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TOEPLITZ-COMPOSITION C^*-ALGEBRAS

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ABSTRACT. Let ζ and η be distinct points on the unit circle and suppose that φ is a linear-fractional self-map of the unit disk $\mathbb{D}$, not an automorphism, with $\varphi(\zeta) = \eta$. We describe the C^*-algebra generated by the associated composition operator C_φ and the shift operator, acting on the Hardy space on $\mathbb{D}$.

KEYWORDS: *Composition operator, Toeplitz operator, C^*-algebra.*

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1. INTRODUCTION

Any analytic self-map φ of the unit disk $\mathbb{D}$ induces a bounded composition operator $C_\varphi : f \to f \circ \varphi$ on the Hardy space H^2. The linear-fractional self-maps of $\mathbb{D}$ form a rich class of examples, and many properties of composition operators are profitably studied in the context of these maps (e.g. cyclicity, spectral properties, subnormality; see [8], [9], [22]). The space H^2 also supports the Toeplitz operators T_w. Here, w is a bounded measurable function on the unit circle $\partial\mathbb{D}$, and T_w acts on H^2 by $T_w f = P(wf)$, where P is the orthogonal projection of L^2 (the Lebesgue space associated with normalized arc-length measure on $\partial\mathbb{D}$) onto H^2. Taking w to be the independent variable z, one obtains the shift operator T_z on H^2. A theorem of L. Coburn [4], [5] and I. Gohberg and I. Fel'dman [12], [13] asserts that $C^*(T_z)$, the unital C^*-algebra generated by T_z, contains the ideal $\mathcal{K}$ of compact operators, as well as all Toeplitz operators T_w with continuous symbol w. Moreover, the map sending w to the coset of T_w is a $*$-isomorphism of $C(\partial\mathbb{D})$, the algebra of continuous functions on $\partial\mathbb{D}$, onto the quotient algebra $C^*(T_z)/\mathcal{K}$. In this article our goal is to replace $C^*(T_z)$ by $C^*(T_z, C_\varphi)$, the unital C^*-algebra generated by T_z and C_φ, for certain linear-fractional φ.

Section 2 presents a characterization of those analytic self-maps φ of $\mathbb{D}$ with $|\varphi(e^{i\theta})| < 1$ a.e. on $\partial\mathbb{D}$ for which C_φ commutes with T_z or T_z^* modulo $\mathcal{K}$. In Section 3 we show that for any linear-fractional self-map φ of the disk which is not an automorphism, there is an associated linear-fractional map σ (the "Krein adjoint" of φ) and a scalar s so that $C_\varphi^* = sC_\sigma + K$ for some compact operator K. Our setting here is primarily that of H^2, although this result is easily extended to the Bergman space. This theorem plays a key role in the work in Section 4, where we study $C^*(T_z, C_\varphi)$. Recent work of M. Jury [17] treats the case where φ is an automorphism (and indeed ranges over a discrete group Γ of automorphisms), showing that the C^*-algebra generated by $\{C_\varphi : \varphi \in \Gamma\}$ contains T_z, and exhibiting the quotient of this algebra by $\mathcal{K}$ as the discrete crossed product $C(\partial\mathbb{D}) \times \Gamma$. In the present article we suppose φ is not an automorphism but does satisfy $\|\varphi\|_\infty = 1$. In the case that φ is a parabolic non-automorphism (see Section 2 for a discussion of this terminology; such maps have a fixed point on $\partial\mathbb{D}$), the work of P. Bourdon, D. Levi, S. Narayan and J. Shapiro in [3] shows that $C_\varphi^* C_\varphi - C_\varphi C_\varphi^*$ is compact. Such a C_φ also commutes with T_z and T_z^* modulo $\mathcal{K}$, so that $C^*(T_z, C_\varphi)/\mathcal{K}$ is commutative, hence describable by Gelfand theory. Here we suppose that φ is neither an automorphism nor a parabolic non-automorphism, but that there exist distinct points ζ, η in $\partial\mathbb{D}$ with $\varphi(\zeta) = \eta$. In this case $C^*(T_z, C_\varphi)/\mathcal{K}$ is not commutative, but we will see that it is tractable. As an application, in Section 4.6 we concretely determine the essential spectrum of any element of $C^*(T_z, C_\varphi)$. Our main tool is the localization theorem of R.G. Douglas [11].

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2. COMPOSITION OPERATORS ESSENTIALLY COMMUTING WITH T_z OR T_z^*

The commutator $AB - BA$ of two bounded operators A and B on a Hilbert space $\mathcal{H}$ is denoted $[A, B]$. An operator is said to be essentially normal if its self-commutator $[A^*, A]$ is compact. In the course of their work on essentially normal linear-fractional composition operators, Bourdon, Levi, Narayan and Shapiro [3] show that if φ is a linear fractional non-automorphism mapping $\mathbb{D}$ into $\mathbb{D}$ and fixing a point of $\partial\mathbb{D}$, then $[T_z^*, C_\varphi]$ is compact, where T_z is the shift on H^2. Here we will give a generalization which is perhaps of independent interest.

For α a complex number of modulus 1, and φ an analytic self-map of $\mathbb{D}$, the real part of $(\alpha + \varphi)/(\alpha - \varphi)$ is a positive harmonic function on $\mathbb{D}$. Necessarily then this function is the Poisson integral of a finite positive Borel measure μ_α on $\partial\mathbb{D}$; $\mu_\alpha, |\alpha| = 1$ are the *Clark measures* for φ. We write $E(\varphi)$ for the closure in $\partial\mathbb{D}$ of the union of the closed supports of the singular parts μ_α^s of the Clark measures as α ranges over the unit circle. For a linear-fractional non-automorphism φ which sends $\zeta \in \partial\mathbb{D}$ to $\eta \in \partial\mathbb{D}$, one has $\mu_\alpha^s = 0$ when $\alpha \neq \eta$ and $\mu_\eta^s = |\varphi'(\zeta)|^{-1}\delta_\zeta$, where δ_ζ is the unit point mass at ζ. We will use the following result, proved in [18]. Here M_w denotes the operator on $L^2 = L^2(\partial\mathbb{D})$ of multiplication by the bounded measurable function w.

THEOREM 2.1. [18] *Let φ be an analytic self-map of $\mathbb{D}$ such that $|\varphi(e^{i\theta})| < 1$ a.e. with respect to Lebesgue measure on $\partial\mathbb{D}$, and suppose that w is a bounded measurable* *function on $\partial\mathbb{D}$ which is continuous at each point of $E(\varphi)$. The weighted composition operator $M_w C_\varphi : H^2 \to L^2$ is compact if and only if $w \equiv 0$ on $E(\varphi)$.*

It will be convenient to recast Theorem 2.1 in terms of Toeplitz operators.

COROLLARY 2.2. *Suppose that φ and w satisfy the hypotheses in the first sentence of Theorem* 2.1. *Then $T_w C_\varphi : H^2 \to H^2$ is compact if and only if $w \equiv 0$ on $E(\varphi)$.*

Proof. It is enough to show that $M_w C_\varphi$ is compact when $T_w C_\varphi$ is compact. Note that

$$M_w C_\varphi = T_w C_\varphi + H_w C_\varphi$$

where $H_w : H^2 \to (H^2)^\perp$ is the Hankel operator defined by $H_w = (I - P)M_w|_{H^2}$. We need only check that $H_w C_\varphi$ is compact. Let $\widetilde{w}$ be a continuous function on $\partial\mathbb{D}$ agreeing with w on $E(\varphi)$. We have

$$H_w C_\varphi = (I - P)M_{(w-\widetilde{w})} C_\varphi + H_{\widetilde{w}} C_\varphi.$$

Since $\widetilde{w}$ is continuous, $H_{\widetilde{w}}$ is compact by Hartman's theorem [14]. On the other hand, $M_{(w-\widetilde{w})} C_\varphi$ is compact by Theorem 2.1, and we are done. ∎

The next result gives the above-mentioned generalization.

THEOREM 2.3. *Let φ be an analytic self-map of $\mathbb{D}$ such that $|\varphi(\mathrm{e}^{\mathrm{i}\theta})| < 1$ a.e. with respect to Lebesgue measure. Suppose that φ agrees almost everywhere on $\partial\mathbb{D}$ with a bounded measurable function $\widehat{\varphi}$ which is continuous at each point of $E(\varphi)$. Then the following are equivalent:*

(i) $[T_z, C_\varphi] \in \mathcal{K}$.

(ii) $[T_z^*, C_\varphi] \in \mathcal{K}$.

(iii) *For each ζ in $E(\varphi)$, $\widehat{\varphi}(\zeta) = \zeta$.*

When these conditions hold, $[T_w, C_\varphi] \in \mathcal{K}$ for every w in $C(\partial\mathbb{D})$.

Proof. We use the following identity from [3]:

$$[T_z^*, C_\varphi] = T_{(\bar{z}\varphi - 1)} C_\varphi T_z^*. \tag{2.1}$$

Since T_z^*, the backward shift, is a partial isometry with range H^2, the operator on the right-hand side of Equation (2.1) is compact exactly when $T_{(\bar{z}\varphi-1)} C_\varphi$ is compact. This operator clearly coincides with $T_{(\bar{z}\widehat{\varphi}-1)} C_\varphi$. Corollary 2.2 gives the equivalence of (ii) and (iii). For the equivalence of (i) and (iii) we easily check that

$$[T_z, C_\varphi] = T_{(z-\varphi)} C_\varphi = T_{(z-\widehat{\varphi})} C_\varphi$$

and again apply Corollary 2.2, with $w = z - \widehat{\varphi}$. The statement about $[T_w, C_\varphi]$ is immediate. ∎

3. THE ADJOINT OF C_φ

In this section we develop some properties of linear-fractional composition operators and their adjoints. To any linear-fractional map

$$\varphi(z) = \frac{az+b}{cz+d} \tag{3.1}$$

we associate another linear-fractional map σ_φ defined as

$$\sigma_\varphi(z) = \frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}. \tag{3.2}$$

The map σ_φ is sometimes referred to as the "Krein adjoint" of φ; for an explanation of this terminology, see [10]. When no confusion can result, we write σ for σ_φ. When φ is a self-map of the disk, σ will be also, and if $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma(\eta) = \zeta$; see [8]. Carl Cowen [8] has shown that the adjoint of any linear-fractional C_φ, acting on H^2, is given by

$$C_\varphi^* = T_g C_\sigma T_h^* \tag{3.3}$$

where $g(z) = (-\overline{b}z+\overline{d})^{-1}$, $h(z) = cz+d$, and T_g, T_h are the analytic Toeplitz operators of multiplication by the H^∞ functions g and h.

