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ON LAX-PHILLIPS SEMIGROUPS

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Dedicated to Friedrich Hirzebruch on his 80th birthday

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ABSTRACT. Lax-Phillips evolutions are described by two-space scattering systems. The canonical identification operator is characterized for Lax-Phillips evolutions, whose outgoing and incoming projections commute. In this case a (generalized) Lax-Phillips semigroup can be introduced and its spectral theory is considered. In the special case, originally considered by Lax and Phillips (where the outgoing and incoming subspaces are mutually orthogonal), this semigroup coincides with that introduced by Lax and Phillips. The basic connection of the Lax-Phillips semigroup to the so-called characteristic semigroup of the reference evolution is emphasized.

KEYWORDS: *Lax-Phillips evolutions, semigroups, scattering theory.*

MSC (2000): 47A40, 47D06.

1. INTRODUCTION

Recently several papers were published where the mathematical framework of the Lax-Phillips (LP) scattering theory [10] is used for the description of resonances in quantum theory, see Strauss [12], [13] and papers quoted there, e.g. Flesia and Piron [6], Horwitz and Piron [8], Eisenberg and Horwitz [5], Strauss, Horwitz and Eisenberg [14]. The reason is the existence of a distinguished semigroup in the LP-scattering theory (the LP-semigroup) and the relation between their eigenvalues and poles of the scattering matrix.

The basic concept in [10] is the (general) LP-*evolution*. Necessarily, its generator has pure absolutely continuous spectrum which coincides with the real line and has constant multiplicity. The LP-semigroup is then established only under the additional assumption that the outgoing and incoming subspaces are mutually orthogonal.

Hamiltonians in quantum scattering theory have absolutely continuous spectrum with constant multiplicity. However, usually they are bounded below.

Therefore there are several serious obstacles for the idea to use LP-theory to solve quantum scattering and resonance problems.

The difficulties due to the property of Hamiltonians to be semibounded can be overcome by using ideas of Halmos [7] (refined by Kato [9]). This approach is pointed out in [3]. A further approach is given by Strauss [13] which is based on the theory of Sz.-Nagy–Foias [15] of contraction operators on Hilbert space.

In this paper the obstacle resulting by the special assumption in [10] that outgoing and incoming subspaces are mutually orthogonal is removed. This requires to investigate carefully the LP-theory from the pure mathematical point of view. This is done in the present paper. It is shown that the LP-semigroup can be established also in the case that the projections onto the outgoing and incoming subspaces commute (which includes the case of mutual orthogonality of these subspaces). However, in this case the holomorphic continuability of the scattering matrix into the whole upper half plane does not follow in general. Despite of this lack the eigenvalue spectrum of the (generalized) LP-semigroup is investigated (see Subsection 3.2). The result shows that for this part of the spectrum there is a similar connection to properties of the scattering matrix as in the original LP-case (in our case for vector functions, see Proposition 3.6). Further results which take into account further assumptions, necessary for the application on the resonance problem in quantum scattering theory, are presented in [3]. (For example, restrictions of the characteristic semigroup (considered in Subsection 2.3) such that the semigroup property is violated, independent analyticity properties of the scattering matrix, such that quantum scattering systems can be handled which have poles in the upper half plane).

A result concerning the resolvent set of the infinitesimal generator in the general case, similar to that of Lax and Phillips (see, for example, Theorem 23 in [4], p. 263), is missing.

2. LP-EVOLUTIONS

A unitary strongly continuous evolution group $U(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ is called an LP-evolution, if there are subspaces $\mathcal{D}_+$, $\mathcal{D}_-$ in $\mathcal{H}$, called outgoing and incoming, such that

$$U(t)\mathcal{D}_+ \subseteq \mathcal{D}_+, \quad t \geqslant 0, \quad U(t)\mathcal{D}_- \subseteq \mathcal{D}_-, \quad t \leqslant 0,$$
$$\bigcap_{t\in\mathbb{R}} U(t)\mathcal{D}_\pm = \{0\}, \quad \mathrm{clo}\Big\{ \bigcup_{t\in\mathbb{R}} U(t)\mathcal{D}_\pm \Big\} = \mathcal{H}.$$

These evolutions were introduced by Lax and Phillips in [10], where the basic theorems are presented and the theory of these evolutions is developed, especially for the case that outgoing and incoming subspaces are mutually orthogonal.

2.1. THE REFERENCE EVOLUTION. Let $\mathcal{H}_0 := L^2(\mathbb{R}, \mathrm{d}x, \mathcal{K})$, where $\mathcal{K}$ is a separable Hilbert space and

$$T(t)f(x) := f(x-t), \quad f \in \mathcal{H}_0,$$

the regular translation group representation on $\mathcal{H}_0$ (where multiplicity $\dim \mathcal{K}$ is taken into account).

For convenience of the reader we recall the properties of this LP-evolution (see e.g. [4], p. 250):

$$(P_\pm f)(x) := \chi_{\mathbb{R}_\pm}(x) f(x), \quad f \in \mathcal{H}_0, \tag{2.1}$$

where $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_- := (-\infty, 0]$, are the projections onto the outgoing/ incoming subspaces:

$$P_\pm(t) := T(-t) P_\pm T(t), \quad t \in \mathbb{R}.$$

The function $t \to P_+(t)$ is monotonically increasing,

$$P_+(t_1) \leqslant P_+(t_2), \quad t_1 \leqslant t_2, \quad \text{and} \quad \text{s-}\lim_{t \to +\infty} P_+(t) = \mathbb{1}_{\mathcal{H}_0}, \quad \text{s-}\lim_{t \to -\infty} P_+(t) = 0.$$

Similarly, $P_-(\cdot)$ is monotonically decreasing and

$$\text{s-}\lim_{t \to +\infty} P_-(t) = 0, \quad \text{s-}\lim_{t \to -\infty} P_-(t) = \mathbb{1}_{\mathcal{H}_0}. \tag{2.2}$$

Furthermore, $T(t)P_+\mathcal{H}_0 \subseteq P_+\mathcal{H}_0$ for $t \geqslant 0$ or $T(t)P_+ = P_+T(t)P_+, t \geqslant 0$ and correspondingly, $T(t)P_- = P_-T(t)P_-, t \leqslant 0$. The unitary evolution group $T(\cdot)$ on $\mathcal{H}_0$ is called the *reference* LP-evolution, $P_+\mathcal{H}_0$ is the *outgoing* and $P_-\mathcal{H}_0$ the *incoming* subspace. In this case $P_+\mathcal{H}_0$ and $P_-\mathcal{H}_0$ are mutually orthogonal and $P_+\mathcal{H}_0 \oplus P_-\mathcal{H}_0 = \mathcal{H}_0$.

