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ESTIMATES OF THE SPECTRAL RADIUS OF REFINEMENT AND SUBDIVISION OPERATORS WITH ISOTROPIC DILATIONS

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ABSTRACT. The paper presents lower bounds for the spectral radii of refinement and subdivision operators with continuous matrix symbols and with dilations from a class of isotropic matrices. This class contains main dilation matrices used in wavelet analysis. After obtaining general formulas, two kinds of estimates for the spectral radii are established: namely, estimates using point values of the symbols, as well as other ones making use of integrals on special subsets of the torus $\mathbb{T}^s$. For some symbol classes the exact value of the spectral radius of the refinement operator is found.

KEYWORDS: *Spectral radius, refinement operator, subdivision operator.*

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INTRODUCTION

Let X be a set, and let $s \geqslant 1$ be a positive integer. As usual, the symbol X^s is used for the Cartesian product of s copies of X. If X is also a normed space, then the set X^s has the norm

$$\|y\|_s = \|y\|_{X^s} := \Big(\sum_{k=1}^{s} \|x_k\|^2 \Big)^{1/2}$$

where $y = (x_1, x_2, \ldots, x_s) \in X^s$ and $\|\cdot\|$ denotes the corresponding norm on X. Moreover, let $X^{s\times s}$ denote the set of all $s \times s$ matrices with entries from X.

Consider the unit circle $\mathbb{T} := \{z \in \mathbb{C} : z = \mathrm{e}^{\mathrm{i}x}, x \in \mathbb{R}\}$, and an essentially bounded measurable matrix-function $a : \mathbb{T}^s \to \mathbb{C}^{m\times m}$ with the Fourier representation $a(x) \sim \sum\limits_{k\in\mathbb{Z}^s} a_k \mathrm{e}^{\mathrm{i}kx}, x \in \mathbb{R}^s$. If M is an $s \times s$ matrix with integer entries, then a and M generate two operators widely used in computer graphics and wavelet analysis: namely, the operator $R_a^M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ and the operator $S_a^M : l_2^m(\mathbb{Z}^s) \to l_2^m(\mathbb{Z}^s)$ defined by

$$R_a^M \varphi := \sum_{k\in\mathbb{Z}^s} a_k \varphi(M\cdot -k); \quad (S_a^M \xi)_j := \sum_{k\in\mathbb{Z}^s} a_{j-Mk}\xi_k, \quad j \in \mathbb{Z}^s.$$

In passing, note that a non-singular integer matrix M is often called a dilation matrix if

$$\lim_{n\to\infty} M^{-n} = 0, \tag{0.1}$$

but our considerations here are not restricted to this condition. Thus let us assume that M is a non-singular matrix with integer entries, and for convenience call such matrices dilation matrices.

The operators R_a^M and S_a^M are referred to as refinement and subdivision operators, respectively. In wavelet literature the matrix-function a is called the mask of the corresponding operator, but throughout this paper it will also be called the symbol of the operator R_a^M or S_a^M. An important role of the refinement and subdivision operators lies in the construction and study of wavelet bases. Thus under some conditions the solutions of the equations

$$\varphi = R_a^M \varphi, \tag{0.2}$$

called refinable or M-refinable vector functions, produce wavelets. The operator S_a^M arises when one applies an iteration procedure to solve equation (0.2), [8]. Spectral radii $\rho(R_a^M)$ and $\rho(S_a^M)$ of these operators are used to establish the convergence of the iterative algorithm mentioned [3], [13], [15], [22], [25], [26] and to study the regularity of the refinable functions [4], [5], [6], [7], [9], [10], [13], [16], [17], [25], [26], [27], [34]. On the other hand, these two operators are closely associated with the transfer operator, also called the Ruelle operator or the Perron–Frobenius–Ruelle operator, which finds various applications in statistical mechanics, dynamical systems and ergodic theory [20], [29], [30], [31], [32], [33].

Despite this, the evaluation of $\rho(R_a^M)$ and $\rho(S_a^M)$ remains an open question. The most essential progress so far has been made for $m = 1$ in the case $s = 1$. However, even in this relatively simple situation the evaluation of the spectral radius heavily depends upon the concrete form of the symbol, and usually the symbol of these operators is assumed to be a polynomial, so the corresponding spectral radius is evaluated using different limit characteristics of auxiliary finite matrices [4], [9], [13], [35], [36]. More detailed results have been obtained for the so-called continuous refinement operator T_a^M defined by

$$T_a^M f := \int_{\mathbb{R}^s} a(\cdot - My) f(y)\mathrm{d}y.$$

Thus it was shown in [14] that for a compactly supported non-negative function $a : \mathbb{R}^s \to \mathbb{C}$ and for a dilation matrix M satisfying condition (0.1) the spectral radius of the operator T_a^M can be found by the formula

$$\rho(T_a^M) = \frac{1}{\sqrt{|\det M|}} \int\limits_{\mathbb{R}^s} a(y)\mathrm{d}y. \tag{0.3}$$

A different approach to the evaluation of the spectral radius of the operator T_a^M has been employed in [12], where formula (0.3) was proved assuming not that a is non-negative and a compactly supported function but rather that $a \in L_1(\mathbb{R}^s)$. Note that for such symbols, the integral in (0.3) has to be replaced by its modulus. Moreover, condition (0.1) can be dropped for some classes of matrices M. It is also worth mentioning that the approach of [12] leads to the same limit expression for $\rho(T_a^M)$ and $\rho(R_a^M)$. Thus one could expect that obtaining more convenient formulas for $\rho(R_a^M)$ would be similar to that problem for $\rho(T_a^M)$. However, in contrast to operator T_a^M the peculiarities of the symbol of the operator R_a^M preclude such an effective formula for $\rho(R_a^M)$ as (0.3) is for $\rho(T_a^M)$. It turns out that evaluation of the spectral radius of the operator R_a^M presents a problem similar to that for the operator S_a^M, so these two operators are considered here together.

Note that even for $m = 1$ and $s = 1$ there are only a few papers devoted to the evaluation of the spectral radius of the operator S_a^M for non-polynomial symbols. Thus for symbols whose Fourier coefficients are rapidly decreasing, two approximate algorithms have been proposed in [5]. Other results concerning continuous symbols can be found in [18], [19]. In [11], estimates for the spectral radii of the operators R_a^M and S_a^M with piecewise continuous symbol a are established. In some cases, the spectral radii of S_a^M and R_a^M have been estimated by using the supremum norm of the function a. As a rule, it is very difficult to extend the methods and results obtained in the case $m = 1$ and $s = 1$ to the multivariate cases $m \geqslant 1$ and $s > 1$, and it appears that only the paper [28] and the recently published paper [4] contain some results concerning the approximate calculation of the spectral radius of S_a^M for the case $m > 1$ and $s > 1$. Note that the dilation matrix M in [28] is supposed to be isotropic, i.e. such that

$$MM^* = \lambda I,$$

where $\lambda > 1$ and I is the identity matrix.