Our first result uses Equation (3.3) to show that when $\|\varphi\|_\infty = 1$ but φ is not an automorphism, the adjoint of C_φ, modulo the ideal $\mathcal{K}$ of compact operators, is a scalar multiple of C_σ.

THEOREM 3.1. *Suppose that φ given by Equation* (3.1) *is a linear-fractional self-map of $\mathbb{D}$, not an automorphism, which satisfies $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. Let $s = (\overline{c}\overline{\zeta}+\overline{d})/(-\overline{b}\eta+\overline{d})$. Then there exists a compact operator K on H^2 so that $C_\varphi^* = sC_\sigma + K$, where σ is as given by Equation* (3.2).

Proof. We first consider the case where $\zeta = \eta$, so that ζ is a fixed point of φ. Let σ, h and g be associated to φ as in Equations (3.2) and (3.3), and note that σ fixes ζ also. It is immediate that $[C_\sigma, T_h^*] = \overline{c}[C_\sigma, T_z^*]$. Invoking Theorem 2.3, it follows that $C_\sigma T_h^* = T_h^* C_\sigma + K_1$ for some compact operator K_1. From Equation (3.3) we then have

$$C_\varphi^* = T_g C_\sigma T_h^* \equiv T_g T_h^* C_\sigma \ (\text{mod } \mathcal{K}) \equiv T_{\overline{h}g} C_\sigma \ (\text{mod } \mathcal{K})$$

where the last equivalence is justified by Proposition 7.22 in [11]. Since $E(\sigma) = \{\eta\} = \{\zeta\}$, we may now apply Corollary 2.2 with $w = \overline{h}g - \overline{h(\zeta)}g(\zeta)$ to see that

$$T_{\overline{h}g} C_\sigma - \overline{h(\zeta)}g(\zeta)C_\sigma = T_{(\overline{h}g - \overline{h(\zeta)}g(\zeta))} C_\sigma \in \mathcal{K},$$

which gives the desired conclusion.

In the case that $\zeta \neq \eta$ we consider the map $\psi(z) = \zeta\overline{\eta}\varphi(z)$ which fixes ζ. Since $C_\varphi^* = C_U C_\psi^*$ where $U(z) = \zeta\overline{\eta}z$, the first part of the argument shows that $C_\varphi^* = C_U C_\psi^* \equiv C_U(\overline{h}_\psi(\zeta)g_\psi(\zeta)C_{\sigma_\psi})$ (mod $\mathcal{K}$). Since $\sigma_\psi \circ U = \sigma_\varphi$ and $\overline{h}_\psi(\zeta)g_\psi(\zeta) = (\overline{c}\overline{\zeta}+\overline{d})/(-\overline{b}\eta+\overline{d})$, the conclusion follows. ∎

REMARK 3.2. An analogue of Theorem 3.1 holds in the Bergman space A^2 of analytic functions in $L^2(\mathbb{D}, dA)$, where dA is normalized area measure on $\mathbb{D}$. If φ given by Equation (3.1) is a self-map of $\mathbb{D}$, then on A^2 we have $C_\varphi^* = T_g C_\sigma T_h^*$, where σ is as in Equation (3.2), $g(z) = (-\overline{b}z + \overline{d})^{-2}$, and $h(z) = (cz + d)^2$ [15]. We follow the outline of the proof of Theorem 3.1 to see that $C_\varphi^* = sC_\sigma + K$ for some compact K on A^2, where now $s = [(\overline{c}\overline{\zeta} + \overline{d})/(-\overline{b}\eta + \overline{d})]^2$. Now the compactness of $[C_\sigma, T_z^*]$ follows from Theorem 3 in [19], and the compactness of $T_{\overline{h}g - \overline{h(\zeta)}g(\zeta)} C_\sigma$ is obtained as an application of Lemma 1 in [20] on compact Carleson measures of the form $W(z)d(A\sigma^{-1})$, with the choice $W(z) = |\overline{h(z)}g(z) - \overline{h(\zeta)}g(\zeta)|^2$. We leave the details to the interested reader.

The scalar $s = (\overline{c}\overline{\zeta} + \overline{d})/(-\overline{b}\eta + \overline{d})$ can equivalently be described as $|\sigma'(\eta)|$ or $|\varphi'(\zeta)|^{-1}$. This will be verified below, in Proposition 3.6. In particular, the scalar s in the statement of Theorem 3.1 is strictly positive.

COROLLARY 3.3. *For φ a linear-fractional self-map of the disk, not an automorphism, with $\|\varphi\|_\infty = 1$, the self-commutator $[C_\varphi^*, C_\varphi]$ is compact if and only if $\varphi \circ \sigma = \sigma \circ \varphi$.*

Proof. We have $[C_\varphi^*, C_\varphi] = s(C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi}) + K$ where s is as in the statement of Theorem 3.1 and K is compact. Since a difference of non-compact linear-fractional composition operators is compact only if it is zero [2], [18], the result follows. ∎

A linear-fractional self-map whose fixed point set, relative to the Riemann sphere, consists of a single point ζ in $\partial\mathbb{D}$ is termed *parabolic*. It is conjugate, via the map $(\zeta + z)/(\zeta - z)$, to a translation by some complex number t, $\operatorname{Re} t \geqslant 0$, in the right half-plane. When $\operatorname{Re} t = 0$ we have a (parabolic) automorphism; otherwise the map is not an automorphism. When the translation number t is strictly positive, we call the associated linear-fractional self-map of $\mathbb{D}$ a *positive parabolic* non-automorphism. Among the linear-fractional non-automorphisms fixing $\zeta \in \partial\mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$. For further details on the classification of linear-fractional self-maps of $\mathbb{D}$, see [3] or Chapter 0 of [22].

A linear-fractional non-automorphism φ with a fixed point ζ on $\partial\mathbb{D}$, which commutes with its Krein adjoint, must be parabolic. This follows by a consideration of fixed points: if φ has another fixed point z_0 in the Riemann sphere, and it commutes with σ, then $\sigma(z_0)$ would also be fixed by φ. Neither $\sigma(z_0) = \zeta$ nor $\sigma(z_0) = z_0$ are possible, since σ fixes the boundary point ζ if φ does, and φ fixes $1/\overline{z}_0$ if σ fixes z_0. Thus Corollary 3.3 gives another view of the main result in [3]: a non-automorphism linear-fractional composition operator C_φ is non-trivially essentially normal if and only if φ is parabolic.

PROPOSITION 3.4. *Suppose φ, not an automorphism, is a linear-fractional self-map of $\mathbb{D}$ with $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$. If σ is the Krein adjoint of φ, then $\varphi'(\zeta)\sigma'(\eta) = 1$ and $\tau \equiv \varphi \circ \sigma$ is a positive parabolic non-automorphism.*

Proof. Using $\widetilde{\ }$ for the Krein adjoint, we have $\widetilde{\varphi \circ \sigma} = \widetilde{\sigma} \circ \widetilde{\varphi} = \varphi \circ \sigma$. Thus the map $\tau = \varphi \circ \sigma$, a non-automorphism fixing $\eta \in \partial\mathbb{D}$, is its own Krein adjoint. By the remark preceeding the statement of Proposition 3.4, this means that τ is parabolic and $\tau(z) = \Phi^{-1}(\Phi(z) + t)$ for $\Phi(z) = (\eta + z)/(\eta - z)$ and some t with $\operatorname{Re} t > 0$. Direct calculation, using $\widetilde{\tau} = \tau$, shows that t must be positive.

Since parabolic non-automorphisms have derivative one at their (boundary) fixed point ([22], p. 3), we have $\varphi'(\sigma(\eta))\sigma'(\eta) = 1$ or $\varphi'(\zeta)\sigma'(\eta) = 1$, as desired. ∎

The spectrum of a composition operator whose symbol is a parabolic non-automorphism has been described in [7]. In particular, we have the following result.

PROPOSITION 3.5. [7] *Let $\tau = \varphi \circ \sigma$, where φ is a non-automorphism with $\varphi(\zeta) = \eta$ for $\zeta, \eta \in \partial\mathbb{D}$. The spectrum $\sigma(C_\tau)$ and essential spectrum $\sigma_{\mathrm{e}}(C_\tau)$ are both equal to* $[0,1]$.

Proof. The map τ fixes $\eta \in \partial\mathbb{D}$, and by conjugating by a rotation, C_τ is unitarily equivalent to a composition operator with positive parabolic symbol fixing 1. Such a map can be written as

$$\frac{(2-t)z + t}{-tz + 2 + t}$$

for some positive t. Applying Corollary 6.2 in [7], we have $\sigma(C_\tau) = [0,1]$. Since every point of $\sigma(C_\tau)$ is a boundary point of the spectrum, and none is isolated, we also have $\sigma_{\mathrm{e}}(C_\tau) = \sigma(C_\tau) = [0,1]$ ([6], Theorem 37.8). ∎

As promised, we can describe the scalar s appearing in Theorem 3.1 in a more useful way:

PROPOSITION 3.6. *Let φ, σ and s be as in the statement of Theorem* 3.1. *We have* $s = |\sigma'(\eta)| = |\varphi'(\zeta)|^{-1}$.

Proof. Direct calculation shows that

$$\frac{\overline{\sigma'(\eta)}}{\overline{\varphi'(\zeta)}} = \Big(\frac{\overline{c}\overline{\zeta} + \overline{d}}{-\overline{b}\eta + \overline{d}}\Big)^2.$$

By Proposition 3.4, $\varphi'(\zeta) = (\sigma'(\eta))^{-1}$, so that $s^2 = |\sigma'(\eta)|^2$. By Theorem 3.1, $C_\varphi C_\varphi^* \equiv sC_\varphi C_\sigma \pmod{\mathcal{K}} = sC_{\sigma\circ\varphi}$, and by Proposition 3.5, the essential spectrum of $C_{\sigma\circ\varphi}$ is $[0,1]$. Since $C_\varphi C_\varphi^*$ is positive, the scalar s must be positive, and we have $s = |\sigma'(\eta)|$. ∎

COROLLARY 3.7. *If φ is a non-automorphism, linear-fractional map with $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial\mathbb{D}$, then $\sigma_{\mathrm{e}}(C_\varphi^* C_\varphi) = \sigma_{\mathrm{e}}(C_\varphi C_\varphi^*) = [0, s]$.*

Proof. We have $C_\varphi^* \equiv sC_\sigma \pmod{\mathcal{K}}$ for $s = 1/|\varphi'(\zeta)|$ by Theorem 3.1 and Proposition 3.6. Thus $C_\varphi C_\varphi^* \equiv sC_{\sigma\circ\varphi} \pmod{\mathcal{K}}$ and $C_\varphi^* C_\varphi \equiv sC_{\varphi\circ\sigma} \pmod{\mathcal{K}}$, and the conclusion follows from Proposition 3.5 and Proposition 3.6. ∎

Note that since the non-zero points in $\sigma(C_\varphi C_\varphi^*)$ and $\sigma(C_\varphi^* C_\varphi)$ are the same, we also have $\sigma(C_\varphi C_\varphi^*) = \sigma(C_\varphi^* C_\varphi)$. Moreover, this common spectrum consists of $[0, s]$ plus at most finitely many eigenvalues greater than s, and of finite multiplicity.

