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FUNCTIONS ACTING ON SYMMETRICALLY NORMED IDEALS AND ON THE DOMAINS OF DERIVATIONS ON THESE IDEALS.

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ABSTRACT. In this paper we study "scalar" and "operator" smoothness conditions for functions to act on symmetrically normed ideals of $B(H)$ and on the domains of weakly* closed derivations of these ideals.

KEYWORDS: *Symmetrically normed ideals, $*$-derivations, J-Lipschitz functions.*

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1. INTRODUCTION AND PRELIMINARIES

Every semisimple Hermitian Banach $*$-algebra $\mathcal{A}$ can be considered as a $*$-subalgebra of the algebra $(B(H), \|\cdot\|)$ of all bounded operators on a Hilbert space H and $\mathrm{Sp}_{\mathcal{A}}(A) = \mathrm{Sp}_{B(H)}(A)$ for each $A \in \mathcal{A}$. Denote by $\mathcal{A}_{\mathrm{n}}$ the set of all normal operators in $\mathcal{A}$. For a subset σ of $\mathbb{C}$, set

$$\mathcal{A}_{\mathrm{n}}(\sigma) = \{A \in \mathcal{A}_{\mathrm{n}} : \mathrm{Sp}(A) \subseteq \sigma\}.$$

Each bounded Borel complex-valued function g on σ defines, via the spectral theorem, a map $g : T \to g(T)$ from the set of all normal operators with spectrum in σ into $B(H)$. If we restrict g to $\mathcal{A}_{\mathrm{n}}$ then $g(T)$ does not necessarily belong to $\mathcal{A}$. A function g is said to *act* on $\mathcal{A}$ if

$$g(T) \in \mathcal{A} \quad \text{for each } T \in \mathcal{A}_{\mathrm{n}}(\sigma). \tag{1.1}$$

It is well known that all continuous functions act on C^*-algebras. On the other hand, only holomorphic functions act on the Wiener-Fourier algebra $l^1(T)$ (see [14]). For many Hermitian subalgebras of $B(H)$, however, the situation is not "black and white", especially if $\sigma \subseteq \mathbb{R}$. Bratteli, Elliott and Jorgensen [6], for example, proved that each continuous function g in $L^1(\mathbb{R})$ whose Fourier transform satisfies

$$\int_{-\infty}^{\infty} |s\widehat{g}(s)|\mathrm{d}s < \infty$$

acts on the domains of all closed $*$-derivations of C^*-algebras. A wide class of semisimple Hermitian Banach $*$-algebras — differential algebras — was introduced by Blackadar and Cuntz in [5] and by Kissin and Shulman in [15]. These algebras are operator analogues of algebras of differentiable functions. In particular, all symmetrically normed ideals of $B(H)$ and all domains of closed derivations of C^*-algebras are differential algebras. In [5] and [15] various smoothness conditions were obtained for functions to act on differential algebras.

In this paper we study "scalar" and "operator" smoothness conditions for functions to act on *symmetrically normed* ideals (for the sake of brevity, s. n. ideals) of $B(H)$ and on the domains of weakly closed derivations of these ideals.

Let H be a separable Hilbert space. A two-sided ideal J of $B(H)$ is an s. n. ideal if it is a Banach space with respect to a norm $\|\cdot\|_J$ and

$$\|AXB\|_J \leqslant \|A\|\|X\|_J\|B\| \quad \text{for } A, B \in B(H) \text{ and } X \in J. \tag{1.2}$$

By Calkin's theorem, $J \subseteq C(H)$, where $C(H)$ is the ideal of all compact operators in $B(H)$.

In Section 2 we study functions g defined on compact subsets α of $\mathbb{C}$ that act on s. n. ideals J. We show that g acts on J if and only if it is Lipschitzian at 0. Moreover, such a function acts *continuously* on all separable s. n. ideals. We also prove that g acts on the unitization $\widetilde{J} = \mathbb{C}\mathbf{1} + J$ of J if and only if it is Lipschitzian at each point of α. It acts *continuously* on the unitization of separable ideals if and only if it is Lipschitzian on α.

For more intricate classes of differential algebras such as the domains of closed $*$-derivations, the "scalar" smoothness conditions are not however, always well-suited to determine the spaces of functions acting on these algebras. In these cases more suitable conditions turn out to be various "operator" smoothness conditions imposed on the maps $T \to g(T)$.

"Operator" smoothness conditions characterize interesting and important classes of functions. For example, the condition that the maps $T \to g(T)$ are Gateaux or Frechet differentiable defines the classes of Gateaux and Frechet operator differentiable functions; the condition that these maps are Lipschitzian defines the class of operator Lipschitz functions. If, apart from the standard operator norm on $B(H)$, one considers symmetric operator norms $\|\cdot\|_J$ and the classes of Gateaux (Frechet) operator differentiable and operator Lipschitz functions with respect to these norms, one obtains a rich variety of functional spaces. Thus the operator theory suggests its own scale of smoothness of functions and defines naturally new functional spaces. Much work has been done to relate the "operator" smoothness and the traditional "scalar" smoothness conditions of functions in papers of Daletskii and Krein [7], Birman and Solomyak [3], [4], Davies [8], Farforovskaya [9], [10], Peller [22] and others.

Let $\mathcal{A}$ be a Banach $*$-algebra. A map δ from a $*$-subalgebra $D(\delta)$ (called the domain of δ) of $\mathcal{A}$ into $\mathcal{A}$ is a *$*$-derivation* if

$$(1.3)\qquad \delta(AB) = A\delta(B) + \delta(A)B \quad \text{and} \quad \delta(A^*) = \delta(A)^*, \quad \text{for } A, B \in D(\delta).$$

A derivation δ is *closed* if the graph $G(\delta) = \{(A, \delta(A)) \colon A \in D(\delta)\}$ of δ is norm closed in $\mathcal{A} \dotplus \mathcal{A}$. If $\mathcal{A}$ is the dual space of a Banach space $\mathcal{A}_*$ and the graph $G(\delta)$ is closed in $\mathcal{A} \dotplus \mathcal{A}$ in the $\sigma(\mathcal{A} \dotplus \mathcal{A}, \mathcal{A}_* \dotplus \mathcal{A}_*)$-topology, then δ is called *weakly closed*.

Similar to (1.1), a function g on a subset σ of $\mathbb{R}$ *acts* on the domain $D(\delta)$ of an unbounded $*$-derivation δ of a Hermitian Banach algebra, if

$$(1.4)\qquad A = A^* \in D(\delta) \quad \text{and} \quad \mathrm{Sp}(A) \subseteq \sigma \quad \text{implies that } g(A) \in D(\delta).$$

The first step in finding "operator" smoothness conditions for functions to act on the domains of closed $*$-derivations was made by Arazy, Barton and Friedman in [2]. They proved that Gateaux operator differentiable functions act on the domains of the generators of one-parameter groups of automorphisms of C^*-algebras. The authors in [16], [18] and Pedersen in [21] extended their result and showed that these functions act on the domains of *all* closed $*$-derivations on C^*-algebras. In [18] it was also established that a function acts on the domains of *all weakly* closed $*$-derivations on C^*-algebras, if and only if it is operator Lipschitzian.

In Section 3 we consider "operator" smoothness conditions for functions to act on the domains of closed and weakly closed unbounded $*$-derivations on s. n. ideals. We prove that if J is a coseparable s. n. ideal, then the operator J-Lipschitz functions act on the domains of all weakly closed $*$-derivations of J. If J is reflexive, then the operator J-Lipschitz functions act on the domains of all closed $*$-derivations on J. We also extend a result of the second author in [25] and show that, if δ is a closed $*$-derivation on a C^*-algebra or on a separable s. n. ideal J and if $\delta(A)$ belongs to a reflexive ideal I, then $g(A) \in D(\delta)$ and $\delta(g(A)) \in I$, for each operator I-Lipschitz function g on $\mathrm{Sp}(A)$.

We apply the above results to extend some operator inequalities in Schatten ideals C^p. It was proved in [8] that, for each $1 < p < \infty$, there is a constant γ_p (evaluated in [1]) such that, for all $T = T^* \in B(H)$ and all $A = A^* \in C^p$,

$$\|[T, |A|]\|_p \leqslant \gamma_p \|[T, A]\|_p, \quad \text{where } [T, A] = TA - AT \quad \text{and } |A| = (A^*A)^{1/2}.$$

We show that this inequality holds for all unbounded symmetric operators T and all $A = A^* \in C^p$ that preserve the domain $D(T)$ of T.

