0. \tag{3.3}$$

But in the interval $\lambda \in (0, \lambda_1)$, $S(\lambda)$ is negative, and thus by (3.3), $f(\lambda) - h(\lambda) \leqslant 0$ there. But this means that $h(\lambda) \geqslant f(\lambda) > 0$, $\lambda \in (0, \lambda_1)$, since f is positive on $\mathbb{R}_+$. Thus (since $h \in P$, and since Pick functions are non-decreasing) we obtain $h > 0$ on $\mathbb{R}_+$, i.e., $h \in P'$ (see (2.2)). Thus f coincides on the set S with a P'-function, and since $S = \{\lambda_i\}_{i=1}^{2n+1} \subseteq \mathbb{R}_+$ was arbitrary, we deduce using Corollary 2.4 that $f \in C_{2n+1}$.

"$C_{2n} \subseteq P'_n$": This is done as in [1], by using Donoghue's trick ([6], pp. 266–267). We include the proof for completeness. Let $f \in C_{2n}$ and let $A, B \in M_n$, $0 < A \leqslant B$. Form the $2n \times 2n$ matrices

$$A_1 = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then

$$T^* A_1 T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = A_1,$$

so we deduce that $T^* f(A_1) T \leqslant f(A_1)$, or

$$\begin{pmatrix} f(A) & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} f(B) & 0 \\ 0 & f(A) \end{pmatrix}.$$

We deduce that $f(A) \leqslant f(B)$, i.e. $f \in P'_n$. This concludes the proof of (3.1).

To prove that $P'_n \subseteq C_n$ for all $n \in \mathbb{N}$ we now use (3.1) in the following way:

$$P'_n \subseteq C_{2n-1} \subseteq C_n, \quad n \geqslant 1.$$

If $n \geqslant 3$, we have furthermore, using (2.3), then (3.1)

$$P'_n \subsetneqq P'_{n-1} \subseteq C_{2n-3} \subseteq C_n.$$

This proves (3.2) for all $n \geqslant 3$.

For $n = 2$, we argue as follows. The function $h(\lambda) = \min(1, \lambda)$ is quasi-concave, and thus is C_2 by Proposition 3.7 below. But by a theorem of Löwner ([13], top of p. 187), a function in P'_2 is either constant or strictly increasing, whence the function h above cannot be in P'_2. This finishes the proof of (3.2) in the case $n = 2$. Finally, for $n = 1$ (3.2) is obvious, because any positive function which is somewhere strictly decreasing belongs to C_1 but not to P'_1. ∎

REMARK 3.2. Combining Theorem 3.1 with Löwner's theorem (equation (2.1)), we obtain a proof of the Foiaş–Lions theorem (equation (2.4)).

REMARK 3.3. We shall prove below that all inclusions in (3.1) are proper for small values of n. (More precisely, we will prove that $C_4 \subsetneqq P'_2 \subsetneqq C_3 \subsetneqq C_2 \subsetneqq P'_1 \subsetneqq C_1$.)

CONJECTURE 3.4. *All inclusions in* (3.1) *are proper for all* n.

Let $S \subseteq \mathbb{R}_+$ be an arbitrary set and $f : S \to \mathbb{R}_+$ a function. We define the *reverse* and *dual* functions f^* and $\check{f}$ on the set $S^{-1} = \{\frac{1}{\lambda} : \lambda \in S\}$ by $f^*(\lambda) = \lambda f(1/\lambda)$ and $\check{f}(\lambda) = \frac{1}{f(1/\lambda)}$. We also define $\widetilde{f} : S \to \mathbb{R}_+$ by $\widetilde{f}(\lambda) = (\check{f})^*(\lambda) = \frac{\lambda}{f(\lambda)}$.

PROPOSITION 3.5. *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class C_n if and only if one (and then all three) of the functions f^*, $\check{f}$, and $\widetilde{f}$ belong to C_n.*

Proof. It suffices to note that a function belongs to C_n if and only if $f|S \in P'|S$ for every n-set $S \subseteq \mathbb{R}_+$ and observe that the class P' is closed under the operations $h \mapsto h^*$ and $h \mapsto \check{h}$. The latter statement is clear if h is a constant, and otherwise if one of the functions h, $\check{h}$ or h^* has positive imaginary part in the upper half plane, then clearly so does the other two. ∎

A result related to Proposition 3.5 is found in Theorem III of [5].

Recall that a function f belongs to C_n if and only if for all subsets $\{\lambda_i\}_{i=1}^n \subseteq \mathbb{R}_+$ consisting of n points, we have that $f|\{\lambda_i\}_{i=1}^n \in P'|\{\lambda_i\}_{i=1}^n$. We shall need the following lemma:

LEMMA 3.6. *A function $h : \{\lambda_i\}_{i=1}^n \to \mathbb{R}_+$ belongs to $P'|\{\lambda_i\}_{i=1}^n$ if and only if for all scalar sequences $(a_i)_{i=1}^n$ holds:*

$$\sum_{i=1}^n a_i \frac{\lambda_i}{t + \lambda_i} \geqslant 0, \quad t > 0 \quad \text{implies} \quad \sum_{i=1}^n a_i h(\lambda_i) \geqslant 0. \tag{3.4}$$

Proof. Our proof follows Lemma 7.1 of [1], and the subsequent remarks.