The general case $s \geqslant 1$ and $m \geqslant 1$ for a class of dilation matrices is considered in the present paper. This class includes known isotropic matrices encountered in wavelet analysis. However, in contrast to [28], approximation methods are not used and the corresponding spectral radius is estimated from below. For $m = 1$, in some cases the estimates obtained yield

$$\rho(R_a^M) = \frac{1}{\sqrt{|\det M|}} \|a\|_\infty, \tag{0.4}$$

and sufficient conditions for equality (0.4) are given below. Estimates for $\rho(S_a^M)$ are also presented in this paper.

Let us describe the set of dilations considered here, starting with an example. An isotropic dilation often employed in work on bivariate refinable functions is the matrix

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

with eigenvalues $\lambda_1 = 1 - \mathrm{i}$ and $\lambda_2 = 1 + \mathrm{i}$. Representing these eigenvalues in the polar form $\lambda_1 = r\exp(\mathrm{i}\pi q_1)$ and $\lambda_2 = r\exp(\mathrm{i}\pi q_2)$ one can pay attention to two remarkable properties they have: namely, the indices $q_1(= -1/4)$ and $q_2(= 1/4)$ are rational, and there exists a positive integer $l(= 2)$ such that $|\lambda_1|^l = |\lambda_2|^l$ is an integer greater than one. It is easily seen that the second property is inherent for any isotropic dilation matrix, but the first is not. This motivates the introduction of a set $\mathfrak{M}^s$ of all isotropic dilation matrices M with the property that for any eigenvalue $\lambda_j = r\exp(\mathrm{i}\pi q_j)$ of M, the index $q_j, j = 1, 2, \ldots, m$ is rational. Note that for $s = 2$ any dilation matrix M of the form

$$M = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad a^2 + b^2 \geqslant 2$$

is in $\mathfrak{M}^s$.

The aim of this paper is to obtain spectral radius estimates for the operators R_a^M and S_a^M with a continuous symbol a and isotropic dilation matrix $M \in \mathfrak{M}^s$. The paper is organized as follows. In Section 1 certain properties of the refinement and subdivision operators are presented, and lower estimates are established for spectral radii in terms of multiplier norms of sequences of matrix-functions. In Section 2 the notion of μ-cyclic p-tuples is introduced and applied to obtain estimates of the spectral radii, using values of their symbols on special sets of points. In Section 3, integral estimates for the spectral radii of the refinement operator R_a^M are established. Section 4 gives sufficient conditions when the spectral radius of the refinement operator can be calculated exactly via formula (0.4).

1. GENERAL ESTIMATES FOR THE SPECTRAL RADIUS OF SUBDIVISION AND REFINEMENT OPERATORS

Let $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ and let $\|a\|_\infty$ denote the multiplier norm of a on $L_2^{m\times m}(\mathbb{T}^s)$ or $L_2^{m\times m}(\mathbb{R}^s)$ — i.e. the norm of the operator of multiplication by a on the corresponding space L_2^m. It is known that

$$\|a\|_\infty = \max_{1\leqslant j\leqslant m} \operatorname*{ess\,sup}_{x\in\mathbb{R}^s} \sqrt{\lambda_j(x)}, \quad t = \mathrm{e}^{\mathrm{i}x} \in \mathbb{T}^s, \tag{1.1}$$

where $\lambda_j(x), j = 1, 2, \ldots, m$ are the eigenvalues of the matrix a^*a. Of course, if a is a constant matrix, then

$$\|a\|_\infty = \max_{1\leqslant j\leqslant m} \sqrt{\lambda_j}$$

and if $a \in \mathbf{C}^{m\times m}(\mathbb{T}^s)$, then for any $t_k \in \mathbb{T}^s$

$$\|a(t_k)\|_\infty \leqslant \|a\|_\infty.$$

Let us also recall that the norm of any linear bounded operator A on $L_2^m(\mathbb{R}^s)$ satisfies the equation

$$\|A\|^2 = \|AA^*\|, \tag{1.2}$$

and the spectral radius of A can be represented in the form

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n}. \tag{1.3}$$

Further, let us adopt the notation $\prod\limits_{k=j}^{n} A_k$ for the following ordered product $A_j A_{j+1} \cdots A_n$ of elements $A_j, A_{j+1}, \ldots, A_n$.

We start with the refinement operator R_a^M.

THEOREM 1.1. *If $\{a_k\} \in l_1^{m\times m}(\mathbb{Z}^s)$ and M is a non-singular integer matrix, then*

$$\rho(R_a^M) = \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \Big\| \prod_{k=0}^{n-1} a((M^T)^k \cdot) \Big\|_\infty^{1/n}. \tag{1.4}$$

Proof. Let $B_M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ be the operator defined by

$$B_M f(x) = f(Mx), \quad x \in \mathbb{R}.$$

To derive (1.4) we use the corresponding formulas [1], [2] for the spectral radius of weighted shift operators. Denote by $\mathfrak{F}$ the Fourier transform on $L_2^m(\mathbb{R}^s)$, i.e.

$$\mathfrak{F} f = \frac{1}{(2\pi)^{s/2}} \int\limits_{\mathbb{R}^s} \mathrm{e}^{-\mathrm{i}\,(\cdot)\,y} f(y)\mathrm{d}y.$$

It is easily seen that for any $b \in \mathbb{R}^s$ $\mathfrak{F}\mathrm{e}^{\mathrm{i}b\cdot}\mathfrak{F}^{-1} f = f(\cdot - b)$. Thus, if $\{a_k\} \in l_1^{m\times m}(\mathbb{Z}^s)$, the matrix a is continuous on $\mathbb{R}^s$ and

$$R_a^M = B_M \mathfrak{F} a \mathfrak{F}^{-1}. \tag{1.5}$$

Consider an $s \times s$ non-singular matrix C, and for any matrix-function $a : \mathbb{R}^s \to \mathbb{C}^{m\times m}$ define the matrix $a_C : \mathbb{R}^s \to \mathbb{C}^{m\times m}$ by

$$a_C(x) := a((C^T)^{-1}x).$$

Now the refinement operator R_a^M can be represented in the form

$$R_a^M = \mathfrak{F}\Big(\frac{a_M}{\sqrt{|\det M|}}\, V_M\Big)\mathfrak{F}^{-1}$$

where the operator $V_M : L_2^m(\mathbb{R}^s) \to L_2^m(\mathbb{R}^s)$ defined by

$$V_M = \frac{1}{\sqrt{|\det M|}}\, B_{(M^T)^{-1}}$$

is an invertible isometry. Since $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ are also isometrical operators on the space $L_2^m(\mathbb{R}^s)$, the equation

$$\rho(R_a^M) = \rho\Big(\frac{a_M}{\sqrt{|\det M|}}\, V_M\Big) \tag{1.6}$$

holds. Applying now the corresponding results [1], [2], [21], [24] to the weighted shift operator $a_M/(\sqrt{|\det M|})\, V_M$ and using relation (1.6), one obtains:

$$\begin{aligned}\rho(R_a^M) &= \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \Big\| \prod_{k=0}^{n-1} a((M^T)^{-k-1}\cdot)\Big\|_\infty^{1/n} \\ &= \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \Big\| \prod_{k=0}^{n-1} a((M^T)^{k}\cdot)\Big\|_\infty^{1/n}. \quad \blacksquare\end{aligned}$$

Let us also consider the weighted shift operator $U_a^M : \mathbf{C}^m(\mathbb{R}^s) \to \mathbf{C}^m(\mathbb{R}^s)$ defined by

$$U_a^M = \frac{a}{\sqrt{|\det M|}}\, B_{M^T}. \tag{1.7}$$

COROLLARY 1.2. *If the matrices a and M satisfy the conditions of Theorem* 1.1, *then*

$$\rho(R_a^M) = \rho(U_a^M). \tag{1.8}$$

With reference to the operator R_a^M below, let us always assume that the sequence of Fourier coefficients of the symbol a belongs to the space $l_1^{m\times m}(\mathbb{Z}^s)$.

