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ON THE SIMPLE C^*-ALGEBRAS ARISING FROM DYCK SYSTEMS

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ABSTRACT. The Dyck shift D_N for $2N$ brackets $(N > 1)$ gives rise to a purely infinite simple C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$, that is not stably isomorphic to any Cuntz-Krieger algebra. It is presented as a unique C^*-algebra generated by N partial isometries and N isometries subject to certain operator relations. The canonical AF subalgebra $F_{\mathfrak{L}^{Ch(D_N)}}$ of $O_{\mathfrak{L}^{Ch(D_N)}}$ has a unique tracial state. For the gauge action on the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$, a KMS state at inverse temperature $\log \beta$ exists if and only if $\beta = N + 1$. The admitted KMS state is unique. The GNS representation of $O_{\mathfrak{L}^{Ch(D_N)}}$ by the KMS state yields a factor of type $\mathrm{III}_{1/(N+1)}$.

KEYWORDS: *C^*-algebra, Dyck shifts, KMS state, AF algebra.*

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1. INTRODUCTION

The theory of symbolic dynamics has a close relationship to automata theory and language theory. In the theory of formal language, there is a family of universal languages, called Dyck languages, introduced in [1] . The symbolic dynamics generated by the languages are called the Dyck shifts D_N, $N \in \mathbb{N}$ (cf. [1], [9], [10], [11]). They are nonsofic subshifts. Their alphabet consists of the $2N$ brackets: $(_1, \dots, (_N,)_1, \dots,)_N$. The forbidden words consist of words that do not obey the standard bracket rules. In [12], the Cantor horizon λ-graph system for D_N has been introduced as an irreducible λ-graph system that presents the subshift D_N. It is a minimal irreducible component of the canonical λ-graph system of D_N. Hence it gives rise to a purely infinite simple C^*-algebra denoted by $O_{\mathfrak{L}^{Ch(D_N)}}$. Its K-groups are realized as the K-groups for the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ and computed to be

$$K_0(O_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \quad K_1(O_{\mathfrak{L}^{Ch(D_N)}}) \cong 0$$

for $1 < N \in \mathbb{N}$, where $C(\mathfrak{K}, \mathbb{Z})$ denotes the abelian group of all integer valued continuous functions on a Cantor discontinuum $\mathfrak{K}$ ([12], Corollary 3.17). For $N = 1$, the algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ goes to the Cuntz algebra O_2.

In this paper we first study the operator relations of the canonical generators of the algebras. We prove the following theorem.

THEOREM 1.1. *For $N \in \mathbb{N}$, the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ associated with the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ for the Dyck shift D_N is unital, separable, nuclear, simple and purely infinite. It is the unique C^*-algebra generated by N partial isometries $S_i, i = 1, \dots, N$ and N isometries $T_i, i = 1, \dots, N$ subject to the following operator relations, where $E_{\mu_1 \cdots \mu_l} = S_{\mu_1}^* \cdots S_{\mu_l}^* S_{\mu_l} \cdots S_{\mu_1}$ for $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$:*

$$\sum_{j=1}^{N} S_j^* S_j = 1, \tag{1.1}$$

$$E_{\mu_1 \cdots \mu_l} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1 \cdots \mu_l} S_j S_j^* + T_{\mu_1} E_{\mu_2 \cdots \mu_l} T_{\mu_1}^*, \quad l = 2, 3, \dots. \tag{1.2}$$

We note that the above relations for $N = 1$ go to the relations of the canonical generators of the Cuntz algebra O_2.

By using the above operator relations, we next study KMS states for gauge action on the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$. The result is the following:

THEOREM 1.2. *For a positive real number β, a* KMS *state on $O_{\mathfrak{L}^{Ch(D_N)}}$ at inverse temperature* $\log \beta$ *exists if and only if $\beta = N + 1$. The admitted* KMS *state is unique.*

The value $\log(N + 1)$ is the topological entropy of the Dyck shift D_N [9]. We will prove that the fixed point algebra ${O_{\mathfrak{L}^{Ch(D_N)}}}^{\alpha^{DN}}$ of $O_{\mathfrak{L}^{Ch(D_N)}}$ under the gauge action α^{D_N} is a simple AF algebra with a unique tracial state (Theorem 5.14). We need a combinatorial argument using Catalan numbers and their generating function for the proof of uniqueness of tracial state on the AF algebra. As a consequence we have

THEOREM 1.3. *Let $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$ be the von Neumann algebra generated by the* GNS-*representation $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})$ of the algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ by the unique* KMS *state φ. Then $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$ is the injective factor of type* $\mathrm{III}_{1/(N+1)}$.

Corresponding results for Cuntz algebras and Cuntz-Krieger algebras are seen in [24] and [4] respectively. Related results for Cuntz-Krieger type algebras are seen in many authors [5], [6], [8], [13], [22], [23], [26], etc.

2. THE λ-GRAPH SYSTEMS FOR THE DYCK SHIFTS

Throughout this paper $N > 1$ is a fixed positive integer. In what follows, $\mathbb{Z}_+$ denotes the set $\{0, 1, \dots\}$ of nonnegative integers.