$\Rightarrow$: Let h be a P'-function and let ϱ be the positive Radon measure on $[0, \infty]$ occurring in the representation (1.2) of h. Assuming that the function $v(t) :=$

$\sum_{i=1}^{n} a_i \frac{\lambda_i}{t+\lambda_i}$ in non-negative for all $t > 0$, we infer that also the function $u(t) := (1+t^{-1})v(t^{-1}) = \sum_{i=1}^{n} a_i \frac{(1+t)\lambda_i}{1+t\lambda_i}$ is non-negative on $[0,\infty]$. The property (3.4) now follows, because

$$\sum_{i=1}^{n} a_i h(\lambda_i) = \sum_{i=1}^{n} a_i \int_{[0,\infty]} \frac{(1+t)\lambda_i}{1+t\lambda_i} \mathrm{d}\varrho(t) = \int_{[0,\infty]} u(t)\mathrm{d}\varrho(t) \geqslant 0.$$

$\Leftarrow$: Suppose that h is any function defined on a given finite subset $\{\lambda_i\}_{i=1}^{n} \subset \mathbb{R}_+$ such that (3.4) holds. We can without loss of generality assume that the point 1 belongs to the set $\{\lambda_i\}$ (replace the function $h(\lambda)$ by $h(c\lambda)$ for some suitable $c > 0$). Let $C = C([0,\infty])$ be the unital C^*-algebra of continuous complex-valued functions on the compact set $[0,\infty]$. Define functions $e_i(t) := \frac{(1+t)\lambda_i}{1+t\lambda_i}$ and let V denote the linear span of the e_i's. Note that V is a finite-dimensional subspace of C, containing the unit $1 = e_1(t) \in C$. The condition (3.4) says precisely that the functional $\phi : V \to \mathbb{C}$ defined by $\phi : \sum a_i e_i \mapsto \sum a_i h(\lambda_i)$ is a positive functional on V in the sense that if $u \in V$ and $u(t) \geqslant 0$ for all $t > 0$, then $\phi(u) \geqslant 0$. By well-known properties of positive functionals this is equivalent to $\|\phi\| = \phi(1)$. Let $\Phi : C \to \mathbb{C}$ be a Hahn-Banach extension of ϕ to C of the same norm. Then $\|\Phi\| = \|\phi\| = \phi(1) = \Phi(1)$, and it follows that Φ is a positive functional on C. But then, by the Riesz Representation Theorem, there exists a positive Radon measure ϱ on $[0,\infty]$ such that $\Phi(u) = \int_{[0,\infty]} u(t)\mathrm{d}\varrho(t)$ for all $u \in C$, and in particular

$$h(\lambda_i) = \phi(e_i) = \Phi(e_i) = \int_{[0,\infty]} \frac{(1+t)\lambda_i}{1+t\lambda_i} \mathrm{d}\varrho(t), \quad i = 1, \dots, n.$$

But in view of the representation (1.2), this latter equation means precisely that $h \in P' | \{\lambda_i\}_{i=1}^{n}$, and our lemma is proved. ∎

3.1. INVESTIGATIONS OF THE CLASSES C_2 AND C_3. We shall now undertake a closer study of the classes C_n for $n = 2$ and $n = 3$. Our point of departure will be Corollary 2.4, a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to C_n if and only if its restriction to any subset of $\mathbb{R}_+$ consisting of n points coincides there with a P' function.

Recall (cf. e.g. [4]) that a positive function h on $\mathbb{R}_+$ is *quasi-concave* if $h(t) \leqslant h(s)\max(1, \frac{t}{s})$ for all $s, t > 0$. Let $\mathcal{Q}$ denote the class of all quasi-concave functions on $\mathbb{R}_+$. We have the following result.

PROPOSITION 3.7. $C_2 = \mathcal{Q}$.

Proof. "$\mathcal{Q} \subseteq C_2$". Take $f \in \mathcal{Q}$. Let $s, t \in \mathbb{R}_+$ and assume with no loss of generality that $t > s$. Since $f(t) \leqslant f(s)\frac{t}{s}$, we can then find an affine positive function h on $\mathbb{R}_+$ such that $h(s) = f(s)$ and $h(t) = f(t)$. (To see this, note that in the extreme case $f(t) = f(s)\frac{t}{s}$, our h is simply the linear function $h(\lambda) = f(s)\frac{\lambda}{s}$.)

But this function h belongs to P'. Thus f coincides at any two points of $\mathbb{R}_+$ with a P'-function, i.e. $f \in C_2$.

"$C_2 \subseteq \mathcal{Q}$". Take $f \in C_2$. Take two points $s, t \in \mathbb{R}_+$. Then there exists a P'-function h such that $f(s) = h(s)$ and $f(t) = h(t)$. But P'-functions are quasi-concave. Thus $f(t) = h(t) \leqslant h(s)\max(1, \frac{t}{s}) = f(s)\max(1, \frac{t}{s})$, i.e. $f \in \mathcal{Q}$. ∎

REMARK 3.8. A function h is quasi-concave if and only if h is increasing and $t \mapsto \frac{h(t)}{t}$ is decreasing on $\mathbb{R}_+$. This yields that quasi-concave functions are continuous on $\mathbb{R}_+$. Thus, by Proposition 3.7, $C_n \subseteq C_2 \subseteq C(\mathbb{R}_+)$ for $n \geqslant 2$, where $C(\mathbb{R}_+)$ is the set of continuous functions on $\mathbb{R}_+$.

We shall now turn to the problem of characterizing the class C_3. To this end, our main tool will be polynomial techniques which essentially go back to Sparr [16].

The important observation now is that the property (3.4) is inherited by C_n-functions in the following sense: f belongs to C_n if and only if for all n-subsets $\{\lambda_i\}_{i=1}^n \subseteq \mathbb{R}_+$ we have

$$(\forall (a_i)_{i=1}^n \in \mathbb{R}^n): \quad \Big(\sum_{i=1}^n a_i \frac{\lambda_i}{t+\lambda_i} \geqslant 0, \forall t > 0\Big) \quad \text{implies} \quad \sum_{i=1}^n a_i f(\lambda_i) \geqslant 0.$$

We shall now use this characterization of C_n functions to prove a more convenient one in the case $n = 3$. In the sequel, we shall denote by $\mathcal{P}_n$ the linear space of real polynomials of degree at most n.