2. FUNCTIONS ACTING ON SYMMETRICALLY NORMED IDEALS

We start this section with a brief discussion of some basic properties of s. n. ideals. Let J be a s. n. ideal of $B(H)$. If V is an isometry from H onto another space K, then the ideal VJV^* of $B(K)$ with the norm $\|VAV^*\|_{VJV^*} = \|A\|_J$, for $A \in J$, does not depend on the choice of V. We will denote it by the same symbol J (or $J(K)$ if it is necessary to underline the space). Therefore when we say that a certain statement holds for an ideal J, we mean that it holds for all ideals $J(K)$ with K being any separable infinite dimensional space.

Let Φ be the set of all positive *symmetric norming* functions on the space c_0 of all sequences of real numbers converging to 0 (see Section III.3 of [11]). The most important classes of s. n. ideals are defined by means of these functions in the following way (see Section III.4 of [11]). For each $A \in C(H)$, the non-increasing sequence $s(A) = \{s_i(A)\}$ of all eigenvalues of $(A^*A)^{1/2}$ belongs to c_0. For $\phi \in \Phi$, the set

$$J^\phi = \{A \in C(H) : \phi(s(A)) < \infty\} \quad \text{with norm } \|A\|_{J^\phi} = \phi(s(A))$$

is an s. n. ideal.

Denote by $\mathcal{F}$ the ideal of all finite rank operators in $B(H)$. The closure J_0^ϕ of $\mathcal{F}$ in $\|\cdot\|_{J^\phi}$ is a separable s. n. ideal and $J_0^\phi \subseteq J^\phi$. It follows from Theorem III.6.2 of [11] that an s. n. ideal is separable if and only if it coincides with some J_0^ϕ.

For many functions $\phi \in \Phi$, the ideals J^ϕ and J_0^ϕ coincide. An important class of such functions consists of the s. n. functions

$$\phi_p(\xi) = \Big(\sum_{i=1}^{\infty} |\xi_i|^p\Big)^{1/p} \quad \text{for } 1 \leqslant p < \infty, \quad \text{and } \phi_\infty(\xi) = \sup |\xi_i|.$$

The corresponding ideals C^p are called Schatten classes. In particular, $C^\infty = C(H)$.

In the following proposition we gathered some known (see, for example, [17]) results that will be used later on.

PROPOSITION 2.1. *Let I and J be s. n. ideals.*

(i) *If $J \subseteq I$ then there exists $c > 0$ such that $\|A\|_I \leqslant c\|A\|_J$, for all $A \in J$.*

(ii) *There is a unique, up to equivalence, function $\phi \in \Phi$ such that $J_0^\phi \subseteq J \subseteq J^\phi$. Moreover, the norms $\|\cdot\|_{J^\phi}$ and $\|\cdot\|_J$ coincide on J_0^ϕ and $\|A\|_{J^\phi} \leqslant \|A\|_J$ for all $A \in J$.*

(iii) *If J is reflexive then there is $\phi \in \Phi$ such that $J_0^\phi = J = J^\phi$.*

For each s. n. function $\phi \neq \phi_1$, there is an *adjoint* s. n. function ψ such that J^ψ can be identified with the dual space of J_0^ϕ in the following way. For each $T \in J^\psi$ and $X \in J_0^\phi$, the operator TX belongs to the trace class C^1, the linear functional

$$F_T(X) = \operatorname{Tr}(TX) \tag{2.1}$$

is bounded on J_0^ϕ with $\|F_T\| = \|T\|_{J^\psi}$, and each bounded functional on J_0^ϕ is of this form. We write $\psi = \phi^*$. This relation is symmetric: $\psi^* = \phi$ if $\phi \neq \phi_1$ and $\phi \neq \phi_\infty$. Thus all ideals J^ϕ, $\phi \neq \phi_\infty$, are dual to separable ideals; we call such ideals *coseparable*. By the weak topology in a coseparable ideal J^ϕ we mean the topology $\sigma(J^\phi, J_0^{\phi^*})$.

Let $\widetilde{J} = \mathbb{C}1 + J$ be the unitization of an s. n. ideal J. Then it is a Banach $*$-algebra in the norm $\|\lambda\mathbf{1}+A\|_{\widetilde{J}} = |\lambda| + \|A\|_J$, for $\lambda \in \mathbb{C}$ and $A \in J$. By J_{n} and $\widetilde{J}_{\mathrm{n}}$ we denote the spaces of all normal operators in J and $\widetilde{J}$.

In this section α will denote a compact subset of $\mathbb{C}$. For any s. n. ideal J, we set

$$J_{\mathrm{n}}(\alpha) = \{A \in J_{\mathrm{n}} : \mathrm{Sp}(A) \subseteq \alpha\} \quad \text{and} \quad \widetilde{J}_{\mathrm{n}}(\alpha) = \{A \in \widetilde{J}_{\mathrm{n}} : \mathrm{Sp}(A) \subseteq \alpha\}.$$

Since 0 lies in the spectra of all compact operators, we always assume that $0 \in \alpha$ when we consider $J_{\mathrm{n}}(\alpha)$. For $\widetilde{J}_{\mathrm{n}}(\alpha)$ we do not need to assume this.

Recall that a function g on α is Lipschitzian at a point $s \in \alpha$, if there is $D > 0$ such that $|g(t) - g(s)| \leqslant D|t - s|$ for all $t \in \alpha$. Clearly, the above condition is non-trivial only if s is a limit point of α. A function g is Lipschitzian on α if $|g(t) - g(s)| \leqslant D|t - s|$ for all $t, s \in \alpha$. It is easy to construct an example of a function Lipschitzian at each point of α but not Lipschitzian on α.

We set

$$\widetilde{g}_s(t) = \begin{cases} \frac{g(t)-g(s)}{t-s} & \text{for } t \neq s, \\ 0 & \text{for } t = s. \end{cases} \tag{2.2}$$

An operator $A \in B(H)$ is diagonal, if there exists an orthonormal basis $\{e_n\}$ in H such that $Ae_n = \lambda_n e_n$ for all n. We will write $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n, \ldots)$ without mentioning the basis.

Before discussing the question which functions act on s. n. ideals, let us note that for the ideal $C(H)$ the answer is well known: any continuous function g acts on $\widetilde{C(H)} = \mathbb{C}\mathbf{1} + C(H)$; it acts on $C(H)$ if and only if $g(0) = 0$.

THEOREM 2.2. *Let g be a bounded Borel function on $\alpha \subset \mathbb{C}$ and let $J \neq C^\infty$. Then:*

(i) *g acts on J if and only if $g(0) = 0$ and g is Lipschitzian at $t = 0$;*

(ii) *g acts on $\widetilde{J}$ if and only if it is Lipschitzian at each point of α.*

Proof. If g is Lipschitzian at 0 and $g(0) = 0$, then $g(t) = t\widetilde{g}_0(t)$ and $\widetilde{g}_0$ defined in (2.2) is a bounded Borel function on α. Hence $\widetilde{g}_0(A) \in B(H)$ so $g(A) = A\widetilde{g}_0(A) \in J$, for each $A \in J_{\mathrm{n}}(\alpha)$.

Conversely, if g acts on J then $g(0) = 0$. If g is not Lipschitzian at 0, then there are $t_k \in \alpha$ such that $|t_k| \leqslant k^{-3}$ and $\left|\frac{g(t_k)}{t_k}\right| \geqslant k^4$. By Proposition 2.1, there is $\phi \in \Phi$ such that $J_0^\phi \subseteq J \subseteq J^\phi$. Set

$$\xi_n = (\underbrace{1, \ldots, 1}_{n}, 0, \ldots) \quad \text{and} \quad R(n) = \phi(\xi_n).$$

As $J^\phi \neq C(H)$ we have (see Section III.3 of [11]) $R(n) \to \infty$, as $n \to \infty$,

$$1 \leqslant R(n) \leqslant R(n-1) + 1 \text{ and } R(n) = \phi(\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{n}, 0, \ldots) \text{ for all } m. \tag{2.3}$$

Hence $R(n) \leqslant n$. Since $1 \leqslant k^{-3}|t_k|^{-1}$ and $R(n) \to \infty$, we can choose n_k such that

$$1 \leqslant (k^3|t_k|)^{-1} \leqslant R(n_k) \leqslant k(k^3|t_k|)^{-1}. \tag{2.4}$$

Let us fix some orthonormal basis in H. The operator $A = \operatorname{diag}(\underbrace{t_1,\dots,t_1}_{n_1},\dots,\underbrace{t_k,\dots,t_k}_{n_k},\dots)$ is normal and $A \in C(H)$. For each k, set

$$A_k = \operatorname{diag}(\underbrace{t_1,\dots,t_1}_{n_1},\dots,\underbrace{t_k,\dots,t_k}_{n_k},0,0,\dots).$$

It follows from Properties II and III in Section III.3.1 of [11] and from (2.3) and (2.4) that $\|A_k\|_{J^\phi} = \phi(\underbrace{|t_1|,\dots,|t_1|}_{n_1},\dots,\underbrace{|t_k|,\dots,|t_k|}_{n_k},0,0,\dots) \leqslant |t_1|R(n_1)+\dots+|t_k|R(n_k) \leqslant \sum_{m=1}^{k} m^{-2}$. Since A_k converge to A in the weak operator topology and $\sup\|A_k\|_{J^\phi} < \infty$, it follows from Theorem III.5.1 of [11] that $A \in (J_0^\phi)_{\mathrm{n}}(\alpha) \subseteq J_{\mathrm{n}}(\alpha)$.