We consider the Dyck shift D_N with alphabet $\Sigma = \Sigma^- \cup \Sigma^+$ where $\Sigma^- = \{\alpha_1, \dots, \alpha_N\}, \Sigma^+ = \{\beta_1, \dots, \beta_N\}$. The symbols α_i, β_i correspond to the brackets $(_i,)_i$ respectively. The Dyck inverse monoid $\mathbb{D}_N$ has the relations

$$\alpha_i \beta_j = \begin{cases} \mathbf{1} & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

for $i, j = 1, \dots, N$ [10], [11]. A word $\omega_1 \cdots \omega_n$ of Σ is admissible for D_N precisely if $\prod_{m=1}^{n} \omega_m \neq 0$. For a word $\omega = \omega_1 \cdots \omega_n$ of Σ, we denote by $\widetilde{\omega}$ its reduced form. Namely $\widetilde{\omega}$ is a word of $\Sigma \cup \{0, \mathbf{1}\}$ obtained after the operations (2.1). Hence a word ω of Σ is forbidden for D_N if and only if $\widetilde{\omega} = 0$.

A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over an alphabet Σ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,l+1} : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. The sets V_l and $E_{l,l+1}$ are finite for each $l \in \mathbb{Z}_+$. An edge $e \in E_{l,l+1}$ has its source vertex $s(e)$ in V_l, its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ. It yields a subshift by taking the set of all label sequences appearing in the labeled Bratteli diagram. There are many λ-graph systems that present a given subshift. Among them the canonical λ-graph system is a generalization of the left-Krieger cover graph for a sofic shift, and its strong shift equivalence class is a complete invariant for topological conjugacy of subshifts [15]. The canonical λ-graph system $\mathfrak{L}^{C(D_N)}$ for the Dyck shift D_N together with its K-groups has been calculated in [18]. One however sees that the λ-graph system $\mathfrak{L}^{C(D_N)}$ is not irreducible, so that the resulting C^*-algebra $O_{\mathfrak{L}^{C(D_N)}}$ is not simple. The Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ for D_N is a minimal irreducible component of the canonical λ-graph system $\mathfrak{L}^{C(D_N)}$. It gives rise to a purely infinite simple C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ that is a quotient of $O_{\mathfrak{L}^{C(D_N)}}$ by an ideal [20]. The K-groups of the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ are realized as the K-groups of the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ and computed to be

$$K_0(O_{\mathfrak{L}^{Ch(D_N)}}) \cong \mathbb{Z}/N\mathbb{Z} \oplus C(\mathfrak{K}, \mathbb{Z}), \quad K_1(O_{\mathfrak{L}^{Ch(D_N)}}) \cong 0 \quad ([12]).$$

Let us describe the Cantor horizon λ-graph system $\mathfrak{L}^{Ch(D_N)}$ of D_N. Let Σ_N be the full N-shift $\{1, \dots, N\}^{\mathbb{Z}}$. We denote by $B_l(D_N)$ and $B_l(\Sigma_N)$ the set of admissible words of length l of D_N and that of Σ_N respectively. The vertices V_l of $\mathfrak{L}^{Ch(D_N)}$ at level l are given by the words of length l consisting of the symbols of Σ^+. That is,

$$V_l = \{(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in B_l(D_A) : \mu_1 \cdots \mu_l \in B_l(\Sigma_N)\}.$$

Hence the cardinal number of V_l is N^l. The mapping $\iota (= \iota_{l,l+1}) : V_{l+1} \to V_l$ deletes the rightmost symbol of a word such as

$$\iota((\beta_{\mu_1} \cdots \beta_{\mu_{l+1}})) = (\beta_{\mu_1} \cdots \beta_{\mu_l}), \quad (\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}.$$

There exists an edge labeled α_j from $(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_l$ to $(\beta_{\mu_0}\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_{l+1}$ precisely if $\mu_0 = j$, and there exists an edge labeled β_j from $(\beta_j\beta_{\mu_1} \cdots \beta_{\mu_{l-1}}) \in V_l$ to $(\beta_{\mu_1} \cdots \beta_{\mu_{l+1}}) \in V_{l+1}$. It is easy to see that the resulting labeled Bratteli diagram with ι-map becomes a λ-graph system over Σ, denoted by $\mathfrak{L}^{Ch(D_N)}$, that presents the Dyck shift D_N [12]. We use the lexicographic order from left on the words in the symbols in Σ^+, that is, we assign to a word $(\beta_{\mu_1} \cdots \beta_{\mu_l})$ of vertex in V_l the number $N(\mu_1 \cdots \mu_l)$

$$N(\mu_1 \cdots \mu_l) = 1 + \sum_{n=1}^{l} (\mu_n - 1) N^{l-n}.$$

Hence the vertex set V_l bijectively corresponds to the set of positive integers less than or equal to N^l. We write the vertex $(\beta_{\mu_1} \cdots \beta_{\mu_l}) \in V_l$ as $v^l_{N(\mu_1 \cdots \mu_l)}$.

3. THE C^*-ALGEBRA $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$

We first see that the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ is simple and purely infinite. This fact is first observed in Proposition 2.2, Corollary 3.13 of [12].

A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ is said to satisfy λ-condition (I) if for every vertex $v \in V_l$ of $\mathfrak{L}$ there exist at least two paths with distinct label sequences starting with the vertex v and terminating with the same vertex. A λ-graph system $\mathfrak{L}$ is said to be λ-irreducible if for an ordered pair of vertices $u, v \in V_l$, there exists a number $L_l(u,v) \in \mathbb{N}$ such that for a vertex $w \in V_{l+L_l(u,v)}$ with $\iota^{L_l(u,v)}(w) = u$, there exists a path π in $\mathfrak{L}$ such that $s(\pi) = v$, $t(\pi) = w$, where $\iota^{L_l(u,v)}$ means the $L_l(u,v)$-times compositions of ι, and $s(\pi), t(\pi)$ denote the source vertex, the terminal vertex of π, respectively [19].

