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INTERPOLATION IN THE NONCOMMUTATIVE SCHUR-AGLER CLASS

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ABSTRACT. The class of Schur-Agler functions over a domain $\mathcal{D} \subset \mathbb{C}^d$ is defined as the class of holomorphic operator-valued functions on $\mathcal{D}$ for which a certain von Neumann inequality is satisfied when a commuting tuple of operators satisfying a certain polynomial norm inequality is plugged in for the variables. There now has been introduced a noncommutative version of the Schur-Agler class which consists of formal power series in noncommuting indeterminates satisfying a noncommutative version of the von Neumann inequality when a tuple of operators (not necessarily commuting) coming from a noncommutative operator ball is plugged in for the formal indeterminates. The purpose of this paper is to extend the previously developed interpolation theory for the commutative Schur-Agler class to this noncommutative setting.

KEYWORDS: *Noncommutative Schur-Agler class, von Neumann inequality, formal power series in noncommutative indeterminates, conservative noncommutative structured multidimensional linear system, interpolation.*

MSC (2000): Primary 47A57; Secondary 47A48, 47A13, 13F25, 93C05.

1. INTRODUCTION

1.1. THE CLASSICAL SETTING. By way of introduction we recall the classical Schur class $\mathcal{S}$ of analytic functions mapping the unit disk $\mathbb{D}$ into the closed unit disk $\overline{\mathbb{D}}$. The operator-valued *Schur class* $\mathcal{S}(\mathcal{U},\mathcal{Y})$ consists, by definition, of analytic functions F on $\mathbb{D}$ with values $F(z)$ equal to contraction operators between two Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. In what follows, the symbol $\mathcal{L}(\mathcal{U},\mathcal{Y})$ stands for the algebra of bounded linear operators mapping $\mathcal{U}$ into $\mathcal{Y}$, and we often abbreviate $\mathcal{L}(\mathcal{U},\mathcal{U})$ to $\mathcal{L}(\mathcal{U})$. The class $\mathcal{S}(\mathcal{U},\mathcal{Y})$ admits several remarkable characterizations. In particular, any $F \in \mathcal{S}(\mathcal{U},\mathcal{Y})$ can be realized in the form

$$F(z) = D + zC(I - zA)^{-1}B \tag{1.1}$$

where the connecting operator (or *colligation*)

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$

is unitary, and where $\mathcal{H}$ is an auxiliary Hilbert space. From the point of view of system theory, the function (1.1) is the *transfer function* of the linear system

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\mathbf{U}) : \left\{ \begin{array}{rcl} x(n+1) & = & Ax(n) + Bu(n) \\ y(n) & = & Cx(n) + Du(n). \end{array} \right.$$

It is also well known that the Schur-class functions satisfy a von Neumann inequality: *if $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{K})$ satisfies $\|T\| < 1$, then $F(T)$ is a contraction operator ($\|F(T)\| \leqslant 1$), where $F(T)$ is defined by*

$$F(T) = \sum_{n=0}^{\infty} F_n \otimes T^n \in \mathcal{L}(\mathcal{U} \otimes \mathcal{K}, \mathcal{Y} \otimes \mathcal{K}) \quad \text{if} \quad F(z) = \sum_{n=0}^{\infty} F_n z^n.$$

There is also a well-developed interpolation theory for the classical Schur class. One convenient formalism which encodes classical Nevanlinna-Pick and Carathéodory-Fejér interpolation (see e.g. [13], [24]) proceeds as follows. Making use of power series expansions one can introduce the left and the right evaluation maps

$$F^{\wedge L}(T_L) = \sum_{n=0}^{\infty} T_L^n F_n \quad \text{and} \quad F^{\wedge R}(T_R) = \sum_{n=0}^{\infty} F_n T_R^n, \tag{1.2}$$

which make sense for $F \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ and for every choice of strictly contractive operators $T_L \in \mathcal{L}(\mathcal{Y})$ and $T_R \in \mathcal{L}(\mathcal{U})$. One can then formulate an interpolation problem with the data set consisting of two Hilbert spaces $\mathcal{K}_L$ and $\mathcal{K}_R$ and operators

$$\begin{aligned} &T_L \in \mathcal{L}(\mathcal{K}_L), \quad T_R \in \mathcal{L}(\mathcal{K}_R), \quad X_L \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_L), \\ &Y_L \in \mathcal{L}(\mathcal{U}, \mathcal{K}_L), \quad X_R \in \mathcal{L}(\mathcal{K}_R, \mathcal{Y}), \quad Y_R \in \mathcal{L}(\mathcal{K}_R, \mathcal{U}), \end{aligned}$$

as follows:

PROBLEM 1.1. *Given the data as above, find necessary and sufficient conditions for existence of a function $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$ such that*

$$(X_L S)^{\wedge L}(T_L) = Y_L \quad \textit{and} \quad (S Y_R)^{\wedge R}(T_R) = X_R. \tag{1.3}$$

The answer is well known: Problem 1.1 has a solution if and only if there exists a positive semidefinite operator $P \in \mathcal{L}(\mathcal{K}_L \oplus \mathcal{K}_R)$ subject to the Stein identity

$$M^* P M - N^* P N = X^* X - Y^* Y,$$

where

$$M = \begin{bmatrix} I_{\mathcal{K}_L} & 0 \\ 0 & T_R \end{bmatrix}, \quad N = \begin{bmatrix} T_L^* & 0 \\ 0 & I_{\mathcal{K}_R} \end{bmatrix}, \quad X = \begin{bmatrix} X_L^* & X_R \end{bmatrix}, \quad Y = \begin{bmatrix} Y_L^* & Y_R \end{bmatrix}. \tag{1.4}$$

1.2. MULTIVARIABLE EXTENSIONS. Multivariable generalizations of these and many other related results have been obtained recently; one very general formulation (see [4], [3], [9]) proceeds as follows. Let $\mathbf{Q}$ be a $m \times k$ matrix-valued polynomial

$$\mathbf{Q}(z) = \begin{bmatrix} \mathbf{q}_{11}(z) & \dots & \mathbf{q}_{1k}(z) \\ \vdots & & \vdots \\ \mathbf{q}_{m1}(z) & \dots & \mathbf{q}_{mk}(z) \end{bmatrix} : \mathbb{C}^d \to \mathbb{C}^{m \times k} \tag{1.5}$$

and let $\mathcal{D}_{\mathbf{Q}} \in \mathbb{C}^d$ be the domain defined by $\mathcal{D}_{\mathbf{Q}} = \{z \in \mathbb{C}^d : \|\mathbf{Q}(z)\| < 1\}$. The *Schur-Agler class* $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ consists, by definition, of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions S holomorphic on $\mathcal{D}_{\mathbf{Q}}$ and satisfying the following $\mathbf{Q}$*-von Neumann inequality*: $\|S(T_1, \dots, T_n)\| \leqslant 1$ for any n-tuple $(T_1, \dots, T_n)$ of commuting operators on a Hilbert space $\mathcal{K}$, subject to $\|\mathbf{Q}(T_1, \dots, T_n)\| < 1$ (the Taylor joint spectrum of such n-tuples is contained in $\mathcal{D}_{\mathbf{Q}}$, so one can use a tensored version of the Taylor functional calculus to define $S(T_1, \dots, T_d)$ — see [4], [10]). The class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ is the subclass of the *Schur class* $\mathcal{S}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ of contractive valued functions analytic on $\mathcal{D}_{\mathbf{Q}}$ and, as was first understood for the tridisk case, it can happen that the containment $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y}) \subset \mathcal{S}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ is strict. It is this smaller class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ which has a characterization analogous to (1.1) and thereby can be interpreted as the set of transfer functions of some type of conservative linear system, namely (see [9], [3]): *an $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued function analytic on $\mathcal{D}_{\mathbf{Q}}$ belongs to the class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ if and only if there exists an auxiliary Hilbert space $\mathcal{H}$ and a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \mathbb{C}^p \otimes \mathcal{H} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \mathbb{C}^q \otimes \mathcal{H} \\ \mathcal{Y} \end{bmatrix}$$

such that

$$S(z) = D + C(I_{\mathbb{C}^p} \otimes I_{\mathcal{H}} - (\mathbf{Q}(z) \otimes I_{\mathcal{H}})A)^{-1}(\mathbf{Q}(z) \otimes I_{\mathcal{H}})B. \tag{1.6}$$

Note that special choices of

$$\mathbf{Q}(z) = \operatorname{diag}(z_1, \dots, z_d) \quad \text{and} \quad \mathbf{Q}(z) = \begin{bmatrix} z_1 & z_2 & \cdots & z_d \end{bmatrix} \tag{1.7}$$

lead to the unit polydisk $\mathcal{D}_{\mathbf{Q}} = \mathbb{D}^d$ and the unit ball $\mathcal{D}_{\mathbf{Q}} = \mathbb{B}^d$ of $\mathbb{C}^d$, respectively. The classes $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ for these two generic cases have been known for a while (see [1], [23] and also [2] for recent developments and further references). Schur-Agler-class functions on $\mathbb{D}^d$ and $\mathbb{B}^d$ arise as the transfer functions of *Givone-Roesser* (see [41], [26]) and *Fornasini-Marchesini* (see [25], [26]) systems, respectively, which satisfy an additional energy-balance relation (see [14]). In the general case, formula (1.6) can be interpreted as representing S as the transfer function of a more general type of multidimensional conservative linear system (see Section 4 of [10] for more detail).

An interpolation problem similar to Problem 1.1 has been studied in [10]. Interpolation conditions for this problem are the same as in (1.3) but T_{L} and T_{R}

are now commuting d-tuples satisfying conditions

$$\|\mathbf{Q}(T_{\rm L})\| < 1 \quad \text{and} \quad \|\mathbf{Q}(T_{\rm R})\| < 1$$

and definitions of the left and the right evaluation maps are more involved and rely on the Martinelli-type formula [44] for the Taylor functional calculus. Similarly to the one variable case, the problem has a solution if and only if there is a positive semidefinite operator $P \in \mathcal{L}((\mathcal{K}_{\rm L})^m \oplus (\mathcal{K}_{\rm R})^k)$ subject to the Stein identity

$$\sum_{j=1}^{m} M_j^* P M_j - \sum_{\ell=1}^{k} N_\ell^* P N_\ell = X^*X - Y^*Y$$

where X and Y are the same as in (1.4) and M_j and N_ℓ are certain operators depending on $T_{\rm L}$ and $T_{\rm R}$ respectively (see Theorem 1.4 of [10]).

1.3. THE NONCOMMUTATIVE SETTING. System theoretical aspects of the above ideas have been extended recently [30], [11], [12] to noncommutative multidimensional linear systems of a certain structure. These systems, called *structured noncommutative multidimensional linear systems* or SNMLSs in [11]) have evolution along a free semigroup rather than along an integer lattice as is usually taken in work in multidimensional linear system theory, and the transfer function is a formal power series in noncommuting indeterminates rather than an analytic function of several complex variables. Furthermore, the transfer function of a *conservative* SNMLS satisfies a certain von Neumann type inequality which leads to the definition of a noncommutative Schur-Agler class associated with certain noncommutative analogues of the domains $\mathcal{D}_{\mathbf{Q}}$ (but where $\mathbf{Q}$ is restricted to be linear).

The precise definitions and constructions involve a certain type of graph (an "admissible graph" as defined below). Let Γ be a graph consisting of a set of vertices $V = V(\Gamma)$ and edges $E = E(\Gamma)$. An edge e connects its *source vertex* s, denoted by $s = \mathbf{s}(e) \in V$, to its *range vertex* r, denoted by $r = \mathbf{r}(e) \in V$. Following [11], we say that Γ is *admissible* if it is a finite (V and E are finite sets) bipartite graph such that each connected component is a complete bipartite graph. The latter means that:

(i) the set of vertices V has a disjoint partitioning $V = S \dot\cup R$ into the set of *source* vertices S and *range* vertices R;

(ii) S and R in turn have disjoint partitionings $S = \dot\cup_{k=1}^K S_k$ and $R = \dot\cup_{k=1}^K R_k$ into nonempty subsets $S_1, \dots, S_K$ and $R_1, \dots, R_K$ such that, for each $s_k \in S_k$ and $r_k \in R_k$ (with the same value of k) there is a unique edge $e = e_{s_k, r_k}$ connecting s_k to r_k ($\mathbf{s}(e) = s_k$, $\mathbf{r}(e) = r_k$);

(iii) *every* edge of Γ is of this form.

If v is a vertex of Γ (so either $v \in S$ or $v \in R$) we denote by $[v]$ the path-connected component p (i.e., the complete bipartite graph $p = \Gamma_k$ with set of source vertices equal to S_k and set of range vertices equal to R_k for some $k = 1, \dots, K$) containing v. Thus, given two distinct vertices $v_1, v_2 \in S \cup R$, there is a path of Γ connecting

v_1 to v_2 if and only if $[v_1] = [v_2]$ and this path has length 2 if both v_1 and v_2 are either in S or in R and has length 1 otherwise. In case $s \in S$ and $r \in R$ are such that $[s] = [r]$, we shall use the notation $e_{s,r}$ for the unique edge having s as source vertex and r as range vertex:

$$e_{s,r} \in E \text{ determined by } \mathbf{s}(e_{s,r}) = s,\ \mathbf{r}(e_{s,r}) = r.$$

Note that $e_{s,r}$ is well defined only for $s \in S$ and $r \in R$ with $[s] = [r]$.

For an admissible graph Γ, let $\mathcal{F}_E$ be the free semigroup generated by the edge set E of Γ. An element of $\mathcal{F}_E$ is then a word w of the form $w = e_N \cdots e_1$ where each e_k is an edge of Γ for $k = 1, \dots, N$. We denote the empty word (consisting of no letters) by $\emptyset$. The semigroup operation is concatenation: if $w = e_N \cdots e_1$ and $w' = e'_{N'} \cdots e'_1$, then ww' is defined to be $ww' = e_N \cdots e_1 e'_{N'} \cdots e'_1$. Note that the empty word $\emptyset$ acts as the identity element for this semigroup. On occasion we shall have use of the notation we^{-1} for a word $w \in \mathcal{F}_E$ and an edge $e \in E$; by this notation we mean

$$(1.8) \qquad we^{-1} = \begin{cases} w' & \text{if } w = w'e, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

with a similar convention for $e^{-1}w$. By $w^\top$ we mean $e_1 \cdots e_N$, the transpose of $w = e_N \cdots e_1$.

For each $e \in E$, we define a matrix $I_{\Gamma,e} = [I_{\Gamma,e;s,r}]_{s\in S, r\in R}$ (with rows indexed by S and columns indexed by R) with matrix entries given by

$$(1.9) \qquad I_{\Gamma,e;s,r} = \begin{cases} 1 & \text{if } (s,r) = (\mathbf{s}(e), \mathbf{r}(e)), \\ 0 & \text{otherwise.} \end{cases}$$

We then define the *structure matrix* $Z_\Gamma(z)$ associated with each admissible graph Γ to be the linear form in the noncommuting indeterminates $z = (z_e : e \in E)$ given by

$$(1.10) \qquad Z_\Gamma(z) = \sum_{e\in E} I_{\Gamma,e}\, z_e.$$

EXAMPLE 1.2 (Structure matrix for the noncommutative ball). In this case, we take the admissible graph Γ^{FM} (where the label "FM" refers to *Fornasini-Marchesini* for system-theoretic reasons explained in [11], [12]) to be a complete bipartite graph having only one source vertex. Thus we take $S^{\mathrm{FM}} = \{1\}$, and $R^{\mathrm{FM}} = E^{\mathrm{FM}} = \{1, \dots, d\}$ with $\mathbf{s}^{\mathrm{FM}}(i) = 1$, $\mathbf{r}^{\mathrm{FM}}(i) = i$, i.e., $n = 1, m = d$. Thus we have

$$I_{\Gamma^{\mathrm{FM}},i} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

where 1 is located in the i-th slot. Thus, the structure matrix is given by

$$Z_{\Gamma^{\mathrm{FM}}}(z) = \sum_{i=1}^{d} z_i I_{\Gamma^{\mathrm{FM}},i} = \begin{bmatrix} z_1 & \cdots & z_d \end{bmatrix}.$$

EXAMPLE 1.3 (Structure matrix for the noncommutative polydisk). In this case, we take the admissible graph $\Gamma^{\rm GR}$ (where the label "GR" refers to *Givone-Roesser* for system-theoretic reasons explained in [11], [12]) to have d path-connected components with each path-connected component containing only one source and one range vertex. Thus, we take $S^{\rm GR} = R^{\rm GR} = E^{\rm GR} = \{1, \dots, d\}$ with $\mathbf{s}^{\rm GR}(i) = i$, $\mathbf{r}^{\rm GR}(i) = i$ and thus $n = d = m$. Then $I_{\Gamma^{\rm GR},i}$ is the $d \times d$ matrix with 1 located at the (i,i)-th entry and with all other entries are zeros. Therefore, the structure matrix has the diagonal form

$$Z_{\Gamma^{\rm GR}}(z) = \sum_{i=1}^{d} z_i I_{\Gamma^{\rm GR},i} = \operatorname{diag}(z_1, \cdots, z_d).$$

EXAMPLE 1.4 (Full matrix block structure matrix). In this case, we take $\Gamma^{\rm full}$ to be a general finite, complete bipartite graph. Thus we take $S = \{1, \dots, n\}$, $R = \{1, \dots, m\}$, and $E = \{(i,j) : i \in S, j \in R\}$ with $\mathbf{s}^{\rm full}(i,j) = i$, $\mathbf{r}^{\rm full}(i,j) = j$ where $d = nm$. Then $I_{\Gamma^{\rm full},(i,j)}$ is the $d \times d$ matrix with 1 located at the (i,j)-th entry and all other entries are zeros. Thus the structure matrix for this case has the full-block structure

$$Z_{\Gamma^{\rm full}}(z) = \begin{bmatrix} z_{1,1} & \cdots & z_{1,m} \\ \vdots & & \vdots \\ z_{n,1} & \cdots & z_{n,m} \end{bmatrix}.$$

It then follows that for a general admissible graph Γ (with path-connected components $\Gamma_1, \dots, \Gamma_k$ equal to complete bipartite graphs), the associated structure matrix $Z_\Gamma(z)$ is given by

$$Z_\Gamma(z) = \operatorname{diag}(Z_{\Gamma_1^{\rm full}}(z^1),\ Z_{\Gamma_2^{\rm full}}(z^2), \dots, Z_{\Gamma_k^{\rm full}}(z^k))$$

where $Z_{\Gamma_j^{\rm full}}(z^j)$ is defined as in Example 1.4 for $j = 1, \dots, k$. Thus, the case considered in the present framework corresponds (in the commutative setting) not to arbitrary polynomials (1.5), but just to homogeneous linear functions, in which case the corresponding domain $\mathcal{D}_{\mathbf{Q}}$ is the Cartesian product of finitely many Cartan domains of type I.

In what follows, $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle_\Gamma$ will stand for the space of formal power series

$$F(z) = \sum_{v \in \mathcal{F}_E} F_v z^v, \quad F_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \tag{1.11}$$

in noncommutative variables $z = \{z_e : \ e \in E\}$ indexed by the edge set E of the admissible graph Γ, with coefficients F_v equal to bounded operators acting between Hilbert spaces $\mathcal{U}$ and $\mathcal{Y}$. Here $z^\emptyset = 1$ and $z^w = z_{e_N} z_{e_{N-1}} \cdots z_{e_1}$ if $w = e_N e_{N-1} \cdots e_1$.

Let $T = (T_e : e \in E)$ be a collection of bounded, linear operators (not necessarily commuting) on some separable infinite-dimensional Hilbert space $\mathcal{K}$

(also indexed by the edge set E of Γ). We define an operator $F(T) : \mathcal{U} \otimes \mathcal{K} \to \mathcal{Y} \otimes \mathcal{K}$ by

$$F(T) := \lim_{N \to \infty} \sum_{v \in \mathcal{F}_E : |v| \leqslant N} F_v \otimes T^v$$
$$\text{where } T^{\emptyset} = I_{\mathcal{K}} \text{ and } T^v = T_{e_N} \cdots T_{e_1} \text{ if } v = e_N \cdots e_1 \tag{1.12}$$

whenever the limit exists in the weak-operator topology. (In [12] the limit is taken in the norm-operator topology; the weak-operator topology is more convenient for our purposes here.) In general there is no reason for the limit in (1.12) to exist; on the other hand if F is a polynomial in z, its action on noncommutative tuples is well defined. Take the function Z_Γ as in (1.10), define (according to (1.12)) the operator

$$Z_\Gamma(T) := \sum_{e \in E} I_{\Gamma;e} \otimes T_e \in \mathcal{L}(\oplus_{r \in R} \mathcal{K}, \oplus_{s \in S} \mathcal{K})$$

and introduce the noncommutative structured ball

$$\mathcal{B}_\Gamma \mathcal{L}(\mathcal{K}) = \{T = (T_e)_{e \in E} : T_e \in \mathcal{L}(\mathcal{K}) \text{ for } e \in E \text{ and } \|Z_\Gamma(T)\| < 1\}. \tag{1.13}$$

Now we are in position to define the noncommutative Schur-Agler class.