4. THE UNITAL C^*-ALGEBRA GENERATED BY C_φ AND T_z

Throughout this section, φ will be a fixed but arbitrary linear-fractional self-map of $\mathbb{D}$ satisfying the following:

(i) φ is not an automorphism.

(ii) $\varphi(\zeta) = \eta$ for some $\zeta \neq \eta \in \partial\mathbb{D}$.

Conditions (i) and (ii) imply that C_φ^2 is compact on H^2, since $\|\varphi \circ \varphi\|_\infty < 1$.

The algebra $C^*(T_z, C_\varphi)$ is the closed linear span of all words in $T_z, T_z^*, C_\varphi, C_\varphi^*$ and I, and contains all Toeplitz operators T_w with w continuous. We set $\mathcal{A} = C^*(T_z, C_\varphi)/\mathcal{K}$, and denote the cosets of C_φ, C_φ^*, and T_w by x, x^*, and t_w, respectively. Let e denote the coset of the identity. A main goal of this section is a description of $\mathcal{A}$. This description will allow us, for example, to determine the essential norm and essential spectrum of any element of $C^*(T_z, C_\varphi)$. For φ as described above, $E(\varphi) = \{\zeta\}$, and Corollary 2.2 implies that $T_{w-w(\zeta)}C_\varphi$ is compact, that is,

$$T_w C_\varphi \equiv w(\zeta) C_\varphi \pmod{\mathcal{K}}.$$

Since $E(\sigma) = \{\eta\}$, we also see from Corollary 2.2, Theorem 3.1, and Proposition 3.6 that

$$C_\varphi T_w = (T_{\overline{w}} C_\varphi^*)^* \equiv s(T_{\overline{w}} C_\sigma)^* \pmod{\mathcal{K}} \equiv s(\overline{w(\eta)} C_\sigma)^* \pmod{\mathcal{K}} \equiv w(\eta) C_\varphi \pmod{\mathcal{K}}$$

where $s = |\varphi(\zeta)|^{-1}$. In addition, $T_v T_w - T_{vw}$ is compact whenever v and w are in $C(\partial\mathbb{D})$. Phrasing these relations in terms of the cosets yields

$$t_w x = w(\zeta)x, \quad x t_w = w(\eta)x, \quad t_w x^* = w(\eta)x^*, \quad x^* t_w = w(\zeta)x^*, \quad t_v t_w = t_{vw},$$

for all w and v in $C(\partial\mathbb{D})$. Since $x^2 = (x^*)^2 = 0$, we generate $\mathcal{A}$ as a Banach space from linear combinations of

$$t_w,\ (x^*x)^m,\ (xx^*)^n,\ x(x^*x)^j,\ x^*(xx^*)^k,$$

where $w \in C(\partial\mathbb{D})$, the integers m, n are positive, and the integers j and k are non-negative.

Let K be a compact subset of the non-negative real numbers which contains $[0, s]$. We write $C_0(K)$ for the space of functions in $C(K)$ which vanish at zero. We will need the next result, which follows easily from the Hahn-Banach Theorem and the Riesz Representation Theorem; here t denotes the independent variable.

LEMMA 4.1. (i) *Let $\mathcal{R}$ and $\mathcal{S}$ be dense linear manifolds in $C_0(K)$ and $C(K)$, respectively. If $\alpha > 0$, then*

$$\overline{t^\alpha \mathcal{R}} = \overline{t^\alpha \mathcal{S}} = C_0(K).$$

(ii) *Suppose* $0 < \lambda \leqslant s$ *and let* $\mathcal{T}$ *be a linear manifold which is dense in the subspace* $\{f \in C(K) : f(\lambda) = 0\}$. *Then*

$$\overline{t^{\alpha}\mathcal{T}} = \{f \in C_0(K) : f(\lambda) = 0\}.$$

We next introduce the various objects which are central to our analysis and record some observations about them.

4.1. THE C^*-ALGEBRA $\mathcal{C}$. It follows from the relations described above that for every continuous function w on $\partial\mathbb{D}$, t_w commutes with x^*x and xx^*. Further, if we let $C_{\zeta,\eta}$ denote the algebra of all w in $C(\partial\mathbb{D})$ satisfying $w(\eta) = w(\zeta)$, then t_w commutes with x and x^* whenever w lies in $C_{\zeta,\eta}(\partial\mathbb{D})$. Finally note that the self-adjoint element $a \equiv xx^* + x^*x$ commutes with both x and x^*. The spectrum of a is easily identified:

PROPOSITION 4.2. *Let x be the coset of C_φ in $\mathcal{A}$, where $\varphi = (az+b)/(cz+d)$ satisfies conditions* (i)–(ii) *stated at the beginning of Section* 4. *If $a = xx^* + x^*x$, then $\sigma(a) = \sigma(xx^*) \cup \sigma(x^*x) = [0, s]$ where $s = 1/|\varphi'(\zeta)|$.*

Proof. The elements x^*x and xx^* generate a commutative C^*-algebra. It follows from Gelfand theory, the facts that $(xx^*)(x^*x) = (x^*x)(xx^*) = 0$, and (by Corollary 3.7) $\sigma(xx^*) = \sigma(x^*x) = [0, s]$, that $\sigma(a) = \sigma(xx^*) \cup \sigma(x^*x)$. ∎

Let $\mathcal{C}$ denote the (necessarily commutative) C^*-algebra generated by a and the Toeplitz cosets $\{t_w : w \in C_{\zeta,\eta}(\partial\mathbb{D})\}$. Clearly, $\mathcal{C}$ lies in the center of $\mathcal{A}$. We next describe the Gelfand theory of $\mathcal{C}$. First we look at the algebra $C_{\zeta,\eta}(\partial\mathbb{D})$.

It is easy to see that the multiplicative linear functionals on $C_{\zeta,\eta}(\partial\mathbb{D})$ are all point evaluations

$$\ell_\lambda : f \to f(\lambda)$$

with the proviso that $\ell_\eta = \ell_\zeta$. Accordingly, the maximal ideal space of $C_{\zeta,\eta}(\partial\mathbb{D})$ is a "figure eight", namely, the circle $\partial\mathbb{D}$ with ζ and η identified. We denote by Λ the disjoint union of $\partial\mathbb{D}$ and $[0, s]$, with ζ, η and 0 identified to a point $\mathbf{p}$ (a figure eight with an interval attached). Given w in $C_{\zeta,\eta}(\partial\mathbb{D})$, let us agree to extend w continuously to Λ by setting $w(\lambda) = w(\zeta) = w(\eta)$ when $\lambda = \mathbf{p}$ or $0 < \lambda \leqslant s$. Similarly, if $f \in C_0([0, s])$, extend f continuously to Λ by putting $f(\mathbf{p}) = f(0) = 0$ and $f(\lambda) = 0$ for $\lambda \in \partial\mathbb{D}\backslash\{\zeta, \eta\}$. With these understandings, which remain in force throughout, we have the following result.

PROPOSITION 4.3. *The algebra $\mathcal{C}$ consists of all elements of the form $b = t_w + f(a)$ where w is in $C_{\zeta,\eta}(\partial\mathbb{D})$ and f is in $C_0([0, s])$. Moreover, b uniquely determines w and f. The maximal ideal space of $\mathcal{C}$ coincides with Λ, and the Gelfand transform from $\mathcal{C}$ to $C(\Lambda)$ has the form*

$$t_w + f(a) \to w + f.$$

Proof. We temporarily write $\mathcal{C}_0$ for $\{t_w + f(a) : w \in C_{\zeta,\eta}(\partial\mathbb{D}) \text{ and } f \in C_0([0, s])\}$. If $w(\zeta) = w(\eta)$ and f is in $C_0([0, s])$, then, since f is a uniform limit of polynomials vanishing at zero (and $(x^*x)(xx^*) = 0$), we have $t_w f(a) =$

$t_w(f(x^*x) + f(xx^*)) = w(\eta)f(x^*x) + w(\zeta)f(xx^*) = w(\zeta)f(a)$. Since $t_w t_v = t_{wv}$ for continuous w and v, we see that $\mathcal{C}_0$ is an algebra.

Suppose ℓ is a multiplicative linear functional on $\mathcal{C}$. Restricting ℓ to

$$\{t_w : w \in C_{\zeta,\eta}(\partial\mathbb{D})\} \cong C_{\zeta,\eta}(\partial\mathbb{D})$$

we see that there is a unique $\alpha \in \partial\mathbb{D}$ with $\ell(t_w) = w(\alpha)$ for all continuous w with $w(\zeta) = w(\eta)$. Restricting ℓ to

$$\{f(a) : f \in C([0,s])\} \cong C([0,s])$$

shows that there is a unique point β in $[0,s]$ with $\ell(f(a)) = f(\beta)$. Thus

$$\ell(t_w f(a)) = \ell(t_w)\ell(f(a)) = w(\alpha)f(\beta).$$

Also, if $f(0) = 0$, then $t_w f(a) = w(\zeta)f(a)$ as seen above, so $\ell(t_w f(a)) = w(\zeta)f(\beta)$. Since any function in $C_0([0,s])$ vanishes at 0, we can have $\alpha \in \partial\mathbb{D}\backslash\{\zeta,\eta\}$ if $\beta = 0$, but if $0 < \beta \leqslant s$, $\alpha \in \{\zeta,\eta\}$. Thus with the understandings stated prior to the statement of the proposition, $\ell(t_w + f(a)) = w(\lambda) + f(\lambda)$ for a unique λ in Λ and any $t_w + f(a)$ in $\mathcal{C}_0$.