On the other hand, by Properties II, III, 1° in Section III.3.1 of [11] and by (2.4),

$$\begin{aligned}\|g(A)\|_{J^\phi} &= \lim_{k\to\infty}\phi(\underbrace{|g(t_1)|,\dots,|g(t_1)|}_{n_1},\dots,\underbrace{|g(t_k)|,\dots,|g(t_k)|}_{n_k},0,\dots)\\ &\geqslant \lim_{k\to\infty}\phi(0,\dots,0,\underbrace{|g(t_k)|,\dots,|g(t_k)|}_{n_k},0,\dots)\\ &= \lim_{k\to\infty}|g(t_k)|R(n_k) \geqslant \lim_{k\to\infty}k^4|t_k|R(n_k) \geqslant \lim_{k\to\infty}k = \infty.\end{aligned}$$

Thus $g(A) \notin J^\phi$. Hence $g(A) \notin J$, so g does not act on J. Part (i) is proved.

For $s \in \alpha$ and $B \in J_{\mathrm{n}}$, let $A = s\mathbf{1} + B \in \widetilde{J}_{\mathrm{n}}(\alpha)$. If g is Lipschitzian at s, then $g(t) = g(s) + (t-s)\widetilde{g}_s(t)$ and $\widetilde{g}_s$ defined in (2.2) is a bounded Borel function on α. Hence $\widetilde{g}_s(A) \in B(H)$ so that $g(A) \in \widetilde{J}$, since

$$g(A) = g(s)\mathbf{1} + (A - s\mathbf{1})\widetilde{g}_s(A) = g(s)\mathbf{1} + B\widetilde{g}_s(A). \tag{2.5}$$

Conversely, let g act on $\widetilde{J}$. First show that g is continuous on α. If g is not continuous at s, there are $s_n \to s$ in α such that the operator $B = \operatorname{diag}(s_1 - s, 0, s_2 - s, 0, \dots, s_n - s, 0, \dots)$ belongs to $J_0^\phi \subseteq J$ and $g(s_n)$ do not converge to $g(s)$. Then

$$A = t\mathbf{1} + B \in \widetilde{J}_{\mathrm{n}}(\alpha) \quad\text{and}\quad g(A) = \operatorname{diag}(g(s_1), g(s), g(s_2), g(s), \dots, g(s_n), g(s), \dots) \in \widetilde{J}.$$

However, since $g(s_n)$ do not converge to $g(s)$, we have $g(A) \notin \mathbb{C}1 + C(H)$ — a contradiction.

Let $A = \operatorname{diag}(t_1, t_2, \dots, t_n, \dots) \in J$ be such that $s\mathbf{1} + A \in \widetilde{J}_{\mathrm{n}}(\alpha)$. Since $t_n \to 0$ and g is continuous, $g(s\mathbf{1} + A) - g(s)\mathbf{1} \in C(H)$. Since g acts on $\widetilde{J}$, $g(s\mathbf{1} + A) - g(s)\mathbf{1} \in J$. Hence the function $h(t) = g(s+t) - g(s)$ defined on the set $\{z - s : z \in \alpha\}$ acts on J. By (i), h is Lipschitzian at $t = 0$. Hence g is Lipschitzian at s. ∎

Denote by $\mathbf{C}_\rho$ the disk of radius ρ with centre in 0.

LEMMA 2.3. *Let $\{A_k\}_{k=0}^\infty$ be normal operators, let $\|A_k - A_0\| \to 0$ and let a compact set α in $\mathbb{C}$ contain all $\mathrm{Sp}(A_k)$. Let $\{h_k\}_{k=1}^\infty$ be Borel functions on α such that $M = \sup_{k,t} |h_k(t)| < \infty$. Let also $h_k \to h_0$ uniformly outside each disk $\mathbf{C}_\rho$, where the function h_0 is continuous on $\alpha \setminus \{0\}$. Then*

$$\|h_k(A_k)x - h_0(A_0)x\| \to 0 \quad \text{for each } x \in \overline{A_0 H}.$$

Proof. Since $\sup_k \|h_k(A_k)\| \leqslant M$, it suffice to prove the lemma for $x \in A_0 H$. We will even show that $\|h_k(A_k)A_0 - h_0(A_0)A_0\| \to 0$.

Setting $f_k(t) = th_k(t)$ and $f_0(t) = th_0(t)$, we have

$$\begin{aligned}\|h_k(A_k)A_0 - h_0(A_0)A_0\| &\leqslant \|h_k(A_k)(A_0 - A_k)\| + \|f_k(A_k) - f_0(A_0)\| \\ &\leqslant M\|A_0 - A_k\| + \|f_k(A_k) - f_0(A_k)\| + \|f_0(A_k) - f_0(A_0)\|.\end{aligned}$$

The first term tends to 0. The second term tends to 0 because $f_k \to f_0$ uniformly on α, the third term tends to 0 because f_0 is continuous on α. ∎

We will consider now the question of which continuous functions act continuously on s. n. ideals.

THEOREM 2.4. *Let g be a continuous function on $\alpha \subset \mathbb{C}$ and let $g(0) = 0$.*

(i) *g acts continuously on a separable ideal $J \neq C^\infty$ if and only if it is Lipschitzian at $t = 0$.*

(ii) *If g is differentiable at $t = 0$, it acts continuously on all s. n. ideals.*

Proof. The part "only if" in (i) follows from Proposition 2.2.

Let operators A and A_k belong to $J_{\mathrm{n}}(\alpha)$ and let $\|A - A_k\|_J \to 0$. The function $\widetilde{g}_0$ defined in (2.2) is continuous on $\alpha \backslash 0$, $|\widetilde{g}_0(t)| < D$ for some $D > 0$, and $g(t) = t\widetilde{g}_0(t)$. Hence

$$\begin{aligned}\|g(A) - g(A_k)\|_J &= \|\widetilde{g}_0(A)A - \widetilde{g}_0(A_k)A_k\|_J \\ &\leqslant \|\widetilde{g}_0(A)A - \widetilde{g}_0(A_k)A\|_J + \|\widetilde{g}_0(A_k)(A - A_k)\|_J,\end{aligned} \tag{2.6}$$

$$\|\widetilde{g}_0(A_k)(A - A_k)\|_J \leqslant \|\widetilde{g}_0(A_k)\| \|A - A_k\|_J \leqslant D\|A - A_k\|_J \to 0. \tag{2.7}$$

Let Q be the projection on $\overline{AH}$. Then $A = QA$. Set all $h_k = \widetilde{g}_0$, $k = 0, 1, \ldots$, in Lemma 2.3. Then the operators $\widetilde{g}_0(A_k)Q$ strongly converge to $\widetilde{g}_0(A)Q$. If J is separable, it follows from Theorem III.6.3 of [11] that

$$\|(\widetilde{g}_0(A) - \widetilde{g}_0(A_k))A\|_J = \|(\widetilde{g}_0(A)Q - \widetilde{g}_0(A_k)Q)A\|_J \to 0.$$

Hence $\|g(A) - g(A_k)\|_J \to 0$ in (2.6) and part (i) is proved.

Let g be differentiable at $t = 0$. Changing the value of $\widetilde{g}_0(0)$ from 0 to $g'(0)$, we may assume that $\widetilde{g}_0$ is continuous on α. If $\|A - A_k\|_J \to 0$ then $\|A - A_k\| \to 0$, so that $\|\widetilde{g}_0(A) - \widetilde{g}_0(A)\| \to 0$. Hence

$$\|(\widetilde{g}_0(A) - \widetilde{g}_0(A_k))A\|_J \leqslant \|\widetilde{g}_0(A) - \widetilde{g}_0(A_k)\| \|A\|_J \to 0.$$

Taking into account (2.6) and (2.7), we have $\|g(A) - g(A_k)\|_J \to 0$. ∎

Although, by Proposition 2.2(ii), functions Lipschitzian at every point of α act on the unitizations of s. n. ideals, only "globally" Lipschitz functions on α act continuously.