DEFINITION 1.5. Given an admissible graph Γ, a formal power series (1.11) is said to belong to the noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ if, for each Hilbert space $\mathcal{K}$ and each $T = (T_e)_{e \in E} \in \mathcal{B}_\Gamma \mathcal{L}(\mathcal{K})$, the limit

$$F(T) = \lim_{N \to \infty} \sum_{v \in \mathcal{F}_E : |v| \leqslant N} F_v \otimes T^v \tag{1.14}$$

exists in the weak-operator topology and defines a contractive operator

$$F(T) : \mathcal{U} \otimes \mathcal{K} \to \mathcal{Y} \otimes \mathcal{K}, \quad \|F(T)\| \leqslant 1.$$

For the particular case of Example 1.2 the associated noncommutative Schur-Agler class appears explicitly already in [35]. System-theory connections for this case are worked out in [18].

The noncommutative analogue of the unitary realization (1.6) for the Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ was obtained in Theorem 5.3 of [12] for the general case and will be recalled in Theorem 1.6 below; in the particular case of Example 1.2, more structure is present and the realization result was obtained in Theorem 5.4.1 of [18], Theorem 3.16 of [17] as well as much earlier in Theorem 5.1 of [33] (for a certain subclass of $\mathcal{S}A_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$) by different methods than those used in [12]). To formulate the result we shall need some additional notation and terminology.

First, given a collection $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ of Hilbert spaces indexed by the set P of path-connected components of Γ, let

$$Z_{\Gamma,\mathcal{H}}(z) = \sum_{e \in E} I_{\Gamma,\mathcal{H};e} z_e \tag{1.15}$$

where $I_{\Gamma,\mathcal{H};e} : \bigoplus_{r\in R} \mathcal{H}_{[r]} \to \bigoplus_{s\in S} \mathcal{H}_{[s]}$ is given via matrix entries

$$[I_{\Gamma,\mathcal{H};e}]_{s,r} = \begin{cases} I_{\mathcal{H}_{[\mathbf{s}(e)]}} = I_{\mathcal{H}_{[\mathbf{r}(e)]}} & \text{if } s = \mathbf{s}(e) \text{ and } r = \mathbf{r}(e), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, let $z' = (z'_e : e \in E)$ be another system of noncommuting indeterminates; while $z_e z_{e'} \neq z_{e'} z_e$ and $z'_e z'_{e'} \neq z'_{e'} z'_e$ unless $e = e'$, we will use the convention that $z_e z'_{e'} = z'_{e'} z_e$ for all $e, e' \in E$. For $F(z)$ of the form (1.11), we will use the convention that

$$F(z)^* = \Big(\sum_{v\in\mathcal{F}_E} F_v z^v \Big)^* := \sum_{v\in\mathcal{F}_E} F_v^* z^{v^\top} = \sum_{v\in\mathcal{F}_E} F_{v^\top}^* z^v. \tag{1.16}$$

We also use the notation

$$\text{Row}_{x\in X} M_x = \begin{bmatrix} M_{x_1} & \cdots & M_{x_N} \end{bmatrix}, \quad \text{Col}_{x\in X} M_x = \begin{bmatrix} M_{x_1} \\ \vdots \\ M_{x_N} \end{bmatrix} \text{ if } X = \{x_1, \dots, x_N\}$$

for block row and column matrices with rows or columns indexed by the set X.

THEOREM 1.6. *Let $F(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z\rangle\rangle_\Gamma$. The following are equivalent:*

(i) *F belongs to the noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$.*

(ii) *There exist a collection $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ of Hilbert spaces indexed by the set P of path-connected components Γ and a unitary operator*

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \begin{bmatrix} \bigoplus_{s\in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{r\in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}$$

such that

$$F(z) = D + C(I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1} Z_{\Gamma,\mathcal{H}}(z)B \tag{1.17}$$

where $Z_{\Gamma,\mathcal{H}}$ is defined in (1.15).

(iii) *There exist a collection of Hilbert spaces $\mathcal{H}=\{\mathcal{H}_p : p\in P\}$ and a formal power series*

$$H(z) = \text{Row}_{s\in S} H_s(z) \in \mathcal{L}(\oplus_{s\in S}\mathcal{H}_{[s]}, \mathcal{Y})\langle\langle z\rangle\rangle \tag{1.18}$$

so that

$$I_{\mathcal{Y}} - F(z)F(z')^* = H(z)(I - Z_{\Gamma,\mathcal{H}}(z) Z_{\Gamma,\mathcal{H}}(z')^*) H(z')^*. \tag{1.19}$$

(iv) *There exist a collection of Hilbert spaces $\mathcal{H}=\{\mathcal{H}_p : p\in P\}$ and a formal power series*

$$G(z) = \text{Col}_{r\in R} G_r(z) \in \mathcal{L}(\mathcal{U}, \oplus_{r\in R}\mathcal{H}_{[r]})\langle\langle z\rangle\rangle \tag{1.20}$$

so that

$$I_{\mathcal{U}} - F(z)^* F(z') = G(z)^* (I - Z_{\Gamma,\mathcal{H}}(z)^* Z_{\Gamma,\mathcal{H}}(z')) G(z'). \tag{1.21}$$

(v) *There exist a collection of Hilbert spaces* $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ *and formal power series* $H(z)$ *and* $G(z)$ *as in* (1.18), (1.20) *so that relations* (1.19), (1.21) *hold along with*

$$F(z) - F(z') = H(z)(Z_{\Gamma,\mathcal{H}}(z) - Z_{\Gamma,\mathcal{H}}(z'))G(z'). \tag{1.22}$$

A representation of the form (1.17) with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is called a *unitary realization* for F, or, in more detail in the terminology from [12], a realization of F as the transfer function for the *conservative Structured Noncommutative Multidimensional Linear System* $\mathbf{\Sigma} = \{\Gamma, \mathcal{H}, \mathcal{U}, \mathcal{Y}, \mathbf{U}\}$ (see Section 2 for further details). Note that if F is of the form (1.17), then representations (1.19), (1.21) and (1.22) are valid with

$$H(z) = C(I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1} \quad \text{and} \quad G(z) = (I - AZ_{\Gamma,\mathcal{H}}(z))^{-1}B \tag{1.23}$$

and the following representations for F hold:

$$F(z) = \sum_{v \in \mathcal{F}_E} F_v z^v = D + H(z)Z_{\Gamma,\mathcal{H}}(z)B = D + CZ_{\Gamma,\mathcal{H}}(z)G(z). \tag{1.24}$$

Now we turn to the subject of the paper. We shall consider bitangential interpolation problems with the data set consisting of two Hilbert spaces $\mathcal{K}_{\rm L}$ and $\mathcal{K}_{\rm R}$, two tuples $T_{\rm L} = \{T_{{\rm L},e} : e \in E\}$ and $T_{\rm R} = \{T_{{\rm R},e} : e \in E\}$ of operators acting on $\mathcal{K}_{\rm L}$ and $\mathcal{K}_{\rm R}$ respectively, and bounded operators

$$X_{\rm L} : \mathcal{Y} \to \mathcal{K}_{\rm L}, \quad Y_{\rm L} : \mathcal{U} \to \mathcal{K}_{\rm L}, \quad X_{\rm R} : \mathcal{K}_{\rm R} \to \mathcal{Y}, \quad Y_{\rm R} : \mathcal{K}_{\rm R} \to \mathcal{U}.$$

The pair $(T_{\rm L}, X_{\rm L})$ will be said to be *left admissible* (with respect to the class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$) if the left-tangential evaluation map (with operator argument)

$$H \mapsto (X_{\rm L}H)^{\wedge {\rm L}}(T_{\rm L}) = \sum_{v \in \mathcal{F}_E} T_{\rm L}^{v^\top} X_{\rm L} H_v \tag{1.25}$$

is well-defined (with convergence of the series in the weak-operator topology) whenever $H(z) = \sum_{v \in \mathcal{F}_E} H_v z^v$ is a formal power series of the form (1.18) appearing in the representation (1.19) for an $F(z) \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$. Whenever this is the case, from identities (1.24) we read off that then the left-tangential map is also well-defined on $F(z)$:

$$(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L}) = \sum_{v \in \mathcal{F}_E} T_{\rm L}^{v^\top} X_{\rm L} F_v = X_{\rm L}D + \sum_{e \in E} T_{{\rm L},e}[(X_{\rm L}H_{\mathbf{s}(e)})^{\wedge {\rm L}}(T_{\rm L})]B_{\mathbf{r}(e)}.$$

Similarly, we say that the pair $(Y_{\rm R}, T_{\rm R})$ is *right admissible* (with respect to $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$) if the right-tangential evaluation map (with operator argument)

$$G \mapsto (GY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = \sum_{v \in \mathcal{F}_E} G_v Y_{\rm R} T_{\rm R}^{v^\top} \tag{1.26}$$

exists (with convergence of the series in the weak-operator topology) whenever $G(z) = \sum_{v \in \mathcal{F}_E} G_v z^v$ is a formal power series of the form (1.20) appearing in the representation (1.21) for an $F(z) \in \mathcal{SA}_{\mathcal{G}}(\mathcal{U},\mathcal{Y})$. Using identities (1.24) we then see that the right-tangential evaluation map is well-defined on $F(z)$ as well:

$$(FY_{\rm R})^{\wedge R}(T_{\rm R}) = \sum_{v \in \mathcal{F}_E} F_v Y_{\rm R} T_{\rm R}^{v^\top} = DY_{\rm R} + \sum_{e \in E} C_{\mathbf{s}(e)} [(G_{\mathbf{r}(e)} Y_{\rm R})^{\wedge R}(T_{\rm R})] T_{{\rm R},e}.$$

We say that the data set

$$\mathcal{D} = \{T_{\rm L},\ T_{\rm R},\ X_{\rm L},\ Y_{\rm L},\ X_{\rm R},\ Y_{\rm R}\} \tag{1.27}$$

is *admissible* (with respect to $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$) if $(T_{\rm L}, X_{\rm L})$ is left admissible and $(Y_{\rm R}, T_{\rm R})$ is right admissible. We shall give examples and further details on admissible interpolation data sets in Section 3 below.

Given an admissible interpolation data set (1.27), the formal statement of the associated bitangential interpolation problem is:

PROBLEM 1.7. *Find necessary and sufficient conditions for existence of a power series $F \in \mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ such that*

$$(X_{\rm L} F)^{\wedge L}(T_{\rm L}) = Y_{\rm L} \quad \textit{and} \quad (FY_{\rm R})^{\wedge R}(T_{\rm R}) = X_{\rm R}. \tag{1.28}$$

To formulate the solvability criterion we need some additional notation. Let $\delta_{s,s'}$ be the Kronecker delta function

$$\delta_{s,s'} = \begin{cases} 1 & \text{if } s = s', \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in S$ and $r \in R$, define operators:

$$E_{{\rm L},s} = \operatorname{Col}_{s' \in S : [s'] = [s]} \delta_{s,s'} I_{\mathcal{K}_{\rm L}} \colon\ \mathcal{K}_{\rm L} \to \bigoplus_{s' \in S : [s'] = [s]} \mathcal{K}_{\rm L}, \tag{1.29}$$

$$E_{{\rm R},r} = \operatorname{Col}_{r' \in R : [r'] = [r]} \delta_{r,r'} I_{\mathcal{K}_{\rm R}} \colon\ \mathcal{K}_{\rm R} \to \bigoplus_{r' \in R : [r'] = [r]} \mathcal{K}_{\rm R}, \tag{1.30}$$

$$\widetilde{N}_r(T_{\rm L}) = \operatorname{Col}_{s' \in S : [s'] = [r]} T^*_{{\rm L}, e_{s',r}} \colon\ \mathcal{K}_{\rm L} \to \bigoplus_{s' \in S : [s'] = [r]} \mathcal{K}_{\rm L}, \tag{1.31}$$

$$\widetilde{M}_s(T_{\rm R}) = \operatorname{Col}_{r' \in R : [r'] = [s]} T_{{\rm R}, e_{s,r'}} \colon\ \mathcal{K}_{\rm R} \to \bigoplus_{r' \in R : [r'] = [s]} \mathcal{K}_{\rm R}. \tag{1.32}$$

Define also the operators:

$$M_s = M_s(T_{\rm R}) = \begin{bmatrix} E_{{\rm L},s} & 0 \\ 0 & \widetilde{M}_s(T_{\rm R}) \end{bmatrix} \quad (s \in S), \tag{1.33}$$

$$N_r = N_r(T_{\rm L}) = \begin{bmatrix} \widetilde{N}_r(T_{\rm L}) & 0 \\ 0 & E_{{\rm R},r} \end{bmatrix} \quad (r \in R). \tag{1.34}$$

THEOREM 1.8. *There is a power series $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ satisfying interpolation conditions* (1.28) *if and only if there exists a collection $\mathbb{K} = \{\mathbb{K}_p : p \in P\}$ of positive semidefinite operators*

$$\mathbb{K}_p \in \mathcal{L}((\oplus_{s\in S:[s]=p}\mathcal{K}_{\rm L}) \oplus (\oplus_{r\in R:[r]=p}\mathcal{K}_{\rm R}))$$

indexed by the set P of path-connected components of Γ, which satisfies the Stein identity

$$\sum_{s\in S} M_s^* \mathbb{K}_{[s]} M_s - \sum_{r\in R} N_r^* \mathbb{K}_{[r]} N_r = X^*X - Y^*Y, \tag{1.35}$$

where M_s and N_r are the operators defined via formulas (1.33), (1.34) *and where*

$$X = \begin{bmatrix} X_{\rm L}^* & X_{\rm R} \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_{\rm L}^* & Y_{\rm R} \end{bmatrix}. \tag{1.36}$$

Let $\mathbb{K} = \{\mathbb{K}_p : p \in P\}$ be any collection of operators satisfying the conditions in Theorem 1.8. Let us represent these operators more explicitly as

$$\mathbb{K}_p = \begin{bmatrix} \mathbb{K}_{p,{\rm L}} & \mathbb{K}_{p,{\rm LR}} \\ \mathbb{K}_{p,{\rm LR}}^* & \mathbb{K}_{p,{\rm R}} \end{bmatrix} \tag{1.37}$$

where

$$\mathbb{K}_{p,{\rm L}} = [\Psi_{s,s'}], \quad \mathbb{K}_{p,{\rm R}} = [\Phi_{r,r'}], \quad \mathbb{K}_{p,{\rm LR}} = [\Lambda_{s,r}] \tag{1.38}$$

for $s, s' \in S$ and $r, r' \in R$ such that $[s] = [s'] = [r] = [r'] = p$ and with

$$\Psi_{s,s'} \in \mathcal{L}(\mathcal{K}_{\rm L}), \quad \Phi_{r,r'} \in \mathcal{L}(\mathcal{K}_{\rm R}), \quad \Lambda_{s,r} \in \mathcal{L}(\mathcal{K}_{\rm R}, \mathcal{K}_{\rm L}). \tag{1.39}$$

It turns out that for every collection $\mathbb{K} = \{\mathbb{K}_p : p \in P\}$ of positive semidefinite operators satisfying (1.35), there is a solution F of the bitangential interpolation Problem 1.7 such that, for some choice of associated functions $H(z)$ and $G(z)$ of the form (1.18) and (1.20) in representations (1.19), (1.21), (1.22), it holds that

$$(X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L})[(X_{\rm L}H_{s'})^{\wedge {\rm L}}(T_{\rm L})]^* = \Psi_{s,s'} \quad \text{for } s, s' \in S : [s] = [s'], \tag{1.40}$$

$$(X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L})(G_rY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = \Lambda_{s,r} \quad \text{for } s \in S;\ r \in R : [s] = [r], \tag{1.41}$$

$$[(G_rY_{\rm R})^{\wedge {\rm R}}(T_{\rm R})]^*(G_{r'}Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = \Phi_{r,r'} \quad \text{for } r, r' \in R : [r] = [r']. \tag{1.42}$$

Furthermore, it turns out that conversely, for every solution F of Problem 1.7 with representations (1.19), (1.21), (1.22) (existence of these representations is guaranteed by Theorem 1.6), the operators $\mathbb{K}_p$ defined via (1.37)–(1.39) and (1.40)–(1.42) satisfy conditions of Theorem 1.8. These observations suggest the following modification of Problem 1.7 with the data set

$$\mathcal{D} = \{T_{\rm L},\ T_{\rm R},\ X_{\rm L},\ Y_{\rm L},\ X_{\rm R},\ Y_{\rm R},\ \Psi_{s,s'},\ \Phi_{r,r'},\ \Lambda_{s,r}\}. \tag{1.43}$$

PROBLEM 1.9. *Given the data $\mathcal{D}$ as in* (1.43)*, find all power series $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ satisfying interpolation conditions* (1.28) *and such that for some choice of associated functions H_s and G_r in the representations* (1.19), (1.21), (1.22)*, the equalities* (1.40)–(1.42) *hold.*

In contrast to Problem 1.7, the solvability criterion for Problem 1.9 can be given explicitly in terms of the interpolation data.

THEOREM 1.10. *Problem* 1.9 *has a solution if and only if the operators* $\mathbb{K}_p$ $(p \in P)$ *given by* (1.37), (1.38) *are positive semidefinite and satisfy the Stein identity* (1.35).

Moreover, there exist Hilbert spaces $\widetilde{\Delta}$ *and* $\widetilde{\Delta}_*$, *a collection of Hilbert spaces* $\widehat{\mathcal{H}} = \{\widehat{\mathcal{H}}_p : \ p \in P\}$, *and a formal power series*

$$\Sigma(z) = \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} : \begin{bmatrix} \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix} \tag{1.44}$$

from the noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U} \oplus \widetilde{\Delta}_*, \ \mathcal{Y} \oplus \widetilde{\Delta})$ *and completely determined by the interpolation data set* $\mathcal{D}$ *so that* F *is a solution of Problem* 1.9 *if and only if* F *has the form*

$$F(z) = \Sigma_{11}(z) + \Sigma_{12}(z)(I_{\widetilde{\Delta}_*} - \mathcal{T}(z)\Sigma_{22}(z))^{-1}\mathcal{T}(z)\Sigma_{21}(z) \tag{1.45}$$

for a power series $\mathcal{T}(z) \in \mathcal{SA}_\Gamma(\widetilde{\Delta}, \widetilde{\Delta}_*)$.

There has been some work on noncommutative interpolation theory of the sort discussed here, but to this point it is not nearly as well developed as the commutative theory. Most of the previous work of which we are aware (with exceptions to be mentioned below) has been in the context of the noncommutative-ball case (see Example 1.2 above). In this case the Schur-Agler class $\mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ can be identified with the space of contractive multipliers on a Fock space of formal power series in noncommuting indeterminates with norm-square-summable vector coefficients, a noncommutative analogue of the unit ball of analytic Toeplitz operators acting on the classical Hardy space (see e.g. [35], [18]). For this setting Popescu [39], [40] and Constantinescu and Johnson [20], using different approaches, formulated and obtained a necessary and sufficient condition (in terms of positivity of an associated Pick matrix) for the existence of solutions for an interpolation problem of the form (when translated to our notation) $F^{\wedge \rm R}(Z_i) = W_i$ $(i = 1, \dots, N)$ for the class $\mathcal{SA}_{\Gamma^{\rm FM}}(\mathbb{C}, \mathbb{C})$. A number of authors (see [33], [5], [21], [38]) have analyzed noncommutative analogues of the Sarason formulation of interpolation as well as the Carathéodory interpolation problem [37] for the noncommutative-ball setting; one approach for these problems is as an application of the Commutant Lifting Theorem developed by Popescu for this setting (see [34]) where a parametrization of the set of contractive intertwining lifts in terms of "choice sequences" is also available (see [36]).

There are also some papers on noncommutative interpolation which are not in the non-commutative ball setting. We mention the paper of Kalyuzhnyĭ-Verbovetzkiĭ [27] which solves a Carathéodory interpolation problem for the noncommutative polydisk setting. Recently there have appeared more abstract settings, namely the setting of a Hardy algebra over a W^*-correspondence of [31], [32] and the Hardy algebra associated with a collection of test functions on an admissible semigroupoid of [22]; the appropriate specializations of these theories have large intersection with the results which we are discussing here, but, on the other hand, these very general frameworks cover many other different kinds of examples as well.

The paper is organized as follows. After the present Introduction, Section 2 derives some consequences of the energy balance relations encoded in the conservative SNMLSs beyond what was derived in [12] which are needed in the sequel. These consequences are then used in Section 3 to derive some necessary conditions for a given pair of operators $(X_{\rm L}, T_{\rm L})$ (or $(T_{\rm R}, Y_{\rm R})$) to induce a well-defined left (or right) tangential point evaluation with operator argument on a given noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$. Section 4 then establishes the criterion for existence of solutions in Theorems 1.8 and 1.10. Section 5 establishes a correspondence between solutions of Problem 1.9 and unitary extensions of a certain partially defined isometry constructed from the data of the problem, while Section 6 then uses the idea of Arov-Grossman and its further elaboration by Katsnelson-Kheifets-Yuditskii (see [6], [28], [29]) to obtain the linear-fractional parametrization for the set of all solutions of Problem 1.9 as described in Theorem 1.10. Sections 4, 5 and 6 closely parallel the analysis of [10] worked out for the commutative case. The final Section 7 discusses various examples and special cases.