PROPOSITION 3.9. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitrary function. The following conditions are equivalent:*

(i) $f \in C_3$*;*

(ii) *for any scalar triple $(a_i)_{i=1}^3 \in \mathbb{R}^3$ holds*

$$\sum_{i=1}^3 a_i \frac{\lambda_i}{t+\lambda_i} \geqslant 0, \quad t > 0 \quad \textit{implies} \quad \sum_{i=1}^3 a_i f(\lambda_i) \geqslant 0; \tag{3.5}$$

(iii) *for any three numbers $\varepsilon, \lambda, \omega \in \mathbb{R}_+$ such that $\varepsilon < \lambda < \omega$, and any polynomial $P \in \mathcal{P}_2$ such that $P(t) \geqslant 0$, $t > 0$ we have*

$$\frac{P(-\varepsilon)}{\varepsilon(\lambda-\varepsilon)(\omega-\varepsilon)} f(\varepsilon) - \frac{P(-\lambda)}{\lambda(\lambda-\varepsilon)(\omega-\lambda)} f(\lambda) + \frac{P(-\omega)}{\omega(\omega-\varepsilon)(\omega-\lambda)} f(\omega) \geqslant 0; \tag{3.6}$$

(iv) *f is concave, and for all $\varepsilon, \lambda, \omega \in \mathbb{R}_+$ such that $\varepsilon < \lambda < \omega$ and all numbers $c > 0$ we have*

$$f(\lambda) \leqslant \Big(\frac{\varepsilon + c}{\lambda + c}\Big)^2 \frac{\lambda(\omega-\lambda)}{\varepsilon(\omega-\varepsilon)} f(\varepsilon) + \Big(\frac{\omega + c}{\lambda + c}\Big)^2 \frac{\lambda(\lambda-\varepsilon)}{\omega(\omega-\varepsilon)} f(\omega); \tag{3.7}$$

(v) *f is concave, and for all $c > 0$, the function $\lambda \mapsto (\lambda + c)^2 \frac{f(\lambda)}{\lambda}$ is convex on $\mathbb{R}_+$.*

Proof. (i) $\Longleftrightarrow$ (ii): This is clear by the preceding remarks.

(ii) $\Longleftrightarrow$ (iii): Take an arbitrary function $f : \mathbb{R}_+ \to \mathbb{R}_+$.

Take $\varepsilon, \lambda, \omega$ as in (iii) and put $L(t) = (t+\varepsilon)(t+\lambda)(t+\omega)$. For $P \in \mathcal{P}_2$, we define $a_i = a_i(P)$, $i = 1, 2, 3$ by

$$\text{(3.8)} \qquad \frac{P(t)}{L(t)} = a_1 \frac{\varepsilon}{t+\varepsilon} + a_2 \frac{\lambda}{t+\lambda} + a_3 \frac{\omega}{t+\omega},$$

where

$$\text{(3.9)} \quad a_1 = \frac{P(-\varepsilon)}{\varepsilon(\lambda-\varepsilon)(\omega-\varepsilon)}, \quad a_2 = -\frac{P(-\lambda)}{\lambda(\lambda-\varepsilon)(\omega-\lambda)}, \quad a_3 = \frac{P(-\omega)}{\omega(\omega-\varepsilon)(\omega-\lambda)}.$$

By (3.9) is defined a linear bijection

$$\mathcal{P}_2 \to \mathbb{R}^3 \quad : \quad P \mapsto a = (a_i)_{i=1}^3.$$

Moreover, by (3.8), it is clear that $P(t) \geqslant 0$, $t > 0$ if and only if the corresponding sum $a_1(P)\frac{\varepsilon}{t+\varepsilon} + a_2(P)\frac{\lambda}{t+\lambda} + a_3(P)\frac{\omega}{t+\omega}$ is $\geqslant 0$ for $t > 0$. Thus for a function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the assertions (3.5) and (3.6) are equivalent, as desired.

(iii) $\Longleftrightarrow$ (iv): Let $\mathcal{C}$ denote the cone of polynomials $P \in \mathcal{P}_2$ such that $P(t) \geqslant 0$ for all $t \geqslant 0$. Let $\mathcal{G}$ denote the subcone of $\mathcal{C}$ consisting of polynomials P of exact degree 2 such that $P(0) > 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an arbitrary function. Since $\mathcal{G}$ is dense in $\mathcal{C}$, it is sufficient to show that the property that (3.6) holds for f is equivalent to (3.7) for polynomials in the cone $\mathcal{G}$. Fix a polynomial $P \in \mathcal{G}$. Multiplying P by a positive constant does not change the problem, so we can assume that the leading coefficient of P is 1. Moreover, since $P(0) > 0$, the constant term of P is of the form c^2 for some number $c > 0$. Thus $P(t) = t^2 + bt + c^2$ for some constants $b \in \mathbb{R}$ and $c > 0$. But since $P(c) = c(b + 2c) \geqslant 0$, this yields

$$b + 2c \geqslant 0.$$

We have thus the following decomposition of a generic polynomial P:

$$P(t) = a((t-c)^2 + (b+2c)t),$$

where a is the leading coefficient of P and the term $b + 2c$ is non-negative.

By these considerations, it is clear that a function f satisfies (3.6) for all polynomials $P \in \mathcal{C}$ if and only if it satisfies that same condition with respect to special polynomials of the form

(I) $P(t) = (t-c)^2$ where $c > 0$ and
(II) $P(t) = t$.

Consider first the case $P(t) = t$. Then (3.6) becomes

$$f(\lambda) \geqslant \frac{\omega-\lambda}{\omega-\varepsilon} f(\varepsilon) + \frac{\lambda-\varepsilon}{\omega-\varepsilon} f(\omega).$$

Setting $\lambda = \alpha\varepsilon + (1-\alpha)\omega$, this means $f(\lambda) \geqslant \alpha f(\varepsilon) + (1-\alpha) f(\omega)$, i.e. f is concave on $\mathbb{R}_+$.

There remains to investigate the case of polynomials of the form $P(t) = (t-c)^2$ where $c > 0$. But in this case, (3.6) becomes

$$\frac{(\varepsilon+c)^2}{\varepsilon(\lambda-\varepsilon)(\omega-\varepsilon)}f(\varepsilon) - \frac{(\lambda+c)^2}{\lambda(\lambda-\varepsilon)(\omega-\lambda)}f(\lambda) + \frac{(\omega+c)^2}{\omega(\omega-\varepsilon)(\omega-\lambda)}f(\omega) \geqslant 0,$$

which is readily seen to be equivalent to (3.7).

(iv) $\Longleftrightarrow$ (v): Let $0 < \varepsilon < \omega$ be given together with a number $\alpha \in (0,1)$, and put $\lambda = \alpha\varepsilon + (1-\alpha)\omega$. Then (3.7) becomes

$$f(\lambda) \leqslant \Big(\frac{\varepsilon+c}{\lambda+c}\Big)^2 \frac{\lambda}{\varepsilon}\alpha f(\varepsilon) + \Big(\frac{\omega+c}{\lambda+c}\Big)^2 \frac{\lambda}{\omega}(1-\alpha) f(\omega),$$

which means precisely that the function $x \mapsto (x+c)^2 \frac{f(x)}{x}$ is convex. ∎

We have the following corollary.