The above arguments show that $C(\Lambda)$ is the Gelfand representation for $\mathcal{C}$. Moreover, the map

$$t_w + f(a) \to w + f$$

from $\mathcal{C}_0$ to $C(\Lambda)$ is an isometric $*$-homomorphism from $\mathcal{C}_0$ to $C(\Lambda)$. But $C(\Lambda)$ consists of exactly such sums $w + f$, so this $*$-homomorphism is onto $C(\Lambda)$. Since $C(\Lambda)$ is complete, so is $\mathcal{C}_0$. Since $\mathcal{C}_0$ is dense in $\mathcal{C}$, we conclude $\mathcal{C}_0 = \mathcal{C}$. ∎

4.2. THE POLAR DECOMPOSITION OF C_φ AND THE ALGEBRA $\mathcal{A}_0$. We begin with some observations on the polar decomposition of any operator T on a Hilbert space $\mathcal{H}$. Suppose that $T = U\sqrt{T^*T}$, where U is a partial isometry with initial space $(\ker T)^\perp = \overline{T^*\mathcal{H}}$ and final space $\overline{T\mathcal{H}} = (\ker T^*)^\perp$. The operators U^*U and UU^* are, respectively, the projections onto $(\ker T)^\perp$ and $\overline{T\mathcal{H}}$. Moreover, $UT^*T = TT^*U$ and so

$$Uf(T^*T) = f(TT^*)U \tag{4.1}$$

for all functions continuous on the spectra of both T^*T and TT^*. Taking f to be the square root function shows that the polar decomposition for T^* is $T^* = U^*\sqrt{TT^*}$. The partial isometry U is unitary if T and T^* are one-to-one. Observe that every non-trivial composition operator is one-to-one, and the adjoint formula of Equation (3.3) guarantees that, for linear-fractional composition operators, the adjoint is also one-to-one. Thus the linear-fractional composition operators under consideration here have the polar decomposition $C_\varphi = U\sqrt{C_\varphi^* C_\varphi}$ where U is *unitary*. If we apply these remarks to $T = C_\varphi = U\sqrt{C_\varphi^* C_\varphi}$, we have $x = u\sqrt{x^*x}$ and $x^* = u^*\sqrt{xx^*}$ where $u = [U]$, the coset of U modulo $\mathcal{K}$, and $x = [C_\varphi]$. Moreover, as observed above, U, and hence u, are unitary. By Corollary 3.7, the sets $\sigma(x^*x) = \sigma_{\mathrm{e}}(C_\varphi^* C_\varphi)$ and $\sigma(xx^*) = \sigma_{\mathrm{e}}(C_\varphi C_\varphi^*)$ both coincide with $[0,s]$, where $s = |\varphi'(\zeta)|^{-1}$.

Now $C^*(T_z, C_\varphi)$ is the closed linear span of elements of the form

$$T_w,\ f(C_\varphi^* C_\varphi),\ g(C_\varphi C_\varphi^*),\ C_\varphi p(C_\varphi^* C_\varphi),\ C_\varphi^* q(C_\varphi C_\varphi^*),\ K,$$

where f, g, p and q are polynomials with $f(0) = g(0) = 0$, w is in $C(\partial\mathbb{D})$, and K is a compact operator. The map $f \to f(C_\varphi^* C_\varphi)$ extends to a $*$-isomorphism of $C_0(\sigma(C_\varphi^* C_\varphi))$ onto the closed subspace $\{f(C_\varphi^* C_\varphi) : f \in C_0(\sigma(C_\varphi^* C_\varphi))\}$ in $\mathcal{B}(H^2)$; the analogous statement holds for the map $g \to g(C_\varphi C_\varphi^*)$. Writing

$$C_\varphi p(C_\varphi^* C_\varphi) = U\sqrt{C_\varphi^* C_\varphi}\, p(C_\varphi^* C_\varphi),$$

we see by Lemma 4.1 that $\overline{\{C_\varphi p(C_\varphi^* C_\varphi) : p \text{ a polynomial}\}} = \{Uh(C_\varphi^* C_\varphi) : h \in C_0(\sigma(C_\varphi^* C_\varphi))\}$; similarly, $\overline{\{C_\varphi^* q(C_\varphi C_\varphi^*) : q \text{ a polynomial}\}} = \{U^* k(C_\varphi C_\varphi^*) : k \in C_0(\sigma(C_\varphi C_\varphi^*))\}$. Thus $\mathcal{A} = C^*(T_z, C_\varphi)/\mathcal{K}$ contains, and is the closure of, the set $\mathcal{A}_0$ of elements of the form

$$b = t_w + f(x^* x) + g(x x^*) + u h(x^* x) + u^* k(x x^*) \tag{4.2}$$

where $w \in C(\partial\mathbb{D})$, and f, g, h and k are in $C_0([0, s])$, with $s = 1/|\varphi'(\zeta)|$. We will see later that $\mathcal{A}_0 = \mathcal{A}$; for now we show that $\mathcal{A}_0$ is an algebra, and each element of $\mathcal{A}_0$ has a unique representation in the above form. To this end, we record some consequences of the next pair of equations, which follow from Equation (4.1) by taking cosets and adjoints:

$$u f(x^* x) = f(x x^*) u \quad \text{and} \quad u^* f(x x^*) = f(x^* x) u^* \tag{4.3}$$

for all $f \in C([0, s])$.

PROPOSITION 4.4. *If $\mathcal{A}_0$ is defined as above, then $\mathcal{A}_0$ is an algebra.*

Proof. We must show that given elements $b_1 \in \mathcal{A}_0$ and $b_2 \in \mathcal{A}_0$ having the form

$$b_j = t_{w_j} + f_j(x^* x) + g_j(x x^*) + u h_j(x^* x) + u^* k_j(x x^*), \quad j = 1, 2$$

with $w_j \in C(\partial\mathbb{D})$ and f_j, g_j, h_j, k_j in $C_0([0, s])$, then $b_1 b_2$ has the same form. To do this, it suffices to show that the product of any of the five terms of b_1 with any of the five terms of b_2 is again in $\mathcal{A}_0$. Some of these verifications are immediate, for example $f_1(x^* x) f_2(x^* x) = f_1 f_2(x^* x)$, where $f_1 f_2$ is in $C_0([0, s])$ if f_1 and f_2 are. For the others, we make use of the basic Equations of (4.3) together with:

$$f(x^* x) g(x x^*) = 0 = g(x x^*) f(x^* x) \tag{4.4}$$

for f and g in $C_0([0, s])$. Equation (4.4) follows by uniformly approximating f and g by polynomials vanishing at 0. From these equations we see that:

$$\begin{aligned}
&g_1(x x^*) u h_2(x^* x) = u g_1(x^* x) h_2(x^* x), \quad u h_1(x^* x) g_2(x x^*) = 0, \\
&u h_1(x^* x) u h_2(x^* x) = u h_1(x^* x) h_2(x x^*) u^* = 0, \\
&u h_1(x^* x) u^* k_2(x x^*) = h_1(x x^*) u u^* k_2(x x^*) = h_1(x x^*) k_2(x x^*), \\
&u^* k_1(x x^*) u h_2(x^* x) = u^* u k_1(x^* x) h_2(x^* x) = k_1(x^* x) h_2(x^* x), \\
&u^* k_1(x x^*) u^* k_2(x x^*) = u^* k_1(x x^*) k_2(x^* x) u^* = 0.
\end{aligned}$$

Similarly we see (using the coset identities preceeding Lemma 4.1) that for f, g, h, and k in $C_0([0,s])$ and $w \in C(\partial\mathbb{D})$,

$$t_w f(x^*x) = w(\eta)f(x^*x), \quad t_w g(xx^*) = w(\zeta)g(xx^*),$$
$$t_w uh(x^*x) = w(\zeta)uh(x^*x), \quad t_w u^*k(xx^*) = w(\eta)u^*k(xx^*).$$

This shows that $\mathcal{A}_0$ is an algebra. ∎

The next result addresses the uniqueness of representation of elements in $\mathcal{A}_0$.

PROPOSITION 4.5. *For an element b in $\mathcal{A}_0$, there is a unique $w \in C(\partial\mathbb{D})$ and unique functions f, g, h and k in $C_0([0,s])$ so that Equation* (4.2) *holds.*

Proof. It suffices to show that if

$$0 = t_w + f(x^*x) + g(xx^*) + uh(x^*x) + u^*k(xx^*), \tag{4.5}$$

then each term on the right-hand side is zero. Multiplying on the right by x^*x yields $0 = t_w x^*x + f(x^*x)x^*x + g(xx^*)x^*x + uh(x^*x)x^*x + u^*k(xx^*)x^*x = w(\eta)x^*x + f(x^*x)x^*x + uh(x^*x)x^*x$ so that

$$uh(x^*x)x^*x = -[w(\eta)x^*x + f(x^*x)x^*x].$$

The right-hand side is normal, and the left-hand side has square zero, so both sides must vanish. Thus $h \equiv 0$ and $f + w(\eta) \equiv 0$ on $[0,s]$; since $f(0) = 0$, we must have $w(\eta) = 0$ and $f \equiv 0$. Thus Equation (4.5) is now

$$0 = t_w + g(xx^*) + u^*k(xx^*).$$

Multiplying on the left by xx^* gives $0 = xx^*t_w + xx^*g(xx^*) + xx^*u^*k(xx^*) = w(\zeta)xx^* + xx^*g(xx^*) + xx^*u^*k(xx^*)$ so that

$$-[w(\zeta)xx^* + xx^*g(xx^*)] = xx^*u^*k(xx^*) = 0.$$

It follows that $g + w(\zeta) \equiv 0$ on $[0,s]$; since $g(0) = 0$, we see that $w(\zeta) = 0$ and $g \equiv 0$ on $[0,s]$. Returning again to Equation (4.5) we have

$$0 = t_w + u^*k(xx^*).$$

Multiplying on the left by x^*x yields

$$0 = x^*xt_w + x^*xu^*k(xx^*) = w(\eta)x^*x + x^*xk(x^*x)u^*.$$

Since $w(\eta) = 0$, this forces $k \equiv 0$, and from this it follows finally that $t_w = 0$. ∎

4.3. LOCALIZATION AND THE STRUCTURE OF $\mathcal{A}$. For λ in Λ, let $\mathcal{I}_\lambda$ denote the closed, two-sided ideal in $\mathcal{A}$ generated by the maximal ideal

$$J_\lambda = \{t_w + f(a) : w \in C_{\zeta,\eta}(\partial\mathbb{D}), f \in C_0([0,s]) \text{ and } w(\lambda) + f(\lambda) = 0\}$$

of $\mathcal{C}$. Here w and f are understood to extend to Λ as described prior to Proposition 4.3. For b in $\mathcal{A}$, we write $[b]_{\mathcal{I}_\lambda}$ for the coset of b in $\mathcal{A}/\mathcal{I}_\lambda$. The localization theorem of R.G. Douglas ([11], p. 196) tells us that

$$\|b\| = \sup_{\lambda\in\Lambda} \|[b]_{\mathcal{I}_\lambda}\|,$$

and the map

$$b \to \{[b]_{\mathcal{I}_\lambda}\}_{\lambda\in\Lambda}$$

is an isometric $*$-homomorphism of $\mathcal{A}$ into $\sum\limits_{\lambda\in\Lambda} \bigoplus \mathcal{A}/\mathcal{I}_\lambda$. Moreover, a given b in $\mathcal{A}$ is invertible if and only if each coset $[b]_\lambda$ is invertible, for $\lambda \in \Lambda$. Our immediate objective is to compute the local algebras $\mathcal{A}/\mathcal{I}_\lambda$.