COROLLARY 1.3. *If a and M satisfy the assumptions of Theorem* 1.1, *then*

$$\rho(R_a^M) \leqslant \frac{\|a\|_\infty}{\sqrt{|\det M|}}. \tag{1.9}$$

On the other hand, for any sequence $x_n \in \mathbb{R}^s$

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \limsup_{n\to\infty} \Big\| \prod_{k=0}^{n-1} a((M^T)^{k} x_n)\Big\|_\infty^{1/n}. \tag{1.10}$$

COROLLARY 1.4. *Let a and M satisfy the assumptions of Theorem* 1.1 *and let* $\|a\|_\infty = \|a(0)\|_\infty$. *If $a(0)$ is a normal matrix, then*

$$\rho(R_a^M) = \frac{\|a(0)\|_\infty}{\sqrt{|\det M|}}. \tag{1.11}$$

Proof. Inequality (1.10) implies that for any subsequence $\{n_j\} \subset \mathbb{N}$

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \lim_{n_j\to\infty} \|(a(0))^{n_j}\|_\infty^{1/n_j}. \tag{1.12}$$

If $a(0)$ is a self-adjoint matrix, then $\|(a(0))^2\|_\infty = \|a(0)\|_\infty^2$ which yields $\|(a(0))^{2^j}\|_\infty = \|(a(0))\|_\infty^{2^j}$ for any $j \in \mathbb{N}$. Now let $a(0)$ be a normal matrix, when

$$\|(a(0))^{2^j}\|_\infty = \|(a(0))^{2^j}(a^*(0))^{2^j}\|_\infty^{1/2} = \|(a(0)a^*(0))^{2^j}\|_\infty^{1/2} = \|a(0)\|_\infty^{2^j}.$$

Thus taking the subsequence $n_j = 2^j, j \in \mathbb{N}$, equality (1.11) follows from (1.9) and (1.12). ∎

Consider now the subdivision operator S_a^M. It is easily seen that the operator S_a^M is isometrically isomorphic to the operator $\widehat{S}_a^M : L_2^m(\mathbb{T}^s) \to L_2^m(\mathbb{T}^s)$,

$$\widehat{S}_a^M f(x) = a(x) f(M^T x), \quad t = \mathrm{e}^{\mathrm{i}2\pi x}, \quad x \in \mathbb{R}^s. \tag{1.13}$$

Let $q \geqslant 2$ denote the absolute value of the determinant of M. For a positive integer n, let $E_j^{(n)} + (M^T)^n \mathbb{Z}^s, j = 0, 1, \ldots, q^n - 1$ be the distinct elements of the quotient space $\mathbb{Z}^s/(M^T)^n\mathbb{Z}^s$ such that $E_0^{(n)} = 0$. For $n = 1$ the corresponding elements $E_j^{(1)}$ are denoted by $E_j, j = 0, 1, \ldots, q-1$.

Moreover, by $\mathcal{A}_M^{(n)} = \mathcal{A}_M^{(n)}(x), x \in \mathbb{T}^s$ we denote the matrix

$$\mathcal{A}_M^{(n)}(x) = \prod_{k=1}^{n} a_{M^k}(x) \prod_{k=0}^{n-1} a^*_{M^{n-k}}(x).$$

THEOREM 1.5. *Let $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ and let M be a dilation matrix. Then*

$$\rho(S_a^M) = \frac{1}{\sqrt{|\det M|}} \lim_{n\to\infty} \Big\| \sum_{j=0}^{q^n-1} \mathcal{A}_M^{(n)}(\cdot + E_j^{(n)}) \Big\|_\infty^{1/2n}. \tag{1.14}$$

Proof. Consider the operator $\widehat{T}_a^M : L_2^m(\mathbb{T}^s) \to L_2^m(\mathbb{T}^s)$ defined by

$$\widehat{T}_a^M g = \frac{1}{|\det M|} \sum_{j=0}^{q-1} a_M(\cdot + E_j) g((M^T)^{-1}(\cdot + E_j)).$$

Then the adjoint for the operator $\widehat{T}_a^M$ has the form

$$(\widehat{T}_a^M)^* = \widehat{S}_{a^*}^M. \tag{1.15}$$

Relation (1.15) is widely used if $m = 1$ and $s \geqslant 1$. For $m > 1$ it can be derived from known results for the scalar case, and to obtain (1.15) one can exploit formula (3.5) of [7]. Taking into account (1.2) and (1.15) one obtains $\|(\widehat{S}_a^M)^n\|^2 = \|(\widehat{S}_{a^*}^M)^n\|^2 = \|(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n\|$. Let $g \in L_2^m(\mathbb{T}^s)$. Then the function $(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n g$ can be expressed in the form

$$
\begin{aligned}
&(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n g(x)\\
&= \frac{1}{|\det M|^n}\sum_{j_n=0}^{q-1}\sum_{j_{n-1}=0}^{q-1}\cdots\sum_{j_1=0}^{q-1}\Big(\prod_{k=1}^{n} a_{M^k}\Big(x+\sum_{l=1}^{k}(M^T)^{l-1}E_{j_{n-l+1}}\Big)\\
&\qquad\qquad\times\prod_{k=1}^{n} a^*_{M^{n-k+1}}\Big(x+\sum_{l=1}^{n-k}(M^T)^{l}E_{j_{n-l+1}}\Big)\Big)g(x)\\
&= \frac{1}{|\det M|^n}\sum_{j_n=0}^{q-1}\sum_{j_{n-1}=0}^{q-1}\cdots\sum_{j_1=0}^{q-1}\Big(\prod_{k=1}^{n} a_{M^k}\Big(x+\sum_{l=1}^{n}(M^T)^{l-1}E_{j_{n-l+1}}\Big)\\
&\qquad\qquad\times\prod_{k=1}^{n} a^*_{M^{n-k+1}}\Big(x+\sum_{l=1}^{n}(M^T)^{l}E_{j_{n-l+1}}\Big)\Big)g(x)\\
&= \Big(\frac{1}{|\det M|^n}\sum_{j=0}^{q^n-1}\mathcal{A}_M^{(n)}(x+E_j^{(n)})\Big)g(x).
\end{aligned}
\tag{1.16}
$$

Note that the lines 4 and 5 in this transformation follow from the assumption that M is an integer matrix, and the final step uses the fact that the complete set of coset representatives $\{E_j^{(n)}\}_{j=0}^{q^n-1}$ and the set $\{\mathcal{E}+M^T\mathcal{E}+\cdots+(M^T)^{n-1}\mathcal{E}\}$, where $\mathcal{E}:=\Big\{E_0,E_1,\dots,E_{q-1}\Big\}$, coincide.