THEOREM 2.5. *Let $J = J_0^{\phi} \neq C^{\infty}$ be a separable s. n. ideal. A function g on α acts continuously on $\widetilde{J}$ if and only if it is Lipschitzian (in the usual sense) on α.*

Proof. Suppose that g is a Lipschitz function on α. Let $B_0, B_k \in J$ and let $A_0 = t_0\mathbf{1} + B_0$, $A_k = t_k\mathbf{1} + B_k$ belong to $\widetilde{J}_{\mathrm{n}}(\alpha)$. Replacing g by the function $g(t+t_0)$ and α by the set $\{z - t_0 \colon z \in \alpha\}$, we may assume that $t_0 = 0$ and $A_0 = B_0$.

Each function $h_k = \widetilde{g}_{t_k}$ defined in (2.2) is continuous on $\alpha \backslash \{t_k\}$. Since g is Lipschitzian on α, there is $D > 0$ (see (2.2)) such that $|h_k(t)| \leqslant D$ for all $t \in \alpha$. We have from (2.5) that $g(A_k) = g(t_k)\mathbf{1} + h_k(A_k)B_k$ and $\|h_k(A_k)\| \leqslant D$. Hence $\|g(A_0) - g(A_k)\|_{\widetilde{J}} = |g(0) - g(t_k)| + \|h_0(A_0)B_0 - h_k(A_k)B_k\|_J$, and

$$\begin{aligned}
&\|h_0(A_0)B_0 - h_k(A_k)B_k\|_J \\
&\leqslant \|h_0(A_0)(B_0 - B_k)\|_J + \|(h_0(A_0) - h_k(A_k))(B_0 - B_k)\|_J + \|(h_0(A_0) - h_k(A_k))B_0\|_J \\
&\leqslant (2\|h_0(A_0)\| + \|h_k(A_k)\|)\|B_0 - B_k\|_J + \|(h_0(A_0) - h_k(A_k))B_0\|_J.
\end{aligned}$$

Hence

$$\|g(A_0) - g(A_k)\|_{\widetilde{J}} \leqslant D|t_k| + 3D\|B_0 - B_k\|_J + \|(h_0(A_0) - h_k(A_k))B_0\|_J. \tag{2.8}$$

Let $\|A_0 - A_k\|_{\widetilde{J}} \to 0$. Since $t_k \to t_0 = 0$, the functions h_k converge uniformly to h_0 on α outside $\mathbf{C}_\rho$ for any $\rho > 0$. Therefore, by Lemma 2.3, the operators $h_k(A_k)$ strongly converge to $h_0(B_0)$ on $\overline{B_0H}$. Let Q be the projection on $\overline{B_0H}$. Then $B_0 = QB_0$ and $h_k(A_k)Q$ strongly converge to $h_0(B_0)Q$. Since J is separable, it follows from Theorem III.6.3 of [11] that

$$\|(h_0(B_0) - h_k(A_k))B_0\|_J = \|(h_0(B_0)Q - h_k(A_k)Q)B_0\|_J \to 0.$$

Hence, by (2.8), $\|g(A_0) - g(A_k)\|_{\widetilde{J}} \to 0$, so g acts continuously on $\widetilde{J}$.

Conversely, let g be a non-Lipschitz function α. There exist $\lambda, t_k, s_k \in \alpha$ such that $t_k \to \lambda$, $|s_k| \leqslant k^{-2}$ and $|g(s_k + t_k) - g(t_k)| \geqslant k^3|s_k|$. Without loss of generality, assume that $\lambda = 0$.

Since $J = J_0^{\phi} \neq C^{\infty}$ and since $1 < |s_k|^{-1}k^{-2}$, one can find n_k, as in the proof of Proposition 2.2, such that $1 \leqslant |s_k|^{-1}k^{-2} \leqslant R(n_k) \leqslant k^{1/2}(|s_k|^{-1}k^{-2})$, so $k^{-2} \leqslant |s_k|R(n_k) \leqslant k^{-3/2}$. Set $A_k = \operatorname{diag}(\underbrace{s_k, \dots, s_k}_{n_k}, 0, \dots)$. Then $\|A_k\|_J = \phi(\underbrace{|s_k|, \dots, |s_k|}_{n_k}, 0, \dots) = |s_k|R(n_k) \to 0$, so that $\|t_k\mathbf{1} + A_k\|_{\widetilde{J}} = |t_k| + \|A_k\|_J \to 0$. Hence $t_k\mathbf{1} + A_k$ converge to $\mathbf{0}$ in $\widetilde{J}$.

On the other hand,

$$g(t_k\mathbf{1}+A_k)=\operatorname{diag}(\underbrace{g(t_k+s_k),\ldots,g(t_k+s_k)}_{n_k},g(t_k),\ldots)=g(t_k)\mathbf{1}+\operatorname{diag}(\underbrace{r_k,\ldots,r_k}_{n_k},0,\ldots),$$

where $r_k = g(t_k+s_k) - g(t_k)$ and $|r_k| \geqslant k^3|s_k|$. Since $g(\mathbf{0}) = g(0)\mathbf{1}$, we have

$$\|g(t_k\mathbf{1}+A_k)-g(\mathbf{0})\|_{\widetilde{J}}=|g(t_k)-g(0)|+\operatorname{diag}(\underbrace{|r_k|,\ldots,|r_k|}_{n_k},0,\ldots)$$
$$=|g(t_k)-g(0)|+|r_k|R(n_k)\geqslant|g(t_k)-g(0)|+k^3|s_k|R(n_k)\to\infty.$$

Therefore g does not act continuously on $\widetilde{J}$. ∎

REMARK 2.6. It follows from (2.8) that, for all (not only separable) ideals, the following statement holds: if g is a Lipschitz function on α, then there is $D > 0$ such that

$$\|g(A) - g(B)\|_{\widetilde{J}} \leqslant 4D\|A - B\|_{\widetilde{J}} + 2D\|A\|_{\widetilde{J}},$$

for $A, B \in \widetilde{J}_{\mathrm{n}}(\alpha)$. Thus the map $A \to g(A)$ is locally bounded on $\widetilde{J}_{\mathrm{n}}$.

Denote by $\mathrm{Lip}(\alpha)$ the Banach algebra of all Lipschitz functions g on α with norm

$$\|g\|_{\mathrm{Lip}} = \sup_{t\in\alpha}|g(t)| + K(g,\alpha), \quad \text{where } K(g,\alpha) = \sup_{t,s\in\alpha}|\widetilde{g}_s(t)|.$$

Denote by $\mathrm{Lip}_0(\alpha)$ the Banach algebra of all continuous functions g on α Lipschitzian at $t = 0$ with $g(0) = 0$ and with norm

$$\|g\|_{\mathrm{Lip}_0} = \sup_{t\in\alpha}|g(t)| + K(g,0), \quad \text{where } K(g,0) = \sup_{t\in\alpha}|\widetilde{g}_0(t)|.$$

PROPOSITION 2.7. *Let $J \neq C^\infty$ be an s. n. ideal.*

(i) *For $A \in J_n(\alpha)$, the homomorphism $g \to g(A)$ from* $\mathrm{Lip}_0(\alpha)$ *into J is bounded.*

(ii) *For $A \in \widetilde{J}_n(\alpha)$, the homomorphism $g \to g(A)$ from* $\mathrm{Lip}(\alpha)$ *into $\widetilde{J}$ is bounded.*

Proof. Let $A \in J_{\mathrm{n}}(\alpha)$. By Proposition 2.2(i), the map $g \to g(A)$ is a homomorphism from $\mathrm{Lip}_0(\alpha)$ into J. The function $\widetilde{g}_0$ is continuous on $\alpha\backslash\{0\}$ and $\|\widetilde{g}_0(A)\| \leqslant \sup\{|\widetilde{g}_0(t)| : t \in \alpha\} = K(g,0)$. Thus

$$\|g(A)\|_J = \|A\widetilde{g}_0(A)\|_J \leqslant \|A\|_J\|\widetilde{g}_0(A)\| \leqslant \|A\|_J\|g\|_{\mathrm{Lip}_0}.$$

Let now $A = s\mathbf{1} + B \in \widetilde{J}_{\mathrm{n}}(\alpha)$, where $s \in \alpha$ and $B \in J_{\mathrm{n}}$. By Proposition 2.2(ii), the map $g \to g(A)$ is a homomorphism from $\mathrm{Lip}(\alpha)$ into $\widetilde{J}$. The function $\widetilde{g}_s(t)$ is continuous on $\alpha\backslash\{s\}$ and $\|\widetilde{g}_s(A)\| \leqslant \sup\{|\widetilde{g}_s(t)| : t \in \alpha\} \leqslant K(g,\alpha)$. Thus $\|\widetilde{g}_s(A)\| \leqslant K(g,\alpha)$, so that, by (2.5),

$$\|g(A)\|_{\widetilde{J}} \leqslant |g(s)| + \|B\|_J\|\widetilde{g}_s(A)\| \leqslant |s|K(g,\alpha) + \|B\|_J K(g,\alpha) \leqslant \|A\|_{\widetilde{J}}\|g\|_{\mathrm{Lip}}.$$ ∎

3. J-LIPSCHITZ FUNCTIONS ACT ON THE DOMAINS OF WEAKLY CLOSED DERIVATIONS

It was proved in [18] that Operator Lipschitz functions act on the domains of all weakly closed $*$-derivations of C^*-algebras. In this section we show that J-Lipschitz functions act on the domains of all weakly closed $*$-derivations of coseparable s. n. ideals J.