For λ in Λ we define a map $\Phi_\lambda : \mathcal{A}_0 \to \mathbb{M}_2$, the algebra of 2×2 matrices, as follows. Let b in $\mathcal{A}_0$ be given by Equation (4.2). We put

$$\Phi_\lambda(b) = \begin{cases} \begin{bmatrix} w(\zeta)+g(\lambda) & h(\lambda) \\ k(\lambda) & w(\eta)+f(\lambda) \end{bmatrix} & \text{if } 0<\lambda\leqslant s, \\ \\ \begin{bmatrix} w(\zeta) & 0 \\ 0 & w(\eta) \end{bmatrix} & \text{if } \lambda = \mathbf{p}, \\ \\ \begin{bmatrix} w(\lambda) & 0 \\ 0 & w(\lambda) \end{bmatrix} & \text{if } \lambda \in \partial\mathbb{D}\backslash\{\zeta,\eta\}. \end{cases} \tag{4.6}$$

We write $I_{2\times 2}$ for the identity matrix in $\mathbb{M}_2$ and $\mathbb{M}_2^{\text{diag}}$ for the algebra of 2×2 diagonal matrices. The range of Φ_λ will be denoted Ran Φ_λ.

PROPOSITION 4.6. *For each λ in Λ, Φ_λ is a $*$-homomorphism from $\mathcal{A}_0$ to $\mathbb{M}_2$ with*

$$\textit{Ran } \Phi_\lambda = \begin{cases} \mathbb{M}_2 & \textit{when } 0<\lambda\leqslant s, \\ \mathbb{M}_2^{\text{diag}} & \textit{when } \lambda=\mathbf{p}, \\ \{cI_{2\times 2} : c\in\mathbb{C}\} & \textit{when } \lambda\in\partial\mathbb{D}\backslash\{\zeta,\eta\}. \end{cases} \tag{4.7}$$

Proof. First consider $\lambda > 0$. Any element b in $\mathcal{A}_0$ has the form $b = t_w + y$, where w is in $C(\partial\mathbb{D})$ and

$$y = f(x^*x) + g(xx^*) + uh(x^*x) + u^*k(xx^*) \tag{4.8}$$

with f, g, h, k in $C_0([0,s])$. Given $b_1 = t_{w_1} + y_1$ and $b_2 = t_{w_2} + y_2$ in $\mathcal{A}_0$,

$$b_1b_2 = t_{w_1}t_{w_2} + y_1t_{w_2} + t_{w_1}y_2 + y_1y_2. \tag{4.9}$$

Taking the notation from Equation (4.8) for y_1 and y_2, we have

$$\begin{aligned} y_1y_2 = {} & [f_1(x^*x)f_2(x^*x) + k_1(x^*x)h_2(x^*x)] + u[g_1(x^*x)h_2(x^*x) + h_1(x^*x)f_2(x^*x)] \\ & + u^*[k_1(xx^*)g_2(xx^*) + f_1(xx^*)k_2(xx^*)] + [g_1(xx^*)g_2(xx^*) + h_1(xx^*)k_2(xx^*)], \end{aligned}$$

where we have used the list of identities in the proof of Proposition 4.4 and collected like terms. Thus

$$\begin{aligned} \Phi_\lambda(y_1y_2) &= \begin{bmatrix} g_1(\lambda)g_2(\lambda)+h_1(\lambda)k_2(\lambda) & g_1(\lambda)h_2(\lambda)+h_1(\lambda)f_2(\lambda) \\ k_1(\lambda)g_2(\lambda)+f_1(\lambda)k_2(\lambda) & f_1(\lambda)f_2(\lambda)+k_1(\lambda)h_2(\lambda) \end{bmatrix} \\ &= \Phi_\lambda(y_1)\Phi_\lambda(y_2). \end{aligned}$$

Now $t_{w_1}y_2 = w_1(\eta)f_2(x^*x) + w_1(\zeta)g_2(xx^*) + w_1(\zeta)uh_2(x^*x) + w_1(\eta)u^*k_2(xx^*)$. Thus

$$\begin{aligned}\Phi_\lambda(t_{w_1}y_2) &= \begin{bmatrix} w_1(\zeta)g_2(\lambda) & w_1(\zeta)h_2(\lambda) \\ w_1(\eta)k_2(\lambda) & w_1(\eta)f_2(\lambda) \end{bmatrix} \\ &= \begin{bmatrix} w_1(\zeta) & 0 \\ 0 & w_1(\eta) \end{bmatrix} \begin{bmatrix} g_2(\lambda) & h_2(\lambda) \\ k_2(\lambda) & f_2(\lambda) \end{bmatrix} = \Phi_\lambda(t_{w_1})\Phi_\lambda(y_2).\end{aligned}$$

Similarly, we find $\Phi_\lambda(y_1t_{w_2}) = \Phi_\lambda(y_1)\Phi_\lambda(t_{w_2})$. Since $t_{w_1}t_{w_2} = t_{w_1w_2}$, it follows that $\Phi_\lambda(t_{w_1}t_{w_2}) = \Phi_\lambda(t_{w_1})\Phi_\lambda(t_{w_2})$. Applying Φ_λ to both sides of Equation (4.9) and invoking the above identities, we see that

$$\begin{aligned}\Phi_\lambda(b_1b_2) &= \Phi_\lambda(t_{w_1})\Phi_\lambda(t_{w_2}) + \Phi_\lambda(y_1)\Phi_\lambda(t_{w_2}) + \Phi_\lambda(t_{w_1})\Phi_\lambda(y_2) + \Phi_\lambda(y_1)\Phi_\lambda(y_2) \\ &= (\Phi_\lambda(t_{w_1}) + \Phi_\lambda(y_1))(\Phi_\lambda(t_{w_2}) + \Phi_\lambda(y_2)) = \Phi_\lambda(b_1)\Phi_\lambda(b_2)\end{aligned}$$

as desired. Clearly the range of Φ_λ is $\mathbb{M}_2$, which yields the conclusion for $0 < \lambda \leqslant s$.

The remaining cases $\lambda = \mathbf{p}$ and $\lambda \in \partial\mathbb{D}\backslash\{\zeta, \eta\}$, which are considerably easier since there one has $\Phi_\lambda(t_w + y) = \Phi_\lambda(t_w)$, are left for the reader. ∎

PROPOSITION 4.7. *For* $\lambda \in \Lambda$, $\overline{\ker \Phi_\lambda} = \mathcal{I}_\lambda$.

Proof. For λ in Λ, denote by $\mathcal{I}_\lambda^{\mathrm{alg}}$ the two-sided algebraic ideal in $\mathcal{A}_0$ generated by J_λ. Since $\ker \Phi_\lambda$ is an ideal containing J_λ, we know $J_\lambda \subset \mathcal{I}_\lambda^{\mathrm{alg}} \subset \ker \Phi_\lambda$. By definition, $\mathcal{I}_\lambda = \overline{\mathcal{I}}_\lambda^{\mathrm{alg}}$. It suffices to show that $\ker \Phi_\lambda \subset \overline{\mathcal{I}}_\lambda^{\mathrm{alg}}$, for then we will have

$$\mathcal{I}_\lambda = \overline{\mathcal{I}}_\lambda^{\mathrm{alg}} \subset \overline{\ker \Phi_\lambda} \subset \overline{\mathcal{I}}_\lambda^{\mathrm{alg}} = \mathcal{I}_\lambda,$$

which gives the desired conclusion.

Consider first the case $0 < \lambda \leqslant s$. An element b in $\mathcal{A}_0$, given by Equation (4.2), lies in $\ker \Phi_\lambda$ exactly when $w(\zeta) + g(\lambda), w(\eta) + f(\lambda), h(\lambda)$ and $k(\lambda)$ are all zero. We claim that the sum of the first three terms on the right side of Equation (4.2) lies in $\mathcal{I}_\lambda^{\mathrm{alg}}$. To see this, pick m and n in $C(\partial\mathbb{D})$ with $m + n \equiv 1, m(\zeta) = 0, m(\eta) = 1$, and $n(\zeta) = 1, n(\eta) = 0$. Then $w = mw + nw$ so that $t_w = t_{mw} + t_{nw}$. To prove the claim, it is enough to show that both $t_{mw} + f(x^*x)$ and $t_{nw} + g(xx^*)$ lie in $\mathcal{I}_\lambda^{\mathrm{alg}}$. Consider $t_{mw} + f(x^*x)$.

Case 1. $w(\eta) \neq 0$.

Putting $m_1 = mw/w(\eta)$, we see that

$$\begin{aligned}t_{mw} + f(x^*x) &= t_{mw} + m_1(\eta)f(x^*x) + m_1(\zeta)f(xx^*) \\ &= t_{m_1w(\eta)} + t_{m_1}(f(x^*x) + f(xx^*)) = t_{m_1}(t_{w(\eta)} + f(a)).\end{aligned}$$

Since $w(\eta)$ is constant (and hence lying in $C_{\zeta,\eta}(\partial\mathbb{D})$) and $w(\eta) + f(\lambda) = 0$, $t_{w(\eta)} + f(a)$ lies in J_λ, so $t_{mw} + f(x^*x) \in \mathcal{I}_\lambda^{\mathrm{alg}}$.

Case 2. $w(\eta) = 0$.