Thus equality (1.16) shows that $(\widehat{T}_a^M)^n(\widehat{S}_{a^*}^M)^n$ is just the operator of multiplication by the matrix $(1/|\det M|^n)\sum_{j=0}^{q^n-1}\mathcal{A}_M^{(n)}(x+E_j^{(n)})$. This implies formula (1.14). ∎

As an immediate consequence of formula (1.14) we note the following estimates.

COROLLARY 1.6. *Let a and M satisfy the assumptions of Theorem* 1.5. *Then*

$$
\rho(S_a^M)\leqslant\lim_{n\to\infty}\Big\|\prod_{k=0}^{n-1}a((M^T)^k\cdot)\Big\|_\infty^{1/n}.
\tag{1.17}
$$

In addition, if $m=1$ and the function a is continuous on $\mathbb{T}^s$, then for any sequence $x_n\in\mathbb{R}^s$

$$
\rho(S_a^M)\geqslant\frac{1}{\sqrt{|\det M|}}\limsup_{n\to\infty}\Big\|\prod_{k=0}^{n-1}a((M^T)^k x_n)\Big\|_\infty^{1/n}.
\tag{1.18}
$$

REMARK 1.7. The important point to note here is that R_a^M and S_a^M as operators are of a very different nature. Thus if the refinement operator R_a^M is generated by an invertible shift operator, the subdivision operator S_a^M is related to a similar but non-invertible operator. It is known [1], [2] that the weighted shift operators generated by non-invertible maps are studied in less detail, and results obtained often depend on the concrete form of the corresponding operator. It seems that formula (1.14) for the spectral radius of the subdivision operator S_a^M acting on the space of vector-valued functions belongs to such results. At least, the author is not aware of similar representations for other classes of weighted shift operators.

2. ESTIMATES USING POINT VALUES OF THE SYMBOL

It is not an easy task to evaluate the spectral radii of the operators R_a^M and S_a^M using relations (1.4) and (1.14). However, for some classes of dilation matrices these formulas lead to estimates of $\rho(R_a^M)$ and $\rho(S_a^M)$ which do not contain a limit but only the spectral norms of certain constant matrices.

For any $a \in L_\infty^{m\times m}(\mathbb{T}^s)$ consider the quantity

$$S_n(a, M) := \Big\| \prod_{k=0}^{n-1} a((M^T)^k \cdot) \Big\|_\infty^{1/n}.$$

The following statement turns out to be useful in the problem under consideration. Its proof follows immediately from (1.1).

LEMMA 2.1. *Let $P = (p, p, \ldots, p), p \in \mathbb{R}^+$ be a fixed vector and let*

$$\widetilde{a}_p(x) := a(Px), \quad x \in \mathbb{R}^s.$$

Then

$$S_n(a, M) = S_n(\widetilde{a}_p, M).$$

A consequence of this result is that while studying the limit in (1.4) one can always assume a to be a 1^s-periodic matrix-functions, so henceforth the symbol a of the corresponding operators will be identified with matrix-functions on the torus $\mathbb{T}^s := \mathbb{R}^s / \mathbb{Z}^s$. Note that the 1^s-periodicity of the symbol a can be assumed from the very beginning, as was done earlier in the case of the subdivision operator. Nevertheless, Lemma 2.1 allows the subdivision and refinement operators to be replaced by operators with the same spectral radii, and this property will be exploited below.

LEMMA 2.2. *Let $M \in \mathfrak{M}^s$. Then there exist numbers $\mu, q \in \mathbb{N}, \mu \geqslant 2$ and matrices $A_0, A_1, \ldots, A_{q-1} \in \mathbb{Z}^{s\times s}$ such that for any number $n \in \mathbb{N}$ the matrix $(M^T)^n$ can be represented in the form*

$$(M^T)^n = A_r\, \mu^l I, \tag{2.1}$$

where $r, l \in \mathbb{N}$ are defined by the equation $n = lq + r, 0 \leqslant r \leqslant q - 1$.

Proof. By Schur's theorem [23], p.176, there exists a unitary matrix U and an upper triangular matrix T such that $M = UTU^*$. However, M is an isotropic dilation matrix, therefore

$$MM^* = \lambda I \tag{2.2}$$

with a $\lambda > 1$, and hence $TT^* = \lambda I$. Since the matrix T is upper triangular, the latter equation implies that T is a diagonal matrix with elements the eigenvalues of M. Let $\lambda_j = \lambda^{1/2} e^{i\pi(p_j/q_j)}, j = 1, 2, \ldots, m$ be the eigenvalues of M, and let $u \in \mathbb{N}$ be the smallest number such that $\lambda^{u/2} \in \mathbb{N}$. Set

$$q := 2 \operatorname{lcm}(q_1, q_2, \ldots, q_m, u).$$

Then $\lambda^q \in \mathbb{N}$ and $qp_j/q_j = 2k_j$ with $k_j \in \mathbb{N}, j = 1, 2, \ldots, m$. Set $\mu := \lambda^q$. Now one can use the representation $M = UTU^*$ to obtain $M^q = UT^qU^* = U\lambda^q \operatorname{diag}(e^{i2\pi k_1}, e^{i2\pi k_2}, \ldots, e^{i2\pi k_m})U^* = \mu I$, so that writing $n \in \mathbb{N}$ in the form $n = ql + r, 0 \leqslant r \leqslant q - 1$ one has

$$M^n = M^{ql+r} = \mu^l M^r.$$

Thus, representation (2.1) is valid with $A_r = (M^T)^r, 0 \leqslant r \leqslant q - 1$, and the proof is complete. ∎

Let us now fix $\mu, p \in \mathbb{N}, \mu \geqslant 2$ and let $d_0, d_1, \ldots, d_{p-1} \in \mathbb{N} \cap [0, \mu - 1]$. Consider a repeating fraction $x^{(p)} = 0.\overline{d_0, d_1, \ldots, d_{p-1}}$ with the base μ, i.e.

$$x^{(p)} = \Big(\frac{d_0}{\mu} + \frac{d_1}{\mu^2} + \cdots + \frac{d_{p-1}}{\mu^p}\Big) + \Big(\frac{d_0}{\mu^{p+1}} + \frac{d_1}{\mu^{p+2}} + \cdots + \frac{d_{p-1}}{\mu^{2p-1}}\Big) + \cdots .$$

Each such point may be associated with an ordered set $[x^{(p)}] = \{x_0^{(p)}, \ldots, x_{p-1}^{(p)}\}$ of p repeating fractions

$$x_0^{(p)} = 0.\overline{d_0 d_1 \cdots d_{p-1}}\,, \quad x_1^{(p)} = 0.\overline{d_1 d_2 \cdots d_0}\,, \quad \ldots, \quad x_{p-1}^{(p)} = 0.\overline{d_{p-1} d_0 \cdots d_{p-2}}\,,$$

with the base μ. The set $[x^{(p)}]$ is called the μ-cyclic p-tuple corresponding to the point $x^{(p)}$ and the set of all μ-cyclic p-tuples is denoted by $\mathcal{C}_\mu^{(p)}$. The number p is referred to as the length of the corresponding tuple.