PROPOSITION 3.1. *The λ-graph system $\mathfrak{L}^{Ch(D_N)}$ is λ-irreducible, and it satisfies λ-condition* (I). *Hence the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ is simple and purely infinite.*

Proof. Let v^l_i be a vertex in V_l. We write $i = N(i_1 \cdots i_l)$ for some $i_1 \cdots i_l \in B_l(\Sigma_N)$. Take $\mu_1, \nu_1 \in \{1, \ldots, N\}$ with $\mu_1 \neq \nu_1$, and $\zeta_2 \cdots \zeta_{2l+2} \in B_{2l+1}(\Sigma_N)$. There exist two distinct paths labeled $\beta_{i_1} \cdots \beta_{i_l} \beta_{\mu_1}$ and $\beta_{i_1} \cdots \beta_{i_l} \beta_{\nu_1}$ whose sources are both v^l_i and terminals are both $v^{2l+1}_{N(\zeta_2 \cdots \zeta_{2l+2})} \in V_{2l+1}$. Hence $\mathfrak{L}^{Ch(D_N)}$ satisfies λ-condition (I).

Let v^l_j be a vertex in V_l so that $j = N(j_1 \cdots j_l)$ for some $j_1 \cdots j_l \in B_l(\Sigma_N)$ respectively. For any vertex $v^{2l}_k \in V_{2l}$ such that $\iota(v^{2l}_k) = v^l_j$, we may write $k = V(j_1 \cdots j_l h_{l+1} \cdots h_{2l})$ for some $h_{l+1} \cdots h_{2l} \in B_l(\Sigma_N)$. Then there exists a path labeled $\beta_{i_1} \cdots \beta_{i_l}$ starting at v^l_i and terminating at v^{2l}_k. This means that $\mathfrak{L}^{Ch(D_N)}$ is λ-irreducible. By [19], the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ is simple and purely infinite. ∎

We will study operator relations of the canonical generators of the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_N)}}$ to prove Theorem 1.1. By Corollary 5.2 of [12] and a general structure result for the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with λ-graph system $\mathfrak{L}$ ([16], Theorem A), we have:

LEMMA 3.2. *The C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ is unital, separable, nuclear, simple and purely infinite. It is the universal unique concrete C^*-algebra generated by partial isometries $s_\gamma, \gamma \in \Sigma$ and projections $e_i^l, i = 1, 2, \dots, N^l$, $l \in \mathbb{Z}_+$ satisfying the following operator relations:*

$$\sum_{\gamma \in \Sigma} s_\gamma s_\gamma^* = 1, \tag{3.1}$$

$$\sum_{i=1}^{N^l} e_i^l = 1, \quad e_i^l = \sum_{j=1}^{N^{l+1}} I_{l,l+1}(i,j) e_j^{l+1}, \tag{3.2}$$

$$s_\gamma s_\gamma^* e_i^l = e_i^l s_\gamma s_\gamma^*, \tag{3.3}$$

$$s_\gamma^* e_i^l s_\gamma = \sum_{j=1}^{N^{l+1}} A_{l,l+1}(i,\gamma,j) e_j^{l+1}, \tag{3.4}$$

for $i = 1, 2, \dots, N^l$, $l \in \mathbb{Z}_+, \gamma \in \Sigma$, where $V_l = \{v_1^l, \dots, v_{N^l}^l\}$,

$$A_{l,l+1}(i,\gamma,j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \gamma, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \dots, N^l$, $j = 1, 2, \dots, N^{l+1}$, $\gamma \in \Sigma$.

Note that the partial isometry s_γ corresponds to the edges labeled γ, and the projection e_i^l corresponds to the vertex v_i^l. We will prove Theorem 1.1 from Lemma 3.2 by using the structure of the λ-graph system $\mathfrak{L}^{Ch(D_N)}$.

Let $s_\gamma, \gamma \in \Sigma$ and $e_i^l, i = 1, \dots, N^l, l \in \mathbb{Z}_+$ be as in Lemma 3.2. Define the operators $S_i, T_i, i = 1, \dots, N$ by setting

$$S_i := s_{\alpha_i}, \quad T_i := s_{\beta_i} \quad \text{for } i = 1, \dots, N.$$

As the word $\alpha_i \beta_j$ is forbidden for $i \neq j$, we note $S_i T_j = 0$ for $i \neq j$. We will first show that the operators $S_i, T_i, i = 1, \dots, N$ satisfy the relations (1.1) and (1.2).

LEMMA 3.3. *The operators $T_i, i = 1, \dots, N$ are isometries.*

Proof. The first equality of (3.2) and (3.4) implies

$$T_i^* T_i = \sum_{k=1}^{N^l} s_{\beta_i}^* e_k^l s_{\beta_i} = \sum_{h=1}^{N^{l+1}} \Big(\sum_{k=1}^{N^l} A_{l,l+1}(k, \beta_i, h) \Big) e_h^{l+1}.$$

For the symbol β_i every vertex v_h^{l+1} has a unique incoming edge labeled β_i. This means $\sum_{k=1}^{N^l} A_{l,l+1}(k, \beta_i, h) = 1$ so that $T_i^* T_i = 1$. ∎

We put $E_{\mu_1 \cdots \mu_l} = S_{\mu_1}^* \cdots S_{\mu_l}^* S_{\mu_l} \cdots S_{\mu_1}$ for $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$.

PROPOSITION 3.4. *The operators $S_i, T_i, i = 1,\dots,N$ satisfy the relations* (1.1) *and* (1.2).