If m and n are as above, mw vanishes at both ζ and η. Fix a closed arc I in $\partial\mathbb{D}$ whose interior contains ζ, but with η not in I. This time, define $m_1 = |mw|^{1/2}$ on I, $m_1 > 0$ on $\partial\mathbb{D}\backslash I$, and $m_1(\eta) = 1$. Let w_1 be $mw/|mw|^{1/2}$ when $mw \neq 0$ and 0 otherwise. Note that w_1 is continuous and $mw = m_1 w_1$ on $\partial\mathbb{D}$. Thus

$$\begin{aligned} t_{mw} + f(x^*x) &= t_{mw} + m_1(\eta)f(x^*x) + m_1(\zeta)f(xx^*) \\ &= t_{m_1w_1} + t_{m_1}(f(x^*x) + f(xx^*)) = t_{m_1}(t_{w_1} + f(a)). \end{aligned}$$

Since $w_1(\zeta) = w_1(\eta) = 0$ and $f(\lambda) = 0$, $t_{w_1} + f(a)$ lies in J_λ. We conclude that $t_{mw} + f(x^*x)$ is in $\mathcal{I}_\lambda^{\text{alg}}$ in Case 2, as well as Case 1. A similar argument shows that $t_{nw} + g(xx^*)$ lies in $\mathcal{I}_\lambda^{\text{alg}}$ in both cases, thus proving the claim.

Next we show that the fourth term in b, $uh(x^*x)$, is in $\overline{\mathcal{I}}_\lambda^{\text{alg}}$. If p is continuous on $[0,s]$, with $p(0) = p(\lambda) = 0$, then $p(a)$ lies in J_λ. Thus $xp(x^*x) = xp(a)$ is in $\mathcal{I}_\lambda^{\text{alg}}$. Writing $x = u\sqrt{x^*x}$, we see that $xp(a) = u\sqrt{x^*x}p(x^*x)$. According to (ii) of Lemma 4.1, the closure of such objects includes our fourth term $uh(x^*x)$, so that $uh(x^*x)$ is in $\overline{\mathcal{I}}_\lambda^{\text{alg}}$. Similarly, $\overline{\mathcal{I}}_\lambda^{\text{alg}}$ contains $u^*k(xx^*)$, the fifth term of b, so that b is in $\overline{\mathcal{I}}_\lambda^{\text{alg}}$ as desired. This completes the proof for $0 < \lambda \leqslant s$.

Next we consider the case $\lambda = \mathbf{p} = \{0, \zeta, \eta\}$, the triple point in Λ. Recall that if f is in $C_0([0,s])$, then $f(\mathbf{p}) = f(0) = 0$, while any w in $C_{\zeta,\eta}(\partial\mathbb{D})$ satisfies $w(\mathbf{p}) = w(\zeta) = w(\eta)$. An element b of $\mathcal{A}_0$, specified by Equation (4.2), lies in the kernel of $\Phi_{\mathbf{p}}$ exactly when $w(\zeta) = w(\eta) = 0$. We want to show that $\ker \Phi_{\mathbf{p}} \subset \overline{\mathcal{I}}_{\mathbf{p}}^{\text{alg}}$. Let m and n be as described above. For f in $C_0([0,s])$,

$$t_m f(a) = t_m(f(x^*x) + f(xx^*)) = m(\eta)f(x^*x) + m(\zeta)f(xx^*) = f(x^*x),$$

and similarly, for $g \in C_0([0,s])$, $t_n g(a) = g(xx^*)$. Thus $f(x^*x)$ and $g(xx^*)$ lie in $\mathcal{I}_{\mathbf{p}}^{\text{alg}}$. If $w(\zeta) = w(\eta) = 0$, then t_w lies in $J_{\mathbf{p}} \subset \mathcal{I}_{\mathbf{p}}^{\text{alg}}$. As noted above for the case $0 < \lambda \leqslant s$, $uh(x^*x)$ and $u^*k(xx^*)$ both lie in $\overline{\mathcal{I}}_\lambda^{\text{alg}}$ and thus so does b, establishing the conclusion for $\lambda = \mathbf{p}$.

Finally, if λ is in $\partial\mathbb{D}\backslash\{\zeta,\eta\}$, note that J_λ consists of those elements $t_w + f(a)$ with $w(\lambda) = 0$, while the elements of $\ker \Phi_\lambda$ have the form given by Equation (4.2), with $w(\lambda) = 0$. It follows easily (and similarly), that $\overline{\mathcal{I}}_\lambda^{\text{alg}}$ contains $\ker \Phi_\lambda$ in this case as well. ∎

PROPOSITION 4.8. *Let $\lambda \in \Lambda$.*

(i) *If $0 < \lambda \leqslant s$, $\mathcal{A}/\mathcal{I}_\lambda$ is $*$-isomorphic to $\mathbb{M}_2$.*

(ii) *$\mathcal{A}/I_{\mathbf{p}}$ is $*$-isomorphic to $\mathbb{M}_2^{\text{diag}}$.*

(iii) *If λ is in $\partial\mathbb{D}\backslash\{\zeta,\eta\}$, $\mathcal{A}/\mathcal{I}_\lambda$ is $*$-isomorphic to $\{cI_{2\times 2} : c \in \mathbb{C}\}$.*

Proof. For an ideal $\mathcal{I}$ in an algebra $\mathcal{B}$, we write $[b]_{\mathcal{I}}$ throughout for the coset in $\mathcal{B}/\mathcal{I}$ of an element b in $\mathcal{B}$. First suppose $0 < \lambda \leqslant s$. Since $\ker \Phi_\lambda \subset \mathcal{A}_0 \cap \mathcal{I}_\lambda$, we may define a $*$-homomorphism

$$\Gamma_\lambda : \mathcal{A}_0/\ker \Phi_\lambda \to \mathcal{A}_0/(\mathcal{A}_0 \cap \mathcal{I}_\lambda) \quad \text{by } \Gamma_\lambda([b]_{\ker \Phi_\lambda}) = [b]_{(\mathcal{A}_0 \cap \mathcal{I}_\lambda)}.$$

By Proposition 4.6 we know that $\mathcal{A}_0/\ker\Phi_\lambda$ is $*$-isomorphic to $\mathbb{M}_2$; write this isomorphism as $T_\lambda : \mathbb{M}_2 \to \mathcal{A}_0/\ker\Phi_\lambda$. Thus we have a sequence of onto $*$-homomorphisms

$$\mathbb{M}_2 \to \frac{\mathcal{A}_0}{\ker\Phi_\lambda} \to \frac{\mathcal{A}_0}{\mathcal{A}_0\cap\mathcal{I}_\lambda} \to \frac{\mathcal{A}_0+\mathcal{I}_\lambda}{\mathcal{I}_\lambda}, \tag{4.10}$$

where the first map is T_λ, the second is Γ_λ and the last, call it R_λ, is provided by the first isomorphism theorem for rings (see, for example, p. 105 in [16]) and has the form $R_\lambda : [b]_{\mathcal{A}_0\cap\mathcal{I}_\lambda} \to [b]_{\mathcal{I}_\lambda}$. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, so is $\mathcal{A}_0+\mathcal{I}_\lambda$, and we have $(\mathcal{A}_0+\mathcal{I}_\lambda)/\mathcal{I}_\lambda$ both dense in $\mathcal{A}/\mathcal{I}_\lambda$ and finite-dimensional. Therefore

$$\frac{\mathcal{A}_0+\mathcal{I}_\lambda}{\mathcal{I}_\lambda} = \frac{\mathcal{A}}{\mathcal{I}_\lambda}.$$

Thus we have a homomorphism $S_\lambda = R_\lambda\circ\Gamma_\lambda\circ T_\lambda$ from $\mathbb{M}_2$ onto $\mathcal{A}/\mathcal{I}_\lambda$. Since $\mathbb{M}_2$ has no non-trivial ideals, the kernel of S_λ is either $\mathbb{M}_2$ or $\{0\}$. Since $\mathcal{A}$ is a C^*-algebra, $\mathcal{I}_\lambda \neq \mathcal{A}$ (see [1], p. 33), and thus our homomorphism is injective; that is $\mathbb{M}_2 \cong \mathcal{A}/\mathcal{I}_\lambda$.

Next consider (ii), with $\lambda = \mathbf{p}$. We repeat the above argument, but this time, by Proposition 4.6, we may replace $\mathbb{M}_2$ on the left side of (4.10) by $\mathbb{M}_2^{\text{diag}}$. Again, the above argument yields a homomorphism $S_\mathbf{p}$ from $\mathbb{M}_2^{\text{diag}}$ onto $\mathcal{A}/I_\mathbf{p}$. However, unlike $\mathbb{M}_2$, $\mathbb{M}_2^{\text{diag}}$ contains two non-trivial ideals, namely

$$\left\{\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix} : a\in\mathbb{C}\right\} \quad\text{and}\quad \left\{\begin{bmatrix} 0 & 0\\ 0 & b\end{bmatrix} : b\in\mathbb{C}\right\}. \tag{4.11}$$

Again, $I_\mathbf{p} \neq \mathcal{A}$ and so $\ker S_\mathbf{p}$ is either $\{0\}$ or one of these two ideals. If it is the first ideal in (4.11), then $S_\mathbf{p}$ induces an isomorphism of $\mathbb{C}$ and $\mathcal{A}/I_\mathbf{p}$ whose inverse has the form $[b]_{I_\mathbf{p}} \to w(\eta)$ when b is given by Equation (4.2). In particular, for $b = t_w$, we see that $\|[t_w]_{I_\mathbf{p}}\| = |w(\eta)|$. However, for $0<\lambda\leqslant s$, we know that

$$\|[t_w]_{\mathcal{I}_\lambda}\| = \left\|\begin{bmatrix} w(\zeta) & 0\\ 0 & w(\eta)\end{bmatrix}\right\|_{\mathbb{M}_2} = \max\{|w(\zeta)|,|w(\eta)|\}.$$

The map $\lambda \to \|[b]_{\mathcal{I}_\lambda}\|$ is known to be upper semi-continuous on Λ (see Theorem 1.34 of [1]), which implies that for each w in $C(\partial\mathbb{D})$,

$$\max\{|w(\zeta)|,|w(\eta)|\} = \limsup_{\lambda\downarrow 0}\|[t_w]_{\mathcal{I}_\lambda}\| \leqslant \|[t_w]_{I_\mathbf{p}}\| = |w(\eta)|.$$

This is clearly impossible. Thus $\ker S_\mathbf{p}$ cannot be the first ideal in (4.11), or similarly, the second. Therefore, $S_\mathbf{p}$ has kernel $\{0\}$ and provides an isomorphism of $\mathbb{M}_2^{\text{diag}}$ and $\mathcal{A}/I_\mathbf{p}$, proving (ii).

Finally, for (iii), one can repeat the general argument from (i), with $\lambda \in \partial\mathbb{D}\backslash\{\zeta,\eta\}$, replacing $\mathbb{M}_2$ in (4.10) by $\{cI_{2\times 2} : d\in\mathbb{C}\}\cong\mathbb{C}$, an algebra with no non-trivial ideals.