By Fourier transformation the representation $T(\mathbb{R})$ is transformed into

$$\widehat{T}(t) := FT(t)F^{-1}, \quad \text{where} \quad (\widehat{T}(t)\widehat{f})(p) = \mathrm{e}^{-\mathrm{i}tp}\widehat{f}(p), \quad \widehat{f} \in \mathcal{H}_0,$$

i.e. the multiplication operator H_0 on $\mathcal{H}_0$ given by

$$(H_0\widehat{f})(p) := p\widehat{f}(p), \quad \widehat{f} \in \operatorname{dom} H_0,$$

is the generator of $\widehat{T}(\mathbb{R})$:

$$\widehat{T}(t) = \mathrm{e}^{-\mathrm{i}tH_0}, \quad t \in \mathbb{R}.$$

$\widehat{T}(\mathbb{R})$ is called the *spectral representation* of the reference evolution. One has $\operatorname{spec} H_0 = \mathbb{R}$ and it is pure absolutely continuous. Note that we use the Fourier transformation in the form

$$(Ff)(p) := (2\pi)^{-1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}px} f(x) \mathrm{d}x.$$

The projection P_+, defined by (2.1), is an element from the spectral measure of H_0, therefore $P_+\widehat{T}(t) = \widehat{T}(t)P_+$, $t \in \mathbb{R}$, and $\widehat{T}(t) \restriction P_+\mathcal{H}_0$ is a positive representation, spec $(H_0 \restriction P_+\mathcal{H}_0) = [0,\infty)$, and it is pure absolutely continuous. The projections

$$Q_\mp := FP_\pm F^{-1}$$

are the projections onto the Hardy spaces $\mathcal{H}^2_\mp(\mathbb{R},\mathcal{K}) =: \mathcal{H}^2_\mp \subset \mathcal{H}_0$ (see e.g. [2]). That is, these spaces are outgoing/incoming subspaces for $\widehat{T}(\mathbb{R})$ and $Q_\pm$ are the corresponding projections.

The projection Q_+ is given by

$$\mathcal{H}_0 \ni g \to (Q_+g)(z) = (2\mathrm{i}\pi)^{-1}\int_{-\infty}^{\infty}\frac{g(\lambda)}{\lambda - z}\mathrm{d}\lambda. \tag{2.3}$$

2.2. THE MAIN THEOREM FOR LP-EVOLUTIONS. Let $U(\mathbb{R})$ be an LP-evolution on $\mathcal{H}$ with outgoing/incoming subspaces $\mathcal{D}_\pm$. Then there are isometric operators $V_\pm$ from $\mathcal{H}$ onto $\mathcal{H}_0$ with an appropriate multiplicity space $\mathcal{K}$ such that

$$V_\pm U(t)V_\pm^* = \mathrm{e}^{-\mathrm{i}tH_0}, \quad t \in \mathbb{R}, \quad \text{and} \quad Q_\mp\mathcal{H}_0 = V_\pm\mathcal{D}_\pm.$$

The isometries $V_\pm$ are unique up to isomorphisms of $\mathcal{K}$. This means, if $V'_\pm$ is a second pair of isometries then there are unitaries $K_\pm$ on $\mathcal{K}$ such that $V'_+ = K_+V_+$, $V'_- = K_-V_-$ where $(K_\pm f)(\lambda) := K_\pm f(\lambda)$ (see Sinai [11] and Lax and Phillips [10], see also [4]). $V_\pm$ maps onto the so-called *outgoing/incoming spectral representation* of $U(\mathbb{R})$. In general $V_+ \neq V_-$.

An important implication of the main theorem is that $U(t) = \mathrm{e}^{-\mathrm{i}tH}$, where spec $H = \mathbb{R}$ and H has constant multiplicity.

We introduce the orthoprojections $D_\pm$ onto the subspaces $\mathcal{D}_\pm$. Then

$$D_+ = V_+^*Q_-V_+, \quad D_- = V_-^*Q_+V_-$$

and $\mathcal{D}_+ = V_+^*\mathcal{H}^2_-$, $\mathcal{D}_- = V_-^*\mathcal{H}^2_+$.

The LP-scattering operator is defined by $S_{\mathrm{LP}} := V_+V_-^{-1}$. S_{LP} commutes with the reference evolution, i.e.

$$S_{\mathrm{LP}}\mathrm{e}^{-\mathrm{i}tH_0} = \mathrm{e}^{-\mathrm{i}tH_0}S_{\mathrm{LP}},$$

therefore S_{LP} acts as

$$(S_{\mathrm{LP}}f)(\lambda) = S_{\mathrm{LP}}(\lambda)f(\lambda), \quad f \in \mathcal{H}_0.$$

The operators $S_{\mathrm{LP}}(\lambda)$ are unitaries on $\mathcal{K}$ a.e. on $\mathbb{R}$. The operator function $S_{\mathrm{LP}}(\cdot)$ is called the LP-scattering matrix.

2.3. SEMIGROUPS CONNECTED WITH THE REFERENCE EVOLUTION. First the semigroup

$$T_+(t) := Q_+ e^{-itH_0} Q_+ = Q_+ e^{-itH_0}, \quad t \geqslant 0, \tag{2.4}$$

is considered, respectively its restriction $T_+(t) \upharpoonright \mathcal{H}^2_+$, which we call the *characteristic semigroup*. It plays an important role as an "intermediate step" to obtain the Lax-Phillips semigroup. It was already introduced by Y. Strauss [3]. Further we need its adjoint

$$T_+(t)^* = Q_+ e^{itH_0} Q_+ = e^{itH_0} Q_+, \quad t \geqslant 0, \tag{2.5}$$

respectively $T_+(t)^* \upharpoonright \mathcal{H}^2_+$. The last equations in (2.4) and (2.5) are true because Q_+ is the incoming projection for $\widehat{T}(\cdot)$, i.e. it is the outgoing projection for $\widehat{T}(\cdot)^*$.

First we recall the properties of $T_+(\cdot)^* \upharpoonright \mathcal{H}^2_+$. It is a strongly continuous and isometric semigroup, i.e.

$$\|T_+(t)^* f\| = \|f\|, \quad f \in \mathcal{H}^2_+, \quad \text{we have} \quad T_+(t)^* \upharpoonright \mathcal{H}^2_+ = e^{itC_-}, \quad t \geqslant 0,$$

and the generator C_-, a closed operator on $\mathcal{H}^2_+$, with domain $\operatorname{dom} C_-$ dense in $\mathcal{H}^2_+$, satisfies

$$\mathbb{C}_- \subset \operatorname{res} C_-, \tag{2.6}$$

where $\mathbb{C}_- := \{\zeta \in \mathbb{C} : \operatorname{Im} \zeta < 0\}$. For a closed operator A with dense domain $\operatorname{dom} A$ on a Hilbert space we denote by $\operatorname{res} A$ the (open) set of all $z \in \mathbb{C}$ (complex plane) such that $(z - A)^{-1}$ exists and is bounded. It is called the resolvent set of A.