Proof. By (3.4) and (3.2), one has for a fixed $l \in \mathbb{Z}_+$,

$$S_i^* S_i = \sum_{j=1}^{N^l} \sum_{k=1}^{N^{l+1}} A_{l,l+1}(j,\alpha_i,k) e_k^{l+1}.$$

Every vertex $v_k^{l+1} \in V_{l+1}$ has a unique incoming edge labeled a symbol in Σ^-, so that $\sum_{i=1}^{N} \sum_{j=1}^{N^l} A_{l,l+1}(j,\alpha_i,k) = 1$ for each $k = 1,\dots,N^{l+1}$. Hence we have $\sum_{i=1}^{N} S_i^* S_i = \sum_{k=1}^{N^{l+1}} e_k^{l+1} = 1$. By using (3.3), (3.4) and the first equality of (3.2) recursively, we know that $E_{\mu_1\cdots\mu_l}$ commutes with both $S_j S_j^*$ and $T_j T_j^*$ for $j = 1,\dots,N$. We note that the relation (3.1) implies $\sum_{j=1}^{N} (S_j S_j^* + T_j T_j^*) = 1$ so that

$$E_{\mu_1\cdots\mu_l} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1\cdots\mu_l} S_j S_j^* + \sum_{j=1}^{N} T_j T_j^* E_{\mu_1\cdots\mu_l} T_j T_j^*.$$

As $S_{\mu_1} T_j = 0$ if $\mu_1 \neq j$, the second summation in the right hand side above goes to $T_{\mu_1} T_{\mu_1}^* E_{\mu_1\cdots\mu_l} T_{\mu_1} T_{\mu_1}^*$. Now by (3.4), one has $E_{\mu_1} = s_{\alpha_{\mu_1}}^* s_{\alpha_{\mu_1}} = \sum_{j=1}^{2} A_{0,1}(1,\alpha_{\mu_1},j) e_j^1$ and $A_{0,1}(1,\alpha_{\mu_1},j) = 1$ if and only if $v_j^1 = (\beta_{\mu_1})$ and hence $j = N(\mu_1)$. This means that $E_{\mu_1} = e_{N(\mu_1)}^1$. Similarly we have

$$\begin{aligned}
E_{\mu_1\cdots\mu_l} &= s_{\alpha_{\mu_1}}^* \cdots s_{\alpha_{\mu_l}}^* s_{\alpha_{\mu_l}} \cdots s_{\alpha_{\mu_1}} \\
&= \sum_{i_{l-1}=1}^{N^{l-1}} \sum_{i_l=1}^{N^l} A_{l-1,l}(i_{l-1},\alpha_{\mu_l},i_l) s_{\alpha_{\mu_1}}^* \cdots s_{\alpha_{\mu_{l-1}}}^* e_{i_l}^l s_{\alpha_{\mu_{l-1}}} \cdots s_{\alpha_{\mu_1}} \\
&= \sum_{i_1=1}^{N} \cdots \sum_{i_{l-1}=1}^{N^{l-1}} \sum_{i_l=1}^{N^l} A_{0,1}(1,\alpha_{\mu_1},i_1) \cdots A_{l-1,l}(i_{l-1},\alpha_{\mu_l},i_l) e_{i_l}^l
\end{aligned}$$

and $\sum_{i_1=1}^{N} \cdots \sum_{i_{l-1}=1}^{N^{l-1}} A_{0,1}(1,\alpha_{\mu_1},i_1) \cdots A_{l-1,l}(i_{l-1},\alpha_{\mu_l},i_l) = 1$ if and only if $v_{i_l}^l = (\beta_{\mu_1} \cdots \beta_{\mu_l})$ and hence $i_l = N(\mu_1 \cdots \mu_l)$. This means that $E_{\mu_1\cdots\mu_l} = e_{N(\mu_1\cdots\mu_l)}^l$. Hence we have $T_{\mu_1}^* E_{\mu_1\cdots\mu_l} T_{\mu_1} = \sum_{j=1}^{N^{l+1}} A_{l,l+1}(N(\mu_1 \cdots \mu_l),\beta_{\mu_1},j) e_j^{l+1}$. Since $A_{l,l+1}(N(\mu_1 \cdots \mu_l),\beta_{\mu_1},j) = 1$ if and only if $j = N(\mu_2 \cdots \mu_l \mu_{l+1} \mu_{l+2})$ for some $\mu_{l+1},\mu_{l+2} \in \{1,\dots,N\}$, and the equality $\sum_{\mu_{l+1},\mu_{l+2}=1,\dots,N} E_{\mu_2\cdots\mu_l\mu_{l+1}\mu_{l+2}} = E_{\mu_2\cdots\mu_l}$ holds, we have

$T^*_{\mu_1} E_{\mu_1\cdots\mu_l} T_{\mu_1} = E_{\mu_2\cdots\mu_l}$ so that

$$T_{\mu_1} T^*_{\mu_1} E_{\mu_1\cdots\mu_l} T_{\mu_1} T^*_{\mu_1} = T_{\mu_1} E_{\mu_2\cdots\mu_l} T^*_{\mu_1}.$$

Thus we conclude that the equality (1.2) holds. ∎

We next study the converse argument. Let $S_i, i = 1,\dots,N$ be partial isometries and $T_i, i = 1,\dots,N$ isometries satisfying the operator relations (1.1) and (1.2).