One easily checks that the isomorphism S_λ^{-1} from $\mathcal{A}/\mathcal{I}_\lambda$ into $\mathbb{M}_2$ is given for b in $\mathcal{A}_0$ by

$$S_\lambda^{-1} : [b]_{\mathcal{I}_\lambda} \to \Phi_\lambda(b).$$

By Equation (4.6), S_λ^{-1}, and thus S_λ, are manifestly $*$-maps. ∎

REMARK 4.9. For future reference, we note that by the above proof, the composition S_λ of the three homomorphisms in (4.10) is an isomorphism, and thus the map Γ_λ is an isomorphism of $\mathcal{A}_0/\ker \Phi_\lambda$ and $\mathcal{A}_0/(\mathcal{A}_0 \cap \mathcal{I}_\lambda)$. In other words, $\ker \Phi_\lambda = \mathcal{A}_0 \cap \mathcal{I}_\lambda$.

By Proposition 4.6 and Proposition 4.8, we have $*$-isomorphisms

$$\mathcal{A}/\mathcal{I}_\lambda \cong \mathcal{A}_0/\ker \Phi_\lambda \cong \begin{cases} \mathbb{M}_2 & \text{when } 0 < \lambda \leqslant s, \\ \mathbb{M}_2^{\mathrm{diag}} & \text{when } \lambda = \mathbf{p}, \\ \{cI_{2\times 2} : c \in \mathbb{C}\} & \text{when } \lambda \in \partial\mathbb{D}\backslash\{\zeta,\eta\}, \end{cases} \tag{4.12}$$

the composition being S_λ^{-1}. The objects on the right are C^*-algebras, so that S_λ^{-1} is isometric. Thus, for $b \in \mathcal{A}_0$,

$$\|[b]_{\mathcal{I}_\lambda}\|_{\mathcal{A}/\mathcal{I}_\lambda} = \|\Phi_\lambda(b)\| \tag{4.13}$$

$$= \begin{cases} \left\| \begin{bmatrix} w(\zeta) + g(\lambda) & h(\lambda) \\ k(\lambda) & w(\eta) + f(\lambda) \end{bmatrix} \right\| & \text{if } 0 < \lambda \leqslant s, \\ \left\| \begin{bmatrix} w(\zeta) & 0 \\ 0 & w(\eta) \end{bmatrix} \right\| & \text{if } \lambda = \mathbf{p}, \\ \left\| \begin{bmatrix} w(\lambda) & 0 \\ 0 & w(\lambda) \end{bmatrix} \right\| & \text{if } \lambda \in \partial\mathbb{D}\backslash\{\zeta,\eta\}, \end{cases}$$

the norm on the right being the operator norm in $\mathbb{M}_2$.

Now we write $B(\Lambda, \mathbb{M}_2)$ for the C^*-algebra of all bounded functions F from Λ to $\mathbb{M}_2$, with norm

$$\|F\| = \sup_{\lambda \in \Lambda} \|F(\lambda)\|_{\mathbb{M}_2}.$$

We can define a $*$-homomorphism Φ from $\mathcal{A}_0$ to $B(\Lambda, \mathbb{M}_2)$ by letting $\Phi(b)$ be the function whose value at λ in Λ is $\Phi_\lambda(b)$. We write $\mathcal{D}$ for the range of Φ. According to the above results and Douglas' theorem, $\|b\|_{\mathcal{A}} = \sup_{\lambda \in \Lambda} \|\Phi_\lambda(b)\|$, so that Φ is an isometric $*$-isomorphism of $\mathcal{A}_0$ onto the $*$-algebra $\mathcal{D}$. It is easy to verify that $\mathcal{D}$ consists of all

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

in $B(\Lambda, \mathbb{M}_2)$ such that each f_{ij} is continuous on $\{\mathbf{p}\} \cup (0,s)$ and $\partial\mathbb{D}\backslash\{\zeta,\eta\}$, f_{12} and f_{21} vanish at $\mathbf{p}$ and on $\partial\mathbb{D}\backslash\{\zeta,\eta\}$, $f_{11} = f_{22}$ on $\partial\mathbb{D}\backslash\{\zeta,\eta\}$, while $f_{11}(\mathbf{p}) = \lim_{\lambda\to\zeta} f_{11}(\lambda)$ and $f_{22}(\mathbf{p}) = \lim_{\lambda\to\eta} f_{22}(\lambda)$, the limits being taken as $\lambda \to \zeta$ or $\lambda \to \eta$ through points in $\partial\mathbb{D}\backslash\{\zeta,\eta\}$. One easily checks that $\mathcal{D}$ is closed in $B(\Lambda, \mathbb{M}_2)$. Since

Φ is isometric, $\mathcal{A}_0$ is complete. Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, we can close the circle to obtain the following result.

PROPOSITION 4.10. *The algebra $\mathcal{A}_0$ coincides with $\mathcal{A}$, and ker $\Phi = \mathcal{I}_\lambda$.*

Let us define two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ in $\mathcal{A}$:

$$\mathcal{M} \equiv \{f(x^*x) : f \in C_0([0,s])\}, \quad \mathcal{N} \equiv \{g(xx^*) : g \in C_0([0,s])\}.$$

We have already seen that $\mathcal{A}_0$ is an algebraic direct sum of the closed subspaces $\{t_w : w \in C(\partial\mathbb{D})\}$, $\mathcal{M}, \mathcal{N}, u\mathcal{M}$ and $u^*\mathcal{N}$. Since $\mathcal{A}_0 = \mathcal{A}$, a Banach space, we have the following corollary.

COROLLARY 4.11. *As a Banach space, $\mathcal{A} = C^*(T_z, C_\varphi)/\mathcal{K}$ has the direct sum decomposition*

$$\mathcal{A} = \{t_w : w \in C(\partial\mathbb{D})\} \oplus \mathcal{M} \oplus \mathcal{N} \oplus u\mathcal{M} \oplus u^*\mathcal{N}.$$

In summary we have the following:

THEOREM 4.12. *The map Φ is a $*$-isomorphism of $\mathcal{A}$ onto $\mathcal{D}$.*

REMARK 4.13. Given the form of the algebra $\mathcal{D}$, it is not hard to show that every irreducible representation of $C^*(T_z, C_\varphi)/\mathcal{K}$ is unitarily equivalent either to one of the two-dimensional representations Φ_λ, λ in $(0, s]$, or to one of the scalar representations $\ell_\lambda : b \to w(\lambda)$, λ in $\partial\mathbb{D}$, where b is given by Equation (4.2).

4.4. $C^*(T_z, C_\varphi)$ REVISITED AND THE MAP Ψ. Let E and F be the spectral projections of $C_\varphi^* C_\varphi$ and $C_\varphi C_\varphi^*$ respectively, which are associated to their common essential spectrum $[0, s]$. We have

$$C_\varphi^* C_\varphi = EC_\varphi^* C_\varphi E + (I - E)C_\varphi^* C_\varphi (I - E) \text{ and } C_\varphi C_\varphi^* = FC_\varphi C_\varphi^* F + (I - F)C_\varphi C_\varphi^* (I - F).$$

Notice that the second term on the right-hand side of each of these expressions is a finite rank operator. Thus if f and g are continuous on $\sigma(C_\varphi^* C_\varphi) = \sigma(C_\varphi C_\varphi^*)$, then

$$f(C_\varphi^* C_\varphi) = f(EC_\varphi^* C_\varphi E) + K_1, \quad g(C_\varphi C_\varphi^*) = g(FC_\varphi C_\varphi^* F) + K_2 \tag{4.14}$$

for finite rank operators K_1 and K_2. Also note that the maps $f \to f(EC_\varphi^* C_\varphi E)$ and $g \to f(FC_\varphi C_\varphi^* F)$ are isometries from $C_0([0, s])$ onto closed subspaces $\mathfrak{M}$ and $\mathfrak{N}$ in $C^*(T_z, C_\varphi)$.

THEOREM 4.14. *As a Banach space, $C^*(T_z, C_\varphi)$ is the direct sum of closed subspaces:*

$$C^*(T_z, C_\varphi) = \{T_w : w \in C(\partial\mathbb{D})\} \oplus \mathfrak{M} \oplus \mathfrak{N} \oplus U\mathfrak{M} \oplus U^*\mathfrak{N} \oplus \mathcal{K}. \tag{4.15}$$

Proof. Given $B \in C^*(T_z, C_\varphi)$, the coset $b = [B]$ satisfies Equation (4.2) for unique $w \in C(\partial\mathbb{D})$ and f, g, h and k in $C_0([0, s])$. Since the coset map $B \to [B]$ is one-to-one when restricted to each of the first five direct summands (for example, $[Uh(C_\varphi^* C_\varphi)] = uh(x^*x)$), we see that

$$B = T_w + f(EC_\varphi^* C_\varphi E) + g(FC_\varphi C_\varphi^* F) + Uh(EC_\varphi^* C_\varphi E) + U^*k(FC_\varphi C_\varphi^* F) + K \tag{4.16}$$

for a unique compact operator K. ∎

Now consider the map $\Psi : C^*(T_z, C_\varphi) \to \mathcal{D}$ defined by $\Psi(B) = \Phi([B])$. Clearly we have the following result.

THEOREM 4.15. *We have a short exact sequence of C^*-algebras,*

$$0 \to \mathcal{K} \xrightarrow{i} C^*(T_z, C_\varphi) \xrightarrow{\Psi} \mathcal{D} \to 0,$$

where i is inclusion.