COROLLARY 3.10. *Let $f \in C_3$. Then f is C^1-smooth on $\mathbb{R}_+$, and moreover*

(i) *the function $\lambda \mapsto \lambda f(\lambda)$ is convex on $\mathbb{R}_+$;*

(ii) *the function $\lambda \mapsto f(\lambda)$ is concave on $\mathbb{R}_+$;*

(iii) *the function $\lambda \mapsto \frac{f(\lambda)}{\lambda}$ is convex on $\mathbb{R}_+$.*

Proof. Let $f \in C_3$. The convexity of all functions $g_c(\lambda) = (\lambda+c)^2 \frac{f(\lambda)}{\lambda}$, $c > 0$ implies that $\lim_{c\to 0} g_c(\lambda) = \lambda f(\lambda)$ is convex and also $\lim_{c\to\infty} \frac{g_c(\lambda)}{c^2} = \frac{f(\lambda)}{\lambda}$ is convex. Thus the properties (i),(ii), (iii) follow from (v) of Proposition 3.9.

We prove that f is C^1-smooth. Fix a point $\lambda \in \mathbb{R}_+$. Since f is concave, the right and left derivatives $f'(\lambda+)$ and $f'(\lambda-)$ exist and satisfy $f'(\lambda-) \geqslant f'(\lambda+)$. Similarly the convex function $g(\lambda) = \lambda f(\lambda)$ is right and left differentiable at λ and $g'(\lambda-) \leqslant g'(\lambda+)$. But since $g'(\lambda\pm) = f(\lambda) + \lambda f'(\lambda\pm)$, this implies $f'(\lambda-) \leqslant f'(\lambda+)$. Therefore, we must have $f'(\lambda-) = f'(\lambda+)$, i.e. $f \in C^1$. ∎

REMARK 3.11. Note that for given $t > 0$ the P'-function $h(\lambda) = \frac{\lambda}{1+t\lambda}$ satisfies

$$\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\Big(\frac{(c+\lambda)^2}{\lambda}\frac{\lambda}{1+t\lambda}\Big) = 2\frac{(ct-1)^2}{(1+t\lambda)^3} \geqslant 0, \quad \lambda > 0.$$

By this observation and a convexity argument, one obtains an alternative proof of the fact that all P' functions fulfill the condition (v) of Proposition 3.9.

EXAMPLE 3.12. Let $\mathcal{F}$ be the convex set of C_3-functions such that $f(1/2) = \frac{1}{2}$ and $f(2) = 1$, and let $\mathcal{F}_1 = \{f(1) : f \in \mathcal{F}\}$. $\mathcal{F}_1$ is a closed convex set of $\mathbb{R}_+$, i.e. an interval of the form $[\theta_0, \theta_1]$ for some $\theta_0, \theta_1 \in \mathbb{R}_+$. Since functions in $\mathcal{F}$ are concave, it becomes obvious that $\theta_0 \geqslant \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot 1 = \frac{2}{3}$. Furthermore, by choosing $f(\lambda) = \frac{1+\lambda}{3}$, we see that this bound is attained, i.e. $\theta_0 = \frac{2}{3}$. Moreover, trivially $\theta_1 \leqslant 1$ because functions in $\mathcal{F}$ are increasing. To determine the precise value of θ_1, we make use of the relation (3.7) with $\varepsilon = \frac{1}{2}$ and $\omega = 2$ and an arbitrary number

$c > 0$. It yields

$$f(1) \leqslant \Big(\frac{\frac{1}{2}+c}{1+c}\Big)^2 \frac{4}{3} f(1/2) + \Big(\frac{2+c}{1+c}\Big)^2 \frac{1}{6} f(2) = \Big(\frac{\frac{1}{2}+c}{1+c}\Big)^2 \frac{2}{3} + \Big(\frac{2+c}{1+c}\Big)^2 \frac{1}{6}, \quad c > 0.$$

Minimizing the expression in the right hand side, one obtains that the infimum is attained for $c = 1$, and equals $\frac{3}{8} + \frac{3}{8} = \frac{3}{4}$. Thus $\theta_1 \leqslant \frac{3}{4}$. But since the P'-function $h(\lambda) = \frac{3}{2}\frac{\lambda}{1+\lambda}$ belongs to $\mathcal{F}_1$, and $h(1) = \frac{3}{4}$, we deduce that $\theta_1 \geqslant \frac{3}{4}$, and $\mathcal{F}_1 = [\frac{2}{3}, \frac{3}{4}]$.

If $f \in \mathcal{F}$ and $f(1) = \theta$, then an explicit P'-function h interpolating f at the points $\frac{1}{2}$, 1 and 2 is given by $h(\lambda) = \frac{(5\theta-3)\lambda+3-4\theta}{(6\theta-4)\lambda+5-6\theta}$. In a similar way, one can deduce that a non-constant C_3-function can be interpolated at an arbitrary 3-subset of $\mathbb{R}_+$ by a linear fractional P'-function.

EXAMPLE 3.13. The conditions (i), (ii), (iii) of Corollary 3.10 are not sufficient to guarantee that a function belongs to C_3. A counterexample is provided by the function

$$f(\lambda) = 2\frac{\lambda}{1+\lambda} + \Big(\frac{\lambda}{1+\lambda}\Big)^2.$$

Indeed, $f''(\lambda) = -2\frac{1+4\lambda}{(1+\lambda)^4}$, $\frac{d^2}{d\lambda^2}\{\lambda f(\lambda)\} = 2\frac{2+5\lambda}{(1+\lambda)^4}$ and $\frac{d^2}{d\lambda^2}\{\frac{f(\lambda)}{\lambda}\} = 6\frac{\lambda}{(1+\lambda)^4}$ i.e. f fulfills conditions (i), (ii) and (iii). However, it turns out that the function $g_{3/2}(\lambda) = (\lambda + \frac{3}{2})^2 \frac{f(\lambda)}{\lambda}$ satisfies $g''_{3/2}(\lambda) = -\frac{1}{2}\frac{4+\lambda}{(1+\lambda)^4}$, i.e. $g_{3/2}$ fails to be convex (it is even concave!) whence $f \notin C_3$ by (v) of Proposition 3.9.