LEMMA 3.5.

$$\sum_{j=1}^{N} (S_j S_j^* + T_j T_j^*) = 1, \tag{3.5}$$

$$S_i^* S_i = \sum_{j=1}^{N} S_j S_j^* S_i^* S_i S_j S_j^* + T_i T_i^*, \quad i = 1,\dots,N. \tag{3.6}$$

Proof. For $l = 2$ at (1.2), we have

$$E_{\mu_1\mu_2} = \sum_{j=1}^{N} S_j S_j^* E_{\mu_1\mu_2} S_j S_j^* + T_{\mu_1} E_{\mu_2} T^*_{\mu_1}.$$

By summing up these relations over $\mu_2 = 1,\dots,N$ and using (1.1), the relation (3.6) follows. The relation (3.5) follows by summing up (3.6) over $i = 1,\dots,N$ and using (1.1). ∎

LEMMA 3.6. *For $i, j, \mu_1,\dots,\mu_l \in \{1,\dots,N\}$, we have:*

(i) $T_i^* S_j^* S_j T_i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$

(ii) $T_i^* E_{\mu_1\cdots\mu_l} T_i = \begin{cases} E_{\mu_2\cdots\mu_l} & \text{if } i = \mu_1, \\ 0 & \text{if } i \neq \mu_1. \end{cases}$

Proof. (i) By (3.6), we have

$$T_i^* S_i^* S_i T_i = \sum_{j=1}^{N} T_i^* S_j S_j^* S_i^* S_i S_j S_j^* T_i + T_i^* T_i T_i^* T_i.$$

The relation (3.5) implies $T_i^* S_j = 0$ for $i, j = 1,\dots,N$ so that the above equality goes to $T_i^* S_i^* S_i T_i = T_i^* T_i = 1$. By the relation (1.1), one has $\sum_{j=1}^{N} T_i^* S_j^* S_j T_i = T_i^* T_i = 1$ so that $T_i^* S_j^* S_j T_i = 0$ for $i \neq j$.

(ii) By (1.2), we have

$$T_i^* E_{\mu_1\cdots\mu_l} T_i = \sum_{j=1}^{N} T_i^* S_j S_j^* E_{\mu_1\cdots\mu_l} S_j S_j^* T_i + T_i^* T_{\mu_1} E_{\mu_2\cdots\mu_l} T^*_{\mu_1} T_i.$$

Since $T_i^* S_j = 0$ for i, j, and $T_i^* T_{\mu_1} = 0$ for $i \neq \mu_1$, the above equality goes to

$$T_i^* E_{\mu_1 \cdots \mu_l} T_i = T_i^* T_{\mu_1} E_{\mu_2 \cdots \mu_l} T_{\mu_1}^* T_i = \begin{cases} E_{\mu_2 \cdots \mu_l} & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases} \quad ∎$$

LEMMA 3.7. *Keep the above notations. The projection $E_{\mu_1 \cdots \mu_l}$ commutes with both $S_j S_j^*$ and $T_j T_j^*$ for $j = 1, \ldots, N$.*

Proof. By (1.2), we have

$$S_i S_i^* E_{\mu_1 \cdots \mu_l} = \sum_{j=1}^N S_i S_i^* S_j S_j^* E_{\mu_1 \cdots \mu_l} S_j S_j^* + S_i S_i^* T_{\mu_1} E_{\mu_2 \cdots \mu_l} T_{\mu_1}^*.$$

By (3.5), $S_i^* S_j = 0$ for $i \neq j$, and $S_i^* T_{\mu_1} = 0$ for all i, μ_1. Hence the above equality goes to $S_i S_i^* E_{\mu_1 \cdots \mu_l} = S_i S_i^* E_{\mu_1 \cdots \mu_l} S_i S_i^*$ so that $S_i S_i^*$ commutes with $E_{\mu_1 \cdots \mu_l}$. By (1.2), we have $T_i T_i^* E_{\mu_1 \cdots \mu_l} = \sum_{j=1}^N T_i T_i^* S_j S_j^* E_{\mu_1 \cdots \mu_l} S_j S_j^* + T_i T_i^* T_{\mu_1} E_{\mu_2 \cdots \mu_l} T_{\mu_1}^*$. By (3.5), $T_i^* T_{\mu_1} = 0$ for $i \neq \mu_1$, and $T_i^* S_j = 0$ for all i, j. Hence we have

$$T_i T_i^* E_{\mu_1 \cdots \mu_l} = \begin{cases} T_{\mu_1} E_{\mu_2 \cdots \mu_l} T_{\mu_1}^* & \text{if } i = \mu_1, \\ 0 & \text{otherwise.} \end{cases}$$

so that $T_i T_i^*$ commutes with $E_{\mu_1 \cdots \mu_l}$. ∎

PROPOSITION 3.8. *Keep the above notations. Define*

$$s_{\alpha_i} := S_i, \quad s_{\beta_i} := T_i \quad \textit{for } i = 1, \ldots, N$$

and

$$e^l_{N(\mu_1 \cdots \mu_l)} := E_{\mu_1 \cdots \mu_l} (= S_{\mu_1}^* \cdots S_{\mu_l}^* S_{\mu_l} \cdots S_{\mu_1}) \quad \textit{for } \mu_1 \cdots \mu_l \in B_l(\Sigma_N).$$

Then the family of operators $s_\gamma, \gamma \in \Sigma$, $e^l_{N(\mu_1 \cdots \mu_l)}, \mu_1 \cdots \mu_l \in B_l(\Sigma_N)$ satisfies the relations (3.1), (3.2), (3.3) *and* (3.4).