LEMMA 2.3. *Let $a \in \mathbf{C}(\mathbb{T})$ and $[x^{(p)}] = \{x_0^{(p)}, x_1^{(p)}, \ldots, x_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p)}$. Then for any $l \in \mathbb{N}$ such that*

$$l \equiv r \mod p, \quad 0 \leqslant r \leqslant p - 1$$

one has

$$a(\mu^l x^{(p)}) = a(x_r^{(p)}).$$

The proof of this result is straightforward from the definition of $[x^{(p)}]$.

Now for any fixed number $p \in \mathbb{N}$ consider a system which consists of s μ-cyclic p-tuples $[x_1^{(p)}], [x_2^{(p)}], \ldots, [x_s^{(p)}]$. Recall that

$$[x_j^{(p)}] = \{x_{j,0}^{(p)}, x_{j,1}^{(p)}, \ldots, x_{j,p-1}^{(p)}\}, \quad j = 1, 2, \ldots, s.$$

Combining the corresponding elements of these tuples, one can form vector-columns $\widetilde{x}_k^{(p)}, k = 0, 1, \ldots, p - 1$ where

$$\widetilde{x}_0^{(p)} = (x_{1,0}^{(p)}, x_{2,0}^{(p)}, \ldots, x_{s,0}^{(p)})^T, \quad \ldots, \quad \widetilde{x}_{p-1}^{(p)} = (x_{1,p-1}^{(p)}, x_{2,p-1}^{(p)}, \ldots, x_{s,p-1}^{(p)})^T.$$

Let $[\widetilde{x}^{(p)}]$ denote the family $\{\widetilde{x}_0^{(p)}, \widetilde{x}_1^{(p)}, \ldots, \widetilde{x}_{p-1}^{(p)}\}$, and the set of all such elements $[\widetilde{x}^{(p)}]$ is $\mathcal{C}_\mu^{(p),s}$. (Note that this notation differs from our previous agreement concerning the sets obtained via Cartesian products.) Let us now introduce the class $\mathbf{W} = \mathbf{W}(\mathbb{T}^s)$ of continuous functions with absolutely convergent Fourier series, and proceed to estimate the spectral radius of the operator R_a^M with symbols from $\mathbf{W}^{m\times m}(\mathbb{T}^s)$. In passing, note that some of the results established remain valid for the matrix-functions a of $L_\infty^{m\times m}(\mathbb{T}^s)$.

Let $M \in \mathfrak{M}^s$ be a dilation matrix and let $A_1, A_2, \ldots, A_{q-1}$ be the matrices introduced in Lemma 2.2. For any matrix-function a we define a matrix $b_{a,M}$ by

$$b_{a,M}(x) := a(x)a(A_1x)\cdots a(A_{q-1}x).$$

The use of sequences generated by μ-cyclic p-tuples leads to the following general estimates for the spectral radius of the refinement operator.

THEOREM 2.4. *Let $a \in \mathbf{W}^{m\times m}(\mathbb{T}^s)$. Then for any $p \in \mathbb{N}$ and for any $[\widetilde{x}^{(p)}] = \{\widetilde{x}_0^{(p)}, \widetilde{x}_1^{(p)}, \ldots, \widetilde{x}_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p),s}$ the following inequality holds:*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \limsup_{l\to\infty} \Big\| \Big(\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)}) \Big)^l \Big\|_\infty^{1/lqp}. \tag{2.3}$$

Proof. Fix $p \in \mathbb{N}$, and for the sequence $\{n_l\}_{l\in\mathbb{N}} = \{lpq\}_{l\in\mathbb{N}}$ set $x_{n_l} := \widetilde{x}_0^{(p)}$. Then by inequality (1.10) of Corollary 1.3 the spectral radius $\rho(R_a^M)$ satisfies the inequality $\rho(R_a^M) \geqslant 1/\sqrt{|\det M|} \limsup\limits_{l\to\infty} \Big\| \prod\limits_{j=0}^{n_l-1} a((M^T)^j x_{n_l}) \Big\|_\infty^{1/n_l}$. Application of Lemma 2.2 and Lemma 2.3 to the matrix $\prod\limits_{j=0}^{n_l-1} a((M^T)^j x_{n_l})$ yields

$$\Big\| \prod_{j=0}^{n_l-1} a((M^T)^j x_{n_l}) \Big\|_\infty^{1/n_l} = \Big\| \prod_{j=0}^{lp-1} b_{a,M}(\mu^j \widetilde{x}_0^{(p)}) \Big\|_\infty^{1/lpq} = \Big\| \prod_{j=0}^{l-1} \prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)}) \Big\|_\infty^{1/lpq},$$

which implies inequality (2.3). ∎

Consider now some consequences of Theorem 2.4.

(1) $m = 1, \quad s = 1.$

In this case, M is the operator of multiplication by an integer which implies that $q = 1, \mu = M$ if $M \geqslant 2$ and $q = 2, \mu = M^2$ if $M \leqslant -2$. Assume first that $M \geqslant 2$. Since all factors in the product (2.3) commute with each other, the following results are obtained (cf. also [11]):

COROLLARY 2.5. *Let $a \in \mathbf{W}(\mathbb{T})$ and $M \geqslant 2$ be a positive integer. Then for any M-cyclic p-tuple $[x^{(p)}]$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{M}} \Big(\prod_{r=0}^{p-1} |a(x_r^{(p)})| \Big)^{1/p}. \tag{2.4}$$

COROLLARY 2.6. *Let $a \in \mathbf{W}(\mathbb{T})$ and $M \geqslant 2$ be a positive integer. Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{M}} \sup_{p\in\mathbb{N}} \max_{[x^{(p)}]\in\mathcal{C}_\mu^{(p)}} \Big(\prod_{r=0}^{p-1} |a(x_r^{(p)})|\Big)^{1/p}.$$

An interesting result appears if one assumes that $M \leqslant -2$. Then $q = 2$ and $\mu = M^2$, so the estimates of the spectral radius of the operator R_a^M undergo a modification.

COROLLARY 2.7. *Let $a \in \mathbf{W}(\mathbb{T})$ and $M \leqslant -2$ be a negative integer. Then for any M^2-cyclic p-tuple $[x^{(p)}]$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|M|}} \Big(\prod_{r=0}^{p-1} |a(x_r^{(p)})a(Mx_r^{(p)})|\Big)^{1/2p}.$$

This result can be considered as a bridge between the multivariate case where matrix dilations with negative determinants are allowed and the one variable case, where mainly positive dilations has been considered. Moreover, it shows that representation (2.1) can be of interest even in the case of functions of one variable.