PROPOSITION 2.1. *The generator C_- satisfies the following properties:*

(i) $\operatorname{dom} C_- = \{f \in \operatorname{dom} H_0 \cap \mathcal{H}^2_+ : H_0 f \in \mathcal{H}^2_+\}$ *and*

$$(C_- f)(z) = z f(z), \quad \operatorname{Im} z > 0, \quad f \in \operatorname{dom} C_-.$$

(ii) *The deficiency space*

$$\mathcal{N}_\zeta := \mathcal{H}^2_+ \ominus (\zeta - C_-)\operatorname{dom} C_-, \quad \operatorname{Im} \zeta > 0$$

is given by

$$\mathcal{N}_\zeta = \{f \in \mathcal{H}^2_+ : f(z) = (z - \overline{\zeta})^{-1} k, \, k \in \mathcal{K}\}. \tag{2.7}$$

Moreover, $(\zeta - C_-)\operatorname{dom} C_-$ *is a subspace and it coincides with*

$$\mathcal{M}_\zeta := \{f \in \mathcal{H}^2_+ : f(\zeta) = 0\}.$$

Proof. (i) is obvious because of (2.5).

(ii) First we prove that $\mathcal{M}_\zeta$ is a subspace. Let $f_n \in \mathcal{H}^2_+$, $f_n(\zeta) = 0$ and $\|f_n - f\| \to 0$ for $n \to \infty$, where $f \in \mathcal{H}^2_+$. We have to show that $f(\zeta) = 0$. We put

$$h_\zeta(x) := \frac{1}{x - \zeta}.$$

Then $h_\zeta \in L^2(\mathbb{R}, \mathrm{d}x)$. According to (2.3) we have $f(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} h_\zeta(x) f(x) \mathrm{d}x$ and $f_n(\zeta) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} h_\zeta(x) f_n(x) \mathrm{d}x$. Then

$$
\begin{aligned}
\|f(\zeta) - f_n(\zeta)\|_{\mathcal{K}} &\leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |h_\zeta(x)| \cdot \|f(x) - f_n(x)\|_{\mathcal{K}} \mathrm{d}x \\
&\leqslant \frac{1}{2\pi} \Big(\int_{-\infty}^{\infty} |h_\zeta(x)|^2 \mathrm{d}x \Big)^{1/2} \cdot \Big(\int_{-\infty}^{\infty} \|f(x) - f_n(x)\|_{\mathcal{K}}^2 \mathrm{d}x \Big)^{1/2}.
\end{aligned}
$$

This implies $\|f(\zeta) - f_n(\zeta)\|_{\mathcal{K}} \to 0$ hence $f(\zeta) = 0$ follows. Now we prove $(\zeta - C_-)\mathrm{dom}\, C_- = \mathcal{M}_\zeta$. The inclusion $\subseteq$ is obvious because for $f \in \mathrm{dom}\, C_-$ the function $g(z) := (\zeta - z) f(z)$ vanishes at the point ζ i.e. $g(\zeta) = 0$. To prove the other inclusion let $f \in \mathcal{M}_\zeta$, i.e. $f(\zeta) = 0$. Then

$$
f(z) = (z - \zeta) g(z), \tag{2.8}
$$

where the function $g(z) := \frac{f(z)}{z - \zeta}$ is holomorphic on the upper half plane. Moreover, one calculates easily that $g \in \mathcal{H}^2_+$. Now from (2.8) one gets

$$
z g(z) = \zeta g(z) + f(z)
$$

and the right hand side is an element of $\mathcal{H}^2_+$. Therefore $g \in \mathrm{dom}\, C_-$ follows, i.e. $f \in (\zeta - C_-)\mathrm{dom}\, C_-$.

Finally we prove (2.7). Let

$$
f_{\overline{\zeta},k}(z) := \frac{k}{z - \overline{\zeta}}, \quad k \in \mathcal{K} \quad \text{and} \quad g \in \mathcal{H}^2_+.
$$

Then

$$
(f_{\overline{\zeta},k}, g) = \int_{-\infty}^{\infty} \Big(\frac{k}{x - \overline{\zeta}}, g(x) \Big)_{\mathcal{K}} \mathrm{d}x = \int_{-\infty}^{\infty} \frac{1}{x - \zeta} (k, g(x))_{\mathcal{K}} \mathrm{d}x = 2\mathrm{i}\pi (k, g(\zeta))_{\mathcal{K}}. \tag{2.9}
$$

Now, if $g \in \mathcal{M}_\zeta$ then $f_{\overline{\zeta},k} \perp g$ follows or $f_{\overline{\zeta},k} \in \mathcal{M}_{\zeta}^{\perp}$. On the other hand, if $(f_{\overline{\zeta},k}, g) = 0$ for all $k \in \mathcal{K}$ then $(k, g(\zeta))_{\mathcal{K}} = 0$ follows, i.e. $g(\zeta) = 0$ or $g \in \mathcal{M}_\zeta$. ∎

Proposition 2.1 implies that the deficiency number $\dim \mathcal{N}_\zeta$ of C_- with respect to the upper half plane coincides with $\dim \mathcal{K}$. (2.6) implies that the deficiency number of C_- for the lower half plane is 0.

C_- is even maximal symmetric, there is no symmetric extension of C_-.

Now let C_-^* be the adjoint of C_-. Then C_-^* is an extension of C_-, $C_- \subset C_-^*$.

PROPOSITION 2.2. *The adjoint C_-^* of C_- satisfies the following properties:*

(i) *One has*

$$
\mathrm{dom}\, C_-^* = \mathrm{dom}\, C_- \oplus \mathcal{N}_{\overline{\zeta}},
$$

where $\operatorname{Im}\zeta < 0$, ζ *fixed but arbitrary and*

$$C_-^* f = \zeta f, \quad f \in \mathcal{N}_{\overline{\zeta}},$$

i.e. each point $\zeta \in \mathbb{C}_-$ *is an eigenvalue of* C_-^* *and the corresponding eigenspace is given by* $\mathcal{N}_{\overline{\zeta}}$ *i.e. all eigenvectors are given by*

$$\mathbb{C}_+ \ni z \to f_{\zeta,k}(z) := \frac{k}{z-\zeta}, \quad k \in \mathcal{K}, \quad \operatorname{Im}\zeta < 0.$$

(ii) $\frac{1}{2i\pi} f_{\zeta,k}$ *coincides with the Dirac linear forms (evaluation forms) for the scalar holomorphic function* $\mathbb{C}_+ \ni z \to (k, f(z))_{\mathcal{K}}$ *on the upper half plane.*

Proof. (i) is obvious because of the formulas of von Neumann (see for example [1], p. 292).