Proof. The relation (3.1) is nothing but (3.5). The relation (1.1) implies that $\sum_{\mu_1 \in B_1(\Sigma_N)} e^1_{N(\mu_1)} = 1$. Suppose that $\sum_{\mu_1 \cdots \mu_l \in B_l(\Sigma_N)} e^l_{N(\mu_1 \cdots \mu_l)} = 1$ holds for $l = k$. As

$$S_{\mu_1}^* \cdots S_{\mu_k}^* S_{\mu_k} \cdots S_{\mu_1} = \sum_{j=1}^N S_{\mu_1}^* \cdots S_{\mu_k}^* S_j^* S_j S_{\mu_k} \cdots S_{\mu_1},$$

the equality $\sum_{\mu_1 \cdots \mu_l \in B_l(\Sigma_N)} e^l_{N(\mu_1 \cdots \mu_l)} = 1$ holds for $l = k+1$ and hence for all l by induction. The above equality implies

$$I_{l,l+1}(N(\mu_1 \cdots \mu_l), N(\nu_1 \cdots \nu_{l+1})) = \begin{cases} 1 & \text{if } \nu_1 \cdots \nu_l = \mu_1 \cdots \mu_l, \\ 0 & \text{otherwise,} \end{cases}$$

so that the relation (3.2) holds. The equality (3.3) comes from the preceding lemma.

We will finally show the equality (3.4). For $\mu_1 \cdots \mu_l \in B_l(\Sigma_N)$ and $\alpha_k \in \Sigma^-$ we have $s^*_{\alpha_k} e^l_{N(\mu_1\cdots\mu_l)} s_{\alpha_k} = S^*_k S^*_{\mu_1} \cdots S^*_{\mu_l} S_{\mu_l} \cdots S_{\mu_1} S_k = A_{l,l+1}(N(\mu_1 \cdots \mu_l), \alpha_k, N(k\mu_1 \cdots \mu_l)) e^{l+1}_{N(k\mu_1\cdots\mu_l)}$. Since $A_{l,l+1}(N(\mu_1 \cdots \mu_l), \alpha_k, j) = 0$ if $j \neq N(k\mu_1 \cdots \mu_l)$, one has

$$s^*_{\alpha_k} e^l_{N(\mu_1\cdots\mu_l)} s_{\alpha_k} = \sum_{\nu_1\cdots\nu_{l+1} \in B_{l+1}(\Sigma_N)} A_{l,l+1}(N(\mu_1 \cdots \mu_l), \alpha_k, N(\nu_1 \cdots \nu_{l+1})) e^{l+1}_{N(\nu_1\cdots\nu_{l+1})}.$$

We also have by Lemma 3.6 (ii)

$$\begin{aligned} & s^*_{\beta_j} e^l_{N(\mu_1\cdots\mu_l)} s_{\beta_j} \\ &= T^*_j E_{\mu_1\cdots\mu_l} T_j = \begin{cases} E_{\mu_2\cdots\mu_l} & \text{if } j = \mu_1, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \displaystyle\sum_{\mu_{l+1},\mu_{l+2} \in B_2(\Sigma_N)} S^*_{\mu_2} \cdots S^*_{\mu_l} S^*_{\mu_{l+1}} S^*_{\mu_{l+2}} S_{\mu_{l+2}} S_{\mu_{l+1}} S_{\mu_l} \cdots S_{\mu_2} & \text{if } j = \mu_1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since $A_{l,l+1}(N(\mu_1 \cdots \mu_l), \beta_j, N(\nu_1 \cdots \nu_{l+1})) = 1$ precisely if $j = \mu_1$ and $\nu_i = \mu_{i+1}$ for $i = 1, \ldots, l-1$, and 0 otherwise, we have

$$s^*_{\beta_j} e^l_{N(\mu_1\cdots\mu_l)} s_{\beta_j} = \sum_{\nu_1\cdots\nu_{l+1} \in B_{l+1}(\Sigma_N)} A_{l,l+1}(N(\mu_1 \cdots \mu_l), \beta_j, N(\nu_1 \cdots \nu_{l+1})) e^{l+1}_{N(\nu_1\cdots\nu_{l+1})}.$$

Therefore (3.4) holds. ∎

By Proposition 3.4 and Proposition 3.8, the family of the operator relations (1.1) and (1.2) is equivalent to the family of the operator relations (3.1), (3.2), (3.3) and (3.4). Thus by Proposition 3.1 and Lemma 3.2 we conclude Theorem 1.1.

4. KMS STATES ON $O_{\mathfrak{L}^{Ch(D_N)}}$

This section is devoted to proving Theorem 1.2. The result in this section for the uniqueness of KMS state on $O_{\mathfrak{L}^{Ch(D_N)}}$ is directly deduced from the result in the next section for the uniqueness of tracial state on the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$. We will in this section give a direct and easy proof of the uniqueness of KMS state on $O_{\mathfrak{L}^{Ch(D_N)}}$.

Suppose that partial isometries $S_i, i=1,\ldots,N$ and isometries $T_i, i=1,\ldots,N$ satisfy the relations (1.1) and (1.2). They generate the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$. By the uniqueness of the algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ subject to the relations (1.1) and (1.2), the correspondence

$$S_i \to zS_i, \quad T_i \to zT_i \quad \text{for } i = 1, \ldots, N$$

for $z \in \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ yields an action of $\mathbb{T}$ on $O_{\mathfrak{L}^{Ch(D_N)}}$. It is called the *gauge action* and is denoted by α^{D_N}. A state φ on $O_{\mathfrak{L}^{Ch(D_N)}}$ is called a *KMS state at* *inverse temperature* $\log \beta$ for $0 < \beta \in \mathbb{R}$ if

$$\varphi(a\alpha^{D_N}_{i\log\beta}(b)) = \varphi(ba)$$

for all a, b in a dense α^{D_N}-invariant $*$-subalgebra of the analytic elements of the action α^{D_N} on $O_{\mathfrak{L}^{Ch(D_N)}}$. The gauge action α^{D_N} on the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ is a full C^*-dynamical system considered in [26] and a quasi-free dynamics of Pimsner algebra considered in [13] (cf. [25], and Proposition 6.1 of [16]). General theory of KMS states for the C^*-algebras constructed from Hilbert C^*-bimodules in [26] and [13] says that a state φ on $O_{\mathfrak{L}^{Ch(D_N)}}$ is a KMS state at inverse temperature $\log \beta$ if and only if φ satisfies the following condition ([26], Lemma 1.2, [13], Theorem 2.5; cf. Theorem 3.6 of [22]):