2. CONSERVATIVE STRUCTURED NONCOMMUTATIVE MULTIDIMENSIONAL LINEAR SYSTEMS

Following [11], [12] we define a *structured noncommutative multidimensional linear system* (SNMLS) to be a collection

$$\boldsymbol{\Sigma} = \{\Gamma, \mathcal{H}, \mathcal{U}, \mathcal{Y}, \mathbf{U}\} \tag{2.1}$$

where Γ is an admissible graph, $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ is a collection of (separable) Hilbert spaces (called state spaces) indexed by the path-connected components p of the graph Γ, where $\mathcal{U}$ and $\mathcal{Y}$ are additional (separable) Hilbert spaces (to be interpreted as the *input space* and the *output space* respectively) and where $\mathbf{U}$ is a *connection matrix* (sometimes also called *colligation*) of the form

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} [A_{r,s}] & [B_r] \\ [C_s] & D \end{bmatrix} : \begin{bmatrix} \bigoplus_{s\in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{r\in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix}. \tag{2.2}$$

In case the connection matrix $\mathbf{U}$ is unitary, we shall say that $\boldsymbol{\Sigma}$ is a *conservative* or *unitary* SNMLS. Associated with any SNMLS $\boldsymbol{\Sigma}$ as in (2.1) is the collection of system equations with evolution along the free semigroup $\mathcal{F}_E$

$$\boldsymbol{\Sigma} : \left\{ \begin{array}{rcl} x_{\mathbf{s}(e)}(ew) & = & \Sigma_{s\in S} A_{\mathbf{r}(e),s} x_s(w) + B_{\mathbf{r}(e)} u(w) \\ x_{s'}(ew) & = & 0 \ \text{ if } \ s' \neq \mathbf{s}(e) \\ y(w) & = & \Sigma_{s\in S} C_s x_s(w) + D u(w) \ \text{ for } \ w \in \mathcal{F}_E \end{array} \right. . \tag{2.3}$$

Let $\widetilde{\mathbf{\Sigma}} = \{\Gamma, \ \widetilde{\mathcal{H}}, \ \mathcal{U}, \ \mathcal{Y}, \ \widetilde{\mathbf{U}}\}$ be another SNMLS with the same structure graph Γ and the same input and output spaces as in (2.1) and with the connecting matrix

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} = \begin{bmatrix} [\widetilde{A}_{r,s}] & [\widetilde{B}_r] \\ [\widetilde{C}_s] & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \bigoplus_{s\in S} \widetilde{\mathcal{H}}_{[s]} \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{r\in R} \widetilde{\mathcal{H}}_{[r]} \\ \mathcal{Y} \end{bmatrix}. \tag{2.4}$$

The colligations $\mathbf{\Sigma}$ and $\widetilde{\mathbf{\Sigma}}$ are said to be *unitarily equivalent* if there is a collection $\Upsilon = \{\Upsilon_p : p \in P\}$ of unitary operators $\Upsilon_p : \mathcal{H}_p \to \widetilde{\mathcal{H}}_p$ (for each path connected component p of Γ) such that

$$\begin{bmatrix} \bigoplus_{r\in R} \Upsilon_{[r]} & 0 \\ 0 & I_{\mathcal{Y}} \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} \bigoplus_{s\in S} \Upsilon_{[s]} & 0 \\ 0 & I_{\mathcal{U}} \end{bmatrix}. \tag{2.5}$$

REMARK 2.1. It is an easy computation to see that unitarily equivalent colligations have the same transfer functions.

It will be convenient to have the notation $p \mapsto s_p$ for a *source-vertex cross-section*, i.e., for each path-connected component p of Γ, s_p is the assignment of a one particular source vertex in the path-connected component p. From the structure of the system equations (2.3) and under the assumption that $\mathbf{U}$ is unitary (or more generally, under the assumption that $\mathbf{U}$ is contractive), we read off the following properties for system trajectories $w \mapsto (u(w), x(w), y(w))$ satisfying equations (2.3):

$$x_s(e_{s,r}w) \text{ is independent of } s \text{ for any given } r \in R \text{ and } w \in \mathcal{F}_E, \tag{2.6}$$

$$\sum_{r\in R} \|x_{s_{[r]}}(e_{s_{[r]},r}w)\|^2 - \|x(w)\|^2 \leqslant \|u(w)\|^2 - \|y(w)\|^2, \tag{2.7}$$

$$x_{s'}(ew) = 0 \ \text{ if } \ s' \neq \mathbf{s}(e). \tag{2.8}$$

We may then compute

$$\begin{aligned} \sum_{e\in E} \|x(ew)\|^2 &= \sum_{s\in S}\sum_{e\in E} \|x_s(ew)\|^2 \\ &= \sum_{e\in E} \|x_{\mathbf{s}(e)}(ew)\|^2 \quad \text{(by (2.8))} \\ &= \sum_{p\in P} \sum_{s\in S, r\in R:[s]=[r]=p} \|x_s(e_{s,r}w)\|^2 \\ &= \sum_{p\in P} \sum_{r\in R:[r]=p} n_{s_p} \|x_{s_p}(e_{s_p,r}w)\|^2 \text{ (by (2.6))} \end{aligned}$$

where we have set n_{s_p} equal to the number of source vertices s in the path-connected component p of Γ. If we now set N_S equal to the maximum number of source vertices in any path-connected component of Γ

$$N_S = \max\{n_{s_p} : p \in P\}, \tag{2.9}$$

then

$$\sum_{e\in E}\frac{1}{N_S}\|x(ew)\|^2=\sum_{p\in P}\sum_{r:[r]=p}\frac{n_{s_p}}{N_S}\|x_{s_p}(e_{s_p,r}w)\|^2\leqslant\sum_{p\in P}\sum_{r:[r]=p}\|x_{s_p}(e_{s_p,r}w)\|^2$$
$$=\sum_{r\in R}\|x_{s_p}(e_{s_p,r}w)\|^2$$
$$\leqslant\|x(w)\|^2+\|u(w)\|^2-\|y(w)\|^2\quad\text{(by (2.7))}.$$

Summing over all words w of a fixed length n followed by multiplying by N_S^{-n} then gives

$$\sum_{w:|w|=n+1}\frac{1}{N_S^{n+1}}\|x(w)\|^2-\sum_{w:|w|=n}\frac{1}{N_S^{n}}\|x(w)\|^2$$
$$\leqslant\sum_{w:|w|=n}\frac{1}{N_S^{n}}\|u(w)\|^2-\sum_{w:|w|=n}\frac{1}{N_S^{n}}\|y(w)\|^2. \tag{2.10}$$

If we now sum over $n=0,1,\dots,N$, the left-hand side of (2.10) telescopes and we arrive at

$$\sum_{w:|w|=N+1}\frac{1}{N_S^{|w|}}\|x(w)\|^2-\|x(\emptyset)\|^2\leqslant\sum_{w:|w|\leqslant N}\frac{1}{N_S^{|w|}}\|u(w)\|^2-\sum_{w:|w|\leqslant N}\frac{1}{N_S^{|w|}}\|y(w)\|^2.$$

In particular, we get the estimate

$$\sum_{w:|w|\leqslant N}\frac{1}{N_S^{|w|}}\|y(w)\|^2\leqslant\|x(\emptyset)\|^2+\sum_{w:|w|\leqslant N}\frac{1}{N_S^{|w|}}\|u(w)\|^2.$$

Letting $N\to\infty$ then gives

$$\sum_{w\in\mathcal{F}_E}\frac{1}{N_S^{|w|}}\|y(w)\|^2\leqslant\|x(\emptyset)\|^2+\sum_{w\in\mathcal{F}_E}\frac{1}{N_S^{|w|}}\|u(w)\|^2 \tag{2.11}$$

for all system trajectories (u,x,y) of the SNMLS $\mathbf{\Sigma}$ as long as the connection matrix $\mathbf{U}$ satisfies $\|\mathbf{U}\|\leqslant 1$.

If $\{u(w)\}_{w\in\mathcal{F}_E}$ is a $\mathcal{U}$-valued input string and $x(\emptyset)$ the initial state fed into the system equations to produce a $\mathcal{Y}$-valued output string $\{y(w)\}_{w\in\mathcal{F}_E}$ and if we introduce the formal Z-transforms of $\{u(w)\}_{w\in\mathcal{F}_E}$ and $\{y(w)\}_{w\in\mathcal{F}_E}$ according to

$$\widehat{u}(z)=\sum_{w\in\mathcal{F}_E}u(w)z^w,\quad\widehat{y}(z)=\sum_{w\in\mathcal{F}_E}y(w)z^w,$$

then it follows that

$$\widehat{y}(z)=C(I-Z_{\Gamma,\mathcal{H}}(z)A)^{-1}x(\emptyset)+F_{\mathbf{\Sigma}}(z)\cdot\widehat{u}(z) \tag{2.12}$$

where $F_{\mathbf{\Sigma}}(z)$ is the formal noncommutative power series given by

$$F_{\mathbf{\Sigma}}(z)=D+C(I-Z_{\Gamma,\mathcal{H}}(z)A)^{-1}Z_{\Gamma,\mathcal{H}}(z)B$$

with $Z_{\Gamma,\mathcal{H}}$ defined as in (1.15). In particular, if we take the initial state $x(\emptyset)$ equal to 0, we obtain the relation $\widehat{y}(z)=F_{\mathbf{\Sigma}}(z)\cdot\widehat{u}(z)$ between the Z-transformed input signal $\widehat{u}(z)$ and the Z-transformed output signal $\widehat{y}(z)$. We shall call $F_{\mathbf{\Sigma}}(z)$ the *transfer function* of the SNMLS $\mathbf{\Sigma}$ (see [11], [12]). The assertion of Theorem 1.6 then is that a power series F belongs to the noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ if and only if it is the transfer function of a conservative SNMLS $\mathbf{\Sigma}$ of the form (2.1).

REMARK 2.2. For future reference, we note that the action of $F_{\mathbf{\Sigma}}(z)$ on a vector $u \in \mathcal{U}$, namely

$$F_{\mathbf{\Sigma}}(z) = D + C(I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1}Z_{\Gamma,\mathcal{H}}(z)B : \; u \to y$$

is the result of the feedback connection

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} h \\ u \end{bmatrix} = \begin{bmatrix} h' \\ y \end{bmatrix}, \quad h = Z_{\Gamma,\mathcal{H}}(z)h'$$

where $h \in \bigoplus_{s\in S} \widetilde{\mathcal{H}}_{[s]}$ and $h' \in \bigoplus_{r\in R} \widetilde{\mathcal{H}}_{[r]}$.

3. ADMISSIBLE INTERPOLATION DATA SETS

With these preliminaries out of the way, we now turn to the issue of identifying large classes of examples of left admissible and right admissible pairs $(T_{\rm L}, X_{\rm L})$ and $(Y_{\rm R}, T_{\rm R})$ for a general admissible graph Γ. In particular, we shall see that the class of interpolation problems covered in Problem 1.7 and 1.9 is nonempty.

By definition, a formal power series $F(z)$ belongs to the Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ if and only if $F(T)$ (defined via (1.14)) is a contraction for all $T \in \mathcal{B}_\Gamma(\mathcal{K})$. Given operators $X_{\rm L} \in \mathcal{L}(\mathcal{Y},\mathcal{K}_{\rm L})$ and $Y_{\rm R} \in \mathcal{L}(\mathcal{K}_{\rm R},\mathcal{U})$ and operator tuples $T_{\rm L} \in \mathcal{L}(\mathcal{K}_{\rm L})^{n_E}$ and $T_{\rm R} \in \mathcal{L}(\mathcal{K}_{\rm R})^{n_E}$ (here we use n_E to denote the number of edges $e \in E$ for the admissible graph Γ), the hope would be that $(T_{\rm L}, X_{\rm L})$ would be left admissible as soon as $T_{\rm L} \in \mathcal{B}_\Gamma\mathcal{L}(\mathcal{K}_{\rm L})$ and that $(Y_{\rm R}, T_{\rm R})$ would be right admissible (with respect to $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$) as soon as $T_{\rm R}$ is in $\mathcal{B}_\Gamma\mathcal{L}(\mathcal{K}_{\rm R})$. As we shall see below, this is indeed correct in some special cases while we obtain only partial results in this direction for the case of a general admissible graph Γ. The statements in the next proposition are immediate consequences of the above definitions.

PROPOSITION 3.1. *Let $F(z)$ belong to $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ and let $\lambda = (\lambda_e)_{e\in E}$ be a tuple of complex numbers.*

(i) *The pair $(T_{\rm L}, X_{\rm L})$ with $T_{\rm L} = \lambda \cdot I_{\mathcal{K}_{\rm L}} = (\lambda_e \cdot I_{\mathcal{K}_{\rm L}})_{e\in E}$ is left admissible whenever $X_{\rm L} \in \mathcal{L}(\mathcal{Y},\mathcal{K}_{\rm L})$ and $\|Z_\Gamma(\lambda)\| < 1$.*

(ii) *The pair $(Y_{\rm R}, T_{\rm R})$ with $T_{\rm R} = \lambda \cdot I_{\mathcal{K}_{\rm R}} = (\lambda_e \cdot I_{\mathcal{K}_{\rm R}})_{e\in E}$ is right admissible whenever $Y_{\rm R} \in \mathcal{L}(\mathcal{K}_{\rm R},\mathcal{U})$ and $\|Z_\Gamma(\lambda)\| < 1$.*

(iii) *If $\|Z_\Gamma(\lambda)\| < 1$, then*

$$F^{\wedge {\rm L}}(\lambda \cdot I_{\mathcal{Y}}) = F^{\wedge {\rm R}}(\lambda \cdot I_{\mathcal{U}}). \tag{3.1}$$

We next explore the function of the scalar-tuple variable $\lambda = (\lambda_1, \dots, \lambda_{n_E})$ a little further. To simplify notation, in the statement of the next result we label the edges of the graph G by the letters $1, 2, \dots, d$ where $d = n_E$ is the number of edges of G. Then words in $\mathcal{F}_E$ have the form $w = i_N i_{N-1} \cdots i_1$ where each $i_\ell \in \{1, \dots, d\}$. If $F(z) = \sum_{v \in \mathcal{F}_E} F_v z^v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$, the function $F^{\mathbf{a}}(\lambda)$ of the scalar d-tuple $(\lambda_1, \dots, \lambda_d)$ given by either the left-hand side or the right-hand side of (3.1) (under the assumption that the series converges) can be expressed as

$$F^{\mathbf{a}}(\lambda) = \sum_{v \in \mathcal{F}_E} F_v (\lambda I_{\mathcal{U}})^v = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} \Big[\sum_{v : v \in \mathbf{a}^{-1}(\mathbf{n})} F_v \Big] \lambda^{\mathbf{n}} =: \sum_{\mathbf{n} \in \mathbb{Z}_+^d} F_{\mathbf{n}}^{\mathbf{a}} \lambda^{\mathbf{n}}$$

where we have introduced the *abelianization map* $\mathbf{a} : \mathcal{F}_d \to \mathbb{Z}_+^d$ given by

$$\mathbf{a}(i_N \cdots i_1) = (n_1, \dots, n_d) \text{ if } n_j = \#\{\ell : i_\ell = j\} \text{ for } j = 1, \dots, d,$$

where $\lambda^v = \lambda_{i_N} \cdots \lambda_{i_1}$ if $v = i_N \cdots i_1$ and where $\lambda^{\mathbf{n}} = \lambda_1^{n_1} \cdots \lambda_d^{n_d}$ if $\mathbf{n} = (n_1, \dots, n_d)$, and where we have set

$$F_{\mathbf{n}}^{\mathbf{a}} = \sum_{v : v \in \mathbf{a}^{-1}(\mathbf{n})} F_v.$$

If $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ then necessarily $F^{\mathbf{a}}$ is analytic on $\mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$ where $Z_\Gamma^{\mathbf{a}}(\lambda)$ is just the abelianization of the structure matrix $Z_\Gamma(z)$ for Γ. For a general matrix-valued polynomial $\mathbf{Q}(\lambda)$ in the commuting variables $\lambda = (\lambda_1, \dots, \lambda_d)$, the associated *commutative* Schur-Agler class $\mathcal{SA}_{\mathbf{Q}}(\mathcal{U}, \mathcal{Y})$ was defined in [10] to consist of holomorphic functions $\lambda \mapsto F(\lambda)$ defined on the domain $\mathcal{D}_{\mathbf{Q}} := \{\lambda \in \mathbb{C}^d : \|\mathbf{Q}(\lambda)\| < 1\}$ such that $\|F(T)\| \leqslant 1$ for any commuting d-tuple of operators $(T_1, \dots, T_d)$ on $\mathcal{K}$ such that $\|\mathbf{Q}(T_1, \dots, T_d)\| < 1$. For the special case where $\mathbf{Q}$ is taken to be the abelianized structure matrix $\mathbf{Q}(\lambda) = Z_\Gamma^{\mathbf{a}}(\lambda)$, then we see that the set of commuting d-tuples T with $\|Z_\Gamma^{\mathbf{a}}(T)\| < 1$ is just the intersection of $\mathcal{B}_\Gamma \mathcal{L}(\mathcal{K})$ with commutative operator tuples. A consequence of Lemma 1 from [4] is that a commuting d-tuple $T = (T_1, \dots, T_d)$ has its Taylor spectrum in the domain $\mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$ whenever $\|Z_\Gamma^{\mathbf{a}}(T)\| < 1$. Moreover, as $Z_\Gamma^{\mathbf{a}}$ is a *linear* polynomial, the associated domain $\mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$ is a logarithmically convex Reinhardt domain, and the functional calculus with operator argument defined via the Taylor functional calculus can equivalently be carried out by using power series centered at the origin (see Remark 2.2 of [10]). Hence, if $T_{\mathrm{L}} = (T_{\mathrm{L},j})_{j=1,\dots,d}$ is a *commuting* d-tuple of operators on $\mathcal{K}_{\mathrm{L}}$ and $X_{\mathrm{L}} \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_{\mathrm{L}})$, then

$$(X_{\mathrm{L}} F)^{\wedge \mathrm{L}}(T_{\mathrm{L}}) = \sum_{v \in \mathcal{F}_E} T_{\mathrm{L}}^v X_{\mathrm{L}} F_v = \sum_{\mathbf{n} \in \mathbb{Z}_+^d} T_{\mathrm{L}}^{\mathbf{n}} X_{\mathrm{L}} F_{\mathbf{n}}^{\mathbf{a}} = (X_{\mathrm{L}} F^{\mathbf{a}})^{\wedge \mathrm{L}}(T_{\mathrm{L}}) \tag{3.2}$$

where $(X_{\mathrm{L}} F^{\mathbf{a}})^{\wedge \mathrm{L}}(T_{\mathrm{L}})$ is the functional calculus with commuting operator argument used in [10]. We conclude that: *if the formal power series $F(z) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$ is in the noncommutative Schur-Agler class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$, then its abelianization $F^{\mathbf{a}}(\lambda)$ is in the commutative Schur-Agler class $\mathcal{SA}_{Z_\Gamma^{\mathbf{a}}}$ associated with $\mathbf{Q}(\lambda) := Z_\Gamma^{\mathbf{a}}(\lambda)$ as defined in* [10]. Moreover, we see that the pair $(X_{\mathrm{L}}, T_{\mathrm{L}})$ is left admissible whenever

$T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ is a commutative operator-tuple in $\mathcal{B}_\Gamma\mathcal{L}(\mathcal{K})$, and then, from the identity (3.2), we see in addition that

$$(X_{\rm L}F)^{\wedge{\rm L}}(T_{\rm L}) = (X_{\rm L}F^{\mathbf{a}})^{\wedge{\rm L}}(T_{\rm L}).$$

More generally, if $T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ is a commuting operator-tuple with Taylor spectrum contained in $\mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$, one can use Theorem 2.1 from [19] to see that then $T_{\rm L}$ is similar to a commuting operator-tuple $T'_{\rm L}$ satisfying $\|Z_\Gamma^{\mathbf{a}}(T'_{\rm L})\| < 1$, and hence $(X_{\rm L}, T_{\rm L})$ is admissible in this case as well. We have arrived at the following result.

PROPOSITION 3.2. *Let Γ be an admissible graph.*

(i) *If $X_{\rm L} \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_{\rm L})$ and $T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ is a commutative tuple of operators on $\mathcal{K}_{\rm L}$ with Taylor joint spectrum $\sigma_{\rm Taylor}(T_{\rm L})$ contained in $\mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$, then the pair $(T_{\rm L}, X_{\rm L})$ is left admissible. In particular, $(T_{\rm L}, X_{\rm L})$ is left admissible whenever $T_{\rm L}$ is a commutative tuple in $\mathcal{B}_\Gamma\mathcal{L}(\mathcal{K}_{\rm L})$.*

(ii) *If $Y_{\rm R} \in \mathcal{L}(\mathcal{K}_{\rm R}, \mathcal{U})$ and $T_{\rm R} = (T_{{\rm R},1}, \dots, T_{{\rm R},d})$ a commutative tuple of operators on $\mathcal{K}_{\rm R}$ with $\sigma_{\rm Taylor}(T_{\rm L}) \subset \mathcal{D}_{Z_\Gamma^{\mathbf{a}}}$, then the pair $(Y_{\rm R}, T_{\rm R})$ is right admissible. In particular, $(Y_{\rm R}, T_{\rm R})$ is right admissible whenever $T_{\rm R}$ is a commutative tuple in $\mathcal{B}_\Gamma\mathcal{L}(\mathcal{K}_{\rm R})$.*

Statement (i) follows from the discussion immediately preceding the statement of the proposition. A completely parallel argument proves statement (ii).

We now give a sufficient condition for left admissibility for the general case.