(ii) follows from the "boundary value formula" (2.9) for Hardy class functions. ∎

Concerning the semigroup (2.4) we obtain

PROPOSITION 2.3. *The semigroup* $t \to T_+(t) \restriction \mathcal{H}_+^2$ *has the following properties:*
(i) *It is strongly continuous and contractive, i.e.*

$$T_+(t) \restriction \mathcal{H}_+^2 = \mathrm{e}^{-\mathrm{i}tC_+}, \quad t \geqslant 0,$$

where the generator C_+ *is closed on* $\mathcal{H}_+^2$, $\operatorname{dom} C_+$ *is dense and* $\mathbb{C}_+ \subset \operatorname{res} C_+$.
(ii) $C_+ = C_-^*$.
(iii) $(T_+(t)f)(z) = \frac{1}{2i\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}t\lambda}}{\lambda - z} f(\lambda)\mathrm{d}\lambda$, $f \in \mathcal{H}_+^2$.
(iv) *One has* $\text{s-}\lim\limits_{t\to\infty} \mathrm{e}^{-\mathrm{i}tC_+} = 0$.

Proof. (i) is obvious.

(ii) One has $\int\limits_0^\infty \mathrm{e}^{\mathrm{i}tz}\mathrm{e}^{-\mathrm{i}tC_+}\mathrm{d}t = \mathrm{i}(z - C_+)^{-1}, z \in \mathbb{C}_+$. Then

$$\int\limits_0^\infty \mathrm{e}^{-\mathrm{i}t\overline{z}}(\mathrm{e}^{-\mathrm{i}tC_+})^*\mathrm{d}t = -\mathrm{i}((z-C_+)^{-1})^* = -\mathrm{i}((z-C_+)^*)^{-1} = -\mathrm{i}(\overline{z} - C_+^*)^{-1}.$$

On the other hand the left hand side equals $\int\limits_0^\infty \mathrm{e}^{-\mathrm{i}t\overline{z}}\mathrm{e}^{\mathrm{i}tC_-}\mathrm{d}t = -\mathrm{i}(\overline{z} - C_-)^{-1}$, hence $(\overline{z} - C_+^*)^{-1} = (\overline{z} - C_-)^{-1}$ follows for all $\overline{z} \in \mathbb{C}_-$. This implies the assertion.

(iii) follows from (2.3).

(iv) One has $T_+(t)^*T_+(t) = \mathrm{e}^{\mathrm{i}tH_0}Q_+\mathrm{e}^{-\mathrm{i}tH_0} = F(T(-t)P_-T(t))F^{-1} = FP_-(t)F^{-1}$, which, according to (2.2), converges strongly to zero for $t \to \infty$, i.e. one has

$$\text{s-}\lim_{t\to\infty} T_+(t)^*T_+(t) \restriction \mathcal{H}_+^2 = 0.$$

However $T_+(t)^* \restriction \mathcal{H}_+^2$ is isometric, therefore $\text{s-}\lim\limits_{t\to\infty} \mathrm{e}^{-\mathrm{i}tC_+} = 0$ follows. ∎

PROPOSITION 2.4. *Let* $T_+(t) \restriction \mathcal{H}^2_+ = Q_+ \mathrm{e}^{-\mathrm{i}tH_0} \restriction \mathcal{H}^2_+$, $t \geqslant 0$, *as before. Then:*

(i) $\operatorname{res} C_+ = \mathbb{C}_+$.

(ii) *The eigenvalue spectrum of* C_+ *coincides with* $\mathbb{C}_-$, *i.e. a real point cannot be an eigenvalue.*

(iii) *The eigenspace of the eigenvalue* $\zeta \in \mathbb{C}_-$ *is given by the following subspace*

$$\mathcal{N}_{\overline{\zeta}} := \left\{ f \in \mathcal{H}^2_+ : f(z) := \frac{k}{z-\zeta}, k \in \mathcal{K} \right\},$$

and one has

$$T_+(t) f = \mathrm{e}^{-\mathrm{i}t\zeta} f, \quad f \in \mathcal{N}_{\overline{\zeta}}. \tag{2.10}$$

Proof. It is obvious because of Proposition 3. The equations

$$\begin{aligned}(T_+(t) f_{\zeta,k}, g) &= (f_{\zeta,k}, T_+(t)^* g) = 2\mathrm{i}\pi (k, \mathrm{e}^{\mathrm{i}t\overline{\zeta}} g(\overline{\zeta}))_{\mathcal{K}} = 2\mathrm{i}\pi e^{\mathrm{i}t\overline{\zeta}} (k, g(\overline{\zeta}))_{\mathcal{K}} \\ &= \mathrm{e}^{\mathrm{i}t\overline{\zeta}} (f_{\zeta,k}, g) = (\mathrm{e}^{-\mathrm{i}t\zeta} f_{\zeta,k}, g)\end{aligned}$$

for $g \in \mathcal{H}^2_+$ and $f_{\zeta,k}(z) = \frac{k}{z-\zeta}$ prove relation (2.10) directly. ∎

The von Neumann characterization of $\operatorname{dom} C_+$ can be rewritten into the following modified one.

PROPOSITION 2.5. $f \in \operatorname{dom} C_+$ *if and only if the function*

$$g_f(z) := z f(z) - \frac{\mathrm{i}}{\sqrt{2\pi}} \lim_{x \to -0} (F^{-1} f)(x)$$

is from $\mathcal{H}^2_+$. *Then* $C_+ f = g_f$.

Proof. Without restriction of generality one can choose $\zeta := -\mathrm{i}$ as the reference point of the von Neumann characterization.

(i) Let $f(z) := a(z) + \frac{k}{\mathrm{i}+z}$, $k \in \mathcal{K}$. Then

$$g_f(z) = z a(z) + k\Big(1 - \frac{\mathrm{i}}{\mathrm{i}+z}\Big) - \frac{\mathrm{i}}{\sqrt{2\pi}} \lim_{x \to -0} (F^{-1} a(x) + k F^{-1}\{(\mathrm{i}+z)^{-1}\}(x)).$$

Using $\frac{\mathrm{i}}{\sqrt{2\pi}} \lim_{x\to -0} F^{-1}\{(\mathrm{i}+z)^{-1}\}(x) = 1$ and $\lim_{x \to -0} (F^{-1} a)(x) = 0$, one obtains $g_f \in \mathcal{H}^2_+$.

(ii) Conversely, let $f \in \mathcal{H}^2_+$ and $g_f \in \mathcal{H}^2_+$. The last term in the expression for g_f is a constant $k \in \mathcal{K}$, i.e. we have $z \to z f(z) - k$ is from $\mathcal{H}^2_+$. Now $z \to b(z) := \frac{k}{z+\mathrm{i}}$ is from $\mathcal{H}^2_+$, hence also $z \to z\big(f(z) - \frac{k}{z+\mathrm{i}}\big)$ is from $\mathcal{H}^2_+$, i.e. the functions $z \to a(z) := f(z) - \frac{k}{z+\mathrm{i}}$ and $z \to z a(z)$ are from $\mathcal{H}^2_+$, i.e. $f = a + b$, where $a \in \operatorname{dom} C_-$ and $b \in \mathcal{N}_i$. ∎

2.4. TWO-SPACE SCATTERING. There is a one-to-one correspondence between LP-evolutions and complete two-space scattering systems $\{H, H_0\}$, whose identification operators satisfy characteristic conditions. H_0 denotes, as before, the generator of the reference LP-evolution.