3.2. THE GAP BETWEEN C_3 AND P'_2. We know from Theorem 3.1 that $P'_2 \subseteq C_3$. Our main result in this subsection is the following.

PROPOSITION 3.14. $P'_2 \subsetneq C_3$.

Recall ([6], Section VII, Theorem III and Section VIII, Theorem IV) that a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is in P'_2 if and only if f is C^1-smooth, the derivative f' is non-negative and convex and the suitably normalized Schwarzian derivative

$$Sf(\lambda) := \frac{2}{3} f'(\lambda) f'''(\lambda) - f''(\lambda)^2 = 4 \det \begin{pmatrix} f'(\lambda) & \frac{f''(\lambda)}{2} \\ \frac{f''(\lambda)}{2} & \frac{f'''(\lambda)}{6} \end{pmatrix} \geqslant 0 \tag{3.10}$$

at all points $\lambda \in \mathbb{R}_+$ where it makes sense (i.e. almost everywhere on $\mathbb{R}_+$ by the convexity of f'). (Similar characterizations of P'_n for every fixed n can be found in Donoghue's book [7], Section VII, Theorem VI and Section VIII, Theorem V.)

We now observe that Hansen, Ji and Tomiyama [12] have recently proved that for every integer $n \geqslant 2$, there exists a positive constant c_n such that the function

$$g_n(\lambda) = \sum_{k=1}^{n} \frac{1}{2k-1} \Big(\frac{c_n \lambda}{1+\lambda}\Big)^{2k-1} \tag{3.11}$$

satisfies $g_n \in P'_n$. Furthermore, they prove that any function g_n of the form (3.11) does not belong to P'_{n+1}. Let α_n denote the supremum of numbers c_n such that the corresponding function g_n (3.11) belongs to P'_n. We have the following lemma.

LEMMA 3.15. $\alpha_2^2 = \frac{1}{2}$.

Proof. Consider for $K \geqslant 0$ the function

$$f_K(\lambda) = K\frac{\lambda}{1+\lambda} + \Big(\frac{\lambda}{1+\lambda}\Big)^3. \tag{3.12}$$

The set of K's such that $f_K \in P'_2$ is easily seen to be the interval $[3\alpha_2^{-2}, \infty)$. Let us compute the first three derivatives of f_K:

$$\begin{cases} f'_K(\lambda) = \frac{(3+K)\lambda^2+2K\lambda+K}{(1+\lambda)^4}, \\ f''_K(\lambda) = -2\frac{(3+K)\lambda^2+(2K-3)\lambda+K}{(1+\lambda)^5}, \\ f'''_K(\lambda) = 6\frac{(3+K)\lambda^2+(2K-6)\lambda+K+1}{(1+\lambda)^6}. \end{cases} \tag{3.13}$$

Note that $f'_K \geqslant 0$ on $\mathbb{R}_+$ for all $K \geqslant 0$, which is a necessary condition for $f_K \in P'_2$ to hold. Recall that if f is any smooth increasing function such that $Sf \geqslant 0$, then f' is necessarily convex (even logarithmically convex, cf. [6], p. 74). The lemma will thus follow if we can prove that the Schwarzian derivative satisfies $Sf_K \geqslant 0$ on $\mathbb{R}_+$ if and only if $K \geqslant 6$. In order to see this, we compute

$$Sf_K(\lambda) = \frac{24(1+2\lambda) + 4(K-6)(\lambda+1)^2}{(1+\lambda)^{10}}.$$

It is evident that this expression is positive for all $\lambda > 0$ if and only if $K \geqslant 6$. ∎

We shall now consider the condition $f_K \in C_3$, where f_K is defined as above (3.12). The set of K's such that this is true is an interval of the form $[\beta, \infty)$ for some $\beta \geqslant 0$. By the preceding proposition and Theorem 3.1, we know that $\beta \leqslant 6$. We shall show that in fact:

LEMMA 3.16. $\beta \leqslant 3$.

Proof. It is immediate from (3.13) that f_K is concave on $\mathbb{R}_+$ if $K \geqslant 3$. Thus by Proposition 3.9, it suffices to prove that, for every $c > 0$, the function

$$g_c(\lambda) := \frac{(\lambda+c)^2 f_3(\lambda)}{\lambda}$$

is convex on $\mathbb{R}_+$. But a direct computation yields:

$$g''_c(\lambda) = \frac{4\lambda^2(2c-3)^2 + 2\lambda((2c-3)^2+3) + (4c-3)^2 + 3}{2(1+\lambda)^5},$$

which is evidently positive for $\lambda > 0$. The proof is finished. ∎

REMARK 3.17. By a slightly longer argument, it is possible to prove that $\beta = 3$. (For each fixed positive number $K < 3$, the corresponding function $g_c(\lambda) = (\lambda + c)^2 \frac{f_K(\lambda)}{\lambda}$ fails to be convex for $c = \frac{3}{2}$. We omit the details.)

Proof of Proposition 3.14. By the foregoing lemmas, the function (for example)

$$f_3(\lambda) = 3\frac{\lambda}{1+\lambda} + \left(\frac{\lambda}{1+\lambda}\right)^3$$

belongs to $C_3 \setminus P_2'$. ∎

3.3. THE GAP BETWEEN P_2' AND C_4. In this subsection, we want to prove the following.

PROPOSITION 3.18. $C_4 \subsetneqq P_2'$.