Let δ be a $*$-derivation on J^ϕ (see (1.3)). It is closed (weakly closed) if its graph $\{(A,\delta(A)) : A \in D(\delta)\}$ is closed in $J^\phi \oplus J^\phi$ in norm (respectively, in the weak topology $\sigma(J^\phi \dotplus J^\phi, J_0^{\phi^*} \dotplus J_0^{\phi^*})$).

In this section we denote by α a compact subset of $\mathbb{R}$ and by Γ an open subset of $\mathbb{R}$. Let J_{sa} be the set of all selfadjoint operators in J. Set

$$J_{\mathrm{sa}}(\alpha) = \{A \in J_{\mathrm{sa}} : \mathrm{Sp}(A) \subseteq \alpha\}.$$

DEFINITION 3.1. Let J be an s. n. ideal or $B(H)$. A continuous function g on α is *J-Lipschitzian*, if there is $D > 0$ such that

$$\|g(A) - g(B)\|_J \leqslant D\|A - B\|_J \quad \text{for } A, B \in J_{\mathrm{sa}}(\alpha). \tag{3.1}$$

A function g on Γ is J-Lipschitzian, if it is J-Lipschitzian on each $\alpha \subset \Gamma$ (the constant D in (3.1) depends on α).

J-Lipschitz functions are Lipschitzian in the usual sense: if Q is a rank one projection then $\|Q\|_J = 1$ and, for $t, s \in \alpha$,

$$|g(t) - g(s)| = \|(g(t) - g(s))Q\|_J = \|g(tQ) - g(sQ)\|_J < D\|tQ - sQ\|_J = D|t - s|.$$

We call $B(H)$-Lipschitz functions Operator Lipschitzian. They are differentiable (see [13]), but not necessarily continuously differentiable (see [17]). The class of Operator Lipschitz functions on $[a,b]$ lies between two Besov spaces $B^1_{\infty 1}(a,b)$ and $B^1_{11}(a,b)$ (see [22]). C^p-Lipschitz functions, for $p \in (1,\infty)$, constitute wider classes than Operator Lipschitz functions; they contain non-differentiable functions, in particular, $g(t) = |t|$ (see [8]).

It was proved in [19] that if g is a J-Lipschitz function, for $J = J^\phi$ or J_0^ϕ, then property (3.1) can be extended from J_{sa} to $B(H)_{\mathrm{sa}}$ in the following way: *there is a minimal constant $k_J(g,\alpha) > 0$ such that the condition $A - B \in J$, for $A, B \in B(H)_{\mathrm{sa}}(\alpha)$, implies*

$$g(A) - g(B) \in J \quad \text{and} \quad \|g(A) - g(B)\|_J \leqslant k_J(g,\alpha)\|A - B\|_J. \tag{3.2}$$

DEFINITION 3.2. Let J be an s. n. ideal or $B(H)$. A continuous function g on Γ is *Gateaux J-differentiable* at $A = A^*$ with $\mathrm{Sp}(A) \subset \Gamma$, if there is a bounded linear operator g_A^∇ from J_{sa} into J such that, for any $X \in J_{\mathrm{sa}}$ and sufficiently small $t \in \mathbb{R}$,

$$g(A + tX) - g(A) \in J \quad \text{and} \quad \left\|\frac{1}{t}(g(A + tX) - g(A)) - g_A^\nabla(X)\right\|_J \to 0, \quad \text{as } t \to 0.$$

If g is Gateaux J-differentiable at each $A \in B(H)_{\mathrm{sa}}(\Gamma)$, it is called Gateaux J-differentiable on Γ.

For separable ideals J the following conditions are equivalent (see [18]):

(i) *g is a differentiable, J-Lipschitz function on an open set Γ;*

(ii) *g is Gateaux J-differentiable on Γ.*

Let g be a Gateaux J-differentiable function on Γ, let α be a compact in Γ and let $\mathrm{Sp}(A)$ lie in the interior $\mathrm{int}(\alpha)$ of α. It follows from (3.2) and Definition 3.2 that, for any $X \in J_{\mathrm{sa}}$,

$$\|g_A^\nabla(X)\|_J \leqslant k_J(g,\alpha)\|X\|_J. \tag{3.3}$$

The next lemma (see, for example, [3],[18]) gives us many examples of Gateaux J-differentiable and J-Lipschitz functions.

LEMMA 3.3. *Let $g \in L^1(\mathbb{R})$ be a continuous function. Let $\widehat{g}(s)$ be its Fourier transform and*

$$\int_{-\infty}^{\infty} |s\widehat{g}(s)|\mathrm{d}s < \infty \tag{3.4}$$

(for example, $g'' \in L^2(\mathbb{R})$). Let J be an s. n. ideal or $B(H)$. Then g is J-Lipschitzian and Gateaux J-differentiable on $\mathbb{R}$. For each $A = A^*$ and $X \in J_{sa}$,

$$g_A^\nabla(X) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-\mathrm{i}s)\widehat{g}(s)\Big(\int_0^1 \mathrm{e}^{-\mathrm{i}stA}X\mathrm{e}^{-\mathrm{i}s(1-t)A}\mathrm{d}t\Big)\mathrm{d}s. \tag{3.5}$$

A stronger fact follows from the results of Peller in [23] that the conclusion of Lemma 3.3 holds for all functions from the Besov space $B^1_{\infty 1}$.

Bratteli, Elliott and Jorgensen [6] proved that each function g that satisfies (3.4) acts on the domains of all closed $*$-derivations δ of C^*-algebras. It was shown in [18] that

$$\delta(g(A)) = g_A^\nabla(\delta(A)) \quad \text{for } A = A^* \in D(\delta).$$

To extend this result to closed $*$-derivations on s. n. ideals we need the following adaptation of Lemma 2 of [24] to derivations on Banach algebras.

LEMMA 3.4. *Let δ be a closed derivation on a unital Banach algebra $\mathcal{A}$. If $A \in D(\delta)$ then*

$$\mathrm{e}^A \in D(\delta) \quad \textit{and} \quad \delta(\mathrm{e}^A) = \int_0^1 \mathrm{e}^{tA}\delta(A)\mathrm{e}^{(1-t)A}\mathrm{d}t. \tag{3.6}$$

Any $*$-derivation δ on J extends to a $*$-derivation $\widetilde{\delta}$ on $\widetilde{J}$ by setting

$$\widetilde{\delta}(\lambda\mathbf{1} + A) = \delta(A) \in J \quad \text{for all } A \in D(\delta), \lambda \in \mathbb{C}. \tag{3.7}$$

It is (weakly) closed if δ is (weakly) closed. If $J = B(H)$, we set $\widetilde{J} = B(H)$ and $\widetilde{\delta} = \delta$.

If J is an s. n. ideal and $A = \mathrm{diag}(a_1, \dots, a_n, \dots) \in J_{\mathrm{sa}}$ with $a_n \in \mathbb{R}$, then $\mathrm{e}^{\mathrm{i}A} - \mathbf{1} = \mathrm{diag}(\mathrm{e}^{\mathrm{i}a_1} - 1, \dots, \mathrm{e}^{\mathrm{i}a_n} - 1, \dots) \in J$ and $|\mathrm{e}^{\mathrm{i}a_n} - 1| = 2\sin(a_n/2) \leqslant a_n$. It follows from III.3.1 in [11] that, for $s \in \mathbb{R}$,

$$\|\mathrm{e}^{\mathrm{i}A} - \mathbf{1}\|_J \leqslant \|A\|_J \quad \text{so } \|\mathrm{e}^{\mathrm{i}sA}\|_{\widetilde{J}} \leqslant 1 + \|sA\|_J \leqslant (1 + \|A\|_J)(1 + |s|). \tag{3.8}$$

Hence, for $\lambda\mathbf{1} + A \in \widetilde{J}$ and $\lambda, s \in \mathbb{R}$,

$$\|\mathrm{e}^{\mathrm{i}s(\lambda\mathbf{1}+A)}\|_{\widetilde{J}} = \|\mathrm{e}^{\mathrm{i}sA}\|_{\widetilde{J}} \leqslant (1 + \|A\|_J)(1 + |s|). \tag{3.9}$$