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{R} \ni t \to U(t) = \mathrm{e}^{-\mathrm{i}tH}$ a strongly continuous unitary group on $\mathcal{H}$. Further let $\mathcal{H}_0$ be as before and

$$J : \mathcal{H}_0 \to \mathcal{H}$$

a bounded linear operator. Then one can consider the two-space wave operators

$$W_\pm := \text{s-}\lim_{t\to\pm\infty} U(-t)J\mathrm{e}^{-\mathrm{i}tH_0},$$

(see e.g. [4], p. 168). Usually J is called the *identification operator*.

Since the aim is to reformulate LP-scattering in the framework of two-space scattering with respect to $\mathcal{H}_0$ and $\mathcal{H}$ we assume a priori that the wave operators $W_\pm : \mathcal{H}_0 \to \mathcal{H}$ are isometric, i.e. $W_\pm^* W_\pm = \mathbb{1}_{\mathcal{H}_0}$ and also *complete*, i.e. $W_\pm W_\pm^* = \mathbb{1}_{\mathcal{H}}$. The scattering operator S is given by $S := W_+^* W_-$.

Two (identification) operators $J, \widetilde{J}$ are called asypmptotically equivalent if $W_\pm(J) = W_\pm(\widetilde{J})$. This condition is equivalent to

$$\|(J - \widetilde{J})\mathrm{e}^{-\mathrm{i}tH_0} f\| \to 0, \quad t \to \pm\infty$$

for all $f \in \mathcal{H}_0$. Now it is always possible to replace J by an equivalent identification operator $\widetilde{J}$ such that

$$W_\pm Q_\mp = \widetilde{J} Q_\mp. \tag{2.11}$$

We put

$$\widetilde{J} := W_+ Q_- + W_- Q_+. \tag{2.12}$$

Then one calculates easily $W_\pm(\widetilde{J}) = W_\pm(J)$ and (2.11). That is, for our purpose without restriction of generality we may assume that the identification operator J is given by (2.12). It is called the *canonical* identification operator. This identification operator satisfies the equations

$$J^* J = \mathbb{1}_{\mathcal{H}_0} + Q_+ S^* Q_- + Q_- S Q_+, \tag{2.13}$$

$$J J^* = W_+ Q_- W_+^* + W_- Q_+ W_-^*. \tag{2.14}$$

Note that $W_+ Q_- W_+^*$, $W_- Q_+ W_-^*$ are projections which do not commute in general. These equations lead to

LEMMA 2.6. *$J^* J$ is asymptotically equivalent to $\mathbb{1}_{\mathcal{H}_0}$, i.e. J^* is an asymptotic left inverse for J, and JJ^* is asymptotically equivalent to $\mathbb{1}_{\mathcal{H}}$, i.e. J is an asymptotic left inverse for J^*.*

Proof. One has $\mathrm{e}^{\mathrm{i}tH_0}(J^* J - \mathbb{1}_{\mathcal{H}_0})\mathrm{e}^{-\mathrm{i}tH_0} = \mathrm{e}^{\mathrm{i}tH_0} Q_+ \mathrm{e}^{-\mathrm{i}tH_0} S^* \mathrm{e}^{\mathrm{i}tH_0} Q_- \mathrm{e}^{-\mathrm{i}tH_0}$ $+ \mathrm{e}^{\mathrm{i}tH_0} Q_- \mathrm{e}^{-\mathrm{i}tH_0} S \mathrm{e}^{\mathrm{i}tH_0} Q_+ \mathrm{e}^{-\mathrm{i}tH_0}$, hence

$$\text{s-}\lim_{t\to\pm\infty} (J^* J - \mathbb{1}_{\mathcal{H}_0})\mathrm{e}^{-\mathrm{i}tH_0} \to 0, \quad t \to \pm\infty$$

follows. Similarly for the second property. ∎

2.5. LP-EVOLUTIONS AS TWO-SPACE SCATTERING SYSTEMS. Let $U(\mathbb{R})$ be an LP-evolution on $\mathcal{H}$, $\mathcal{D}_\pm$ the outgoing/incoming subspaces, $V_\pm$ the isometric operators from $\mathcal{H}$ onto $\mathcal{H}_0$ (with an appropriate multiplicity space $\mathcal{K}$) such that $V_\pm U(t) V_\pm^* = \mathrm{e}^{-\mathrm{i}tH_0}$. Then one has

PROPOSITION 2.7. *Let $U(\mathbb{R}), \mathcal{H}, \mathcal{H}_0, \mathcal{D}_\pm, V_\pm$ as above. Put*

$$J := V_+^* Q_- + V_-^* Q_+.$$

Then

$$U(t) J Q_- = J \mathrm{e}^{-\mathrm{i}tH_0} Q_-, \quad t \geqslant 0, \quad \textit{and} \quad U(t) J Q_+ = J \mathrm{e}^{-\mathrm{i}tH_0} Q_+, \quad t \leqslant 0,$$

and the two-space wave operators exist and are given by

$$W_+ = V_+^*, \quad W_- = V_-^*,$$

i.e. they are isometric and complete. That is: with respect to J the given LP*-evolution $U(\mathbb{R})$ forms, together with the reference evolution, a complete two-space scattering system and its scattering operator S coincides with the* LP*-scattering operator S_{LP}.*

The proof is given by straightforward calculation (see e.g. [4], p. 255, where only the case $\mathcal{D}_+ \perp \mathcal{D}_-$ is considered). Conversely, one has

PROPOSITION 2.8. *Let $\{H, H_0; J\}$ be a complete two-space scattering system with (isometric) wave operators $W_\pm$, such that J can be given by*

$$J := W_+ Q_- + W_- Q_+. \tag{2.15}$$

Then $\{U(\mathbb{R}), \mathcal{D}_\pm\}$, where $U(t) := \mathrm{e}^{-\mathrm{i}tH}$, is an LP*-evolution where the outgoing/incoming subspaces are given by $\mathcal{D}_+ := W_+ \mathcal{H}_-^2$, $\mathcal{D}_- := W_- \mathcal{H}_+^2$, i.e. their projections by*

$$D_+ := W_+ Q_- W_+^* = J Q_- J^*, \quad D_- := W_- Q_+ W_-^* = J Q_+ J^*. \tag{2.16}$$