PROPOSITION 3.3. *Let Γ be an admissible graph and let us assume that the tuple $T_{\rm L} = \{T_{{\rm L},e} : e \in E\}$ of bounded operators on a Hilbert space $\mathcal{K}_{\rm L}$ and $X_{\rm L} \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_{\rm L})$ satisfy*

$$\sum_{v \in \mathcal{F}_E} \rho_{\Gamma,{\rm L}}^{-|v|} \|X_{\rm L}^* T_{\rm L}^{*v} k\|_{\mathcal{Y}}^2 < \infty \quad \text{for all } k \in \mathcal{K}_{\rm L}, \tag{3.3}$$

where $\rho_{\Gamma,{\rm L}} = \frac{1}{N_S}$ with N_S defined as in (2.9). *Then the pair $(T_{\rm L}, X_{\rm L})$ is left admissible with respect to $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$.*

Proof. Suppose that $H(z) = \sum_{v \in \mathcal{F}_E} H_v z^v$ is of the form (1.18) in a representation (1.19) for an $F(z) \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$. Then $H(z)$ can be represented as

$$H(z) = C(I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1}$$

in terms of a unitary realization (1.17) for $F(z)$. Let $x \in \bigoplus_{s \in S} \mathcal{H}_{[s]}$. Then from (2.12) which now takes the form

$$\widehat{y}(z) = H(z)x(\emptyset) + F(z) \cdot \widehat{u}(z)$$

we see that the coefficients $H_v x$ of $H(z)x$ amount to the output string $y(v) = H_v x$ associated with running the SNMLS $\mathbf{\Sigma} = (\Gamma, \mathcal{H}, \mathcal{U}, \mathcal{Y}, \mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right])$ with zero input string $u(v) = 0$ for all $v \in \mathcal{F}_E$ and with initial state $x(\emptyset) = x_0$. Hence, from

(2.11) we see that $\sum_{v\in\mathcal{F}_E} \rho_{\Gamma,\mathrm{L}}^{|v|} \|H_v x\|^2_{\mathcal{Y}} \leqslant \|x_0\|^2 < \infty$. Hence, by (3.3),

$$\sum_{v\in\mathcal{F}_E} |\langle T_{\mathrm{L}}^{v^\top} X_{\mathrm{L}} H_v x_0,\ k\rangle_{\mathcal{K}_{\mathrm{L}}}| = \sum_{v\in\mathcal{F}_E} |\langle \rho_{\Gamma,\mathrm{L}}^{|v|/2} H_v x_0,\ \rho_{\Gamma,\mathrm{L}}^{-|v|/2} X_{\mathrm{L}}^* T_{\mathrm{L}}^{*v} k\rangle_{\mathcal{Y}}|$$

$$\leqslant \Big(\sum_{v\in\mathcal{F}_E} \rho_{\Gamma,\mathrm{L}}^{|v|} \|H_v x_0\|^2_{\mathcal{Y}} \Big)^{1/2} \cdot \Big(\sum_{v\in\mathcal{F}_E} \rho_{\Gamma,\mathrm{L}}^{-|v|} \|X_{\mathrm{L}}^* T_{\mathrm{L}}^{*v} k\|^2_{\mathcal{Y}} \Big)^{1/2} < \infty$$

and it follows that $(T_{\mathrm{L}}, X_{\mathrm{L}})$ is left admissible as wanted. ∎

Given an admissible graph Γ, we can always associate a new graph Γ^{FM} of Fornasini-Marchesini type (as in Example 1.2) by letting Γ^{FM} be the admissible graph of Fornasini-Marchesini type having the same edge set E as Γ. This notation appears in the next corollary.

COROLLARY 3.4. *Let Γ be an admissible graph with associated $\rho_{\Gamma,\mathrm{L}} = \frac{1}{N_S}$ given by* (2.9)*, let $T_{\mathrm{L}} = (T_{\mathrm{L},e})_{e\in E}$ be a tuple of operators in $\mathcal{L}(\mathcal{K}_{\mathrm{L}})$ and let $X_{\mathrm{L}} \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_{\mathrm{L}})$. Then a sufficient condition for $(T_{\mathrm{L}}, X_{\mathrm{L}})$ to be left admissible with respect to $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ is that $\|Z_{\Gamma^{\mathrm{FM}}}(T_{\mathrm{L}})\| < \sqrt{\rho_{\Gamma,\mathrm{L}}}$, i.e., that*

$$\|\operatorname{Row}_{e\in E} T_{\mathrm{L},e}\| < \sqrt{\rho_{\Gamma,\mathrm{L}}}.$$

In particular, if Γ is a Fornasini-Marchesini graph (see Example 1.2), then $(T_{\mathrm{L}}, X_{\mathrm{L}})$ is left admissible with respect to $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ whenever $T_{\mathrm{L}} \in \mathcal{B}_\Gamma\mathcal{L}(\mathcal{K}_{\mathrm{L}})$.

Proof. Set $r := \|Z_{G^{\mathrm{FM}}}(T_{\mathrm{L}})\| = \| \begin{bmatrix} T_{\mathrm{L},e_1} & T_{\mathrm{L},e_2} & \cdots & T_{\mathrm{L},e_d} \end{bmatrix} \|$. Then the operator

$$Z_{G^{\mathrm{FM}}}(T_{\mathrm{L}})^* = \operatorname{Col}_{e\in E} T_{\mathrm{L},e}^* : \mathcal{K} \to \bigoplus_{e\in E} \mathcal{K}$$

also has norm r. Hence, for each $k \in \mathcal{K}_{\mathrm{L}}$ we have $\sum_{e\in E} \|T_{\mathrm{L},e}^* k\|^2 \leqslant r^2 \|k\|^2$ and, more generally,

$$\sum_{v:|v|=N+1} \|T_{\mathrm{L}}^{*v} k\|^2 \leqslant r^2 \sum_{v\in\mathcal{F}_E:|v|=N} \|T_{\mathrm{L}}^{*v} k\|^2.$$

An easy induction argument then gives $\sum_{v\in\mathcal{F}_E:|v|=N} \|T_{\mathrm{L}}^{*v} k\|^2 \leqslant r^{2N} \|k\|^2$ and hence also $\sum_{v\in\mathcal{F}_E:|v|=N} \|X_{\mathrm{L}}^* T_{\mathrm{L}}^N k\|^2 \leqslant r^{2N} \|X_{\mathrm{L}}^*\|^2 \|k\|^2$. Hence

$$\sum_{N=0}^{\infty} \sum_{v\in\mathcal{F}_E:|v|=N} \rho_{\Gamma,\mathrm{L}}^{-|v|} \|X_{\mathrm{L}}^* T_{\mathrm{L}}^{*v} k\|^2 \leqslant \|X_{\mathrm{L}}^*\|^2 \|k\|^2 \cdot \sum_{N=0}^{\infty} \Big(\frac{r}{\sqrt{\rho_{\Gamma,\mathrm{L}}}} \Big)^{2N} < \infty,$$

if $r < \sqrt{\rho_{\Gamma,\mathrm{L}}}$. An application of the criterion (3.3) from Proposition 3.3 now completes the proof of Corollary 3.4. ∎

Given an admissible graph Γ together with a tuple of operators $T_{\mathrm{R}} = (T_{\mathrm{R},e})_{e\in E}$ on a Hilbert space $\mathcal{K}_{\mathrm{R}}$ and an operator $Y_{\mathrm{R}} \in \mathcal{L}(\mathcal{K}_{\mathrm{R}}, \mathcal{U})$, there is a sufficient condition for right admissibility of the $(Y_{\mathrm{R}}, T_{\mathrm{R}})$ in the sense of (1.26) dual to condition (3.3) which can be obtained as follows. Note that weak convergence of the series $\sum_{v\in\mathcal{F}_E} F_v Y_{\rm R} T_{\rm R}^{v^\top}$ is equivalent to weak convergence of the adjoint series

$$\sum_{v\in\mathcal{F}_e} T_{\rm R}^{*v} Y_{\rm R}^* F_v^* = \sum_{v\in\mathcal{F}_e} T_{\rm R}^{*v^\top} Y_{\rm R}^* F_{v^\top}^*$$

which has the same form as (1.25) with $T_{\rm R}^*$ in place of $T_{\rm L}$, $Y_{\rm R}^*$ in place of $X_{\rm L}$ and $F_{v^\top}^*$ in place of F_v. To apply the results on left admissibility to get results on right admissibility, we wish to consider $(T_{\rm R}^*, Y_{\rm R}^*)$ as a left pair acting on the formal power series $F(z)^*$ defined in (1.16). By Theorem 1.6, the formal power series $F(z)$ is in $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ if and only if it admits a representation (1.17) with $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right]$ unitary. In this case we compute

$$\begin{aligned} F(z)^* &= D^* + B^* Z_{\Gamma,\mathcal{H}}(z)^*(I - A^* Z_{\Gamma,\mathcal{H}}(z)^*)^{-1} C^* \\ &= D^* + B^*(I - Z_{\Gamma,\mathcal{H}}(z)^* A^*)^{-1} Z_{\Gamma,\mathcal{H}}(z)^* C^*. \end{aligned}$$

This suggests that, given a SNMLS $\boldsymbol{\Sigma} = (\Gamma, \mathcal{H}, \mathcal{U}, \mathcal{Y}, \mathbf{U})$ as defined in (2.1), we define a dual SNMLS $\boldsymbol{\Sigma}' = (\Gamma', \mathcal{H}, \mathcal{Y}, \mathcal{U}, \mathbf{U}')$ where

(i) the admissible graph Γ' for $\boldsymbol{\Sigma}'$ is the same graph as the admissible graph Γ, but with the source vertices for Γ taken to be the range vertices for Γ' and with the range vertices for Γ taken to be the source vertices for Γ'; thus the set of path-components remains unchanged: $P' = P$, and

(ii) the connection matrix $\mathbf{U}'$ for $\boldsymbol{\Sigma}'$ is simply the adjoint

$$\mathbf{U}' = \mathbf{U}^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} : \begin{bmatrix} \bigoplus_{r\in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{s\in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix}$$

of the connection matrix U for $\boldsymbol{\Sigma}$.

Then it is easily checked: *if $F(z)$ is the transfer function of the* SNMLS $\boldsymbol{\Sigma}$, *then $F(z)^*$ is the transfer function of the* SNMLS $\boldsymbol{\Sigma}'$. Moreover $\boldsymbol{\Sigma}$ *is conservative* (i.e., $\mathbf{U}$ is unitary) *if and only if $\boldsymbol{\Sigma}'$ is conservative* (i.e., $\mathbf{U}' = \mathbf{U}^*$ is unitary). By the equivalence (i) $\Longleftrightarrow$ (ii) in Theorem 1.6, we conclude that:

$$F(z) \in \mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y}) \iff F(z)^* \in \mathcal{SA}_{\Gamma'}(\mathcal{Y},\mathcal{U})$$

where Γ' is the reflection of Γ induced by interchanging source vertices with range vertices. A consequence of this analysis is that we have the following analogues of Proposition 3.3 and Corollary 3.4. We leave the details of the proof to the reader. In the statement of the theorem we use the notation

$$n_{\rm R} = \max\{n_{r_p} : p \in P\} \tag{3.4}$$

where n_{r_p} is the number of range vertices in component P of the graph G.

PROPOSITION 3.5. *Let Γ be an admissible graph with associated constant $\rho_{\Gamma,\rm R} := \frac{1}{n_{\rm R}}$ with $n_{\rm R}$ as in* (3.4), *let $T_{\rm R} = (T_{{\rm R},e})_{e\in E}$ be a tuple of operators acting on a Hilbert space*

$\mathcal{K}_{\mathrm{R}}$ and let $Y_{\mathrm{R}} \in \mathcal{L}(\mathcal{K}_{\mathrm{R}}, \mathcal{U})$. Then a sufficient condition for the pair $(Y_{\mathrm{R}}, T_{\mathrm{R}})$ to be right admissible in the sense of (1.26) *is that*

$$\sum_{v \in \mathcal{F}_E} \rho_{\Gamma,\mathrm{R}}^{-|v|} \|Y_{\mathrm{R}} T_{\mathrm{R}}^{v} k\|^2 < \infty \quad \text{for all } \; k \in \mathcal{K}_{\mathrm{R}}. \tag{3.5}$$

For the statement of the following corollary, we use the notation $\Gamma^{\mathrm{FM}'}$ to denote the dual of the Fornasini-Marchesini graph Γ^{FM} associated with Γ; thus $\Gamma^{\mathrm{FM}'}$ has a single range vertex $\{r_0\}$, the same edge set E as does Γ and the source-vertex set taken also equal to E and with each edge e considered to have source itself e and range r_0. The associated structure matrix $Z_{\Gamma}(z)$ is then a column

$$Z_{\Gamma^{\mathrm{FM}'}}(z) = \mathrm{Col}_{e \in E}\, z_e.$$

COROLLARY 3.6. *Let Γ be an admissible graph with associated $\rho_{\Gamma,\mathrm{R}} = \frac{1}{N_{\mathrm{R}}}$ given by* (3.4)*, let $T_{\mathrm{R}} = (T_{\mathrm{R},e})_{e \in E}$ be a tuple of operators in $\mathcal{L}(\mathcal{K}_{\mathrm{R}})$ and let $X_{\mathrm{R}} \in \mathcal{L}(\mathcal{K}_{\mathrm{R}}, \mathcal{U})$. Then a sufficient condition for $(Y_{\mathrm{R}}, T_{\mathrm{R}})$ to be right admissible is that $\|Z_{G^{\mathrm{FM}'}}(T_{\mathrm{R}})\| < \sqrt{\rho_{\Gamma,\mathrm{R}}}$. In particular, if $\Gamma = \Gamma^{\mathrm{FM}'}$ is itself the reflection of a Fornasini-Marchesini graph, then $(Y_{\mathrm{R}}, T_{\mathrm{R}})$ is right admissible whenever $T_{\mathrm{R}} \in \mathcal{B}_{\Gamma}\mathcal{L}(\mathcal{K})$.*

4. THE SOLVABILITY CRITERION

In this section we prove the necessity part of Theorem 1.8. First we need to note the following elementary properties of evaluations (1.25) and (1.26).

LEMMA 4.1. *Let $T = \{T_e : e \in E\}$ and $T' = \{T'_e : e \in E\}$ be tuples of bounded linear operators acting on Hilbert spaces $\mathcal{K}$ and $\mathcal{K}'$, respectively.*

(i) *For every constant function $W(z) \equiv W \in \mathcal{L}(\mathcal{K}', \mathcal{K})$,*

$$(W)^{\wedge \mathrm{L}}(T) = (W)^{\wedge R}(T') = W. \tag{4.1}$$

(ii) *For every $F \in \mathcal{L}(\mathcal{U}, \mathcal{K})\langle\langle z \rangle\rangle$, $\widetilde{F} \in \mathcal{L}(\mathcal{K}', \mathcal{Y})\langle\langle z \rangle\rangle$, $W \in \mathcal{L}(\mathcal{U}', \mathcal{U})$ and $\widetilde{W} \in \mathcal{L}(\mathcal{Y}, \mathcal{Y}')$*

$$(F \cdot W)^{\wedge \mathrm{L}}(T) = F^{\wedge \mathrm{L}}(T) \cdot W \quad \text{and} \quad (\widetilde{W} \cdot \widetilde{F})^{\wedge \mathrm{R}}(T') = \widetilde{W} \cdot \widetilde{F}^{\wedge \mathrm{R}}(T') \tag{4.2}$$

whenever $F^{\wedge \mathrm{L}}(T)$ and $\widetilde{F}^{\wedge R}(T')$ are defined.

(iii) *For every F, $\widetilde{F}$ and T as in part* (ii) *and every $e \in E$,*

$$(F(z) z_e)^{\wedge \mathrm{L}}(T) = T_e \cdot F^{\wedge \mathrm{L}}(T) \quad \text{and} \quad (z_e \widetilde{F}(z))^{\wedge \mathrm{R}}(T') = \widetilde{F}^{\wedge \mathrm{R}}(T') \cdot T'_e. \tag{4.3}$$

(iv) *For every choice of $F \in \mathcal{L}(\mathcal{U}, \mathcal{K})\langle\langle z \rangle\rangle$ and of $\widetilde{F} \in \mathcal{L}(\mathcal{U}', \mathcal{U})\langle\langle z \rangle\rangle$*

$$(F \cdot \widetilde{F})^{\wedge \mathrm{L}}(T) = (F^{\wedge \mathrm{L}}(T) \cdot \widetilde{F})^{\wedge \mathrm{L}}(T) \tag{4.4}$$

whenever $F^{\wedge \mathrm{L}}(T)$ and $(F^{\wedge \mathrm{L}}(T) \cdot \widetilde{F})^{\wedge \mathrm{L}}(T)$ are defined.

(v) *For every choice of $F \in \mathcal{L}(\mathcal{Y}', \mathcal{Y})\langle\langle z \rangle\rangle$ and of $\widetilde{F} \in \mathcal{L}(\mathcal{K}', \mathcal{Y})\langle\langle z \rangle\rangle$,*

$$(F \cdot \widetilde{F})^{\wedge \mathrm{R}}(T') = (F \cdot \widetilde{F}^{\wedge \mathrm{R}}(T'))^{\wedge \mathrm{R}}(T') \tag{4.5}$$

whenever $\widetilde{F}^{\wedge\mathrm{R}}(T')$ *and* $(F \cdot \widetilde{F}^{\wedge\mathrm{R}}(T'))^{\wedge\mathrm{R}}(T')$ *are defined.*

Proof. The two first statements follow immediately from definitions (1.25) and (1.26). To prove (4.4), take F and $\widetilde{F}$ in the form $F(z) = \sum_{v\in\mathcal{F}_E} F_v z^v$ and $\widetilde{F}(z) = \sum_{v\in\mathcal{F}_E} \widetilde{F}_v z^v$. Then $F(z)\cdot\widetilde{F}(z) = \sum_{v\in\mathcal{F}_E}\Big(\sum_{uw=v} F_u\widetilde{F}_w\Big)z^v$ and therefore, according to (1.25),

$$(F\cdot\widetilde{F})^{\wedge\mathrm{L}}(T) = \sum_{v\in\mathcal{F}_E} T^{v^\top}\Big(\sum_{uw=v} F_u\widetilde{F}_w\Big). \tag{4.6}$$

On the other hand, again by (1.25),

$$\begin{aligned}(F^{\wedge\mathrm{L}}(T)\cdot\widetilde{F})^{\wedge\mathrm{L}}(T) &= \sum_{w\in\mathcal{F}_E} T^{v^\top}F^{\wedge\mathrm{L}}(T)\widetilde{F}_w = \sum_{w\in\mathcal{F}_E} T^{w^\top}\Big(\sum_{u\in\mathcal{F}_E} T^{u^\top}F_u\Big)\widetilde{F}_w \\ &= \sum_{w,u\in\mathcal{F}_E} T^{(uw)^\top}F_u\widetilde{F}_w = \sum_{v\in\mathcal{F}_E} T^{v^\top}\Big(\sum_{uw=v} F_u\widetilde{F}_w\Big).\end{aligned}$$

Comparison of the last equality with (4.6) gives (4.4). Equality (4.5) is obtained in much the same way. The first equality in (4.3) follows from (4.4) for the special case of $\widetilde{F}(z) = z_e I_{\mathcal{U}}$. The second equality in (4.3) follows from (4.5) for the special case of $F(z) = z_e I_{\mathcal{Y}}$. ∎

Proof of the necessity part in Theorems 1.8 *and* 1.10. Let F belong to $\mathcal{SA}_\Gamma(\mathcal{U},\mathcal{Y})$ and suppose that F is a solution of Problem 1.7. Choose formal power series H and G of the form (1.18) and (1.20) so that the representations (1.19), (1.21), (1.22) hold. Use (1.40)–(1.42) to define operators $\Psi_{s,s'}$, $\Lambda_{s,r}$ and $\Phi_{r,r'}$ for $s,s'\in S$ and $r,r'\in R$. Then use equations (1.37)–(1.39) to define the block operator matrix $\mathbb{K}_p$. If F is assumed to be a solution of Problem 1.9 then we are given $\mathbb{K}_p$ via (1.37)–(1.39) where $\Psi_{s,s'}$, $\Lambda_{s,r}$ and $\phi_{r,r'}$ are part of the interpolation data and (1.40)–(1.42) hold as part of the interpolation conditions for some choice of H and G associated with the representations (1.19), (1.21), (1.22) for F. In any case, the conditions (1.40)–(1.42) hold and imply that $\mathbb{K}_p$ can be represented as

$$\mathbb{K}_p = \begin{bmatrix}\mathbb{T}_{p,\mathrm{L}}^* \\ \mathbb{T}_{p,\mathrm{R}}^*\end{bmatrix}\begin{bmatrix}\mathbb{T}_{p,\mathrm{L}} & \mathbb{T}_{p,\mathrm{R}}\end{bmatrix} \tag{4.7}$$

where the operators $\mathbb{T}_{p,\mathrm{L}}$ and $\mathbb{T}_{p,\mathrm{R}}$ are given by

$$\mathbb{T}_{p,\mathrm{L}} = \mathrm{Row}_{s\in S:[s]=p}[(X_\mathrm{L}H_s)^{\wedge\mathrm{L}}(T_\mathrm{L})]^* : \bigoplus_{s\in S:[s]=p}\mathcal{K}_\mathrm{L}\to\mathcal{H}_p, \tag{4.8}$$

$$\mathbb{T}_{p,\mathrm{R}} = \mathrm{Row}_{r\in R:[r]=p}(G_rY_\mathrm{R})^{\wedge\mathrm{R}}(T_\mathrm{R}) : \bigoplus_{r\in R:[r]=p}\mathcal{K}_\mathrm{R}\to\mathcal{H}_p. \tag{4.9}$$