Proof. (Cf. Sparr [16], p. 274). We know that $C_4 \subseteq P_2'$. To prove that the inclusion is proper, we shall exploit a fact from Donoghue's book ([7], Section VII, Theorem IV and Section VIII, Theorem III) that a non-constant function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $f \in P_2'$ if and only if f is of class C^1 and the derivative f' is of the form

$$f'(\lambda) = \frac{1}{c(\lambda)^2}$$

with some concave function $c : \mathbb{R}_+ \to \mathbb{R}_+$. We choose $c(x) = \min(1 + x, 2)$ and

$$f(\lambda) = \int_0^\lambda \frac{\mathrm{d}x}{c(x)^2} = \begin{cases} \frac{\lambda}{1+\lambda} & \lambda \leqslant 1, \\ \frac{1}{4}(1+\lambda) & \lambda \geqslant 1. \end{cases}$$

Then $f \in P_2'$. We shall show that $f \notin C_4$. Indeed let $\lambda_i = i$, $i = 1, 2, 3$ and $\lambda_4 \in \mathbb{R}_+$ an arbitrary point. If it were true that $f \in C_4$, we could find a P'-function h interpolating f at the points λ_i. However, the only P'-function interpolating f at the points λ_1, λ_2 and λ_3 is the affine function $h(\lambda) = \frac{1}{4}(1+\lambda)$. Thus $f(\lambda_4) = h(\lambda_4) = \frac{1}{4}(1 + \lambda_4)$ for all points $\lambda_4 \in \mathbb{R}_+$, a contradiction. This shows that $f \notin C_4$. ∎

REMARK 3.19. It is a simple consequence of the above proof that a C_4 function is either affine or is strictly concave on $\mathbb{R}_+$.

3.4. INTERPOLATION FUNCTIONS ON UNITAL C^*-ALGEBRAS. In this subsection, we prove three propositions, which allow us to transport results from the theory of interpolation functions to unital C^*-algebras (other than $B(H)$). The corresponding problem for monotone functions was considered in [12].

Let $\mathcal{A}$ be a unital C^*-algebra. We will denote by $\widehat{\mathcal{A}}$ a complete collection of representatives of the unitary equivalence classes of non-zero irreducible representations of $\mathcal{A}$.

For a fixed strictly positive element a of $\mathcal{A}$, we define the *a-norm* on $\mathcal{A}$ by

$$\|x\|_a = \|a^{1/2}xa^{-1/2}\|.$$

Our goal in this section is to characterize the strictly positive elements $b \in \mathcal{A}$ such that the interpolation inequality

$$\|x\|_b \leqslant \max(\|x\|, \|x\|_a), \quad \forall x \in \mathcal{A} \tag{3.14}$$

is satisfied.

It is sometimes convenient to reformulate the condition (3.14) in the following way:

$$\forall x \in \mathcal{A}: \quad x^*x \leqslant 1 \quad \text{and} \quad x^*ax \leqslant a \quad \text{imply} \quad x^*bx \leqslant b. \tag{3.15}$$

The set of b's such that (3.15) (or, equivalently, (3.14)) holds form a convex cone. Below, we shall address the problem of finding necessary and sufficient conditions for an element b to belong to that cone.

3.4.1. A SUFFICIENT CONDITION. We have the following proposition.

PROPOSITION 3.20. *Assume that a and b are fixed strictly positive elements of a unital C^*-algebra $\mathcal{A}$. Suppose that for each irreducible representation $\pi \in \widehat{\mathcal{A}}$ there exists a function $h_\pi \in P'$ such that $\pi(b) = h_\pi(\pi(a))$. Then the interpolation inequality* (3.14) *holds.*

Proof. Let φ be a pure state on $\mathcal{A}$ and let $\{H_\varphi, \pi_\varphi, \xi_\varphi\}$ be the corresponding GNS representation, i.e.,

$$\varphi(x) = (\pi_\varphi(x)\xi_\varphi, \xi_\varphi)_{H_\varphi}, \quad x \in \mathcal{A}.$$

Since φ is pure, π_φ is irreducible whence by assumption $\pi_\varphi(b) = h_\varphi(\pi_\varphi(a))$ for some function $h_\varphi \in P'|\sigma(\pi_\varphi(a))$. We conclude that $h_\varphi \in C_{\pi_\varphi(a)}$ by (2.7). In particular, the following implication holds:

$$x \in \mathcal{A}, \quad x^*x \leqslant 1\,, \; x^*ax \leqslant a,$$

implies $\pi_\varphi(x)^*\pi_\varphi(x) \leqslant 1$ and $\pi_\varphi(x)^*\pi_\varphi(a)\pi_\varphi(x) \leqslant \pi_\varphi(a)$, and so

$$\pi_\varphi(x)^*h_\varphi(\pi_\varphi(a))\pi_\varphi(x) \leqslant h_\varphi(\pi_\varphi(a)).$$

This yields

$$\varphi(b - x^*bx) = (\pi_\varphi(b)\xi_\varphi, \xi_\varphi)_{H_\varphi} - (\pi_\varphi(x)^*\pi_\varphi(b)\pi_\varphi(x)\xi_\varphi, \xi_\varphi)_{H_\varphi} \geqslant 0 \tag{3.16}$$

for every pure state φ. But since all states belong to the weak* closed convex hull of the pure states, the conclusion of (3.16) remains true for all states φ, i.e.,

$$b - x^*bx \geqslant 0.$$

The proposition follows. ∎

3.4.2. A NECESSARY CONDITION. We shall now prove a partial converse to Proposition 3.20. In order to formulate our result, we need to make some preliminary remarks.

Fix a strictly positive element a of a unital C^*-algebra $\mathcal{A}$. We shall operate under the following "technical" assumption on a and $\mathcal{A}$:

$$\overline{\pi\{x\in\mathcal{A}: \|x\|\leqslant 1, \|x\|_a\leqslant 1\}}^{\,\mathrm{st}}=\{T\in B(H_\pi): \|T\|\leqslant 1, \|T\|_{\pi(a)}\leqslant 1\} \quad \forall \pi\in\widehat{\mathcal{A}}, \tag{3.17}$$

where "st" denotes the closure with respect to the strong operator topology on $B(H_\pi)$.

REMARK 3.21. When $a = 1$, the statement (3.17) holds; indeed, it is equivalent to the Kaplansky Density Theorem. Moreover, (3.17) is trivially satisfied e.g. for C^*-algebras having the property that every irreducible representation is finite-dimensional. At present, we do not know whether or not (3.17) holds in general.

We have the following proposition.