(2) $m = 1, \quad s > 1.$

In this case the factors in (2.3) commute with each other, but the expression in the right-hand side of (2.3) becomes more complicated.

COROLLARY 2.8. *Let $a \in \mathbf{W}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. Then for any $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \Big(\prod_{j=0}^{q-1}\prod_{r=0}^{p-1} |a(A_j\widetilde{x}_r^{(p)})|\Big)^{1/qp} \tag{2.5}$$

where the matrices $A_0, A_1, \dots, A_{q-1}$ are defined in Lemma 2.2.

(3) $m > 1, \quad s > 1.$

In this case the factors in (2.3) generally do not commute, but a simplification of relation (2.3) is sometimes possible.

COROLLARY 2.9. *Let $a \in \mathbf{W}^{m\times m}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. If there is $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ such that $\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)})$ is a normal matrix, then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \Big\|\prod_{r=0}^{p-1} b_{a,M}(\widetilde{x}_r^{(p)})\Big\|_\infty^{1/qp}. \tag{2.6}$$

The proof of inequality (2.6) mainly follows the arguments used in the proof of Corollary 1.4.

As we have seen, it is possible to get lower estimates for the spectral radius of the operator R_a^M by using the multiplier norm of certain constant matrices. However, from a practical point of view it is preferable to work with supremum norms of functions instead of matrix multiplier norms, and a simplification can be achieved in some cases where matrix a has special properties. For example, assume that a is unitarily equivalent to a functional diagonal matrix, i.e. there exists a constant unitary matrix U such that $a(x) = Ud(x)U^*, x \in \mathbb{R}^s$ where $d(x) = \operatorname{diag}(\lambda_1(x), \lambda_2(x), \ldots, \lambda_s(x))$. Moreover, let us also assume that the matrix a possesses a dominant eigenvalue, i.e. an eigenvalue λ_{j_0} such that

$$|\lambda_{j_0}(x)| \geqslant |\lambda_j(x)|, \quad j \in \{1, 2, \ldots, m\}$$

for all $x \in \mathbb{R}^s$. In this case the spectral radius of R_a^M can be estimated by using values of the function λ_0 only, since under this condition, the matrices $a((M^T)^j\cdot)$ and $a((M^T)^k\cdot)$ commute for any non-negative integers j, k. Therefore for any $[\widetilde{x}^{(p)}] \in \mathcal{C}_\mu^{(p),s}$ the spectral radius of the operator R_a^M satisfies the inequality

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \Big(\prod_{r=0}^{p-1} |\Lambda_{a,M}(\widetilde{x}_r^{(p)})| \Big)^{1/qp} \tag{2.7}$$

where $\Lambda_{a,M}(x) = \lambda_{j_0}(x)\lambda_{j_0}(A_1 x) \cdots \lambda_{j_0}(A_{q-1}x)$.

Taking into account Corollary 1.6 one can get lower bounds for the spectral radius of the subdivision operator S_a^M which are analogous to the estimates for $\rho(R_a^M)$ established above.

3. INTEGRAL ESTIMATES

As was mentioned in Corollary 1.2, the spectral radius of the refinement operator $R_a^M : L_2^m(\mathbb{T}^s) \to L_2^m(\mathbb{T}^s)$ is equal to the spectral radius of the weighted shift operator

$$U_a^M := \frac{a}{\sqrt{|\det M|}} B_{M^T}$$

considered on the space $\mathbf{C}^m(\mathbb{T}^s)$. Such operators play an important role in various fields of mathematics, so they have attracted meticulous attention in literature, both generally and also for operators belonging to special classes [1], [2]. In particular, it is known that the spectral radius of the weighted shift operator can be expressed in terms of probabilistic measures, invariant with respect to the corresponding shift [21], [24]. Thus the construction of appropriate invariant measures is of great importance, and leads to estimates of the spectral radius for the operator under consideration. Let us assume that $a \in \mathbf{W}(\mathbb{T}^s)$, $M \in \mathfrak{M}^s$ and consider the operator R_a^M again. By Corollary 1.2 and by Theorem 4.4 of [2] (see also Theorem 6.1 of [1]) one obtains

$$\rho(R_a^M) = \frac{1}{\sqrt{|\det M|}} \max_{\nu \in \mathcal{I}_{M^T}} \exp\Big(\int_{\mathbb{T}^s} \ln|a(x)| d\nu_x\Big), \tag{3.1}$$

where $\mathcal{I}_{M^T}$ denotes the set of all probabilistic Borel measures on the torus $\mathbb{T}^s$ invariant with respect to the automorphism $x \to M^T x$ of this torus. Let us recall that a measure ν is M^T-invariant if for every ν-measurable set $S \subset \mathbb{T}^s$ the equality $\nu(M_T^{-1}(S)) = \nu(S)$ holds.

Equation (3.1) is a source of a variety of integral estimates for the spectral radii of the operator R_a^M and some of these inequalities are given below.

Consider first estimates of the spectral radius of the refinement operator R_a^M that use integrals over one-dimensional subsets. Let E be a subset of $\mathbb{T}^s$ defined by

$$E := \{y \in \mathbb{T}^s : y = (x, x, \dots, x), \quad x \in \mathbb{T}\}.$$

THEOREM 3.1. *Let $a \in W$, $M \in \mathfrak{M}^s$ and let q be the smallest positive integer from Lemma* 2.2. *Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \exp\Big(\frac{1}{q} \sum_{j=0}^{q-1} \int_{\mathbb{T}} \ln|\widetilde{a}((M^T)^j x)| dx\Big), \tag{3.2}$$

where $\widetilde{a}((M^T)^j x) := a((M^T)^j(x, x, \dots, x))$ and dx *is the normalized Lebesgue measure on the circle $\mathbb{T}$.*

Proof. Consider the sets $E_j = (M^T)^j E \subset \mathbb{T}^s, j \in \mathbb{N}^+$ where $E_0 := E$ and $\mathbb{N}^+ := \mathbb{N} \cup \{0\}$. By Lemma 2.2, for every $n_1, n_2 \in \mathbb{N}^+$ such that $n_1 - n_2 = qm$, $m \in \mathbb{Z}$, the subsets E_{n_1} and E_{n_2} belong to the same equivalency class of the torus $\mathbb{T}^s$.

Let m denote the normalized Lebesgue measure on the circle $\mathbb{T}$ and let $\psi : \mathbb{T} \to \mathbb{T}^s$ be the mapping defined by

$$\psi(x) = (x, x, \dots, x), \quad x \in \mathbb{T}.$$

Now one can introduce a measure μ on $\mathbb{T}^s$ in the following way:

(i) If S is a subset of E_j for some $j = 0, 1, \dots, q-1$, then

$$\mu(S) = \frac{m(\widetilde{S})}{q},$$

where $\widetilde{S}$ is the pre-image of the set S under the mapping $(M^T)^j \psi : \mathbb{T} \to E_j$.