$$\sum_{j=1}^{N} \varphi(S_j^* x S_j + T_j^* x T_j) = \beta\varphi(x) \quad \text{for } x \in A_{\mathfrak{L}^{Ch(D_N)}} \tag{4.1}$$

where $A_{\mathfrak{L}^{Ch(D_N)}}$ is the commutative C^*-subalgebra generated by the projections $E_{\mu_1\cdots\mu_l} = S^*_{\mu_1}\cdots S^*_{\mu_l} S_{\mu_l}\cdots S_{\mu_1}$, $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$. By Lemma 1.2 of [26], by [13] or a similar result of Proposition 3.4 in [22], the condition (4.1) is equivalent to the condition:

$$\begin{cases} \varphi(S_{\mu_k}\cdots S_{\mu_1} x S^*_{\nu_1}\cdots S^*_{\nu_k}) & = \delta_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k} \frac{1}{\beta^k}\varphi(x S^*_{\nu_1}\cdots S^*_{\nu_k} S_{\mu_k}\cdots S_{\mu_1}), \\ \varphi(T_{\mu_k}\cdots T_{\mu_1} x T^*_{\nu_1}\cdots T^*_{\nu_k}) & = \delta_{\mu_1\cdots\mu_k,\nu_1\cdots\nu_k} \frac{1}{\beta^k}\varphi(x T^*_{\nu_1}\cdots T^*_{\nu_k} T_{\mu_k}\cdots T_{\mu_1}), \end{cases} \tag{4.2}$$

for $x \in A_{\mathfrak{L}^{Ch(D_N)}}$ and $\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k \in \{1, \dots, N\}$.

Put $T_0 = \sum_{j=1}^{N} S_j$. We first note the following proposition.

PROPOSITION 4.1. *The C^*-subalgebra $C^*(T_0, T_1, \dots, T_N)$ of $O_{\mathfrak{L}^{Ch(D_N)}}$ generated by $T_0, T_1, \dots, T_N$ is canonically isomorphic to the Cuntz algebra O_{N+1}. Hence there exists a unital embedding $\iota : O_{N+1} \to O_{\mathfrak{L}^{Ch(D_N)}}$ satisfying $\alpha_z^{D_N} \circ \iota = \iota \circ \alpha_z^{\Sigma_{N+1}}$ for $z \in \mathbb{T}$ where $\alpha^{\Sigma_{N+1}}$ is the gauge action on O_{N+1}.*

Proof. The equality (1.1) implies that T_0 is an isometry. Hence the equality (3.5) implies the relation $\sum_{i=0}^{N} T_i T_i^* = 1$. ∎

We henceforth assume that φ is a KMS state on $O_{\mathfrak{L}^{Ch(D_N)}}$ at inverse temperature $\log \beta$ for some $0 < \beta \in \mathbb{R}$.

LEMMA 4.2. (i) $\beta = N + 1$, *and*

(ii) $\sum_{j=1}^{N} \varphi(S_j S_j^*) = \varphi(T_i T_i^*) = \frac{1}{N+1}$ *for* $i = 0, 1, \dots, N$.

Proof. Since there exists a unital embedding $\iota : O_{N+1} \hookrightarrow O_{\mathfrak{L}^{Ch(D_N)}}$ compatible to their gauge actions, the restriction of φ to the subalgebra O_{N+1} yields a KMS state on O_{N+1} at inverse temperature $\log \beta$ for the gauge action. Hence by [24] β

must be $N+1$ and the admitted KMS state is unique such that $\varphi(T_iT_i^*) = \frac{1}{N+1}$ for $i = 0, 1, \dots, N$. As $T_0T_0^* = \sum_{j=1}^{N} S_jS_j^*$, we get the desired equalities. ∎

LEMMA 4.3. (i) $\varphi(E_{\mu_1}) = \frac{1}{N+1} + \frac{1}{N+1} \sum_{\mu_0=1}^{N} \varphi(E_{\mu_0\mu_1})$.

(ii) $\varphi(E_{\mu_1\cdots\mu_l}) = \frac{1}{N+1}\varphi(E_{\mu_2\cdots\mu_l}) + \frac{1}{N+1} \sum_{\mu_0=1}^{N} \varphi(E_{\mu_0\mu_1\cdots\mu_l})$ *for* $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$.

Proof. (i) By (4.2) one sees

$$\varphi(S_{\mu_0}E_{\mu_0\mu_1}S_{\mu_0}^*) = \frac{1}{N+1}\varphi(E_{\mu_0\mu_1}S_{\mu_0}^*S_{\mu_0}) = \frac{1}{N+1}\varphi(E_{\mu_0\mu_1}).$$

The relation (3.6) together with the previous lemma implies

$$\varphi(E_{\mu_1}) = \frac{1}{N+1}\sum_{\mu_0=1}^{N}\varphi(E_{\mu_0\mu_1}) + \frac{1}{N+1}.$$