Comparing (4.7) with (1.37) we see that

$$\mathbb{K}_{p,\mathrm{L}} = \mathbb{T}_{p,\mathrm{L}}^*\mathbb{T}_{p,\mathrm{L}},\quad \mathbb{K}_{p,\mathrm{R}} = \mathbb{T}_{p,\mathrm{R}}^*\mathbb{T}_{p,\mathrm{R}},\quad \mathbb{K}_{p,\mathrm{LR}} = \mathbb{T}_{p,\mathrm{L}}^*\mathbb{T}_{p,\mathrm{R}}. \tag{4.10}$$

It follows from (4.7) that $\mathbb{K}_p \geqslant 0$ and thus it remains to show that these operators satisfy the Stein identity (1.35). To this end, note that by (1.15) and (1.18), $H(z)Z_{\Gamma,\mathcal{H}}(z) = \mathrm{Row}_{r\in R} \sum_{s\in S:[s]=[r]} H_s(z)z_{e_{s,r}}$ and therefore, by the first equality in (4.3), $(X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L}) = \mathrm{Row}_{r\in R} \sum_{s\in S:[s]=[r]} T_{{\rm L},e_{s,r}}(X_{\rm L}H_s)^{\wedge \rm L}(T_{\rm L})$ which can be written in terms of (1.31) and (4.8) as

$$(X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L}) = \mathrm{Row}_{r\in R}\, \widetilde{N}_r(T_{\rm L})^*\mathbb{T}^*_{[r],{\rm L}}. \tag{4.11}$$

Note also that according to decompositions (1.18) and (1.29),

$$(X_{\rm L}H)^{\wedge \rm L}(T_{\rm L}) = \mathrm{Row}_{s\in S}\, E^*_{{\rm L},s}\mathbb{T}^*_{[s],{\rm L}}. \tag{4.12}$$

Similarly, by (1.15) and (1.20), $Z_{\Gamma,\mathcal{H}}(z)G(z) = \mathrm{Col}_{s\in S} \sum_{r\in R:[r]=[s]} z_{e_{s,r}}G_r(z)$ and therefore, by the second equality in (4.3),

$$(Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge \rm R}(T_{\rm R}) = \mathrm{Col}_{s\in S} \sum_{r\in R:[r]=[s]} (G_rY_{\rm R})^{\wedge \rm R}(T_{\rm R})T_{{\rm R},e_{s,r}},$$

which can be written in terms of (1.32) and (4.9) as

$$(Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge \rm R}(T_{\rm R}) = \mathrm{Col}_{s\in S}\, \mathbb{T}_{[s],{\rm R}}\widetilde{M}_s(T_{\rm R}). \tag{4.13}$$

Finally, by decompositions (1.20) and (1.30),

$$(GY_{\rm R})^{\wedge \rm R}(T_{\rm R}) = \mathrm{Col}_{r\in R}\, \mathbb{T}_{[r],{\rm R}}E_{{\rm R},r}. \tag{4.14}$$

Substituting the partitionings (1.33), (1.34), (1.36) and (1.37) into (1.35) we conclude that (1.35) is equivalent to the following three equalities:

$$\sum_{s\in S} E^*_{{\rm L},s}\mathbb{K}_{[s],{\rm L}}E_{{\rm L},s} - \sum_{r\in R} \widetilde{N}_r(T_{\rm L})^*\mathbb{K}_{[r],{\rm L}}\widetilde{N}_r(T_{\rm L}) = X_{\rm L}X^*_{\rm L} - Y_{\rm L}Y^*_{\rm L}, \tag{4.15}$$

$$\sum_{s\in S} E^*_{{\rm L},s}\mathbb{K}_{[s],{\rm LR}}\widetilde{M}_s(T_{\rm R}) - \sum_{r\in R} \widetilde{N}_r(T_{\rm L})^*\mathbb{K}_{[r],{\rm LR}}E_{{\rm R},r} = X_{\rm L}X_{\rm R} - Y_{\rm L}Y_{\rm R}, \tag{4.16}$$

$$\sum_{s\in S} \widetilde{M}_s(T_{\rm R})^*\mathbb{K}_{[s],{\rm R}}\widetilde{M}_s(T_{\rm R}) - \sum_{r\in R} E^*_{{\rm R},r}\mathbb{K}_{[r],{\rm R}}E_{{\rm R},r} = X^*_{\rm R}X_{\rm R} - Y^*_{\rm R}Y_{\rm R}. \tag{4.17}$$

To check (4.15) we consider the equality

$$X_{\rm L}X^*_{\rm L} - X_{\rm L}F(z)F(z')^*X^*_{\rm L} = X_{\rm L}H(z)(I - Z_{\Gamma,\mathcal{H}}(z)Z_{\Gamma,\mathcal{H}}(z')^*)H(z')^*X^*_{\rm L} \tag{4.18}$$

which is an immediate corollary of (1.19). We may consider each side of (4.18) as a formal power series in z' with coefficients equal to formal power series in z, i.e., we have a natural identification

$$\mathcal{L}(\mathcal{K}_{\rm L})\langle\langle z,z'\rangle\rangle \cong (\mathcal{L}(\mathcal{K}_{\rm L})\langle\langle z\rangle\rangle)\langle\langle z'\rangle\rangle.$$

We then apply the left evaluation map (applied to formal power series in the variable z) to each coefficient of the resulting formal power series in the variable z'. The result amounts to applying left evaluation to both sides of (4.18) in the variable z with the formal variable z' considered as fixed. Making use of properties (4.1), (4.2) and of relation (4.11) and taking into account the first interpolation condition in (1.28), we get $X_L X_L^* - Y_L F(z')^* X_L^* = (X_L H)^{\wedge L}(T_L) \cdot H(z')^* X_L^* - (X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L) \cdot Z_{\Gamma,\mathcal{H}}(z')^* H(z')^* X_L^*$. This equality holds as an identity in $\mathcal{L}(\mathcal{K}_L)\langle\langle z'\rangle\rangle$. Taking adjoints and replacing z' by z, we get $X_L X_L^* - X_L F(z) Y_L^* = X_L H(z)((X_L H)^{\wedge L}(T_L))^* - X_L H(z) Z_{\Gamma,\mathcal{H}}(z)((X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L))^*$. Applying the left evaluation to the latter equality we get $X_L X_L^* - Y_L Y_L^* = (X_L H)^{\wedge L}(T_L)((X_L H)^{\wedge L}(T_L))^* - (X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L)((X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L))^*$. Substituting (4.11) and (4.12) into the right hand side expression we come to

$$X_L X_L^* - Y_L Y_L^* = \sum_{s\in S} E_{L,s}^* \mathbb{T}_{[s],L}^* \mathbb{T}_{[s],L} E_{L,s} - \sum_{r\in R} \widetilde{N}_r(T_L)^* \mathbb{T}_{[s],L}^* \mathbb{T}_{[s],L} \widetilde{N}_r(T_L)$$

which is equivalent to (4.15), since $\mathbb{T}_{[s],L}^* \mathbb{T}_{[s],L} = \mathbb{K}_{[s],L}$ and $\mathbb{T}_{[r],L}^* \mathbb{T}_{[r],L} = \mathbb{K}_{[r],L}$, by the first equality in (4.10).

To prove (4.16) we start with equality $X_L F(z) Y_R - X_L F(z') Y_R = X_L H(z) (Z_{\Gamma,\mathcal{H}}(z) - Z_{\Gamma,\mathcal{H}}(z')) G(z') Y_R$ which is a consequence of (1.22). We apply the left evaluation in the z variable: by the first interpolation condition in (1.28) we have $Y_L Y_R - X_L F(z') Y_R = (X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L) G(z') Y_R - (X_L H)^{\wedge L}(T_L) Z_{\Gamma,\mathcal{H}}(z') G(z') Y_R$. The last equality holds true as an identity between formal power series in the variable z'; we then apply the right evaluation (1.26) to both sides. In view of the second interpolation condition in (1.28) and of properties (4.1), (4.2), we obtain $Y_L Y_R - X_L X_R = (X_L H Z_{\Gamma,\mathcal{H}})^{\wedge L}(T_L)(G Y_R)^{\wedge R}(T_R) - (X_L H)^{\wedge L}(T_L)(Z_{\Gamma,\mathcal{H}} G Y_R)^{\wedge R}(T_R)$. Substituting equalities (4.11)–(4.14) into the right-hand side expression in the last equality we come to

$$X_L X_R - Y_L Y_R = \sum_{s\in S} E_{L,s}^* \mathbb{T}_{[s],L}^* \mathbb{T}_{[s],R} \widetilde{M}_s(T_R) - \sum_{r\in R} \widetilde{N}_r(T_L)^* \mathbb{T}_{[r],L}^* \mathbb{T}_{[r],R} E_{R,r}$$

which is equivalent to (4.16), since $\mathbb{T}_{[s],L}^* \mathbb{T}_{[s],R} = \mathbb{K}_{[s],LR}$ and $\mathbb{T}_{[r],L}^* \mathbb{T}_{[r],R} = \mathbb{K}_{[r],LR}$, by (4.10). The proof of (4.17) is quite similar: we start with the equality

$$Y_R^* Y_R - Y_R^* F(z)^* F(z') Y_R = Y_R^* G(z)^* (I - Z_{\Gamma,\mathcal{H}}(z)^* Z_{\Gamma,\mathcal{H}}(z')) G(z') Y_R$$

(which follows from (1.21)) and apply the right evaluation in the z' variable. Then we take adjoints in the resulting formal power series identity (in the variable z) and apply again the right evaluation map. The obtained equality together with relations (4.13) and (4.14) leads to (4.17). This completes the proof of necessity in both Theorem 1.8 and Theorem 1.10. ∎

5. SOLUTIONS TO THE INTERPOLATION PROBLEM AND UNITARY EXTENSIONS

In this section we shall show that there is a correspondence between solutions to Problem 1.9 and unitary extensions of a partially defined isometry determined by the problem data set $\mathcal{D}$.

From now on we assume that we are given an interpolation data set $\mathcal{D}$ as in (1.43) and that the necessary conditions for Problem 1.9 to have a solution are in force: the operators $\mathbb{K}_p$ defined in (1.37), (1.38) are each positive semidefinite on the space

$$\mathcal{H}_p^{\circ} = \Big(\bigoplus_{s\in S:[s]=p} \mathcal{K}_{\mathrm{L}}\Big) \oplus \Big(\bigoplus_{r\in R:[r]=p} \mathcal{K}_{\mathrm{R}}\Big) \tag{5.1}$$

and satisfy the Stein identity (1.35) which we write now as

$$\sum_{s\in S} M_s^* \mathbb{K}_{[s]} M_s + Y^*Y = \sum_{r\in R} N_r^* \mathbb{K}_{[r]} N_r + X^*X. \tag{5.2}$$

For every $p \in P$, we introduce the equivalence $\overset{p}{\sim}$ on $\mathcal{H}_p^{\circ}$ by

$$h_1 \overset{p}{\sim} h_2 \ \text{ if and only if } \ \langle \mathbb{K}_p(h_1 - h_2),\ y\rangle_{\mathcal{H}_p^{\circ}} = 0 \ \text{ for all } \ y \in \mathcal{H}_p^{\circ},$$

denote $[h]_p$ the equivalence class of h with respect to the above equivalence and endow the linear space of equivalence classes with the inner product

$$\langle [h]_p,\ [y]_p\rangle = \langle \mathbb{K}_p h,\ y\rangle_{\mathcal{H}_p^{\circ}}. \tag{5.3}$$

We get a prehilbert space whose completion is denoted by $\widehat{\mathcal{H}}_p$. It is readily seen from definitions (1.33), (1.34) of operators M_s and N_r that $M_s f$ and $N_r f$ belong to $\mathcal{H}_{[s]}^{\circ}$ and $\mathcal{H}_{[r]}^{\circ}$, respectively, for every $f \in \mathcal{K}_{\mathrm{L}} \oplus \mathcal{K}_{\mathrm{R}}$. Furthermore, identity (5.2) can be written as $\sum\limits_{s\in S} \|[M_s f]_{[s]}\|^2_{\widehat{\mathcal{H}}_{[s]}} + \|Yf\|^2_{\mathcal{U}} = \sum\limits_{r\in R} \|[N_r f]_{[r]}\|^2_{\widehat{\mathcal{H}}_{[r]}} + \|Xf\|^2_{\mathcal{Y}}$, holding for every choice of $f \in \mathcal{K}_{\mathrm{L}} \oplus \mathcal{K}_{\mathrm{R}}$. Therefore the linear map defined by the rule

$$\mathbf{V}:\ \begin{bmatrix} \mathrm{Col}_{s\in S}[M_s f]_{[s]} \\ Yf \end{bmatrix} \leftarrow \begin{bmatrix} \mathrm{Col}_{r\in R}[N_r f]_{[r]} \\ Xf \end{bmatrix} \tag{5.4}$$

extends by linearity to define an isometry from

$$\mathcal{D}_{\mathbf{V}} = \mathrm{Clos}\Big\{ \begin{bmatrix} \mathrm{Col}_{s\in S}[M_s f]_{[s]} \\ Yf \end{bmatrix},\ f \in \mathcal{K}_{\mathrm{L}} \oplus \mathcal{K}_{\mathrm{R}} \Big\} \subset \begin{bmatrix} \bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]} \\ \mathcal{U} \end{bmatrix} \tag{5.5}$$

onto

$$\mathcal{R}_{\mathbf{V}} = \mathrm{Clos}\Big\{ \begin{bmatrix} \mathrm{Col}_{r\in R}[N_r f]_{[r]} \\ Xf \end{bmatrix},\ f \in \mathcal{K}_{\mathrm{L}} \oplus \mathcal{K}_{\mathrm{R}} \Big\} \subset \begin{bmatrix} \bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]} \\ \mathcal{Y} \end{bmatrix}. \tag{5.6}$$

The next two lemmas establish a correspondence between solutions F to Problem 1.9 and unitary extensions of the partially defined isometry $\mathbf{V}$ given in (5.4).

LEMMA 5.1. *Any solution F to Problem* 1.9 *is a characteristic function of a unitary colligation*

$$\widetilde{\mathbf{\Sigma}} = \{\Gamma,\ \widetilde{\mathcal{H}},\ \mathcal{U},\ \mathcal{Y}, \widetilde{\mathbf{U}}\} \tag{5.7}$$

with the state space

$$\widetilde{\mathcal{H}} = \widehat{\mathcal{H}} \oplus \mathcal{H}' := \{\widetilde{\mathcal{H}}_p = \widehat{\mathcal{H}}_p \oplus \mathcal{H}'_p :\ p \in P\}$$

and the connecting operator

$$\widetilde{\mathbf{U}} = \begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} : \begin{bmatrix} \bigoplus_{s\in S} (\widehat{\mathcal{H}}_{[s]} \oplus \widetilde{\mathcal{H}}_{[s]}) \\ \mathcal{U} \end{bmatrix} \to \begin{bmatrix} \bigoplus_{r\in R} (\widehat{\mathcal{H}}_{[r]} \oplus \widetilde{\mathcal{H}}_{[r]}) \\ \mathcal{Y} \end{bmatrix} \tag{5.8}$$

being an extension of the isometry $\mathbf{V}$ *given in* (5.4).

Proof. Let F be a solution to Problem 1.9. In particular, F belongs to the noncommutative Schur-Agler class $\mathcal{SA}_{\Gamma}(\mathcal{U}, \mathcal{Y})$ and, by Theorem 1.6, it is the characteristic function of some unitary colligation $\mathbf{\Sigma}$ of the form (2.1). In other words, F admits a unitary realization (1.17) with the state space $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ and representations (1.19), (1.21), (1.22) hold for power series H and G defined via (1.23) and decomposed as in (1.18) and (1.20). These series lead to representations (1.24) of F, each of which is equivalent to (1.17).

The interpolation conditions (1.28) and (1.40)–(1.42) which hold for F by assumption force certain restrictions on the connecting operator $\mathbf{U} = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Substituting (1.24) into (1.28) we get equalities $(X_{\rm L}D + X_{\rm L}HZ_{\Gamma,\mathcal{H}}B)^{\wedge{\rm L}}(T_{\rm L}) = Y_{\rm L}$, and $(DY_{\rm R} + CZ_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) = X_{\rm R}$ which are equivalent, due to properties (4.1), (4.2), to

$$X_{\rm L}D + (X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge{\rm L}}(T_{\rm L})B = Y_{\rm L}, \tag{5.9}$$

$$DY_{\rm R} + C(Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) = X_{\rm R}, \tag{5.10}$$

respectively. It also follows from (1.23) that

$$C + H(z)Z_{\Gamma,\mathcal{H}}(z)A = H(z), \quad B + AZ_{\Gamma,\mathcal{H}}(z)G(z) = G(z)$$

and therefore that

$$X_{\rm L}C + (X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge{\rm L}}(T_{\rm L})A = (X_{\rm L}H)^{\wedge{\rm L}}(T_{\rm L}), \tag{5.11}$$

$$BY_{\rm R} + A(Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) = (GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}). \tag{5.12}$$

The equalities (5.9) and (5.11) can be written in matrix form as

$$\begin{bmatrix}(X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge{\rm L}}(T_{\rm L}) & X_{\rm L}\end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix}(X_{\rm L}H)^{\wedge{\rm L}}(T_{\rm L}) & Y_{\rm L}\end{bmatrix}, \tag{5.13}$$

whereas the equalities (5.10) and (5.12) are equivalent to

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix}(Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) \\ Y_{\rm R}\end{bmatrix} = \begin{bmatrix}(GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) \\ X_{\rm R}\end{bmatrix}. \tag{5.14}$$

Since the operator $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ is unitary, we conclude from (5.13) that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix}[(X_{\rm L}H)^{\wedge{\rm L}}(T_{\rm L})]^* \\ Y_{\rm L}^*\end{bmatrix} = \begin{bmatrix}[(X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge{\rm L}}(T_{\rm L})]^* \\ X_{\rm L}^*\end{bmatrix}. \tag{5.15}$$

Combining (5.14) and (5.15) we conclude that for every choice of $f \in \mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \left[(X_{\rm L}H)^{\wedge{\rm L}}(T_{\rm L})\right]^* & (Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) \\ Y_{\rm L}^* & Y_{\rm R} \end{bmatrix} f$$

$$= \begin{bmatrix} \left[(X_{\rm L}HZ_{\Gamma,\mathcal{H}})^{\wedge{\rm L}}(T_{\rm L})\right]^* & (GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R}) \\ X_{\rm L}^* & X_{\rm R} \end{bmatrix} f. \tag{5.16}$$

Let $\mathbb{T}_{p,{\rm L}}$ and $\mathbb{T}_{p,{\rm R}}$ be the operators given by (4.8) and (4.9), respectively, and let

$$\mathbb{T}_p := \begin{bmatrix} \mathbb{T}_{p,{\rm L}} & \mathbb{T}_{p,{\rm R}} \end{bmatrix} : \mathcal{H}_p^\circ \to \mathcal{H}_p. \tag{5.17}$$

Now we use the interpolation conditions (1.40)–(1.42), which provide the factorization (4.7) of the operator $\mathbb{K}_p$. Thus, $\mathbb{K}_p = \mathbb{T}_p^*\mathbb{T}_p$ and $\langle [h]_p, [y]_p \rangle_{\widehat{\mathcal{H}}_p} = \langle \mathbb{K}_p h, y\rangle_{\mathcal{H}_p^\circ} = \langle \mathbb{T}_p h, \mathbb{T}_p y\rangle_{\mathcal{H}_p}$ for every $h, y \in \mathcal{H}_p^\circ$. Therefore, the linear transformation U_p defined by the rule

$$U_p : \mathbb{T}_p h \leftarrow [h]_p \quad (h \in \mathcal{H}_p^\circ) \tag{5.18}$$

can be extended to the unitary map (which still is denoted by U_p) from $\overline{\operatorname{Ran} \mathbb{T}_p}$ onto $\widehat{\mathcal{H}}_p$. Noticing that $\overline{\operatorname{Ran} \mathbb{T}_p}$ is a subspace of $\mathcal{H}_p$ and setting

$$\mathcal{N}_p := \mathcal{H}_p \ominus \overline{\operatorname{Ran} \mathbb{T}_p} \quad \text{and} \quad \widetilde{\mathcal{H}}_p := \widehat{\mathcal{H}}_p \oplus \mathcal{N}_p,$$

we define the unitary map $\widetilde{U}_p : \mathcal{H}_p \leftarrow \widetilde{\mathcal{H}}_p$ by the rule

$$\widetilde{U}_p\, g = \begin{cases} U_p g & \text{for } g \in \overline{\operatorname{Ran} \mathbb{T}_p}, \\ g & \text{for } g \in \mathcal{N}_p. \end{cases} \tag{5.19}$$

Introducing the operators

$$\widetilde{A} = [\widetilde{U}_{[r]} A_{r,s} \widetilde{U}_{[s]}^*]_{r\in R, s\in S}, \ \ \widetilde{B} = \operatorname{Col}_{r\in R} \widetilde{U}_{[r]} B_r, \ \ \widetilde{C} = \operatorname{Row}_{s\in S} C_s \widetilde{U}_{[s]}^*, \ \ \widetilde{D} = D, \tag{5.20}$$

we construct the colligation $\widetilde{\mathbf{\Sigma}}$ via (5.7) and (5.8). By definition, $\widetilde{\mathbf{\Sigma}}$ is unitarily equivalent to the initial colligation $\mathbf{\Sigma}$ defined in (2.1). By Remark 2.1, $\widetilde{\mathbf{\Sigma}}$ has the same characteristic function as $\mathbf{\Sigma}$, that is, $F(z)$. It remains to check that the connecting operator of $\widetilde{\mathbf{\Sigma}}$ is an extension of $\mathbf{V}$, that is

$$\begin{bmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{bmatrix} \begin{bmatrix} \operatorname{Col}_{s\in S}[M_s f]_{[s]} \\ Yf \end{bmatrix} = \begin{bmatrix} \operatorname{Col}_{r\in R}[N_r f]_{[r]} \\ Xf \end{bmatrix} \quad \text{for every } f \in \mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}. \tag{5.21}$$