PROPOSITION 3.22. *Let $\mathcal{A}$ be a unital C^*-agebra and $a > 0$ a fixed element of $\mathcal{A}$ such that the condition* (3.17) *is satisfied. Let b be another strictly positive element of $\mathcal{A}$ such that the interpolation inequality* (3.14) *holds for all $x \in \mathcal{A}$. Then, for every irreducible representation $\pi \in \widehat{\mathcal{A}}$, there exists a function $h_\pi \in P'$ such that $\pi(b) = h_\pi(\pi(a))$.*

We shall need a simple lemma.

LEMMA 3.23. *Let $\mathcal{A}$ be a unital C^*-algebra and $\pi : \mathcal{A} \to B(H)$ a representation of $\mathcal{A}$ on some Hilbert space H. Let $a \in \mathcal{A}$ be a fixed element such that $a > 0$, and put $A = \pi(a)$. Let $\varepsilon > 0$ be given. Suppose that an operator $T \in \pi(\mathcal{A})$ satisfies $\|T\| \leqslant 1$ and $\|T\|_A \leqslant 1$. Then there exists an element $x \in \mathcal{A}$ such that $\pi(x) = T$, $\|x\| \leqslant 1+\varepsilon$ and $\|x\|_a \leqslant 1+\varepsilon$.*

Proof. Let u_λ be an approximate unit for the ideal $\pi^{-1}(\{0\})$, and take $x_0 \in \mathcal{A}$ such that $\pi(x_0) = T$. Put $x_\lambda = x_0(1-u_\lambda)$. Then $\pi(x_\lambda) = T$, and moreover by standard facts about approximate units ([14], Section 3)

$$\|T\| = \lim \|x_\lambda\| \quad \text{and} \quad \|A^{1/2}TA^{-1/2}\| = \lim \|a^{1/2}x_\lambda a^{-1/2}\|.$$

Thus, letting $x = x_\lambda$ with some sufficiently large λ, we obtain an element with desired properties. ∎

Proof of Proposition 3.22. Let $\pi : \mathcal{A} \to B(H)$ be an irreducible representation of $\mathcal{A}$. Put $A = \pi(a)$ and $B = \pi(b)$. Fix $T \in B(H)$ such that $\|T\| \leqslant 1$ and $\|T\|_A \leqslant 1$. Take $\varepsilon > 0$ and let ξ be a unit vector of H.

By the assumption (3.17), there exists $S \in \pi(\mathcal{A})$ such that $\|S\| \leqslant 1$ and $\|S\|_A \leqslant 1$ and also

$$\|B^{1/2}(T-S)B^{-1/2}\xi\| < \frac{\varepsilon}{2}. \tag{3.18}$$

(We have here used the simple fact that the map $X \mapsto B^{1/2}XB^{-1/2}$ is a homeomorphism with respect the strong topology on $B(H)$.)

We now use Lemma 3.23 to find a lifting $x \in \mathcal{A}$ of S such that $\|x\| \leqslant 1 + \frac{\varepsilon}{2}$ and $\|x\|_a \leqslant 1 + \frac{\varepsilon}{2}$. By the condition (3.14), then $\|x\|_b \leqslant 1 + \frac{\varepsilon}{2}$. Applying the representation π it yields $\|S\|_B \leqslant 1 + \frac{\varepsilon}{2}$. Combining this estimate with (3.18), we obtain $\|B^{1/2}TB^{-1/2}\xi\| \leqslant \|B^{1/2}(T-S)B^{-1/2}\xi\| + \|B^{1/2}SB^{-1/2}\xi\| < 1 + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, it yields $\|B^{1/2}TB^{-1/2}\xi\| \leqslant 1$, and since the unit vector ξ was arbitrary, we get $\|T\|_B \leqslant 1$. We infer that $\|T\|_B \leqslant \max(\|T\|, \|T\|_A)$ for all $T \in B(H)$. We may thus apply Donoghue's Theorem (Fact 2.1 and Theorem 2.3). It yields that $B = h(A)$ for some function $h \in P'|\sigma(A)$, as desired. ∎

3.4.3. INTERPOLATION FUNCTIONS. Let $C_{\mathcal{A}}$ be the set of all continuous positive functions h on $\mathbb{R}_+$ such that $\|x\|_{h(a)} \leqslant \max(\|x\|, \|x\|_a)$ for all $x, a \in \mathcal{A}$ such that $a > 0$. It makes sense to refer to $C_{\mathcal{A}}$ as the class of *interpolation functions* with respect to $\mathcal{A}$. In this notation, of course, C_{M_n} coincides with the class C_n of interpolation functions of order n. It will be convenient to define C_n also for $n = \infty$. We make the following convention

$$C_\infty = P'. \tag{3.19}$$

Let $C(\mathbb{R}_+)$ denote the class of continuous functions on $\mathbb{R}_+$. We have the following proposition:

PROPOSITION 3.24. *Let $\mathcal{A}$ be a unital C^*-algebra and let $n = \sup\{\dim(\pi) : \pi \in \widehat{\mathcal{A}}\}$. Then $C_n \cap C(\mathbb{R}_+) \subseteq C_{\mathcal{A}}$. Moreover, if the condition* (3.17) *is satisfied for all $a \in \mathcal{A}$ such that $a > 0$, then $C_{\mathcal{A}} = C_n \cap C(\mathbb{R}_+)$.*

REMARK 3.25. If $n \geqslant 2$, then $C_n \subseteq C(\mathbb{R}_+)$ by Remark 3.8. Taking the intersection with $C(\mathbb{R}_+)$ in Proposition 3.24 is thus only necessary when $n = 1$.

Proof of Proposition 3.24. Fix a strictly positive element $a \in \mathcal{A}$ and a function $f \in C_n \cap C(\mathbb{R}_+)$. For every irreducible representation π of $\mathcal{A}$ we have that $\dim(\pi) \leqslant n$, whence there is a function $h_\pi \in P'$ such that $f = h_\pi$ on $\sigma(\pi(a))$. It follows that $f(\pi(a)) = h_\pi(\pi(a))$. Applying Proposition 3.20, we conclude that (3.14) is valid, i.e., $\|x\|_{f(a)} \leqslant \max(\|x\|, \|x\|_a)$ for all $x \in \mathcal{A}$. Since $a > 0$ was arbitrary, $f \in C_{\mathcal{A}}$.