(ii) $\mu(\mathbb{T}^s \setminus \bigcup_{j=0}^{q-1} E_j) = 0$.

It is evident that μ is a normalized measure on $\mathbb{T}^s$, so one only needs to check that this measure is M^T-invariant. If $q > 1$, the M^T-invariance of μ is obvious. For $q = 1$, let us fix a positive integer λ and consider the mapping φ_λ defined by

$$\varphi_\lambda(x) = \lambda x, \quad x \in \mathbb{T}.$$

A straightforward computation shows that the Lebesgue measure is φ_λ-invariant, and so is μ.

Thus relation (3.1) yields $\rho(R_a^M) \geqslant 1/\sqrt{|\det M|} \exp\Big(\int\limits_{\mathbb{T}^s} \ln|a(x)| \mathrm{d}\mu_x\Big) = 1/\sqrt{|\det M|} \exp\Big(\frac{1}{q}\sum\limits_{j=0}^{q-1}\int\limits_{E_j} \ln|a(x)|\mathrm{d}\mu_x\Big)$. Using now the M^T-invariance of the measure μ, one obtains

$$\begin{aligned}\rho(R_a^M) &\geqslant \frac{1}{\sqrt{|\det M|}} \exp\Big(\frac{1}{q}\sum_{j=0}^{q-1}\int\limits_{E_j} \ln|a(x)|\mathrm{d}\mu_x\Big)\\ &= \frac{1}{\sqrt{|\det M|}} \exp\Big(\frac{1}{q}\sum_{j=0}^{q-1}\int\limits_{\mathbb{T}} \ln|\widetilde{a}((M^T)^j x)|\mathrm{d}x\Big),\end{aligned}$$

so the proof is complete. ∎

Note that there are many other lower estimates for $\rho(R_a^M)$ that can be expressed in terms of one-dimensional integrals. For example, let $k_1, k_2, \ldots, k_s$ be positive integers and let D_K^μ denote the diagonal matrix

$$D_K^\mu = \operatorname{diag}(\mu^{k_1}, \mu^{k_2}, \ldots, \mu^{k_s}),$$

when the following theorem provides another option for obtaining a lower estimate of the spectral radius $\rho(R_a^M)$ via one-dimensional integrals.

THEOREM 3.2. *Let $a \in \mathbf{W}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. Then the spectral radius of the operator R_a^M satisfies the inequality*

$$\rho(R_a^M) \geqslant \frac{1}{\sqrt{|\det M|}} \exp\Big(\frac{1}{q}\sum_{j=0}^{q-1}\int\limits_{\mathbb{T}} \ln|\widetilde{a}((M^T)^j D_K^\mu x)|\mathrm{d}x\Big). \tag{3.3}$$

The proof is similar to that of Theorem 3.1 and is omitted here.

In addition to the estimates represented by one-dimensional integrals, there exists a variety of estimates of $\rho(R_a^M)$ given by integrals of higher dimensions. These lower bounds can be obtained in the same way as estimates (3.2), (3.3). For example, let us formulate a result similar to Theorem 3.1. Thus consider a partition Π of the set of the positive integers $I := \{1, 2, \ldots, s\}$ into l subsets $I_1, I_2, \ldots, I_l$ such that $I = \bigcup\limits_{k=1}^{l} I_k$, and $I_{k_1} \cap I_{k_2} = \varnothing$ if $k_1 \neq k_2$. For each such partition Π let E_Π^k denote the subset of $\mathbb{T}^s$ defined by

$$E_\Pi^k := \{(x_1, x_2, \ldots, x_s) \in \mathbb{T}^s : x_{i_1} = x_{i_2} \text{ if } i_1, i_2 \in I_k, k = 1, 2, \ldots, l\}.$$

In particular, if Π consists of only one set $I_1 = I$, then $E^1_\Pi = E$ and we are in the situation considered in Theorem 3.1. Assume now that the set I_k has d_k elements where $1 \leqslant d_k \leqslant s$. Thus on the set E^k_Π there are only $s - d_k + 1$ independent variables within the set $\{x_1, x_2, \ldots, x_s\}$, so there is a natural mapping $\psi_{E^k_\Pi} : \mathbb{T}^{s-d_k+1} \to E^k_\Pi$ that identifies the torus $\mathbb{T}^{s-d_k+1}$ with the set E^k_Π.

THEOREM 3.3. *Let $a \in \mathbf{W}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. Then the spectral radius of the operator R^M_a satisfies the inequality*

$$\rho(R^M_a) \geqslant \frac{1}{\sqrt{|\det M|}} \exp\Big(\frac{1}{q}\sum_{j=0}^{q-1} \int\limits_{\mathbb{T}^{s-d_k+1}} \ln |a((M^T)^j \psi_{E^k_\Pi}(x))| \mathrm{d}x\Big), \tag{3.4}$$

where $\mathrm{d}x$ is the normalized Lebesgue measure on the torus $\mathbb{T}^{s-d_k+1}$.

As before, for the proof one has to use an appropriate M^T-invariant normalized measure. This can be done analogous to the proof of Theorem 3.1.

REMARK 3.4. Estimates (3.2)–(3.4) can also be proved by using the properties of μ-cyclic tuples $\mathcal{C}^p_\mu$ only. In the case $s = 1$, such a method was employed in [11].

4. EXACT VALUES OF THE SPECTRAL RADII FOR REFINEMENT OPERATORS

Let us now consider some classes of symbols where one can provide the exact value of the spectral radius for the operator R^M_a. Let A denote the maximum of the function $|a|$, and let $\mathcal{M}(a)$ be the set of points of $\mathbb{T}^s$ where $|a|$ attains its maximum.

THEOREM 4.1. *Let $a \in \mathbf{W}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. If there exists an element $[\widetilde{x}^{(p)}] = \{\widetilde{x}^{(p)}_0, \ldots, \widetilde{x}^{(p)}_{p-1}\} \in \mathcal{C}^{(p),s}_\mu$ such that for all $k = 0, \ldots, q-1$ and for all $j = 0, \ldots, p-1$*

$$(M^T)^k \widetilde{x}^{(p)}_j \in \mathcal{M}(a), \tag{4.1}$$

then

$$\rho(R^M_a) = \frac{A}{\sqrt{|\det M|}}, \tag{4.2}$$

$$\frac{A}{\sqrt{|\det M|}} \leqslant \rho(S^M_a) \leqslant A. \tag{4.3}$$

Proof. Equality (4.2) follows from (1.9) and (2.5). For inequality (4.3) one has to invoke relations (1.17), (1.18) and the proof of Theorem 2.4 where the limit $\limsup\limits_{n\to\infty} \Big\| \prod\limits_{k=0}^{n-1} a((M^T)^k x_n) \Big\|^{1/n}_\infty$ has been estimated.