To this end, note that by (5.18), (5.19) and block partitionings (1.33) and (5.17) of M_s and $\mathbb{T}$, it holds that $\widetilde{U}_{[s]}^*[M_s f]_{[s]} = \mathbb{T}_{[s]}(M_s f) = \left[\mathbb{T}_{[s,],{\rm L}} E_{{\rm L},s} \quad \mathbb{T}_{[s],{\rm R}} \widetilde{M}_s(T_{\rm R})\right] f$ for every $f \in \mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}$ and for every $s \in S$. Therefore,

$$\operatorname{Col}_{s\in S} \widetilde{U}_{[s]}^*[M_s f]_{[s]} = \operatorname{Col}_{s\in S} \left[\mathbb{T}_{[s],{\rm L}} E_{{\rm L},s} \quad \mathbb{T}_{[s],{\rm R}} \widetilde{M}_s(T_{\rm R})\right] f \tag{5.22}$$

which, on account of (4.12) and (4.13) can be written as

$$\operatorname{Col}_{s\in S} \widetilde{U}_{[s]}^*[M_s f]_{[s]} = \left[[(X_{\rm L}H)^{\wedge{\rm L}}(T_{\rm L})]^* \quad (Z_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge{\rm R}}(T_{\rm R})\right] f. \tag{5.23}$$

Similarly, by (5.18), (5.19) and block partitionings (1.34) and (5.17) of N_r and $\mathbb{T}$, it holds that $[N_r f]_{[r]} = \widetilde{U}_{[r]}\mathbb{T}_{[r]}(N_r f) = \widetilde{U}_{[r]}\begin{bmatrix}\mathbb{T}_{[r],\mathrm{L}}\widetilde{N}_r(T_{\mathrm{L}}) & \mathbb{T}_{[r],\mathrm{R}}E_{\mathrm{R},r}\end{bmatrix} f$ $(r \in R)$. Therefore,

$$\operatorname{Col}_{r\in R}\widetilde{U}_{[r]}^*[N_r f]_{[r]} = \operatorname{Col}_{r\in R}\begin{bmatrix}\mathbb{T}_{[r],\mathrm{L}}\widetilde{N}_r(T_{\mathrm{L}}) & \mathbb{T}_{[r],\mathrm{R}}E_{\mathrm{R},r}\end{bmatrix} f \tag{5.24}$$

which, on account of (4.11) and (4.14) can be written as

$$\operatorname{Col}_{r\in R}\widetilde{U}_{[r]}^*[N_r f]_{[r]} = \begin{bmatrix}[(X_{\mathrm{L}}HZ_{\Gamma,\mathcal{H}})^{\wedge\mathrm{L}}(T_{\mathrm{L}})]^* & (GY_{\mathrm{R}})^{\wedge\mathrm{R}}(T_{\mathrm{R}})\end{bmatrix} f. \tag{5.25}$$

Thus, by (5.16) and in view of (1.36), (5.23) and (5.25),

(5.26)

$$\begin{aligned}
&\begin{bmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{bmatrix}\begin{bmatrix}\operatorname{Col}_{s\in S}[M_s f]_{[s]}\\ Yf\end{bmatrix}\\
&= \begin{bmatrix}\bigoplus_{r\in R}\widetilde{U}_{[r]} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\operatorname{Col}_{s\in S}\widetilde{U}_{[s]}^*[M_s f]_{[s]}\\ Yf\end{bmatrix}\\
&= \begin{bmatrix}\bigoplus_{r\in R}\widetilde{U}_{[r]} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}[(X_{\mathrm{L}}H)^{\wedge\mathrm{L}}(T_{\mathrm{L}})]^* & (Z_{\Gamma,\mathcal{H}}GY_{\mathrm{R}})^{\wedge\mathrm{R}}(T_{\mathrm{R}})\\ Y_{\mathrm{L}}^* & Y_{\mathrm{R}}\end{bmatrix} f\\
&= \begin{bmatrix}\bigoplus_{r\in R}\widetilde{U}_{[r]} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}[(X_{\mathrm{L}}HZ_{\Gamma,\mathcal{H}})^{\wedge\mathrm{L}}(T_{\mathrm{L}})]^* & (GY_{\mathrm{R}})^{\wedge\mathrm{R}}(T_{\mathrm{R}})\\ X_{\mathrm{L}}^* & X_{\mathrm{R}}\end{bmatrix} = \begin{bmatrix}\operatorname{Col}_{r\in R}[N_r f]_{[r]}\\ Xf\end{bmatrix},
\end{aligned}$$

which proves (5.21) and completes the proof of the lemma. ∎

LEMMA 5.2. *Let* $\widetilde{\mathbf{U}}$ *of the form* (5.8) *be a unitary extension of the isometry* $\mathbf{V}$ *given in* (5.4). *Then the characteristic function* F *of the unitary colligation of the form* (5.7),

$$F(z) = \widetilde{D} + \widetilde{C}(I - Z_{\Gamma,\widetilde{\mathcal{H}}}(z)\widetilde{A})^{-1}Z_{\Gamma,\widetilde{\mathcal{H}}}(z)\widetilde{B},$$

is a solution to Problem 1.9.

Proof. We use the arguments from the proof of the previous lemma in the reverse order. We start with positive semidefinite operators $\mathbb{K}_p \in \mathcal{L}(\mathcal{H}_p^\circ)$ (the spaces $\mathcal{H}_p^\circ$ are given in (5.1)) and fix their factorizations

$$\mathbb{K}_p = \mathbb{T}_p^*\mathbb{T}_p \quad \text{with} \quad \mathbb{T}_p = \begin{bmatrix}\mathbb{T}_{p,\mathrm{L}} & \mathbb{T}_{p,\mathrm{R}}\end{bmatrix} : \mathcal{H}_p^\circ \to \mathcal{H}_p \tag{5.27}$$

where $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ is a collection of auxiliary Hilbert spaces. Comparing (5.27) with (1.37) we get factorizations

$$\mathbb{K}_{p,\mathrm{L}} = \mathbb{T}_{p,\mathrm{L}}^*\mathbb{T}_{p,\mathrm{L}}, \quad \mathbb{K}_{p,\mathrm{R}} = \mathbb{T}_{p,\mathrm{R}}^*\mathbb{T}_{p,\mathrm{R}}, \quad \mathbb{K}_{p,\mathrm{LR}} = \mathbb{T}_{p,\mathrm{L}}^*\mathbb{T}_{p,\mathrm{R}}.$$

for the block entries in $\mathbb{K}_p$ and more detailed decompositions (1.38) lead us to equalities:

$$E^*_{\mathrm{L},s}\mathbb{T}^*_{[s],\mathrm{L}}\mathbb{T}_{[s'],\mathrm{L}}E_{\mathrm{L},s'} = E^*_{\mathrm{L},s}\mathbb{K}_{[s],\mathrm{L}}E_{\mathrm{L},s'} = [\mathbb{K}_{[s],\mathrm{L}}]_{s,s'} = \Psi_{s,s'}, \tag{5.28}$$

$$E^*_{\mathrm{L},s}\mathbb{T}^*_{[s],\mathrm{L}}\mathbb{T}_{[r],\mathrm{R}}E_{\mathrm{L},r} = E^*_{\mathrm{L},s}\mathbb{K}_{[s],\mathrm{LR}}E_{\mathrm{R},r} = [\mathbb{K}_{[s],\mathrm{LR}}]_{s,r} = \Lambda_{s,r}, \tag{5.29}$$

$$E^*_{\mathrm{R},r}\mathbb{T}^*_{[r],\mathrm{R}}\mathbb{T}_{[r'],\mathrm{R}}E_{\mathrm{R},r'} = E^*_{\mathrm{R},r}\mathbb{K}_{[r],\mathrm{R}}E_{\mathrm{R},r'} = [\mathbb{K}_{[r],\mathrm{L}}]_{r,r'} = \Phi_{r,r'} \tag{5.30}$$

(where $E_{\mathrm{L},s}$ and $E_{\mathrm{R},r}$ are given by (1.29), (1.30)) holding for every choice of $s, s' \in S$ and $r, r' \in R$ so that $[s] = [s'] = [r] = [r']$. The latter equalities suggest the introduction of the operators

$$\mathbb{F}_{\mathrm{L}} = \mathrm{Col}_{s\in S}\,\mathbb{T}_{[s],\mathrm{L}}E_{\mathrm{L},s} : \mathcal{K}_{\mathrm{L}} \to \bigoplus_{s\in S}\mathcal{H}_{[s]}, \tag{5.31}$$

$$\mathbb{F}_{\mathrm{R}} = \mathrm{Col}_{r\in R}\,\mathbb{T}_{[r],\mathrm{R}}E_{\mathrm{R},r} : \mathcal{K}_{\mathrm{R}} \to \bigoplus_{r\in R}\mathcal{H}_{[r]}. \tag{5.32}$$

We note the following two formulas

$$\mathrm{Col}_{r\in R}\,\mathbb{T}_{[r],\mathrm{L}}\widetilde{N}_r(T_{\mathrm{L}}) = [(\mathbb{F}^*_{\mathrm{L}}\cdot Z_{\Gamma,\mathcal{H}})^{\wedge\mathrm{L}}(T_{\mathrm{L}})]^*, \tag{5.33}$$

$$\mathrm{Col}_{s\in S}\,\mathbb{T}_{[s],\mathrm{R}}\widetilde{M}_s(T_{\mathrm{R}}) = (Z_{\Gamma,\mathcal{H}}\cdot\mathbb{F}_{\mathrm{R}})^{\wedge\mathrm{R}}(T_{\mathrm{R}}), \tag{5.34}$$

which are similar to formulas (4.11) and (4.13) and are verified in much the same way.

Let $\widetilde{U} = \{\widetilde{U}_p : p \in P\}$ be the collection of unitary maps indexed by the set of path-connected components P of Γ and defined via formulas (5.18), (5.19). Then relations (5.22) and (5.24) hold by construction; in view of (5.31)–(5.34) these relations can be written as

$$\mathrm{Col}_{s\in S}\,\widetilde{U}^*_{[s]}[M_s f]_{[s]} = \begin{bmatrix}\mathbb{F}_{\mathrm{L}} & (Z_{\Gamma,\mathcal{H}}\cdot\mathbb{F}_{\mathrm{R}})^{\wedge\mathrm{R}}(T_{\mathrm{R}})\end{bmatrix} f, \tag{5.35}$$

$$\mathrm{Col}_{r\in R}\,\widetilde{U}^*_{[r]}[N_r f]_{[r]} = \begin{bmatrix}[(\mathbb{F}^*_{\mathrm{L}}\cdot Z_{\Gamma,\mathcal{H}})^{\wedge\mathrm{L}}(T_{\mathrm{L}})]^* & \mathbb{F}_{\mathrm{R}}\end{bmatrix} f. \tag{5.36}$$

Now we define the operator

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix} [A_{r,s}] & [B_r] \\ [C_s] & D\end{bmatrix} : \begin{bmatrix}\bigoplus_{s\in S}\mathcal{H}_{[s]} \\ \mathcal{U}\end{bmatrix} \to \begin{bmatrix}\bigoplus_{r\in R}\mathcal{H}_{[r]} \\ \mathcal{Y}\end{bmatrix}$$

in accordance to (5.20) by

$$\mathbf{U} = \begin{bmatrix} A & B \\ C & D\end{bmatrix} = \begin{bmatrix}\bigoplus_{r\in R}\widetilde{U}^*_{[r]} & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix}\widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{bmatrix}\begin{bmatrix}\bigoplus_{s\in S}\widetilde{U}_{[s]} & 0 \\ 0 & I\end{bmatrix}.$$

By the assumption of the lemma, $\widetilde{\mathbf{U}}$ extends $\mathbf{V}$:

$$\begin{bmatrix}\widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D}\end{bmatrix}\begin{bmatrix}\mathrm{Col}_{s\in S}[M_s f]_{[s]} \\ Yf\end{bmatrix} = \begin{bmatrix}\mathrm{Col}_{r\in R}[N_r f]_{[r]} \\ Xf\end{bmatrix} \quad \text{for every } f \in \mathcal{K}_{\mathrm{L}}\oplus\mathcal{K}_{\mathrm{R}},$$

which can be written in terms of **U** as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathrm{Col}_{s\in S}\, \widetilde{U}^*_{[s]}[M_s f]_{[s]} \\ Yf \end{bmatrix} = \begin{bmatrix} \mathrm{Col}_{r\in R}\, \widetilde{U}^*_{[r]}[N_r f]_{[r]} \\ Xf \end{bmatrix} \quad (f \in \mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}).$$

Upon substituting equalities (5.35) and (5.36) and block decompositions (1.36) for X and Y in the latter equality we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \mathbb{F}_{\rm L} & (Z_{\Gamma,\mathcal{H}} \cdot \mathbb{F}_{\rm R})^{\wedge \rm R}(T_{\rm R}) \\ Y_{\rm L}^* & Y_{\rm R} \end{bmatrix} = \begin{bmatrix} [(\mathbb{F}_{\rm L}^* \cdot Z_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L})]^* & \mathbb{F}_{\rm R} \\ X_{\rm L}^* & X_{\rm R} \end{bmatrix}. \tag{5.37}$$

By Remark 2.1, the colligations $\boldsymbol{\Sigma}$ and $\widetilde{\boldsymbol{\Sigma}}$ defined in (2.1) and (5.7) have the same characteristic functions and thus F can be taken in the form (1.17). Let $H(z)$ and $G(z)$ be defined as in (1.23) and decomposed as in (1.18) and (1.20). We shall use the representations (1.24) of $F(z)$ which are equivalent to (1.17).

Since **U** is unitary, it follows from (5.37) that

$$A^*[(\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L})]^* + C^* X_{\rm L}^* = \mathbb{F}_{\rm L}, \tag{5.38}$$

$$B^*[(\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L})]^* + D^* X_{\rm L}^* = Y_{\rm L}^*, \tag{5.39}$$

$$A(Z_{\Gamma,\mathcal{H}}\mathbb{F}_{\rm R})^{\wedge \rm R}(T_{\rm R}) + BY_{\rm R} = \mathbb{F}_{\rm R}, \tag{5.40}$$

$$C(Z_{\Gamma,\mathcal{H}}\mathbb{F}_{\rm R})^{\wedge \rm R}(T_{\rm R}) + DY_{\rm R} = X_{\rm R}. \tag{5.41}$$

Taking adjoints in (5.38) we get $X_{\rm L}C = \mathbb{F}_{\rm L}^* - (\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L})A$ which can be written, by properties (4.1) and (4.2) of the left evaluation map, as

$$X_{\rm L}C = (\mathbb{F}_{\rm L}^*(I - Z_{\Gamma,\mathcal{H}}A))^{\wedge \rm L}(T_{\rm L}).$$

Multiplying both sides in the last equality by $(I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1}$ on the right and applying the left evaluation map to the resulting identity, we get

$$\begin{aligned} (X_{\rm L}H)^{\wedge \rm L}(T_{\rm L}) &= ((\mathbb{F}_{\rm L}^*(I - Z_{\Gamma,\mathcal{H}}A))^{\wedge \rm L}(T_{\rm L})(I - Z_{\Gamma,\mathcal{H}}A)^{-1})^{\wedge \rm L}(T_{\rm L}) \\ &= (\mathbb{F}_{\rm L}^*(I - Z_{\Gamma,\mathcal{H}}A)(I - Z_{\Gamma,\mathcal{H}}A)^{-1})^{\wedge \rm L}(T_{\rm L}) = (\mathbb{F}_{\rm L}^*)^{\wedge \rm L}(T_{\rm L}) = \mathbb{F}_{\rm L}^*. \end{aligned} \tag{5.42}$$

Note that the second equality in the last chain has been obtained upon applying (4.4) to $T(z) = \mathbb{F}_{\rm L}^*(I - Z_{\Gamma,\mathcal{H}}(z)A)$ and $\widetilde{T}(z) = (I - Z_{\Gamma,\mathcal{H}}(z)A)^{-1}$, whereas the third equality follows by the property (4.1).

Next we take adjoints in (5.39) to get

$$Y_{\rm L} = (\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}})^{\wedge \rm L}(T_{\rm L})B + X_{\rm L}D = (\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L}) + X_{\rm L}D. \tag{5.43}$$

By (5.42), $(\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L}) = ((X_{\rm L}H)^{\wedge \rm L}(T_{\rm L}) \cdot Z_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L})$ and using (4.4) to $T(z) = X_{\rm L}H(z)$ and $\widetilde{T}(z) = Z_{\Gamma,\mathcal{H}}(z)B$ leads us to

$$(\mathbb{F}_{\rm L}^* Z_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L}) = ((X_{\rm L}HZ_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L}).$$

Substituting the latter equality into the left hand side expression in (5.43) and making use of the first representation of S in (1.24), we get

$$Y_{\rm L} = (X_{\rm L}HZ_{\Gamma,\mathcal{H}}B)^{\wedge \rm L}(T_{\rm L}) + X_{\rm L}D = (X_{\rm L}HZ_{\Gamma,\mathcal{H}}B + X_{\rm L}D)^{\wedge \rm L}(T_{\rm L}) = (X_{\rm L}S)^{\wedge \rm L}(T_{\rm L}),$$

which proves the first interpolation condition in (1.28).

To get the second interpolation condition in (1.28) write (5.40) in the form

$$BY_{\rm R} = (I - AZ_{\Gamma,\mathcal{H}})\mathbb{F}_{\rm R})^{\wedge {\rm R}}(T_{\rm R}),$$

multiply the latter equality by $(I - AZ_{\Gamma,\mathcal{H}}(z))^{-1}$ on the left and apply the right evaluation map to the resulting identity $G(z)Y_{\rm R} = (I - AZ_{\Gamma,\mathcal{H}}(z))^{-1}(I - AZ_{\Gamma,\mathcal{H}})$ $\mathbb{F}_{\rm R})^{\wedge {\rm R}}(T_{\rm R})$. We have

$$\begin{aligned}(GY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= ((I - AZ_{\Gamma,\mathcal{H}})^{-1}((I - AZ_{\Gamma,\mathcal{H}})\mathbb{F}_{\rm R})^{\wedge {\rm R}}(T_{\rm R}))^{\wedge {\rm R}}(T_{\rm R}) \\ &= ((I - AZ_{\Gamma,\mathcal{H}})^{-1}(I - AZ_{\Gamma,\mathcal{H}})\mathbb{F}_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = (\mathbb{F}_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = \mathbb{F}_{\rm R}. \end{aligned} \tag{5.44}$$

Note that the third equality in the last chain has been obtained upon applying (4.5) to $T(z) = (I - AZ_{\Gamma,\mathcal{H}}(z))^{-1}$ and $\widetilde{T}(z) = (I - AZ_{\Gamma,\mathcal{H}}(z))\mathbb{F}_{\rm R}$. Substituting (5.44) into (5.41) and applying (4.5) to $T(z) = CZ_{\Gamma,\mathcal{H}}(z)$ and $\widetilde{T}(z) = G(z)Y_{\rm R}$, we get, by the second representation formula in (1.24),

$$\begin{aligned} X_{\rm R} &= (CZ_{\Gamma,\mathcal{H}}(GY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}))^{\wedge {\rm R}}(T_{\rm R}) + DY_{\rm R} = (CZ_{\Gamma,\mathcal{H}}GY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) + DY_{\rm R} \\ &= (CZ_{\Gamma,\mathcal{H}}GY_{\rm R} + DY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = (SY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}), \end{aligned}$$

which proves the second equality in (1.28).

Thus, F belongs to $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ as the characteristic function of a unitary colligation (2.1) and satisfies interpolation conditions (1.28). It remains to show that it satisfies also conditions (1.40)–(1.42). But it follows from (5.42), (5.44) and (5.31) that $(X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L}) = E^*_{{\rm L},s}\mathbb{T}^*_{[s],{\rm L}}$ and $(G_rY_{\rm R})^{\wedge {\rm L}}(T_{\rm R}) = \mathbb{T}_{[r],{\rm R}}E_{{\rm R},r}$ for $s \in S$ and $r \in R$. Now we pick any $s, s' \in S$ and $r, r' \in R$ so that $[s] = [s'] = [r] = [r']$ and combine the two latter equalities with (5.28)–(5.30) to get (1.40)–(1.42):

$$\begin{aligned} (X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L})[(X_{\rm L}H_{s'})^{\wedge {\rm L}}(T_{\rm L})]^* &= E^*_{{\rm L},s}\mathbb{T}^*_{[s],{\rm L}}\mathbb{T}_{[s'],{\rm L}}E_{{\rm L},s'} = \Psi_{s,s'}, \\ (X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L})(G_rY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= E^*_{{\rm L},s}\mathbb{T}^*_{[s],{\rm L}}\mathbb{T}_{[r],{\rm R}}E_{{\rm R},r} = \Lambda_{s,r}, \\ [(G_rY_{\rm R})^{\wedge {\rm R}}(T_{\rm R})]^*(G_{r'}Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= E^*_{{\rm R},r}\mathbb{T}^*_{[r],{\rm R}}\mathbb{T}_{[r'],{\rm R}}E_{{\rm R},r'} = \Phi_{r,r'}, \end{aligned}$$

and complete the proof. ∎

6. THE UNIVERSAL UNITARY COLLIGATION ASSOCIATED WITH THE INTERPOLATION PROBLEM

A general result of Arov and Grossman (see [6], [7]) describes how to parametrize the set of all unitary extensions of a given partially defined isometry **V**. This technique has been developed further in the general setting of the so-called Abstract Interpolation Problem by Katsnelson, Kheifets and Yuditskii (see [28], [29]) and more recently has been extended to the setting of (commutative) multivariable interpolation problems in [15], [16], [10]. In this section we extend this technique to the setting of noncommutative power series.