In the other direction, if $f \in C_{\mathcal{A}}$, then $f \in C(\mathbb{R}_+)$ by definition. Fix an element $a \in \mathcal{A}$ such that $a > 0$. If the condition (3.17) is satisfied, then Proposition 3.22 yields that $\pi(f(a)) = h_\pi(\pi(a))$ for a P'-function h_π. Since f and h_π are continuous, this yields that $f = h_\pi$ on $\sigma(\pi(a))$. If n is finite, then $\sigma(\pi(a))$ can be taken to be any n-subset of $\mathbb{R}_+$, and it follows that $h \in C_n$. On the other hand, if $n = \infty$, the same argument shows that $h \in C_k$ for all finite k, and thus $h \in P'$ by (2.4). ∎

3.5. COMPLETELY POSITIVE MAPS AND A THEOREM OF HANSEN. Let $\mathcal{A}$ be a C^*-algebra and let $\varphi : \mathcal{A} \to B(H)$ be a completely positive map from $\mathcal{A}$ to $B(H)$

for some Hilbert space H. Then, by Stinespring's Theorem, there exists a Hilbert space K, a representation $\pi : \mathcal{A} \to B(K)$ and a map $V \in B(H, K)$ such that $\varphi(x) = V^*\pi(x)V$, $x \in \mathcal{A}$, and moreover $\|\varphi\|_{\mathrm{cb}} = \|\varphi\| = \|\varphi(1)\| = \|V^*V\| = \|V\|^2$ cf. [3]. Thus if φ is contractive, $\|V\| \leqslant 1$. Fix an element $a \in \mathcal{A}$, $a > 0$. We shall associate to φ the following operators in $B(H \oplus K)$:

$$A = \begin{pmatrix} \varphi(a) & 0 \\ 0 & \pi(a) \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ V & 0 \end{pmatrix}.$$

Evidently, $A \geqslant 0$, and if we moreover require that $0 \notin \sigma(\varphi(a))$, then $A > 0$. Moreover,

$$T^*T \leqslant 1 \quad \text{and} \quad T^*AT = \begin{pmatrix} \varphi(a) & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} \varphi(a) & 0 \\ 0 & \pi(a) \end{pmatrix} = A. \tag{3.20}$$

Let n be the dimension of $H \oplus K$, where we allow the case $n = \infty$. We shall make use of the convention (3.19). We have the following result.

PROPOSITION 3.26. *In the above situation holds: if $h \in C_n$, then $\varphi(h(a)) \leqslant h(\varphi(a))$.*

Proof. The case $n = \infty$ is (the corollary in [10]), so we may assume that n is finite. It then follows from (3.20) and the assumption $h \in C_n$ that $T^*h(A)T \leqslant h(A)$, or

$$\begin{pmatrix} \varphi(h(a)) & 0 \\ 0 & 0 \end{pmatrix} \leqslant \begin{pmatrix} h(\varphi(a)) & 0 \\ 0 & h(\pi(a)) \end{pmatrix}$$

and the proposition follows. ∎

EXAMPLE 3.27. Positive linear functionals are completely positive. In this example, we shall consider the algebra $\mathcal{A} = C(X)$ where X is compact. Let $x_1, x_2 \in X$, $0 < \lambda < 1$, and consider the positive functional

$$\varphi(f) = \lambda f(x_1) + (1 - \lambda) f(x_2), \quad f \in C(X).$$

Then $\varphi(f) = V^*\pi(f)V$, where

$$\pi(f) = \begin{pmatrix} f(x_1) & 0 \\ 0 & f(x_2) \end{pmatrix}, \quad V = \begin{pmatrix} \lambda^{1/2} \\ (1 - \lambda)^{1/2} \end{pmatrix}.$$

In this case, $n = \dim(H \oplus K) = 3$. Thus, Proposition 3.26 yields that if $a > 0$ and $h \in C_3$, then $\varphi(h(a)) \leqslant h(\varphi(a))$, or

$$\lambda h(a(x_1)) + (1 - \lambda) h(a(x_2)) \leqslant h(\lambda a(x_1) + (1 - \lambda) a(x_2)).$$

This is an alternative way to see that C_3 functions are concave.

3.6. A FURTHER PROPERTY OF INTERPOLATION FUNCTIONS. Let H be a Hilbert space and $N \in \mathbb{N}$ a fixed number. Let $A \in B(H)$ be a fixed strictly positive operator. Let us say that a function $h : \sigma(A) \to \mathbb{R}_+$ belongs to the class C_A^N if and only if

$$\forall(\{T_k\}_{k=1}^N \subseteq B(H)) : \quad \sum_{k=1}^N T_k^* T_k \leqslant 1 \quad \text{and} \quad \sum_{k=1}^N T_k^* A T_k \leqslant A \tag{3.21}$$

implies

$$\sum_{k=1}^N T_k^* h(A) T_k \leqslant h(A). \tag{3.22}$$

This definition actually coincides with the previous definition of the class C_A, i.e. we have:

PROPOSITION 3.28. $C_A = C_A^N$.

Proof. It is clear that $C_A^N \subseteq C_A$ (choose $T_i = 0$ for $i \geqslant 2$) . We show the reverse inclusion. Consider the following operators in $B(\ell_2^N(H))$:

$$T = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ T_2 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_N & 0 & \cdots & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A \end{pmatrix}.$$

Evidently the condition (3.21) implies that $T^*T \leqslant 1$ and $T^*A_1T \leqslant A_1$. Moreover the operators A and A_1 have the same spectra. We infer by Theorem 2.3 that $C_A = P'|\sigma(A) = C_{A_1}$. In particular, if $h \in C_A$, it yields that $T^*h(A_1)T \leqslant h(A_1)$, which is readily seen to imply the condition (3.22), i.e. we have $h \in C_A^N$. ∎

We note the following corollary.

COROLLARY 3.29. *A function f belongs to C_n if and only if for every positive definite matrix $A \in M_n$, and every finite set of matrices $\{T_i\}_{i=1}^N \subseteq M_n$, we have the implication $\sum_1^N T_i^* T_i \leqslant 1$ and $\sum_1^N T_i^* A T_i \leqslant A$ implies $\sum_1^N T_i^* f(A) T_i \leqslant f(A)$.*

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