The corresponding transformations to the out/in spectral representations are given by $V_+ := W_+^*$, $V_- := W_-^*$. The LP*-scattering operator S_{LP} and S coincide.*

Proof. The equation (2.15) implies

$$\mathrm{e}^{-\mathrm{i}tH} J Q_- = J \mathrm{e}^{-\mathrm{i}tH_0} Q_-, \quad t \geqslant 0, \quad \text{and} \quad \mathrm{e}^{-\mathrm{i}tH} J Q_+ = J \mathrm{e}^{-\mathrm{i}tH_0} Q_+, \quad t \leqslant 0.$$

and the equations in (2.16). Further, the equation

$$U(-t) D_+ U(t) = U(-t) W_+ Q_- W_+^* U(t) = W_+ \mathrm{e}^{\mathrm{i}tH_0} Q_- \mathrm{e}^{-\mathrm{i}tH_0} W_+^*, \quad t \in \mathbb{R}$$

shows that D_+ is an outgoing projection with respect to $U(\cdot)$. Similarly for D_-. ∎

3. LAX-PHILLIPS EVOLUTIONS WITH COMMUTING OUTGOING/INCOMING PROJECTIONS

3.1. IDENTIFICATION OPERATORS. Let $\{H, H_0; J\}$ and the associated LP-evolution $\{U(\mathbb{R}), \mathcal{D}_\pm\}$ be as in Proposition 2.8, in particular J is given by formula (2.15). Then the question arises in which case D_+ and D_- commute, $D_+D_- = D_-D_+$. First we consider the special case that $D_+D_- = 0$, i.e. $\mathcal{D}_+$ and $\mathcal{D}_-$ are mutually orthogonal.

In this case Lax and Phillips introduced in Chapter III 0f [10] their famous semigroup, which is a special restriction of the semigroup (2.4) in Subsection 2.3.

Later on we show that also in the case of commuting projections D_+, D_- the corresponding restriction leads to a semigroup (see Subsection 3.2).

PROPOSITION 3.1. *Let $\{U(\mathbb{R}), \mathcal{D}_\pm\}$ be as before. Then the following conditions are equivalent:*
(i) *J is isometric;*
(ii) *$\mathcal{D}_+ \perp \mathcal{D}_-$;*
(iii) $SQ_+ = Q_+SQ_+$.

Proof. (i)$\Leftrightarrow$ (iii): One calculates

$$J^*J = (W_+Q_- + W_-Q_+)^*(W_+Q_- + W_-Q_+) = Q_- + Q_+S^*Q_- + Q_-SQ_+ + Q_+.$$

If $J^*J = 1\!\!1_{\mathcal{H}_0}$ then $Q_+S^*Q_- + Q_-SQ_+ = 0$ follows, i.e. $Q_-SQ_+ = 0$ or (iii) and vice versa.

(ii)$\Leftrightarrow$ (iii): Using $D_\pm = V_\pm^* Q_\mp V_\pm$ one obtains

$$D_+D_- = V_+^*Q_-V_+V_-^*Q_+V_- = V_+^*Q_-SQ+V_-$$

and the assertion is obvious. ∎

The characterization of J in the general case (commuting outgoing and incoming projections) is given by

THEOREM 3.2. *Let $\{U(\mathbb{R}), \mathcal{D}_\pm\}$ be as before. Then*

$$D_+D_- = D_-D_+ \quad \textit{if and only if} \quad J^*J = 1\!\!1_{\mathcal{H}_0} + E - F,$$

where E, F are selfadjoint projections with $EF = 0$.

Moreover either $E = F = 0$ or both projections are nonzero, $E \neq 0$, $F \neq 0$.

Note that the first case of the last statement corresponds to $\mathcal{D}_+ \perp \mathcal{D}_-$, the second one to $D_+D_- \neq 0$.

Proof. (i) Assume $D_+D_- = D_-D_+$. Then a straightforward calculation yields that this is equivalent to

$$Q_-SQ_+S^* = SQ_+S^*Q_-. \tag{3.1}$$

Using (2.13) we have $J^*J = 1\!\!1_{\mathcal{H}_0} + A + A^*$, where $A := Q_+S^*Q_-$. That is, we have to prove $A + A^* = E - F$, where E, F have the mentioned properties. Note that

$(A+A^*)^2 = AA^* + A^*A$, and $(AA^* + A^*A)^2 = AA^*AA^* + A^*AA^*A$. Now

$$\begin{aligned} AA^*A &= Q_+S^*Q_- \cdot Q_-SQ_+ \cdot Q_+S^*Q_- = Q_+S^* \cdot Q_-SQ_+S^* \cdot Q_- \\ &= Q_+S^* \cdot SQ_+S^* \cdot Q_- = Q_+S^*Q_- = A, \end{aligned}$$

hence $A^*AA^* = A^*$ and

$$(AA^* + A^*A)^2 = AA^* + A^*A =: P,$$

i.e. P is a selfadjoint projection and $(A+A^*)^2 = P$. Put $A + A^* =: V$. Then $V = V^*$ and $V^2 = P$. This implies $V = E - F$ with selfadjoint projections E, F, where $EF = 0$ and $E + F = PP$.

(ii) Conversely, assume $J^*J = \mathbb{1}_{\mathcal{H}_0} + E - F$. Then we have to prove $D_+D_- = D_-D_+$, or, equivalently, $Q_- \cdot SQ_+S^* = SQ_+S^* \cdot Q_-$. Put $E + F =: P$. We have $A + A^* = E - F$. Then $(E-F)^2 = E + F = P$, i.e. $(A+A^*)^2 = P$ or $AA^* + A^*A = P$. Put $X := AA^*$, $Y := A^*A$. Then $X + Y = P$ and $XY = 0$. This implies $X^2 = XP = PX$ and $X^2(\mathbb{1}_{\mathcal{H}_0} - P) = (X(\mathbb{1}_{\mathcal{H}_0} - P))^2 = 0$, hence $X(\mathbb{1}_{\mathcal{H}_0} - P) = 0$ or $X = XP$ follows. Thus we get

$$X^2 = X, \tag{3.2}$$

i.e. X is a selfadjoint projection. Correspondingly, Y is a selfadjoint projection, too. Recall that $X = Q_+S^*Q_- \cdot Q_-SQ_+ = Q_+S^*Q_-SQ_+$. Then (3.2) yields

$$Q_+S^*Q_-SQ_+S^*Q_-SQ_+ = Q_+S^*Q_-SQ_+,$$

or, by multiplication with S^*Q_-S from the right, $(Q_+ \cdot S^*Q_-S)^3 = (Q_+ \cdot S^*Q_-S)^2$. For brevity put $Q_+S^*Q_-S =: B$. Then $(B^2 - B)^2 = 0$ follows. This implies $|B^2 - B| = 0$ and $B^2 = B$. Therefore we obtain

$$\text{s-}\lim_{n\to\infty} (Q_+ \cdot S^*Q_-S)^n = Q_+ \cdot S^*Q_-S.$$

Since the left hand side is a selfadjoint projection (onto the intersection subspace $Q_+\mathcal{H}_0 \cap S^*Q_-S\mathcal{H}_0$), finally we get $Q_+S^*Q_-S = S^*Q_-SQ_+$ or

$$Q_- \cdot SQ_+S^* = SQ_+S^* \cdot Q_-,$$

and this is the assertion.