Let $\mathbf{V}: \mathcal{D}_{\mathbf{V}} \leftarrow \mathcal{R}_{\mathbf{V}}$ be the isometry given in (5.4) with $\mathcal{D}_{\mathbf{V}}$ and $\mathcal{R}_{\mathbf{V}}$ given in (5.5) and (5.6). Introduce the defect spaces

$$\Delta = \begin{bmatrix} \bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]} \\ \mathcal{U} \end{bmatrix} \ominus \mathcal{D}_{\mathbf{V}} \quad \text{and} \quad \Delta_* = \begin{bmatrix} \bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]} \\ \mathcal{Y} \end{bmatrix} \ominus \mathcal{R}_{\mathbf{V}}$$

and let $\widetilde{\Delta}$ to be another copy of Δ and $\widetilde{\Delta}_*$ to be another copy of Δ_* with unitary identification maps

$$i: \ \Delta \leftarrow \widetilde{\Delta} \quad \text{and} \quad i_*: \ \Delta_* \leftarrow \widetilde{\Delta}_*.$$

Define a unitary operator $\mathbf{U}_0$ from $\mathcal{D}_{\mathbf{V}} \oplus \Delta \oplus \widetilde{\Delta}_*$ onto $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ by the rule

$$\mathbf{U}_0 x = \begin{cases} \mathbf{V}x, & \text{if } x \in \mathcal{D}_{\mathbf{V}}, \\ i(x) & \text{if } x \in \Delta, \\ i_*^{-1}(x) & \text{if } x \in \widetilde{\Delta}_*. \end{cases} \tag{6.1}$$

Identifying $\begin{bmatrix} \mathcal{D}_{\mathbf{V}} \\ \Delta \end{bmatrix}$ with $\begin{bmatrix} \bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]} \\ \mathcal{U} \end{bmatrix}$ and $\begin{bmatrix} \mathcal{R}_{\mathbf{V}} \\ \Delta_* \end{bmatrix}$ with $\begin{bmatrix} \bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]} \\ \mathcal{Y} \end{bmatrix}$, we decompose $\mathbf{U}_0$ defined by (6.1) according to

$$\mathbf{U}_0 = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & 0 \end{bmatrix} : \begin{bmatrix} \bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]} \\ \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix} \to \begin{bmatrix} \bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]} \\ \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}. \tag{6.2}$$

The $(3,3)$ block in this decomposition is zero, since (by definition (6.1)), for every $x \in \widetilde{\Delta}_*$, the vector $\mathbf{U}_0 x$ belongs to Δ, which is a subspace of $\begin{bmatrix} \bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]} \\ \mathcal{Y} \end{bmatrix}$ and therefore, is orthogonal to $\widetilde{\Delta}$ (in other words $\mathbf{P}_{\widetilde{\Delta}} \mathbf{U}_0|_{\widetilde{\Delta}_*} = 0$, where $\mathbf{P}_{\widetilde{\Delta}}$ stands for the orthogonal projection of $\mathcal{R}_{\mathbf{V}} \oplus \Delta_* \oplus \widetilde{\Delta}$ onto $\widetilde{\Delta}$).

The unitary operator $\mathbf{U}_0$ is the connecting operator of the unitary colligation

$$\mathbf{\Sigma}_0 = \left\{ \Gamma, \ \widehat{\mathcal{H}}, \ \begin{bmatrix} \mathcal{U} \\ \widetilde{\Delta}_* \end{bmatrix}, \ \begin{bmatrix} \mathcal{Y} \\ \widetilde{\Delta} \end{bmatrix}, \ \mathbf{U}_0 \right\}, \tag{6.3}$$

which is called *the universal unitary colligation* associated with Problem 1.9.

Let $\widetilde{\mathbf{\Sigma}}$ be any colligation of the form

$$\widetilde{\mathbf{\Sigma}} = \{\Gamma, \ \widetilde{\mathcal{H}}, \ \widetilde{\Delta}, \ \widetilde{\Delta}_*, \ \widetilde{\mathbf{U}}\}. \tag{6.4}$$

We define another colligation $\mathcal{F}_{\mathbf{\Sigma}_0}[\widetilde{\mathbf{\Sigma}}]$, called the *coupling* of $\mathbf{\Sigma}_0$ and $\widetilde{\mathbf{\Sigma}}$, to be the colligation of the form

$$\mathcal{F}_{\mathbf{\Sigma}_0}[\widetilde{\mathbf{\Sigma}}] = \{\Gamma, \ \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}, \ \mathcal{U}, \ \mathcal{Y}, \ \mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]\}$$

with the connecting operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ defined as follows:

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]: \begin{bmatrix} c \\ h \\ u \end{bmatrix} \to \begin{bmatrix} c' \\ h' \\ y \end{bmatrix} \tag{6.5}$$

if the system of equations

$$\mathbf{U}_0: \begin{bmatrix} c \\ u \\ \widetilde{d}_* \end{bmatrix} \to \begin{bmatrix} c' \\ y \\ \widetilde{d} \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{U}}: \begin{bmatrix} h \\ \widetilde{d} \end{bmatrix} \to \begin{bmatrix} h' \\ \widetilde{d}_* \end{bmatrix} \tag{6.6}$$

is satisfied for some choice of $\widetilde{d} \in \widetilde{\Delta}$ and $\widetilde{d}_* \in \widetilde{\Delta}_*$. To show that the operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is well defined, i.e., that for every triple (c, h, u), there exist $\widetilde{d}$ and $\widetilde{d}_*$ for which the system (6.6) is consistent and the resulting triple (c', h', y) does not depend on the choice of $\widetilde{d}$ and $\widetilde{d}_*$, we note first that, on account of (6.1) and (6.2), the bottom component of the first equation in (6.6) determines $\widetilde{d}$ uniquely by

$$\widetilde{d} = \mathbf{P}_{\widetilde{\Delta}}(\mathbf{V}\mathbf{P}_{\mathcal{D}_{\mathbf{V}}} + \mathbf{i}\mathbf{P}_{\Delta}) \begin{bmatrix} c \\ u \end{bmatrix} = \mathbf{i}\mathbf{P}_{\Delta} \begin{bmatrix} c \\ u \end{bmatrix}.$$

With this $\widetilde{d}$, the bottom component of the second equation in (6.6) determines uniquely $\widetilde{d}_*$ and h'. Using $\widetilde{d}_*$ one can recover now c' and y from the first and second components of the first equation in (6.6).

Since operators $\mathbf{U}_0$ and $\widetilde{\mathbf{U}}$ are unitary, it follows from (6.6) that $\|c\|^2 + \|u\|^2 + \|\widetilde{d}_*\|^2 = \|c'\|^2 + \|y\|^2 + \|\widetilde{d}\|^2$, $\|h\|^2 + \|\widetilde{d}\|^2 = \|h'\|^2 + \|\widetilde{d}_*\|^2$ and therefore, that $\|c\|^2 + \|u\|^2 + \|h\|^2 = \|c'\|^2 + \|y\|^2 + \|h'\|^2$, which means that the coupling operator $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is isometric. A similar argument can be made with the adjoints of $\mathbf{U}_0$, $\widetilde{\mathbf{U}}$ and $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$, and hence $\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]$ is unitary. Furthermore, by (6.5) and (6.6),

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]|_{(\oplus_{s\in S}\widehat{\mathcal{H}}_{[s]})\oplus\mathcal{U}} = \mathbf{U}_0|_{(\oplus_{s\in S}\widehat{\mathcal{H}}_{[s]})\oplus\mathcal{U}}$$

and since $\mathcal{D}_{\mathbf{V}} \subset \Big(\bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]}\Big) \oplus \mathcal{U}$, it follows that

$$\mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}]|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{U}_0|_{\mathcal{D}_{\mathbf{V}}} = \mathbf{V}. \tag{6.7}$$

Thus, the coupling of the connecting operator $\mathbf{U}_0$ of the universal unitary colligation associated with Problem 1.9 and any other unitary operator is a unitary extension of the isometry $\mathbf{V}$ defined in (5.4). Conversely for every unitary colligation $\mathbf{\Sigma} = \{\Gamma, \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}, \mathcal{U}, \mathcal{Y}, \mathbf{U}\}$ with the connecting operator being a unitary extension of $\mathbf{V}$, there exists a unitary colligation $\widetilde{\mathbf{\Sigma}}$ of the form (6.4) such that $\mathbf{\Sigma} = \mathcal{F}_{\mathbf{\Sigma}_0}[\widetilde{\mathbf{\Sigma}}]$ (the proof is the same as in Theorem 6.2 of [15]). Thus, all unitary extensions $\mathbf{U}$ of the isometry $\mathbf{V}$ defined in (5.4) are parametrized by the formula

$$\mathbf{U} = \mathcal{F}_{\mathbf{U}_0}[\widetilde{\mathbf{U}}], \quad \widetilde{\mathbf{U}}: \Big(\bigoplus_{s\in S} \widehat{\mathcal{H}}_{[s]}\Big) \oplus \widetilde{\Delta} \to \Big(\bigoplus_{r\in R} \widehat{\mathcal{H}}_{[r]}\Big) \oplus \widetilde{\Delta}_* \tag{6.8}$$

and $\widetilde{\mathcal{H}} = \{\widetilde{\mathcal{H}} : p \in P\}$ is a collection of auxiliary Hilbert spaces indexed by the path-connected components $p \in P = P(\Gamma)$ of the admissible graph Γ.

The characteristic function of the colligation $\boldsymbol{\Sigma}_0$ defined in (6.3) with the connecting operator $\mathbf{U}_0$ partitioned as in (6.2), is given by

$$\begin{aligned}\Sigma(z) &= \begin{bmatrix} \Sigma_{11}(z) & \Sigma_{12}(z) \\ \Sigma_{21}(z) & \Sigma_{22}(z) \end{bmatrix} \\ &= \begin{bmatrix} U_{22} & U_{23} \\ U_{32} & 0 \end{bmatrix} + \begin{bmatrix} U_{21} \\ U_{31} \end{bmatrix} (I - Z_{\Gamma,\widehat{\mathcal{H}}}(z)U_{11})^{-1} Z_{\Gamma,\widehat{\mathcal{H}}}(z) \begin{bmatrix} U_{12} & U_{13} \end{bmatrix}\end{aligned} \tag{6.9}$$

and belongs to the class $\mathcal{SA}_\Gamma(\mathcal{U} \oplus \widetilde{\Delta}_*, \ \mathcal{Y} \oplus \widetilde{\Delta})$ by Theorem 1.6.

THEOREM 6.1. *Let* $\mathbf{V}$ *be the isometry defined in* (5.4), *let* Σ *be constructed as above and let* F *be an element in* $\mathcal{L}(\mathcal{U}, \mathcal{Y})\langle\langle z \rangle\rangle$. *Then the following are equivalent:*

(i) *F is a solution of Problem* 1.9.

(ii) *F is a characteristic function of a colligation* $\boldsymbol{\Sigma} = \{\Gamma, \ \widehat{\mathcal{H}} \oplus \widetilde{\mathcal{H}}, \ \mathcal{U}, \ \mathcal{Y}, \ \mathbf{U}\}$ *with the connecting operator* $\mathbf{U}$ *being a unitary extension of* $\mathbf{V}$.

(iii) *F is of the form*

$$F(z) = \Sigma_{11}(z) + \Sigma_{12}(z)(I_{\widetilde{\Delta}_*} - \mathcal{T}(z)\Sigma_{22}(z))^{-1}\mathcal{T}(z)\Sigma_{21}(z) \tag{6.10}$$

where $\mathcal{T}(z)$ *is a formal power series from the class* $\mathcal{SA}_\Gamma(\widetilde{\Delta}, \ \widetilde{\Delta}_*)$.

Proof. The equivalence (i)$\Longleftrightarrow$(ii) follows by Lemmas 5.1 and 5.2.

(ii)$\Longrightarrow$ (iii). By the preceding analysis, the colligation $\boldsymbol{\Sigma}$ is the coupling of the universal colligation $\boldsymbol{\Sigma}_0$ defined in (6.3) and some unitary colligation $\widetilde{\boldsymbol{\Sigma}}$ of the form (6.4). The connecting operators $\mathbf{U}$, $\mathbf{U}_0$ and $\widetilde{\mathbf{U}}$ of these colligations are related as in (6.8). Let F, Σ and $\mathcal{T}$ be characteristic functions of $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}_0$ and $\widetilde{\boldsymbol{\Sigma}}$, respectively. Applying Remark 2.2 to (6.5) and (6.6), we get

$$F(z)e = e_*, \quad \Sigma(z)\begin{bmatrix} u \\ \widetilde{d}_* \end{bmatrix} = \begin{bmatrix} y \\ \widetilde{d} \end{bmatrix}, \quad \mathcal{T}(z)\widetilde{d} = \widetilde{d}_*. \tag{6.11}$$

Substituting the third relation in (6.11) into the second we get $\Sigma(z)\begin{bmatrix} u \\ \mathcal{T}(z)\widetilde{d} \end{bmatrix} = \begin{bmatrix} y \\ \widetilde{d} \end{bmatrix}$, which in view of the block decomposition (6.9) of Σ splits into

$$\Sigma_{11}(z)u + \Sigma_{12}(z)\mathcal{T}(z)\widetilde{d} = y \quad \text{and} \quad \Sigma_{21}(z)u + \Sigma_{22}(z)\mathcal{T}(z)\widetilde{d} = \widetilde{d}.$$

The second from the two last equalities gives $\widetilde{d} = (I - \Sigma_{22}(z)\mathcal{T}(z))^{-1}\Sigma_{21}(z)u$ which, being substituted into the first equality, implies $(\Sigma_{11}(z) + \Sigma_{12}(z)\mathcal{T}(z)(I - \Sigma_{22}(z)\mathcal{T}(z))^{-1}\Sigma_{21}(z))u = y$. The latter is equivalent to

$$(\Sigma_{11}(z) + \Sigma_{12}(z)(I - \mathcal{T}(z)\Sigma_{22}(z))^{-1}\mathcal{T}(z)\Sigma_{21}(z))u = y$$

and the comparison of the last equality with the first relation in (6.11) leads to representation (6.10) of F, since a vector $u \in \mathcal{U}$ is arbitrary.

(iii)$\Longrightarrow$(ii). Let F be of the form (6.10) for some $\mathcal{T} \in \mathcal{SA}_\Gamma(\widetilde{\Delta}, \widetilde{\Delta}_*)$. By Theorem 1.6, $\mathcal{T}$ is the characteristic function of a unitary colligation $\widetilde{\mathbf{\Sigma}}$ of the form (6.4). Let $\mathbf{\Sigma}$ be the unitary colligation defined by $\mathbf{\Sigma} = \mathcal{F}_{\mathbf{\Sigma}_0}[\widetilde{\mathbf{\Sigma}}]$. By the preceding "(ii)$\Longrightarrow$(iii)" part, F of the form (6.10) is the characteristic function of $\mathbf{\Sigma}$. It remains to note that the colligation $\mathbf{\Sigma}$ is of required the form: its input and output spaces coincide with $\mathcal{U}$ and $\mathcal{Y}$, respectively (by the definition of coupling) and its connecting operator is an extension of $\mathbf{V}$, by (6.7). ∎

As a corollary we obtain the sufficiency part in both Theorem 1.8 and Theorem 1.10, including the parametrization of the set of all solutions of Problem 1.9 in Theorem 1.10.

7. EXAMPLES AND SPECIAL CASES

For certain special cases of Problems 1.7 and 1.9, the general interpolation results stated in Theorems 1.8 and 1.10 become much more transparent. Moreover, some of these particular cases are quite important for applications and are interesting in their own right; it seems reasonable therefore to display them in more detail.

7.1. LEFT SIDED INTERPOLATION PROBLEMS. The left sided problem can be considered as the special case of Problem 1.7 when $T_{\rm R}$ is a tuple of operators acting on the space of dimension zero.

PROBLEM 7.1. *Given an admissible data set $\mathcal{D} = \{T_{\rm L}, X_{\rm L}, Y_{\rm L}\}$, find necessary and sufficient conditions for existence of a power series $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ such that*

$$(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L}) = Y_{\rm L}. \tag{7.1}$$

The answer follows immediately from Theorem 1.8.

THEOREM 7.2. *There is a power series $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ satisfying interpolation condition* (7.1) *if and only if there exists a collection $\mathbb{K}_{\rm L} = \{\mathbb{K}_{p,{\rm L}} : p \in P\}$ of positive semidefinite operators on the space $\bigoplus_{s\in S:[s]=p} \mathcal{K}_{\rm L}$ indexed by the set of path-connected components P of Γ, which satisfies the Stein identity*

$$\sum_{s\in S} E_{{\rm L},s}^* \mathbb{K}_{[s],{\rm L}} E_{{\rm L},s} - \sum_{r\in R} \widetilde{N}_r(T_{\rm L})^* \mathbb{K}_{[r],{\rm L}} \widetilde{N}_r(T_{\rm L}) = X_{\rm L}X_{\rm L}^* - Y_{\rm L}Y_{\rm L}^*,$$

where $E_{{\rm L},s}$ and $\widetilde{N}_r$ are the operators defined via formulas (1.29) *and* (1.31), *respectively.*

Furthermore, it follows by Theorem 1.10 that for every choice of a tuple $\mathbb{K}_{\rm L}$ satisfying the conditions of Theorem 7.2, there exists a power series $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ satisfying (besides the left interpolation condition (7.1)) supplementary interpolation conditions

$$(X_{\rm L}H_s)^{\wedge {\rm L}}(T_{\rm L})[(X_{\rm L}H_{s'})^{\wedge {\rm L}}(T_{\rm L})]^* = \Psi_{s,s'} \quad \text{for } s,s' \in S: \ [s]=[s'],$$

for some choice of associated function $H(z)$ in representation (1.19) of F. Furthermore, all such F can be parametrized by a linear fractional transformation. We leave to the reader to formulate the right sided interpolation problem and to derive the right sided version of Theorem 7.2 from Theorem 1.8.

Parallel results hold for right sided interpolation problems; we leave the formulation of explicit statements to the reader.

7.2. THE CASE OF THE NONCOMMUTATIVE BALL. Now we consider the Fornasini-Marchesini case (see Example 1.2 above) where $S = \{1\}$ and $R = E = \{1, \dots, d\}$. In this case, from Corollaries 3.4 and 3.6 we see that a sufficient condition for $T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ to be left admissible is that $T_{\rm L}$ be a strict row contraction and that a sufficient condition for $T_{\rm R} = (T_{{\rm R},1}, \dots, T_{{\rm R},d})$ to be right admissible is that $T_{\rm R}$ be a strict column contraction:

$$\sum_{j=1}^{d} T_{{\rm L},j} T_{{\rm L},j}^{*} < I_{\mathcal{K}_{\rm L}} \quad \text{and} \quad \sum_{j=1}^{d} T_{{\rm R},j}^{*} T_{{\rm R},j} < I_{\mathcal{K}_{\rm R}}.$$

The left sided problem is of special interest.

PROBLEM 7.3. *Given an admissible data set $\mathcal{D} = \{T_{\rm L}, X_{\rm L}, Y_{\rm L}\}$, find necessary and sufficient conditions for existence of a power series $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ satisfying the left sided interpolation condition* (7.1).

In this particular case $E_{\rm L} = I_{\mathcal{K}_{\rm L}}$, $\widetilde{N}_j(T_{\rm L}) = T_{{\rm L},j}^{*}$, and we conclude by Theorem 7.2 that there exists a power series $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ satisfying (7.1) if and only if there exists a positive semidefinite operator $\mathbb{K}_{\rm L}$ subject to the Stein identity $\mathbb{K}_{\rm L} - \sum_{j=1}^{d} T_{{\rm L},j} \mathbb{K}_{\rm L} T_{{\rm L},j}^{*} = X_{\rm L} X_{\rm L}^{*} - Y_{\rm L} Y_{\rm L}^{*}$. Since the d-tuple $T_{\rm L}$ is a strict row contraction, the latter Stein equation has a unique solution given in terms of convergent series by

$$\mathbb{K}_{\rm L} = \sum_{v \in \mathcal{F}_E} T_{\rm L}^{v} (X_{\rm L} X_{\rm L}^{*} - Y_{\rm L} Y_{\rm L}^{*}) (T_{\rm L}^{*})^{v} \tag{7.2}$$

and we come to the following.