4.5. THE DENSE SEMI-POLYNOMIAL SUBALGEBRA $\mathcal{P}$. We write $\mathcal{P}$ for the dense non-commutative semi-polynomial $*$-algebra consisting of finite linear combinations of all T_w, w in $C(\partial\mathbb{D})$, all words in C_φ and C_φ^*, and all compact operators. Every element of $\mathcal{P}$ has the form

$$B = T_w + f(C_\varphi^* C_\varphi) + g(C_\varphi C_\varphi^*) + C_\varphi p(C_\varphi^* C_\varphi) + C_\varphi^* q(C_\varphi C_\varphi^*) + K, \tag{4.17}$$

where w is in $C(\partial\mathbb{D})$, f, g, p and q are polynomials with $f(0) = 0 = g(0)$, and K is compact. Cutting $C_\varphi^* C_\varphi$ and $C_\varphi C_\varphi^*$ down by the spectral projections E and F respectively, we find $B = T_w + f(EC_\varphi^* C_\varphi E) + g(FC_\varphi C_\varphi^* F) + UE\sqrt{C_\varphi^* C_\varphi}\, p(C_\varphi^* C_\varphi)E + U^*F\sqrt{C_\varphi C_\varphi^*}\, q(C_\varphi C_\varphi^*)F + K'$, where we have absorbed each of the finite ranks arising from Equations (4.14) into the new compact operator K'. By Theorem 4.14, B determines each of the six summands here. Since f, g, p and q are polynomials, and so are determined by their restrictions to $[0, s]$, the decomposition of B in Equation (4.17) is unique. Since $C_\varphi^* C_\varphi - sC_{\varphi\circ\sigma}$ and $C_\varphi C_\varphi^* - sC_{\sigma\circ\varphi}$, are compact, we see that Equation (4.17) becomes

$$B = T_w + A_1 + A_2 + A_3 + A_4 + K''$$

where K'' is compact, and A_1, A_2, A_3, A_4 are finite linear combinations of composition operators whose associated self-maps of $\mathbb{D}$ are taken from the respective lists $(\varphi\circ\sigma)_{n_1}$, $(\sigma\circ\varphi)_{n_2}$, $(\varphi\circ\sigma)_{n_3}\circ\varphi$, and $(\sigma\circ\varphi)_{n_4}\circ\sigma$, for integers $n_1, n_2 \geqslant 1$ and $n_3, n_4 \geqslant 0$, where τ_n denotes the n^{th} iterate of the map τ. Since all of these self-maps are distinct, Corollary 5.17 in [18] says the corresponding composition operators are linearly independent modulo $\mathcal{K}$. Thus the operator B determines the coefficients in each of the sums A_1, A_2, A_3, A_4, and w and K'' as well. We summarize these observations in the following theorem.

THEOREM 4.16. *Every operator in $\mathcal{P}$ is a sum of a unique Toeplitz operator with continuous symbol, a unique compact operator and a unique finite linear combination of composition operators with associated disk maps taken from the set*

$$\{(\varphi\circ\sigma)_{n_1}, (\sigma\circ\varphi)_{n_2}, (\varphi\circ\sigma)_{n_3}\circ\varphi, (\sigma\circ\varphi)_{n_4}\circ\sigma\}$$

where $n_k \geqslant 1$ for $k = 1, 2$ and $n_k \geqslant 0$ for $k = 3, 4$.

For an operator B given by Equation (4.17), the matrix function $\Psi(B)$ can properly be called the "symbol of B". In particular, if r is the function defined on

Λ by $r(\lambda) = \sqrt{\lambda}$ for $0 < \lambda \leqslant s$ and $r(\lambda) = 0$ otherwise, then

$$\Psi(C_\varphi) = \begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix}.$$

4.6. ESSENTIAL SPECTRA AND ESSENTIAL NORMS IN $C^*(T_z, C_\varphi)$.

THEOREM 4.17. *Let B in $C^*(T_z, C_\varphi)$ be given by Equation* (4.16). *The essential spectrum of B is the union of $w(\partial\mathbb{D})$ with the image of*

$$\frac{1}{2}\left[f(t) + w(\eta) + g(t) + w(\zeta) \pm \sqrt{(f(t) + w(\eta) - g(t) - w(\zeta))^2 + 4h(t)k(t)} \right]$$

as t ranges over $[0, s]$.

Proof. By Theorem 4.12 or Theorem 4.15, the essential spectrum of B is

$$\{z \in \mathbb{C} : \det\,(\Phi_\lambda([B]) - zI_{2\times 2}) = 0 \text{ for some } \lambda \in \Lambda\}.$$

Evaluating this determinant via Equation (4.6) gives the desired result. ∎

We start with some examples of Theorem 4.17 in which $w = 0$.

EXAMPLE 4.18. The essential spectrum of the real part of C_φ is the interval $[-\sqrt{s}/2, \sqrt{s}/2]$, where $s = |\varphi'(\zeta)|^{-1}$. This follows from using $f(t) = g(t) = 0$ and $h(t) = k(t) = \sqrt{t}$ in Theorem 4.17 to see that

$$\sigma_{\mathrm{e}}(C_\varphi + C_\varphi^*) = [-\sqrt{s}, \sqrt{s}].$$

EXAMPLE 4.19. The essential spectrum of the self-commutator $[C_\varphi^*, C_\varphi]$ is $[-s, s]$. This is obtained from Theorem 4.17, using $f(t) = t$, $g(t) = -t$, and $k(t) = h(t) = 0$. Similarly, the anti-commutator $C_\varphi^* C_\varphi + C_\varphi C_\varphi^*$ has essential spectrum $[0, s]$.

EXAMPLE 4.20. Let

$$B_1 = C_{\varphi\circ\sigma} + C_{\sigma\circ\varphi} + C_\varphi - C_\sigma,$$

so that $f(t) = t/s = g(t)$, $h(t) = \sqrt{t}$ and $k(t) = -\sqrt{t}/s$. Then $\sigma_{\mathrm{e}}(B_1)$ is the parabolic curve $y^2 + \mathrm{i}y$, $-1 \leqslant y \leqslant 1$.

EXAMPLE 4.21. Let

$$B_2 = C_{\varphi\circ\sigma} - C_{\sigma\circ\varphi} + \frac{1}{2}C_\varphi - C_\sigma,$$

so that $f(t) = t/s$, $g(t) = -t/s$, $h(t) = \sqrt{t}/2$ and $k(t) = -\sqrt{t}/s$. Then $\sigma_{\mathrm{e}}(B_2)$ is the union of two complex line segments, $[-1/\sqrt{2}, 1/\sqrt{2}]$ and $[-\mathrm{i}/4, \mathrm{i}/4]$.

EXAMPLE 4.22. Let

$$B_3 = 2C_{\varphi\circ\sigma} + C_\varphi - C_\sigma,$$

so that $f(t) = 2t/s$, $g(t) = 0$, $h(t) = \sqrt{t}$ and $k(t) = -\sqrt{t}/s$. Here $\sigma_{\mathrm{e}}(B_3)$ is the circle of radius $1/2$ centered at $z = 1/2$.

Next we look at the effect of adding a Toeplitz operator. Consider an operator $B = T_w + Y$ given by Equation (4.16), with

$$Y = f(EC_\varphi C_\varphi^* E) + g(FC_\varphi C_\varphi^* F) + Uh(EC_\varphi^* C_\varphi E) + U^* k(FC_\varphi C_\varphi^* F) + K.$$

According to Theorem 4.17, adding Y to T_w does not affect the part of the essential spectrum coming from $\sigma_{\mathrm{e}}(T_w) = w(\partial\mathbb{D})$. If w takes a common value c at the points ζ and η, Theorem 4.17 also implies that

$$\sigma_{\mathrm{e}}(B) = \sigma_{\mathrm{e}}(T_w) \cup \sigma_{\mathrm{e}}(cI + Y).$$

In this case, the effect of adding T_w, on the part of the essential spectrum coming from Y, is to merely translate it by c. However if $w(\zeta) \neq w(\eta)$, adding T_w can non-trivially deform $Y's$ contribution to $\sigma_{\mathrm{e}}(B)$.

EXAMPLE 4.23. For $r \geqslant 0$, suppose w in $C(\partial\mathbb{D})$ satisfies

$$w(\eta) = r\frac{1+\mathrm{i}}{\sqrt{2}}, \quad w(\zeta) = -r\frac{1+\mathrm{i}}{\sqrt{2}}.$$

Let $B = T_w + Y$ where $Y = C_\varphi + C_\varphi^*$. Taking f, g, h, and k as in Example 4.18, we see from Theorem 4.17 that

$$\sigma_{\mathrm{e}}(B) = w(\partial\mathbb{D}) \cup \{\pm\sqrt{t + r^2\mathrm{i}} : 0 \leqslant t \leqslant s\}.$$

Thus when $r = 0$ (so that $w(\zeta) = w(\eta) = 0$),

$$\sigma_{\mathrm{e}}(B) = w(\partial\mathbb{D}) \cup [-\sqrt{s}, \sqrt{s}] = \sigma_{\mathrm{e}}(T_w) \cup \sigma_{\mathrm{e}}(Y).$$

However, when $r > 0$, adding T_w to Y disconnects the essential spectrum of the latter operator, deforming the two halves of $\sigma_{\mathrm{e}}(Y)$, $[0, \sqrt{s}]$ and $[-\sqrt{s}, 0]$, into the curves $\{\sqrt{t + r^2\mathrm{i}} : 0 \leqslant t \leqslant s\}$ and $\{-\sqrt{t + r^2\mathrm{i}} : 0 \leqslant t \leqslant s\}$, respectively. The first of these curves lies in the open first quadrant, is convex, and falls downhill to the right. The second, of course, is its reflection through the origin.

Finally, we consider essential norms. If B in $C^*(T_z, C_\varphi)$ is given by Equation (4.16), we know that the essential norm $\|B\|_{\mathrm{e}}$ is given by

$$\|B\|_{\mathrm{e}} = \sup_{\lambda \in \Lambda} \|\Phi_\lambda([B])\|_{\mathbb{M}_2}.$$

EXAMPLE 4.24. Let $B = T_z + C_\varphi + C_\varphi^*$. Here we have $w(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}, f(t) = g(t) = 0$ and $h(t) = k(t) = \sqrt{t}$. If λ is in $\partial\mathbb{D} \backslash \{\zeta, \eta\}$ or $\lambda = \mathbf{p}$, then $\Phi_\lambda([B])$ is a diagonal unitary matrix. For $0 < \lambda \leqslant s$,

$$\Phi_\lambda([B]) = \begin{bmatrix} \zeta & \sqrt{\lambda} \\ \sqrt{\lambda} & \eta \end{bmatrix}.$$

A well-known formula for the operator norm on $\mathbb{M}_2$ (see [21], p. 17) gives

$$\|B\|_{\mathrm{e}}^2 = \sup_{0<\lambda\leqslant s} \left\| \begin{bmatrix} \zeta & \sqrt{\lambda} \\ \sqrt{\lambda} & \eta \end{bmatrix} \right\|^2 = \sup_{0<\lambda\leqslant s} \left\{ 1+\lambda+\sqrt{(1+\lambda)^2-|\zeta\eta-\lambda|^2} \right\}$$
$$= 1+\frac{1}{|\varphi'(\zeta)|}+\sqrt{\frac{2}{|\varphi'(\zeta)|}}\sqrt{1+\mathrm{Re}(\zeta\eta)}\,.$$

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