Now we prove the last statement. First we assume $E = 0$. Then $F = P$ and

$$J^*J = \mathbb{1}_{\mathcal{H}_0} - P. \tag{3.3}$$

Then also $JJ^* = D_+ + D_- = W_+Q_-W_+^* + W_-Q_+W_-^* = W_+(Q_- + SQ_+S^*)W_+^*$ is a projection, i.e. $Q_- + SQ_+S^*$ is a projection. This gives $SQ_+S^*Q_- + Q_-SQ_+S^* = 0$. But (3.3) implies

$$Q_+S^*Q_-SQ_+ + Q_-SQ_+S^*Q_- = -Q_+S^*Q_- - Q_-SQ_+,$$

hence $Q_-SQ_+S^*Q_- = -Q_-SQ_+$ and $Q_-SQ_+ = 0$ follows. Since $P = -(Q_+S^*Q_- + Q_-SQ_+)$, we get $P = F = 0$.

On the other hand, if $F = 0$, i.e. $E = P$, we have $J^*J = \mathbb{1}_{\mathcal{H}_0} + P$ and $P = Q_+S^*Q_- + Q_-SQ_+$. Now, together with S also $-S$ is an admissible scattering operator, assigned to a complete two-space scattering system $\{\widetilde{H}, H_0\}$ (see [4], p. 238). The corresponding identification operator $\widetilde{J}$ satisfies $\widetilde{J}^*\widetilde{J} = \mathbb{1}_{\mathcal{H}_0} - P$ and $\widetilde{J}\widetilde{J}^* = \widetilde{W}_+(Q_- + SQ_+S^*)\widetilde{W}_+^*$. That is, also in this case $Q_- + SQ_+S^*$ is a projection and we obtain, by similar arguments as before, that $P = F = 0$. ∎

3.2. THE LAX-PHILLIPS SEMIGROUP. As it is mentioned in Subsection 3.1 in the case $\mathcal{D}_+ \perp \mathcal{D}_-$ Lax and Phillips introduced an important semigroup by a characteristic restriction of the LP-evolution.

In this subsection we show that also in the case of commuting outgoing/incoming projections by an analogous restriction a semigroup can be introduced which in the special case of mutually orthogonal outgoing and incoming subspaces coincides with the LP-semigroup.

We start with the semigroup

$$D_+^\perp \mathrm{e}^{-\mathrm{i}tH} D_+^\perp = D_+^\perp \mathrm{e}^{-\mathrm{i}tH}, \quad t \geqslant 0. \tag{3.4}$$

Its transformation into the outgoing spectral representation yields the characteristic semigroup $T_+(\cdot)$ (see Subsection 2.3). Now we define a second restriction of (3.4) by

$$Z(t) := D_+^\perp \mathrm{e}^{-\mathrm{i}tH} D_-^\perp, \quad t \geqslant 0.$$

A straightforward calculation gives $Z(t) = W_+Q_+\mathrm{e}^{-\mathrm{i}tH_0}SQ_-W_-^*$, i.e. the transformation into the outgoing spectral representation yields

$$Z_+(t) = W_+^* Z(t) W_+ = Q_+\mathrm{e}^{-\mathrm{i}tH_0}Q_+ \cdot SQ_-S^*.$$

Recall that the condition $D_+D_- = D_-D_+$ is equivalent with (3.1). Then we have

THEOREM 3.3. *If D_+ and D_- commute then $Z_+(\cdot)$ hence $Z(\cdot)$ is a semigroup for $t \geqslant 0$.*

Proof. We calculate

$$\begin{aligned} Z_+(t_1)Z_+(t_2) &= Q_+\mathrm{e}^{-\mathrm{i}t_1H_0}Q_+SQ_-S^*Q_+\mathrm{e}^{-\mathrm{i}t_2H_0}Q_+SQ_-S^* \\ &= Q_+\mathrm{e}^{-\mathrm{i}t_1H_0}SQ_-S^*\mathrm{e}^{-\mathrm{i}t_2H_0}SQ_-S^* = Q_+S\mathrm{e}^{-\mathrm{i}t_1H_0}Q_-\mathrm{e}^{-\mathrm{i}t_2H_0}Q_-S^* \\ &= Q_+S\mathrm{e}^{-\mathrm{i}t_1H_0}\mathrm{e}^{-\mathrm{i}t_2H_0}Q_-S^* = Q_+\mathrm{e}^{-\mathrm{i}(t_1+t_2)H_0}Q_+ \cdot SQ_-S^* \\ &= Z_+(t_1+t_2). \quad ∎ \end{aligned}$$

Note that $Q_+ \cdot SQ_-S^*$ is the projection of the subspace $Q_+\mathcal{H}_0 \cap SQ_-\mathcal{H}_0$ hence we obtain

$$Q_+SQ_-S^*\mathcal{H}_0 = Q_+\mathcal{H}_0 \cap SQ_-\mathcal{H}_0 = \mathcal{H}_+^2 \cap S\mathcal{H}_-^2 = \mathcal{H}_+^2 \cap S(\mathcal{H}_+^2)^\perp = \mathcal{H}_+^2 \cap (S\mathcal{H}_+^2)^\perp.$$

This means: the elements of this subspace are exactly those vectors $f \in \mathcal{H}_+^2$ which are orthogonal with respect to $S\mathcal{H}_+^2$, i.e. $f \perp S\mathcal{H}_+^2$.

According to Theorem 3.3 this subspace is invariant with respect to the semigroup $Z_+(\cdot)$. Moreover the semigroup vanishes on the orthogonal complement. The restriction

$$Z_+(t) \upharpoonright \mathcal{H}^2_+ \cap (S\mathcal{H}^2_+)^\perp, \quad t \geqslant 0 \tag{3.5}$$

is a strongly continuous contractive semigroup which is a restriction of the characteristic semigroup $T_+(\cdot) \upharpoonright \mathcal{H}^2_+$ considered in Subsection 2.3. This restriction we call the *generalized Lax-Phillips semigroup*.