Note that the estimates in (4.3) are sharp. The upper bound is achieved for any constant symbol a whereas the lower bound appears in examples considered in [13], [15]. ∎

COROLLARY 4.2. *If the symbol a of the refinement operator R_a^M has non-negative Fourier coefficients, then*

$$\rho(R_a^M) = \frac{|a(0)|}{\sqrt{|\det M|}}.$$

Proof. If the Fourier coefficients of a are non-negative, then

$$|a(x)| \leqslant \sum_{k\in\mathbb{Z}^s} a_k = |a(0)|$$

and the results follows from the fact that 0 is a μ-cyclic 0-tuple. ∎

To study a more general situation, let us fix an $\varepsilon > 0$ and consider the set

$$E_a^\varepsilon := \{x \in \mathbb{T}^s : |a(x)| \geqslant A - \varepsilon\},$$

where A means the modulus maximum of the function a on $\mathbb{T}^s$ as before. Having defined the set E_a^ε, for each positive integer p let us introduce a set $\mathcal{N}_p^M(E_a^\varepsilon)$ by

$$\mathcal{N}_p^M(E_a^\varepsilon) := \Big\{j \in \mathbb{Z}^s : \frac{j}{\mu^p - 1} \in E_a^\varepsilon\Big\},$$

and recall that any μ-cyclic p-tuple $[x^{(p)}] = \{x_0^{(p)}, x_1^{(p)}, \ldots, x_{p-1}^{(p)}\} \in \mathcal{C}_\mu^{(p)}$ can be represented as

$$[x^{(p)}] = \Big\{\frac{r_0}{\mu^p - 1}, \frac{r_1}{\mu^p - 1}, \ldots, \frac{r_{p-1}}{\mu^p - 1}\Big\} \tag{4.4}$$

where $r_0, r_1, \ldots, r_{p-1}$ are non-negative integers which do not exceed $\mu^p - 2$, [11]. Moreover, the integer r_0 defines the numbers $r_1, r_2, \ldots, r_{p-1}$ and their order. Thus, if a column of s non-negative integers $\mathbf{r} = (r_0^{(1)}, r_0^{(2)}, \ldots, r_0^{(s)})^T, 0 \leqslant r_0^{(i)} \leqslant \mu^p - 2$ is given, then there are p columns $\mathbf{r}_l, l = 0, 1, \ldots, p-1$ consisting of non-negative integers which appear in the representations (4.4) of the corresponding μ-cyclic p-tuples, defined by $\mathbf{r}_l := (r_l^{(1)}, r_l^{(2)}, \ldots, r_l^{(s)})^T, l = 0, 1, \ldots, p-1$ with $\mathbf{r}_0 := \mathbf{r}$. Recall that $0 \leqslant r_l^{(i)} \leqslant \mu^p - 2$ for all $l = 0, 1, \ldots, p-1$ and for all $i = 1, 2, \ldots, s$. For convenience, the set of such families of columns $\{\mathbf{r}_0, \ldots, \mathbf{r}_{p-1}\}$ is denoted by $\mathcal{R}$.

Consider now a system of congruences

$$\begin{aligned} &(M^T)^k \mathbf{r}_l u = \mathbf{n}_{kl} \mod \mu^p - 1 \\ &k = 0, 1, \ldots, q-1;\ l = 0, 1, \ldots, p-1 \end{aligned} \tag{4.5}$$

where $u \in \mathbb{N}$ is an unknown positive integer and where the vectors $\mathbf{n}_{kl}^T = (n_{kl}^{(1)}, n_{kl}^{(2)}, \ldots, n_{kl}^{(s)})$ are supposed to be in $(\mathcal{N}_p^M(E_a^\varepsilon))^s$ for all $k = 0, 1, \ldots, q-1$ and $l = 0, 1, \ldots, p-1$.

THEOREM 4.3. *Let $a \in \mathbf{W}(\mathbb{T}^s)$, $M \in \mathfrak{M}^s$ and μ, q, ε be as above. If there exists $p \in \mathbb{N}$ such that system* (4.5) *is solvable for at least one family of columns $\{\boldsymbol{r}_0, \dots, \boldsymbol{r}_{p-1}\} \in \mathcal{R}$ and for at least one choice of vectors $\boldsymbol{n}_{kl}^T \in (\mathcal{N}_p^M(E_a^\varepsilon))^s$, then the spectral radii of the operators S_a^M and R_a^M satisfy the inequalities*

$$\rho(S_a^M) \geqslant \frac{A}{\sqrt{|\det M|}} - \varepsilon, \quad \rho(R_a^M) \geqslant \frac{A}{\sqrt{|\det M|}} - \varepsilon. \tag{4.6}$$

Proof. Let $\mathbf{n}_{kl}^T, k = 0, 1, \dots, q-1; l = 0, 1, \dots, p-1$ and $\{\mathbf{r}_0, \dots, \mathbf{r}_{p-1}\}$ be, respectively, vectors from $\mathcal{N}_p^M(E_a^\varepsilon)^s$ and the family of columns of non-negative integers from $\mathcal{R}$ such that system of congruences (4.5) is solvable. For any solution u^* of such system define a function a_{u^*} by

$$a_{u^*} := a(u^* \cdot).$$

Then a_{u^*} is a 1^s-periodic function, and Lemma 2.1 implies $\rho(R_a^M) = \rho(R_{a_{u^*}}^M)$. However, by Corollary 2.8 the spectral radius of the operator $R_{a_{u^*}}^M$ can be estimated as follows

$$\begin{aligned}
\rho(R_{a_{u^*}}^M) &\geqslant \frac{1}{\sqrt{|\det M|}} \Big(\prod_{k=0}^{q-1} \prod_{l=0}^{p-1} \Big| a\Big(\frac{u^*(M^T)^k \mathbf{r}_l}{\mu^p - 1} \Big) \Big| \Big)^{1/qp} \\
&\geqslant \frac{1}{\sqrt{|\det M|}} \Big(\prod_{k=0}^{q-1} \prod_{l=0}^{p-1} \Big| a\Big(\frac{\mathbf{n}_{kl}}{\mu^p - 1} \Big) \Big| \Big)^{1/qp} \geqslant \frac{A}{\sqrt{|\det M|}} - \varepsilon.
\end{aligned}$$

This completes the proof for the operator R_a^M. The estimates for the spectral radius of the operator S_a^M can be obtained similarly. ∎

Combining the last result with Corollary 1.3 leads to another sufficient condition for the equality (0.4).

THEOREM 4.4. *Let $a \in \mathbf{W}(\mathbb{T}^s)$ and $M \in \mathfrak{M}^s$. If for any $\varepsilon > 0$ system* (4.5) *is solvable in the sense of Theorem* 4.3*, then $\rho(R_a^M) = A/\sqrt{|\det M|}$.*

Thus, knowledge of the maximum of the symbol often allows us to obtain the exact value of the spectral radius for the refinement operator. A natural question to ask is whether the spectral radius of the refinement operator can always be given by formula (0.4). The answer to this question is no. A counterexample can be constructed using results of [35].

As far as the subdivision operators S_a^M are concerned, the location of exact values of their spectral radii is more complicated, cf. (4.3) and the remark in the proof of Theorem 4.1. Thus to improve estimate (4.3) one has to study expression (1.14) in more detail.

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