LEMMA 3.5. *Let J be an s. n. ideal or $B(H)$ and let δ be a closed $*$-derivation on J. If g satisfies* (3.4), *then, for each $A = A^* \in D(\widetilde{\delta})$,*

$$g(A) \in D(\widetilde{\delta}) \quad \textit{and} \quad \widetilde{\delta}(g(A)) = g_A^{\nabla}(\widetilde{\delta}(A)). \tag{3.10}$$

Proof. It follows from (3.9) and (3.4) that

$$g(A) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}sA} \widehat{g}(s) \mathrm{d}s = \frac{1}{2\pi} \lim_{n\to\infty} \int_{-n}^{n} \mathrm{e}^{-\mathrm{i}sA} \widehat{g}(s) \mathrm{d}s.$$

Since g is Lipschitzian (in the usual sense), we have $g(A) \in \widetilde{J}$, by Proposition 2.2. We have from (3.7) and Lemma 3.4 that $\widetilde{\delta}(A) \in J$ and $\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \in J$, for $s \in \mathbb{R}$, and

$$\begin{aligned} \|\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA})\|_J &= \Big\| \int_0^1 \mathrm{e}^{-\mathrm{i}stA} \widetilde{\delta}(-\mathrm{i}sA) \mathrm{e}^{-\mathrm{i}s(1-t)A} \mathrm{d}t \Big\|_J \\ &\leqslant \int_0^1 \|\mathrm{e}^{-\mathrm{i}stA}\| \|s\widetilde{\delta}(A)\|_J \|\mathrm{e}^{-\mathrm{i}s(1-t)A}\|_J dt = |s| \|\widetilde{\delta}(A)\|_J. \end{aligned} \tag{3.11}$$

This implies that $\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA})$ is continuous in norm $\|\cdot\|_J$ with respect to s, since

$$\begin{aligned} \|\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) - \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}tA})\|_J &= \|\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}(\mathbf{1} - \mathrm{e}^{-\mathrm{i}(t-s)A}))\|_J \\ &\leqslant \|\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA})\|_J \|\mathbf{1} - \mathrm{e}^{-\mathrm{i}(t-s)A}\|_{\widetilde{J}} + \|\mathrm{e}^{-\mathrm{i}sA}\|_{\widetilde{J}} \|\widetilde{\delta}(\mathrm{e}^{-\mathrm{i}(t-s)A})\|_J \to 0, \end{aligned}$$

as $t \to s$. Since $g \in L^1(\mathbb{R})$, the function $\widehat{g}$ is continuous. Hence the integral $\int_{-n}^{n} \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \widehat{g}(s) \mathrm{d}s$ is the limit of Riemann sums. Since $\widetilde{\delta}$ is a closed derivation, we have $\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}sA} \widehat{g}(s) \mathrm{d}s \in D(\widetilde{\delta})$ and $\int_{-n}^{n} \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \widehat{g}(s) \mathrm{d}s = \widetilde{\delta}\Big(\int_{-n}^{n} \mathrm{e}^{-\mathrm{i}sA} \widehat{g}(s) \mathrm{d}s \Big)$. Taking (3.4) and (3.11) into account, we have that the integral $\int_{-\infty}^{\infty} \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \widehat{g}(s) \mathrm{d}s = \lim_{n\to\infty} \int_{-n}^{n} \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \widehat{g}(s) \mathrm{d}s$ converges. Since $\widetilde{\delta}$ is a closed derivation, we have that

$$g(A) \in D(\widetilde{\delta}) \quad \text{and} \quad \widetilde{\delta}(g(A)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{\delta}(\mathrm{e}^{-\mathrm{i}sA}) \widehat{g}(s) \mathrm{d}s.$$

Substituting (3.6) in this formula and comparing it with (3.5), we have

$$\widetilde{\delta}(g(A)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-\mathrm{i}s)\widehat{g}(s)\Big(\int_0^1 \mathrm{e}^{-\mathrm{i}stA}\widetilde{\delta}(A)\mathrm{e}^{-\mathrm{i}s(1-t)A}\mathrm{d}t\Big)\mathrm{d}s = g_A^{\nabla}(\widetilde{\delta}(A)). \quad ∎$$

COROLLARY 3.6. *Let g have a continuous second derivative on Γ. If J is an s. n. ideal or $B(H)$ then:*

(i) *g is J-Lipschitzian and Gateaux J-differentiable on Γ;*

(ii) *for any closed $*$-derivation δ on J and for each $A = A^* \in D(\widetilde{\delta})$ with $\mathrm{Sp}(A) \subset \Gamma$,*

$$g(A) \in D(\widetilde{\delta}) \quad \textit{and} \quad \widetilde{\delta}(g(A)) = g_A^{\nabla}(\widetilde{\delta}(A)) \in J.$$

Proof. Let $\alpha \subset \Gamma$ and let h be an infinitely differentiable function on $\mathbb{R}$ such that $\mathrm{supp}(h) \subset \Gamma$ and $h(t) = 1$ in some neighbourhood U of α. The function gh has a continuous second derivative on Γ and $\mathrm{supp}(gh) \subset \Gamma$. Extend gh to $\mathbb{R}$ by setting $(gh)(t) = 0$, for $t \notin \Gamma$, and denote this extension also by gh. By Lemma 3.3, gh is J-Lipschitzian and Gateaux J-differentiable on $\mathbb{R}$. Since $(gh)(B) = g(B)h(B) = g(B)$, for $B = B^*$ with $\mathrm{Sp}(B) \subset U$, we have that g is J-Lipschitzian on α, Gateaux J-differentiable on U and $g_B^{\nabla} = (gh)_B^{\nabla}$. Hence g is J-Lipschitzian and Gateaux J-differentiable on Γ.

Let $A = A^* \in D(\widetilde{\delta})$ with $\mathrm{Sp}(A) \subseteq \alpha$. By Lemma 3.5, $g(A) = (gh)(A) \in D(\widetilde{\delta})$ and

$$\widetilde{\delta}(g(A)) = \widetilde{\delta}((gh)(A)) = (gh)_A^{\nabla}(\widetilde{\delta}(A)) = g_A^{\nabla}(\widetilde{\delta}(A)) \in J$$

which completes the proof. ∎

Let g and φ be continuous functions on $\mathbb{R}$ and φ have a compact support. The convolution

$$(g * \varphi)(t) = \int_{-\infty}^{\infty} g(t-r)\varphi(r)\mathrm{d}r$$

is a continuous function. For $h(t) = t^n$,

$$(h * \varphi)(t) = \int_{-\infty}^{\infty}(t-r)^n\varphi(r)\mathrm{d}r = \sum_{i=0}^{n}\binom{n}{i} t^{n-i}\int_{-\infty}^{\infty} r^i\varphi(r)\mathrm{d}r.$$

For a bounded operator A,

$$(h * \varphi)(A) = \sum_{i=0}^{n}\binom{n}{i} A^{n-i}\int_{-\infty}^{\infty} r^i\varphi(r)\mathrm{d}r = \int_{-\infty}^{\infty}(A - r\mathbf{1})^n\varphi(r)\mathrm{d}r.$$

Therefore for any polynomial h,

$$(h * \varphi)(A) = \int_{-\infty}^{\infty} h(A - r\mathbf{1})\varphi(r)\mathrm{d}r.$$

Let g be a continuous function on Γ and let $A = A^*$ with $\mathrm{Sp}(A) \subset \Gamma$. For a compact β in Γ with $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$, choose α in Γ and $a > 0$ such that

$$\mathrm{Sp}(A) \subset \mathrm{int}(\alpha),\ \alpha \subset \mathrm{int}(\beta) \quad \text{and} \quad t + r \in \beta, \quad \text{for } t \in \alpha \text{ and } r \in [-a, a]. \tag{3.12}$$

Choose a non-negative infinitely differentiable function φ with $\mathrm{supp}(\varphi) = [-a, a]$ and

$$\int_{-\infty}^{\infty} \varphi(r)\mathrm{d}r = 1. \tag{3.13}$$

Considering polynomials that uniformly converge to g on β, we obtain

$$(g * \varphi)(A) = \int_{-\infty}^{\infty} g(A - r\mathbf{1})\varphi(r)\mathrm{d}r. \tag{3.14}$$

Set

$$\varphi_n(t) = n\varphi(nt) \quad \text{and} \quad g_n = g * \varphi_n.$$

Then

$$\int_{-\infty}^{\infty} \varphi_n(t)\mathrm{d}t = 1 \quad \text{and} \quad \gamma_n = \ \mathrm{supp}(\varphi_n) = [-\frac{a}{n}, \frac{a}{n}]. \tag{3.15}$$