REMARK 3.4. If even $D_+D_- = 0$, i.e. $\mathcal{D}_+$ and $\mathcal{D}_-$ are orthogonal then Proposition 3.1 yields $SQ_+ = Q_+SQ_+$. This means $S\mathcal{H}^2_+ \subseteq \mathcal{H}^2_+$. In this case we obtain

$$\mathcal{H}^2_+ \cap (S\mathcal{H}^2_+)^\perp = \mathcal{H}^2_+ \ominus S\mathcal{H}^2_+,$$

i.e. in this case $Z_+(\cdot)$ acts on $\mathcal{H}^2_+ \ominus S\mathcal{H}^2_+$ and it is nothing else than the original Lax-Phillips semigroup. Further it turns out that in this case $S(\cdot)$ is holomorphic in $\mathbb{C}_+$ with $\sup_{z\in\mathbb{C}_+} \|S(z)\| \leqslant 1$ such that $S(\lambda) = \text{s-}\lim_{\varepsilon\to+0} S(\lambda + \mathrm{i}\varepsilon)$. That is, in this case the existence of the Lax-Phillips semigroup is simultaneously coupled with strong implications on the analytic continuability of the scattering matrix.

Next we study the spectral theory of (3.5). It is a restriction of the characteristic semigroup $T_+(\cdot) \upharpoonright \mathcal{H}^2_+$ whose spectral theory is already known. Therefore, in view of the problem to characterize the eigenvalue spectrum of (3.5) the crucial question is: Which eigenvalues of the characteristic semigroup, i.e. of $T_+(\cdot)$ on $\mathcal{H}^2_+$, *survive* the restriction to the subspace $\mathcal{H}^2_+ \cap (S\mathcal{H}^2_+)^\perp$? That is, for $f_{\zeta,k} \in \mathcal{N}_{\overline{\zeta}}$, $\zeta \in \mathbb{C}_-$, i.e.

$$f_{\zeta,k}(\lambda) := \frac{k}{\lambda - \zeta}, \quad 0 \neq k \in \mathcal{K},$$

one has to analyze the condition $f_{\zeta,k} \perp S\mathcal{H}^2_+$ or, equivalently,

$$S^* f_{\zeta,k} \in \mathcal{H}^2_-. \tag{3.6}$$

We have $(S^* f_{\zeta,k})(\lambda) = S(\lambda)^* f_{\zeta,k}(\lambda) = \frac{S(\lambda)^* k}{\lambda - \zeta}$. Therefore (3.6) is equivalent to

$$\int_{-\infty}^{\infty} \frac{S(\lambda)^* k}{(\lambda - \zeta)(\lambda - z)} \mathrm{d}\lambda = 0, \quad z \in \mathbb{C}_+,$$

because of (2.3). In particular, (3.6) implies that $(S^* f_{\zeta,k})(\cdot)$ has a holomorphic continuation into $\mathbb{C}_-$. Then

$$\|(S^* f_{\zeta,k})(z)\|_{\mathcal{K}} \leqslant \frac{\|k\|}{|\operatorname{Im}\zeta|}, \quad z \in \mathbb{C}_-, \tag{3.7}$$

follows. On the other hand, $\mathbb{C}_- \ni z \to (z - \zeta)(S^* f_{\zeta,k})(z)$ is the holomorphic continuation of $\mathbb{R} \ni \lambda \to S(\lambda)^* k$ into $\mathbb{C}_-$ and ζ is a zero of this function. This implies

$$|z-\zeta| \cdot \|(S^* f_{\zeta,k})(z)\|_{\mathcal{K}} \leqslant \sup_{\lambda\in\mathbb{R}} \|S(\lambda)^* k\| = \|k\|$$

or

$$\|(S^* f_{\zeta,k})(z)\|_{\mathcal{K}} \leqslant \frac{\|k\|}{|z-\zeta|}, \quad \zeta \neq z \in \mathbb{C}_-. \tag{3.8}$$

Therefore we obtain

PROPOSITION 3.5. *Let $(S^* f_{\zeta,k})(\cdot)$ be holomorphically continuable into $\mathbb{C}_-$. Then $S^* f_{\zeta,k} \in \mathcal{H}^2_-$ follows, i.e. the condition of holomorphic continuability of $S^* f_{\zeta,k}(\cdot)$ into $\mathbb{C}_-$ is sufficient for* (22).

Proof. Choose a square $\mathbb{C}_- \supset G_\varepsilon := \{z : |\mathrm{Re}\, z - \mathrm{Re}\, \zeta| \leqslant \varepsilon, |\mathrm{Im}\, z - \mathrm{Im}\, \zeta| \leqslant \varepsilon\}$, $\varepsilon > 0$, and let $y > 0$. If $(\mathbb{R} - \mathrm{i}y) \cap G_\varepsilon = \emptyset$ then $\int_{-\infty}^{\infty} \|(S^* f_{\zeta,k})(x - \mathrm{i}y)\|^2_{\mathcal{K}} \mathrm{d}x \leqslant \|k\|^2 \frac{\pi}{\varepsilon}$, where we have used (3.8). If $(\mathbb{R} - \mathrm{i}y) \cap G_\varepsilon \neq \emptyset$ then

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\mathrm{Re}\,\zeta-\varepsilon} + \int_{\mathrm{Re}\,\zeta-\varepsilon}^{\mathrm{Re}\,\zeta+\varepsilon} + \int_{\mathrm{Re}\,\zeta+\varepsilon}^{\infty} .$$

To estimate the first and the third term we use (3.8), for the second term we use (3.7). Thus in this case we obtain

$$\int_{-\infty}^{\infty} \|(S^* f_{\zeta,k})(x - \mathrm{i}y)\|^2_{\mathcal{K}} \mathrm{d}x \leqslant \|k\|^2 \Big(\frac{2}{\varepsilon} + \frac{2\varepsilon}{|\mathrm{Im}\,\zeta|^2}\Big),$$

i.e. $\sup_{y>0} \int_{-\infty}^{\infty} \|(S^* f_{\zeta,k})(x - \mathrm{i}y)\|^2_{\mathcal{K}} \mathrm{d}x < \infty$. Therefore, according to the Paley-Wiener theorem, the assertion follows. ∎

PROPOSITION 3.6. *The vector $f_{\zeta,k} \in \mathcal{H}^2_+$ is an eigenvector of the (generalized)* LP*-semigroup, i.e. $f_{\zeta,k} \in S\mathcal{H}^2_-$ if and only if the vector function $\mathbb{R} \ni \lambda \to S(\lambda)^{-1}k$ is holomorphically continuable into $\mathbb{C}_-$ and $S(\zeta)^{-1}k = 0$.*

REMARK 3.7. In the case $\mathcal{D}_+ \perp \mathcal{D}_-$ the operator function $S(\cdot)^{-1}$ is a priori holomorphic in $\mathbb{C}_-$. Then $S^* f_{\zeta,k}(\cdot)$ is holomorphic in $\mathbb{C}_-$ if and only if $S(\zeta)^{-1}k = 0$. But this means that $S(\cdot)$, which is also analytically continuable int $\mathbb{C}_-$, has necessarily a pole at ζ (see Lax and Phillips [10]).

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