THEOREM 7.4. *Assume that $T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ is a strict row contraction. Then there is a power series $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ satisfying interpolation condition* (7.1) *if and only if the operator $\mathbb{K}_{\rm L}$ defined in* (7.2) *is positive semidefinite.*

A remarkable part about the left sided interpolation for the Fornasini-Marchesini case is that no supplementary conditions are needed to get a parametrization of the solution set: since the operator $\mathbb{K}_{\rm L}$ is uniquely determined by the interpolation data, it follows by Theorem 1.10 that for every $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ satisfying (7.1), the function $H(z)$ associated with F via representation (1.19), satisfies

$$(X_{\rm L} H)^{\wedge {\rm L}}(T_{\rm L}) [(X_{\rm L} H)^{\wedge {\rm L}}(T_{\rm L})]^{*} = \mathbb{K}_{\rm L}.$$

Furthermore, in this case the power series Σ defined in (1.44) depends on the data $\{T_{\rm L}, X_{\rm L}, Y_{\rm L}\}$ only and the linear fractional formula (1.45) parametrizes the solution set to Problem 7.3.

The two sided problem in the Fornasini-Marchesini case is less remarkable.

PROBLEM 7.5. *Given an admissible interpolation data set* (1.27), *find necessary and sufficient conditions for existence of a power series* $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ *such that*

$$(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L}) = Y_{\rm L} \quad and \quad (FY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = X_{\rm R}. \tag{7.3}$$

The formulas (1.33) and (1.34) read

$$M = \begin{bmatrix} I_{\mathcal{K}_{\rm L}} & 0 \\ 0 & T_{{\rm R},1} \\ \vdots & \vdots \\ 0 & T_{{\rm R},d} \end{bmatrix} \quad \text{and} \quad N_j = \begin{bmatrix} T_{{\rm L},j}^* & 0 \\ 0 & E_j \end{bmatrix} \quad (j = 1, \dots, d) \tag{7.4}$$

where $E_1 = \begin{bmatrix} I_{\mathcal{K}_{\rm R}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 \\ I_{\mathcal{K}_{\rm R}} \\ \vdots \\ 0 \end{bmatrix}, \dots, E_d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_{\mathcal{K}_{\rm R}} \end{bmatrix}$. Now Theorem 1.8 leads us to the following conclusion:

THEOREM 7.6. *There is a power series* $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ *satisfying interpolation conditions* (7.3) *if and only if there exists a positive semidefinite operator*

$$\mathbb{K} = \begin{bmatrix} \mathbb{K}_{\rm L} & \mathbb{K}_{\rm LR} \\ \mathbb{K}_{\rm LR}^* & \mathbb{K}_{\rm R} \end{bmatrix} \in \mathcal{L}(\mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}^d) \tag{7.5}$$

subject to the Stein identity

$$M^*\mathbb{K}M - \sum_{j=1}^d N_j^*\mathbb{K}N_j = X^*X - Y^*Y, \tag{7.6}$$

where M, N_j, X *and* Y *are defined in* (7.4) *and* (1.36).

Since the block $\mathbb{K}_{\rm L}$ in (7.5) is uniquely determined from the left interpolation data via the Stein identity (7.6), the latter result can be displayed more explicitly in terms of a structured positive completion problem.

THEOREM 7.7. *There is a power series* $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U}, \mathcal{Y})$ *satisfying interpolation conditions* (7.3) *if and only if there exist operators* $\Lambda_j \in \mathcal{L}(\mathcal{K}_{\rm R}, \mathcal{K}_{\rm L})$ *and* $\Phi_{ij} \in \mathcal{L}(\mathcal{K}_{\rm R})$ *for* $i, j = 1, \dots, d$ *subject to Stein identities*

$$\sum_{j=1}^d (T_{{\rm L},j}\Lambda_j - \Lambda_j T_{{\rm R},j}) = Y_{\rm L}Y_{\rm R} - X_{\rm L}X_{\rm R}, \quad \sum_{j=1}^d \Phi_{jj} - \sum_{i,j=1}^d T_{{\rm R},i}^*\Phi_{ij}T_{{\rm R},j} = Y_{\rm R}^*Y_{\rm R} - X_{\rm R}^*X_{\rm R}$$

and such that the following operator is positive semidefinite, with $\mathbb{K}_{\rm L}$ *as defined in* (7.2)*:*

$$\begin{bmatrix} \mathbb{K}_{\rm L} & \Lambda_1 & \dots & \Lambda_d \\ \Lambda_1^* & \Phi_{11} & \dots & \Phi_{1d} \\ \vdots & \vdots & & \vdots \\ \Lambda_d^* & \Phi_{d1} & \dots & \Phi_{dd} \end{bmatrix}.$$

To get Theorem 7.7 from Theorem 7.6, it suffices to let $\mathbb{K}_{\rm LR} = \begin{bmatrix} \Lambda_1 & \cdots & \Lambda_d \end{bmatrix}$ and $\mathbb{K}_{\rm R} = [\Phi_{ij}]_{i,j=1}^d$ and to make use of block decompositions (7.4) and (7.5).

7.3. THE CASE OF THE NONCOMMUTATIVE POLYDISK. Here we consider the Givone-Roesser case (see Example 1.3 above) where $S = R = E = \{1, \dots, d\}$ and the tuples $T_{\rm L}$ and $T_{\rm R}$ are just d-tuples $T_{\rm L} = (T_{{\rm L},1}, \dots, T_{{\rm L},d})$ and $T_{\rm R} = (T_{{\rm R},1}, \dots, T_{{\rm R},d})$ of contractive operators acting on $\mathcal{K}_{\rm L}$ and $\mathcal{K}_{\rm R}$, respectively.

PROBLEM 7.8. *Given an admissible interpolation data set* (1.27)*, find necessary and sufficient conditions for existence of a power series* $F \in \mathcal{SA}_{\Gamma^{\rm GR}}(\mathcal{U}, \mathcal{Y})$ *such that*

$$(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L}) = Y_{\rm L} \quad and \quad (FY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = X_{\rm R}. \tag{7.7}$$

The formulas (1.29)–(1.32) read $E_{{\rm L},j} = I_{\mathcal{K}_{\rm L}}$, $E_{{\rm R},j} = I_{\mathcal{K}_{\rm R}}$, $\widetilde{N}_j(T_{\rm L}) = T_{{\rm L},j}^*$, $\widetilde{M}_j(T_{\rm R}) = T_{{\rm R},j}$ $(j = 1, \dots, d)$ and therefore, formulas (1.33) and (1.34) take the form

$$M_j = \begin{bmatrix} I_{\mathcal{K}_{\rm L}} & 0 \\ 0 & T_{{\rm R},j} \end{bmatrix}, \quad N_j = \begin{bmatrix} T_{{\rm L},j}^* & 0 \\ 0 & I_{\mathcal{K}_{\rm R}} \end{bmatrix} \quad (j = 1, \dots, d). \tag{7.8}$$

Theorem 1.8 now reduces to

THEOREM 7.9. *There is a power series* $F \in \mathcal{SA}_{\Gamma}^{\rm GR}(\mathcal{U}, \mathcal{Y})$ *satisfying interpolation conditions* (7.7) *if and only if there exist positive semidefinite operators*

$$\mathbb{K}_j = \begin{bmatrix} \mathbb{K}_{j,{\rm L}} & \mathbb{K}_{j,{\rm LR}} \\ \mathbb{K}_{j,{\rm LR}}^* & \mathbb{K}_{j,{\rm R}} \end{bmatrix} \in \mathcal{L}(\mathcal{K}_{\rm L} \oplus \mathcal{K}_{\rm R}) \quad for \;\; j = 1, \dots, d, \tag{7.9}$$

that satisfy the following Stein identity, where M_j *and* N_j *are the operators defined via formulas* (7.8)*, and* X *and* Y *are the same as in* (1.36)*:*

$$\sum_{j=1}^d (M_j^* \mathbb{K}_j M_j - N_j^* \mathbb{K}_j N_j) = X^*X - Y^*Y. \tag{7.10}$$

Furthermore, it follows by Theorem 1.10 that for every choice of positive semidefinite operators $\mathbb{K}_1, \dots, \mathbb{K}_d$ of the form (7.9), satisfying the Stein identity (7.10), there exists a power series $F \in \mathcal{SA}_{\Gamma}^{\rm GR}(\mathcal{U}, \mathcal{Y})$ satisfying (besides (7.7)) supplementary interpolation conditions

$$\begin{aligned} (X_{\rm L}H_j)^{\wedge {\rm L}}(T_{\rm L})[(X_{\rm L}H_j)^{\wedge {\rm L}}(T_{\rm L})]^* &= \mathbb{K}_{j,{\rm L}}, \\ (X_{\rm L}H_j)^{\wedge {\rm L}}(T_{\rm L})(G_jY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= \mathbb{K}_{j,{\rm LR}}, \\ [(G_jY_{\rm R})^{\wedge {\rm R}}(T_{\rm R})]^*(G_jY_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= \mathbb{K}_{j,{\rm R}} \end{aligned} \tag{7.11}$$

for $j = 1, \dots, d$ and for some choice of associated functions $H(z)$ and $G(z)$ in representations (1.19), (1.21), (1.22) of F. Furthermore, all such F can be parametrized by a linear fractional transformation.

COROLLARY 7.10. *There is a power series $F \in \mathcal{SA}_\Gamma^{\rm GR}(\mathcal{U}, \mathcal{Y})$ satisfying the left interpolation condition*

$$(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L}) = Y_{\rm L} \tag{7.12}$$

if and only if there exist positive semidefinite operators $\mathbb{K}_{1,{\rm L}}, \dots, \mathbb{K}_{d,{\rm L}} \in \mathcal{L}(\mathcal{K}_{\rm L})$ that satisfy the Stein identity

$$\sum_{j=1}^{d} (\mathbb{K}_{j,{\rm L}} - N_j^*\mathbb{K}_{j,{\rm L}}N_j) = X_{\rm L}^*X_{\rm L} - Y_{\rm L}^*Y_{\rm L}. \tag{7.13}$$

Again, for every choice of operators $\mathbb{K}_{1,{\rm L}}, \dots, \mathbb{K}_{d,{\rm L}}$ meeting conditions of Corollary 7.10, there exists $F \in \mathcal{SA}_\Gamma^{\rm GR}(\mathcal{U}, \mathcal{Y})$ satisfying (besides the left condition (7.12)) conditions (7.11) for $j = 1, \dots, d$ and for some choice of associated function $H(z)$ in representation (1.19) of F.

7.4. THE SCHUR INTERPOLATION PROBLEM. The classical Schur problem [43] is concerned with necessary and sufficient conditions for existence of a (scalar valued) Schur function S with the preassigned first n Taylor coefficients at the origin.

Let Γ be an admissible graph and let $\mathcal{F}_E$ be the free semigroup generated by the edge set E of Γ. A subset $\mathcal{F} \subset \mathcal{F}_E$ will be called *lower inclusive* if whenever $v \in \mathcal{F}$ and $v = uw$ for some $u, w \in \mathcal{F}_E$, then it is the case that also $u \in \mathcal{F}$. A natural noncommutative analogue of the Schur problem is the following:

NSP: Let Γ be an admissible graph, let $\mathcal{F}_E$ be the free semigroup generated by the edge set E of Γ and let $\mathcal{F}$ be a finite lower inclusive subset of $\mathcal{F}_E$. Given a collection of operators $\{S_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) : \ v \in \mathcal{F}\}$, find necessary and sufficient conditions for a noncommutative Schur-Agler function $F(z) = \sum_{v \in \mathcal{F}_E} F_v z^v \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ to exist such that

$$F_v = S_v \quad \text{for every} \ \ v \in \mathcal{F}. \tag{7.14}$$

We will show that conditions (7.14) can be written in the form

$$(X_{\rm L}F)^{\wedge {\rm L}}(T) = Y_{\rm L} \tag{7.15}$$

for an appropriate choice of $X_{\rm L}$, $Y_{\rm L}$ and $T = \{T_e : \ e \in E\}$; in other words we will show that the **NSP** is a particular left sided case of Problem 1.7. The construction does not depend on the structure of the graph Γ and proceeds as follows.

We are given a lower inclusive subset $\mathcal{F}$ of the free semigroup $\mathcal{F}_E$ together with an operator $F_v \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ for each $v \in \mathcal{F}$. We let $\ell^2(\mathcal{F})$ be the Hilbert space with orthonormal basis $\{\delta_v : v \in \mathcal{F}\}$ indexed by $\mathcal{F}$ and set $\mathcal{K}_{\rm L} = \ell^2(\mathcal{F}) \otimes \mathcal{Y}$. Note that elements of $\mathcal{K}_{\rm L}$ can also be viewed as functions $v \mapsto f(v)$ on $\mathcal{F}$ with values in $\mathcal{Y}$ subject to $\sum_{v\in\mathcal{F}} \|f(v)\|^2_{\mathcal{Y}} < \infty$. Note that the empty word $\emptyset$ is in $\mathcal{F}$ since $\mathcal{F}$ is lower-inclusive. Define an operator $X_{\rm L} \in \mathcal{L}(\mathcal{Y}, \mathcal{K}_{\rm L})$ by

$$X_{\rm L} : y \mapsto \delta_\emptyset \otimes y. \tag{7.16}$$

For each $e \in E$, we define an operator $T_{{\rm L},e}$ on $\mathcal{K}_{\rm L}$ in terms of matrix entries $T_{{\rm L},e} = [T_{{\rm L},e;v,w}]_{v,w\in\mathcal{F}}$ (where each $T_{{\rm L},e;v,w} \in \mathcal{L}(\mathcal{Y})$) by

$$T_{{\rm L},e;v,w} = \begin{cases} I_{\mathcal{Y}} & \text{if } v = we, \\ 0 & \text{otherwise,} \end{cases}$$

or via the equivalent functional form $(T_{{\rm L},e}f)(v) = f(v \cdot e^{-1})$ for $f \in \mathcal{K}_{\rm L}$ where we use the convention (1.8) and we declare $f(\text{undefined}) = 0$. Then it is easily checked that, given a formal power series $F(z) = \sum_{v\in\mathcal{F}_E} F_v z^v$, the left evaluation with operator argument $(X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L})$ works out to be given by $((X_{\rm L}F)^{\wedge {\rm L}}(T_{\rm L})u)(v) = F_v u$ for $v \in \mathcal{F}$ and all $u \in \mathcal{U}$. Hence, if we define $Y_{\rm L} : \mathcal{U} \to \mathcal{K}_{\rm L}$ by

$$(Y_{\rm L}u)(v) = S_v u,$$

then the left tangential interpolation problem with operator argument associated with the data set $\mathcal{D} = (T_{\rm L}, X_{\rm L}, Y_{\rm L})$ is exactly equivalent to **NSP**, and hence necessary and sufficient conditions for the **NSP** to have a solution can be derived from Theorem 7.2.

7.5. INTERPOLATION WITH COMMUTATIVE DATA. For this example we consider the general Problems 1.7 and 1.9 when the tuples $T_{\rm L}$ and $T_{\rm R}$ are commutative. As explained in Section 3, the interpolation conditions (1.3) imposed on a formal power series $S \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ associated with Problem 1.7 can be expressed as interpolation conditions on the abelianized function $F^{\mathbf{a}}$ of commuting variables $\lambda_{e_1}, \ldots, \lambda_{e_d}$:

$$(X_{\rm L}F^{\mathbf{a}})^{\wedge {\rm L}}(T_{\rm L}) = Y_{\rm L}, \quad (F^{\mathbf{a}}Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) = X_{\rm R}. \tag{7.17}$$

Similarly, the additional interpolation conditions (1.40)–(1.42) imposed on $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ by Problem 1.9 can be expressed as interpolation conditions on the abelianized function $F^{\mathbf{a}}$:

$$\begin{aligned} (X_{\rm L}H^{\mathbf{a}}_s)^{\wedge {\rm L}}(T_{\rm L})[(X_{\rm L}H^{\mathbf{a}}_{s'})^{\wedge {\rm L}}(T_{\rm L})]^* &= \Psi_{s,s'} \quad \text{for } s, s' \in S: \ [s] = [s'], \\ (X_{\rm L}H^{\mathbf{a}}_s)^{\wedge {\rm L}}(T_{\rm L})(G^{\mathbf{a}}_r Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= \Lambda_{s,r} \quad \text{for } s \in S; \ r \in R: \ [s] = [r], \\ [(G^{\mathbf{a}}_r Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R})]^*(G^{\mathbf{a}}_{r'} Y_{\rm R})^{\wedge {\rm R}}(T_{\rm R}) &= \Phi_{r,r'} \quad \text{for } r, r' \in R: \ [r] = [r']. \end{aligned} \tag{7.18}$$

From the characterization of the class $\mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ as transfer functions of conservative SNMLSs with structure graph Γ and the counterpart of this result for the commutative Schur-Agler class $\mathcal{SA}_{Z^{\mathbf{a}}_\Gamma}(\mathcal{U}, \mathcal{Y})$ found in [9], it is clear that the abelianization $F^{\mathbf{a}}$ of any element $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ is an element of $\mathcal{SA}_{Z^{\mathbf{a}}_\Gamma}(\mathcal{U}, \mathcal{Y})$ as studied in [9], [10], and, conversely, any element S of $\mathcal{SA}_{Z^{\mathbf{a}}_\Gamma}(\mathcal{U}, \mathcal{Y})$ lifts to an element $F \in \mathcal{SA}_\Gamma(\mathcal{U}, \mathcal{Y})$ (so $S = F^{\mathbf{a}}$). The results of [10] can be applied to the abelianized problems involving interpolation conditions (7.17) (and possibly also (7.18)) for a function $F^{\mathbf{a}}$ in the commutative Schur-Agler class $\mathcal{SA}_{Z_{\Gamma}^{\mathbf{a}}}(\mathcal{U},\mathcal{Y})$. When this is done, the Stein equation (1.35) is the same as the Stein equation in [10] where it was shown to be the necessary and sufficient condition for the abelianized interpolation problem to have a solution in the commutative Schur-Agler class $\mathcal{SA}_{Z_{\Gamma}^{\mathbf{a}}}(\mathcal{U},\mathcal{Y})$. In this way, we see that establishing necessary and sufficient conditions for interpolation problems for formal power series in noncommuting indeterminates involving commutative data reduces to the more standard interpolation problems for analytic functions in commuting variables. However, parametrizing of the solution set for an interpolation problem is the point at which commutative and noncommutative interpolation problems got some distinctions. Roughly speaking, to get all solutions $F \in \mathcal{SA}_{Z_{\Gamma}^{\mathbf{a}}}(\mathcal{U},\mathcal{Y})$ of an interpolation problem with the commutative data, it suffices to describe all lifts to $\mathcal{SA}_{\Gamma}(\mathcal{U},\mathcal{Y})$ for every solution $S \in \mathcal{SA}_{Z_{\Gamma}^{\mathbf{a}}}(\mathcal{U},\mathcal{Y})$ of the corresponding commutative interpolation problem. The problem to describe of all such lifts for a given function F in the commutative Schur-Agler class is of certain interest. For simplicity we will discuss only the Fornasini-Marchesini case, where the commutative Schur-Agler class $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$ consists of contractive multipliers of the Arveson space.

PROBLEM 7.11. *Given a function S in the Schur-Agler class $\mathcal{S}_d(\mathcal{U},\mathcal{Y})$, find all formal power series $F \in \mathcal{SA}_{\Gamma^{\rm FM}}(\mathcal{U},\mathcal{Y})$ such that*

$$F^{\mathbf{a}} = S. \tag{7.19}$$

This problem can be treated as a left-tangential version of the corresponding Schur problem. Let now $\mathcal{K}_{\rm L} = \ell^2(\mathcal{F}_E) \otimes \mathcal{Y}$ and let $X_{\rm L} \in \mathcal{L}(\mathcal{Y},\mathcal{K}_{\rm L})$ be defined as in (7.16). We again label the edges of the graph G by the letters $1,2,\dots,d$ where $d = n_E$ is the number of edges of G. For each $j \in \{1,\dots,d\}$ we define an operator $T_{{\rm L},j}$ on $\mathcal{K}_{\rm L}$ in terms of matrix entries $T_{{\rm L},j} = [T_{{\rm L},j;v,w}]_{v,w\in\mathcal{F}}$ by

$$T_{{\rm L},j;v,w} = \begin{cases} I_{\mathcal{Y}} & \text{if } \mathbf{a}^{-1}(v) = \mathbf{a}^{-1}(w) + \mathbf{e}_j, \\ 0 & \text{otherwise,} \end{cases}$$

(where $\mathbf{e}_j$ is the unit vector with i-th component equal to 1 and all other components equal to 0) or via the equivalent functional form $(T_{{\rm L},j}f)(v) = \sum_{w\in\mathbf{a}^{-1}(\mathbf{a}(v)-\mathbf{e}_i)} f(w)$ for $f \in \mathcal{K}_{\rm L}$. Then it is easily checked that the d-tuple $T_{\rm L} = (T_1,\dots,T_d)$ is commutative and that given a formal power series $F(z) = \sum_{v\in\mathcal{F}_E} F_v z^v$, the left evaluation with operator argument $(X_{\rm L}F)^{\wedge{\rm L}}(T_{\rm L})$ works out to be given by

$$((X_{\rm L}F)^{\wedge{\rm L}}(T_{\rm L})u)(v) = \sum_{w\in\mathbf{a}^{-1}(\mathbf{a}(v))} F_w u \quad \text{for } v \in \mathcal{F} \text{ and all } u \in \mathcal{U}.$$

Hence, if we define $Y_{\rm L} : \mathcal{U} \to \mathcal{K}_{\rm L}$ by

$$(Y_{\rm L}u)(v) = S_{\mathbf{a}(v)}u,$$

then the left tangential interpolation problem with operator argument associated with the data set $\mathcal{D} = (T_{\rm L}, X_{\rm L}, Y_{\rm L})$ is exactly equivalent to Problem 7.11, which allows us to get a linear fractional parametrization of its solution set.

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