LEMMA 3.7. *Let g be a Lipschitz function* (*in the usual sense*) *on each compact subset of Γ.*

(i) *If J is a separable s. n. ideal or $B(H)$, then $\|g(A) - g_n(A)\|_{\widetilde{J}} \to 0$ for each $A \in \widetilde{J}_{\mathrm{sa}}(\Gamma)$.*

(ii) *If J is a coseparable s. n. ideal, then $g(A) - g_n(A) \to 0$ in the weak topology on $\widetilde{J}$, for each $A \in \widetilde{J}_{\mathrm{sa}}(\Gamma)$.*

(iii) *Let J be an s. n. ideal or $B(H)$ and let δ be a closed $*$-derivation on J. If g is a J-Lipschitz function on Γ and $A = A^* \in D(\delta)$ with $\mathrm{Sp}(A) \subset \Gamma$, then*

$$g_n(A) \in D(\widetilde{\delta}) \quad \text{and} \quad \widetilde{\delta}(g_n(A)) = (g_n)^{\nabla}_A(\delta(A)) \in J. \tag{3.16}$$

For each compact subset β of Γ such that $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$,

$$\|\widetilde{\delta}(g_n(A))\|_J \leqslant k_J(g, \beta)\|\delta(A)\|_J. \tag{3.17}$$

Proof. For $A \in \widetilde{J}_{\mathrm{sa}}(\Gamma)$, let $\alpha \subset \beta$ in Γ and $a > 0$ be as in (3.12) and let φ_n be as in (3.15). Since g and all g_n are Lipschitzian in the usual sense on α, we have from Proposition 2.2 that $g(A), g_n(A) \in \widetilde{J}$. By (3.14),

$$\|g(A) - g_n(A)\|_{\widetilde{J}} = \Big\| \int_{-\infty}^{\infty} (g(A) - g(A - r\mathbf{1}))\varphi_n(r)\mathrm{d}r \Big\|_{\widetilde{J}}$$

$$\leqslant \int_{-\infty}^{\infty} \|g(A) - g(A - r\mathbf{1})\|_{\widetilde{J}} \varphi_n(r) \mathrm{d}r \leqslant \sup_{r \in \gamma_n} \|g(A) - g(A - r\mathbf{1})\|_{\widetilde{J}}. \tag{3.18}$$

If $J \neq B(H)$ is separable, it follows from Theorem 2.5 that $\|g(A) - g(A - r\mathbf{1})\|_{\widetilde{J}} \to 0$, as $r \to 0$.

If $J = B(H)$, then $\|g(A) - g(A - r\mathbf{1})\| \to 0$, as $r \to 0$. Thus in both cases $\|g(A) - g_n(A)\|_{\widetilde{J}} \to 0$. Part (i) is proved.

Let $J = J^{\phi}$ be a coseparable s. n. ideal. Set $B_n = g(A) - g_n(A)$. It follows from Remark 2.6 that $\sup_{r\in\beta} \|g(A) - g(A - r\mathbf{1})\|_{\widetilde{J}} < \infty$. Hence, by (3.18), $\sup_n \|B_n\|_{\widetilde{J}} \leqslant \infty$.

Similar to (3.18), we have the following estimate for the operator norm of B_n: $\|B_n\| = \|g(A) - g_n(A)\| \leqslant \sup_{r\in\gamma_n} \|g(A) - g(A - r\mathbf{1})\|$, where $\gamma_n = \mathrm{supp}(\varphi_n) = [-\frac{a}{n}, \frac{a}{n}]$. Since

$$\|g(A) - g(A - r\mathbf{1})\| = \sup_{\lambda \in \mathrm{Sp}(A)} |g(\lambda) - g(\lambda - r)| \leqslant K(g, \beta)|r|,$$

where $K(g, \beta)$ was defined after Remark 2.6, we have $\|B_n\| \leqslant K(g, \beta)\frac{a}{n} \to 0$, as $n \to \infty$, where the constant was defined after Remark 2.6.

Let $B_n = \lambda_n \mathbf{1} + A_n \in \widetilde{J}$. Since $\sup_n \|B_n\|_{\widetilde{J}} \leqslant \infty$ and $\|B_n\| \to 0$, we have $\lambda_n \to 0$, $M = \sup_n \|A_n\|_J < \infty$ and $\|A_n\| \to 0$. Let $I = J_0^{\phi^*}$, $T \in I$ and $\varepsilon > 0$. Since I is a separable ideal, there is a finite rank operator T_ε such that $\|T - T_\varepsilon\|_I \leqslant \frac{\varepsilon}{M}$. By (2.1), $|F_T(A_n)| \leqslant |F_{T_\varepsilon}(A_n)| + |F_{T - T_\varepsilon}(A_n)| \leqslant |F_{T_\varepsilon}(A_n)| + \|T - T_\varepsilon\|_I \|A_n\|_J \leqslant |F_{T_\varepsilon}(A_n)| + \varepsilon$. Since $\|A_n\| \to 0$, we have $F_{T_\varepsilon}(A_n) \to 0$, so $F_T(A_n) \to 0$. Thus $B_n \to 0$ in the weak topology on $\widetilde{J}$. Part (ii) is proved.

Let now J be an s. n. ideal or $B(H)$. Let $A = A^* \in D(\delta)$ with $\mathrm{Sp}(A) \subset \mathrm{int}(\beta) \subset \Gamma$. Choose a compact subset α of β and $a > 0$ as in (3.12). We have

$$\mathrm{Sp}(A - r\mathbf{1}) \subseteq \beta, \quad \text{for } r \in \gamma_n = \mathrm{supp}(\varphi_n) = [-\frac{a}{n}, \frac{a}{n}] \text{ and all } n.$$

Since g_n are infinitely differentiable functions on $\mathrm{int}(\alpha)$, it follows from Corollary 3.6(ii) that (3.16) holds for them.

If g is J-Lipschitzian on Γ, we have from (3.3) and (3.16) that

$$\|\widetilde{\delta}(g_n(A))\|_J = \|(g_n)_A^{\nabla}(\delta(A))\|_J \leqslant k_J(g_n, \alpha)\|\delta(A)\|_J. \tag{3.19}$$

We obtain from (3.2) and (3.14) that, if $B, C \in B(H)_{\mathrm{sa}}(\alpha)$ and $B - C \in J$, then

$$\|g_n(B) - g_n(C)\|_J \leqslant \int_{-\infty}^{\infty} \|g(B - r\mathbf{1}) - g(C - r\mathbf{1})\|_J \varphi_n(r) \mathrm{d}r$$

$$\leqslant \int_{-\infty}^{\infty} k_J(g,\beta)\|B-C\|_J\varphi_n(r)\mathrm{d}r = k_J(g,\beta)\|B-C\|_J.$$

Hence $k_J(g_n,\alpha) \leqslant k_J(g,\beta)$, so (3.17) follows from (3.19). ∎

THEOREM 3.8. *Let J be a coseparable ideal. Each J-Lipschitz function g on $\Gamma \subseteq \mathbb{R}$ with $g(0)=0$ acts on the domain of every weakly closed $*$-derivation δ on J:*

$$\text{if } A = A^* \in D(\delta) \text{ and } \mathrm{Sp}(A) \subset \Gamma \text{ then } g(A) \in D(\delta).$$

Moreover, for each compact β in Γ such that $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$,

$$\|\delta(g(A))\|_J \leqslant k_J(g,\beta)\|\delta(A)\|_J.$$

Proof. Let β be a compact in Γ such that $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$. Choose α in Γ and $a > 0$ satisfying (3.12) and let φ satisfy (3.13). For $g_n = g * \varphi_n$, it follows from (3.16) and Lemma 3.7(ii) and (iii) that

$$g_n(A) \in D(\widetilde{\delta}), \quad \widetilde{\delta}(g_n(A)) \in J, \quad \|\widetilde{\delta}(g_n(A))\|_J \leqslant k_J(g,\beta)\|\delta(A)\|_J$$

and $g_n(A) \to g(A)$ in the weak topology on $\widetilde{J}$.

Since the sequence $\{\widetilde{\delta}(g_n(A))\}$ is bounded in J, it has a weak cluster point $R \in J$ and $\|R\|_J \leqslant k_J(g,\beta)\|\delta(A)\|_J$. Hence the pair $(g(A), R)$ belongs to the weak closure of the graph of $\widetilde{\delta}$. Since δ is weakly closed, the derivation $\widetilde{\delta}$ is also weakly closed. Thus $g(A) \in D(\widetilde{\delta})$ and $\delta(g(A)) = R$. Since $g(0) = 0$, we have $g(A) \in J$. Hence $g(A) \in D(\delta)$. ∎

For $J = B(H)$, Theorem 3.8 was proved in [18].

COROLLARY 3.9. *Let J be a reflexive ideal (for example, $J = C^p$, $1 < p < \infty$). Each J-Lipschitz function g on $\Gamma \subseteq \mathbb{R}$ with $g(0) = 0$ acts on the domains of all closed $*$-derivations on J.*

Proof. By Proposition 2.1, J is coseparable. Since J is reflexive, each closed $*$-derivation on J is weakly closed. Thus the proof follows from Theorem 3.8. ∎

For a bounded derivation δ on $B(H)$ and $A = A^* \in B(H)$, Kittaneh obtained in [20] (see also [12]) that $\delta(A) \in C^2$ implies $\delta(g(A)) \in C^2$ for all Lipschitz (in the usual sense) functions g on $\mathrm{Sp}(A)$. In Section 4 of [19] this result was generalized to all separable and coseparable s. n. ideals J : $\delta(A) \in J$ implies $\delta(g(A)) \in J$ for all J-Lipschitz functions g on $\mathrm{Sp}(A)$.

In [25] the second author obtained an analogue of these results for unbounded closed $*$-derivations δ on C^*-subalgebras of $B(H)$: if $A = A^* \in D(\delta)$ and $\delta(A) \in C^p$, for $1 < p < \infty$, then $g(A) \in D(\delta)$ and $\delta(g(A)) \in C^p$, for all functions g on $\mathrm{Sp}(A)$ from a special subclass of C^p-Lipschitz functions introduced in [8]. The next theorem extends this result to all reflexive ideals I and all I-Lipschitz functions.

THEOREM 3.10. *Let J be a separable s. n. ideal or $B(H)$ and let δ be a closed $*$-derivation on J. Let I be a reflexive s. n. ideal and $I \subseteq J$. Suppose that $\delta(A) \in I$, for some $A = A^* \in D(\delta)$, and $\mathrm{Sp}(A) \subset \Gamma$. Then for each I-Lipschitz function g on Γ (assume that $g(0) = 0$ if $J \neq B(H)$)*

$$g(A) \in D(\delta) \quad \textit{and} \quad \delta(g(A)) \in I.$$

Moreover, for each compact subset β of Γ such that $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$,

$$\|\delta(g(A))\|_I \leqslant k_I(g,\beta)\|\delta(A)\|_I.$$

Proof. Let α, β in Γ and $a > 0$ be as in (3.12) and let φ_n be as in (3.15). Since g is a Lipschitz function in the usual sense, it follows from Lemma 3.7(i) that $\|g(A) - g_n(A)\|_{\widetilde{J}} \to 0$, for $g_n = g * \varphi_n$. Since g is an I-Lipschitz function, we have from (3.16) and (3.17) that

$$g_n(A) \in D(\widetilde{\delta}),\ \widetilde{\delta}(g_n(A)) \in I \quad \text{and} \quad \|\widetilde{\delta}(g_n(A))\|_I \leqslant k_I(g,\beta)\|\delta(A)\|_I.$$

Let I^* be the dual space of I. Since I is reflexive and the sequence $\{\widetilde{\delta}(g_n(A))\}$ is bounded in I, it has a cluster point $R \in I$ in the $\sigma(I, I^*)$-topology and $\|R\|_I \leqslant k_I(g,\beta)\|\delta(A)\|_I$. Hence there are linear finite combinations

$$R_k = \sum_{n=m_k}^{n_k} \lambda_n \widetilde{\delta}(g_n(A)) \in I \text{ with } \sum_{n=m_k}^{n_k} \lambda_n = 1, \quad \text{where } m_k < n_k,$$

converging to R in $\|\cdot\|_I$ and $m_k \to \infty$, as $k \to \infty$. Set $B_k = \sum_{n=m_k}^{n_k} \lambda_n g_n(A)$. Then $B_k \in D(\widetilde{\delta})$, $R_k = \widetilde{\delta}(B_k)$ and

$$\|g(A) - B_k\|_{\widetilde{J}} \leqslant \sum_{n=m_k}^{n_k} \lambda_n \|g(A) - g_n(A)\|_{\widetilde{J}} < \sup_{m_k \leqslant n \leqslant n_k} \|g(A) - g_n(A)\|_{\widetilde{J}} \to 0.$$

Since $I \subseteq J$, we have from Proposition 2.1(i) that there is $c > 0$ such that $\|X\|_J \leqslant c\|X\|_I$, for $X \in I$. Hence $R_k = \widetilde{\delta}(B_k)$ converge to R in $\|\cdot\|_J$. Since $\widetilde{\delta}$ is closed, $g(A) \in D(\widetilde{\delta})$ and $\widetilde{\delta}(g(A)) = R \in I$.

If $J = B(H)$, then $\widetilde{\delta} = \delta$ and the theorem is proved. If $J \neq B(H)$, then the condition $g(0) = 0$ implies $g(A) \in J$. Hence $g(A) \in D(\delta)$ and $\delta(g(A)) = R \in I$. ∎

Repeating the proof of Theorem 3.10 and using part (ii) of Lemma 3.7 instead of part (i), we obtain the following result.

THEOREM 3.11. *Let J be a coseparable s. n. ideal and let δ be a weakly closed $*$-derivation on J. Let I be a reflexive s. n. ideal and $I \subseteq J$. Suppose that $\delta(A) \in I$, for some $A = A^* \in D(\delta)$, and $\mathrm{Sp}(A) \subset \Gamma$. Then for each I-Lipschitz function g on Γ (assume that $g(0) = 0$ if $J \neq B(H)$)*

$$g(A) \in D(\delta) \quad \textit{and} \quad \delta(g(A)) \in I.$$

Moreover, for each compact subset β of Γ such that $\mathrm{Sp}(A) \subset \mathrm{int}(\beta)$,

$$\|\delta(g(A))\|_I \leqslant k_I(g,\beta)\|\delta(A)\|_I.$$

It was proved in [8] that, for each $1 < p < \infty$, the function $h(t) = |t|$ is C^p-Lipschitzian on $\mathbb{R}$ and there is a universal constant γ_p such that, for all bounded selfadjoint operators T and all selfadjoint $A \in C^p$,

$$\|[T,|A|]\|_p \leqslant \gamma_p\|[T,A]\|_p. \tag{3.20}$$

The constants γ_p were evaluated in [1] and it was shown there in Theorem 1 that

$$k_{C^p}(h,\beta) \leqslant \gamma_p, \tag{3.21}$$

for every compact subset β of $\mathbb{R}$. Using Theorem 3.10, we will extend now inequality (3.20) to unbounded operators.

Each symmetric densely defined operator T with domain $D(T)$ defines a closed $*$-derivation δ_T on C^p by the formula:

$$\delta_T(A) = i[T,A], \quad \text{where } [T,A] = \text{ Closure}((TA - AT)|_{D(T)}), \tag{3.22}$$

with domain

$$D(\delta_T) = \{A \in C^p : AD(T) \subseteq D(T),\ A^*D(T) \subseteq D(T) \text{ and } [T,A] \in C^p\}.$$

Applying (3.21) and Theorem 3.10 to such derivations, we have

COROLLARY 3.12. *Let T be an unbounded symmetric operator. If $A = A^* \in C^p$, $AD(T) \subseteq D(T)$ and $[T,A] \in C^p$ (see* (3.22)*), then $|A|D(T) \subseteq D(T)$, $[T,|A|] \in C^p$ and*

$$\|[T,|A|]\|_p \leqslant \gamma_p\|[T,A]\|_p.$$

Problem 3.13. It was proved in [18] that functions act on the domains of all weakly closed $*$-derivations on C^*-algebras *if and only if* they are Operator Lipschitzian. It follows from Theorem 3.8 that, for coseparable J, J-Lipschitz functions act on the domains of weakly closed $*$-derivations on J. *Do only J-Lipschitz functions act on all weakly closed $*$-derivations on J*?

Problem 3.14. It was shown in [18] that Gateaux $B(H)$-differentiable functions act on the domains of all closed $*$-derivations δ on C^*-algebras and

$$\delta(g(A)) = g_A^{\nabla}(\delta(A)). \tag{3.23}$$

It follows from Corollary 3.9 that J-Lipschitz functions and, therefore, Gateaux J-differentiable functions act on the domains of all closed $*$-derivations on reflexive ideals J.

1. *Do Gateaux J-differentiable functions act on the domains of all closed $*$-derivations on all s. n. ideals J*?

2. All functions g satisfying (3.4) act on the domains of all closed $*$-derivations on all s. n. ideals and (3.23) holds for them. *Does* (3.23) *hold for every Gateaux J-differentiable function that acts on the domain of a closed $*$-derivation δ on J*?

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