(ii) One similarly sees that by (4.2)

$$\begin{aligned}
\varphi(S_{\mu_0}E_{\mu_0\mu_1\cdots\mu_l}S_{\mu_0}^*) &= \frac{1}{N+1}\varphi(E_{\mu_0\mu_1\cdots\mu_l}S_{\mu_0}^*S_{\mu_0}) = \frac{1}{N+1}\varphi(E_{\mu_0\mu_1\cdots\mu_l}),\\
\varphi(T_{\mu_1}E_{\mu_2\cdots\mu_l}T_{\mu_1}^*) &= \varphi(E_{\mu_2\cdots\mu_l}T_{\mu_1}^*T_{\mu_1}) = \frac{1}{N+1}\varphi(E_{\mu_2\cdots\mu_l}),
\end{aligned}$$

so that the desired equalities hold by (1.2). ∎

Keep the above notations. We set $u^l_{\mu_1\cdots\mu_l} = \varphi(E_{\mu_1\cdots\mu_l})$ for $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$ and $l \in \mathbb{N}$ so that

$$\begin{aligned}
&\sum_{\mu_1,\dots,\mu_l=1}^{N} u^l_{\mu_1\cdots\mu_l} = 1, \quad 0 \leqslant u^l_{\mu_1\cdots\mu_l} \leqslant 1,\\
&u^1_{\mu_1} = \frac{1}{N+1} + \frac{1}{N+1}\sum_{\mu_0=1}^{N} u^2_{\mu_0\mu_1},\\
&u^l_{\mu_1\cdots\mu_l} = \frac{1}{N+1}u^{l-1}_{\mu_2\cdots\mu_l} + \frac{1}{N+1}\sum_{\mu_0=1}^{N} u^{l+1}_{\mu_0\mu_1\cdots\mu_l}.
\end{aligned}$$

We put for $i = 1, \dots, N$,

$$V_i^1 = u_i^1, \quad V_i^2 = \sum_{\mu_0=1}^{N} u^2_{\mu_0 i}, \quad \dots, \quad V_i^l = \sum_{\mu_0,\dots,\mu_{l-2}=1}^{N} u^l_{\mu_0\mu_1\cdots\mu_{l-2}i}, \quad \dots$$

They satisfy the relations:

(i) $V_i^1 = \frac{1}{N+1} + \frac{1}{N+1}V_i^2$ for $i = 1, \dots, N$.

(ii) $V_i^l = \frac{N}{N+1}V_i^{l-1} + \frac{1}{N+1}V_i^{l+1}$ for $i = 1, \dots, N$.

If a sequence $\{x_l\}_{l=1}^{\infty}$ of nonzero real numbers satisfies the conditions:

(1) $x_1 = \frac{1}{N+1} + \frac{1}{N+1}x_2$,

(2) $x_l = \frac{N}{N+1}x_{l-1} + \frac{1}{N+1}x_{l+1}$ for $l = 2, 3, \dots$,

then $x_l = (N^{l-1} + N^{l-2} + \cdots + N^2 + N + 1)x_1 - (N^{l-2} + N^{l-3} + \cdots + N^2 + N + 1)$. As $x_l \geqslant 0$ for all $l \in \mathbb{N}$, we have $x_1 \geqslant \frac{1}{N}$. Here we note the next elementary fact:

LEMMA 4.4. *If sequences $\{x_{l,i}\}_{l=1}^{\infty}, i = 1, \dots, N$ of nonzero real numbers satisfy the following conditions:*

(i) $x_{1,i} = \frac{1}{N+1} + \frac{1}{N+1}x_{2,i}$;

(ii) $x_{l,i} = \frac{N}{N+1}x_{l-1,i} + \frac{1}{N+1}x_{l+1,i}$;

(iii) $\sum_{i=1}^{N} x_{l,i} = 1$ *for all* $i = 1, \dots, N$;

then we have $x_{l,i} = \frac{1}{N}$ *for all* $l = 1, 2, \dots,$ *and* $i = 1, \dots, N$.

Lemma 4.4 directly implies that $V_i^l = \frac{1}{N}$ for all $l \in \mathbb{N}$ and $i = 1, \dots, N$. We put for $k \in \mathbb{N}$ and $i, \nu_1, \dots, \nu_k \in \{1, \dots, N\}$,

$$
\begin{aligned}
V^1_{i,\nu_1\cdots\nu_k}(k) &= N^k u^{k+1}_{i\nu_1\cdots\nu_k},\\
V^2_{i,\nu_1\cdots\nu_k}(k) &= N^k \sum_{\mu_1=1}^{N} u^{k+2}_{\mu_1 i\nu_1\cdots\nu_k},\\
&\cdots\\
V^l_{i,\nu_1\cdots\nu_k}(k) &= N^k \sum_{\mu_1,\dots,\mu_{l-1}=1}^{N} u^{k+l}_{\mu_1\cdots\mu_{l-1} i\nu_1\cdots\nu_k},\\
&\cdots
\end{aligned}
$$

We put $V_i^l(0) = V_i^l$ for $k = 0$. By induction on k, the following relations hold:

LEMMA 4.5. (i) $V^1_{i,\nu_1\cdots\nu_k}(k) = \frac{1}{N+1} + \frac{1}{N+1}V^2_{i,\nu_1\cdots\nu_k}(k)$ *for* $i = 1, \dots, N$.

(ii) $V^l_{i,\nu_1\cdots\nu_k}(k) = \frac{N}{N+1}V^{l-1}_{i,\nu_1\cdots\nu_k}(k) + \frac{1}{N+1}V^{l+1}_{i,\nu_1\cdots\nu_k}(k)$ *for* $i = 1, \dots, N$.

(iii) $\sum_{i=1}^{N} V^l_{i,\nu_1\cdots\nu_k}(k) = 1$.

Therefore we have

COROLLARY 4.6. *For* $k = 0, 1, \dots,$ *and* $l = 1, 2, \dots$ *and* $i, \nu_1, \dots, \nu_k \in \{1, \dots, N\}$, *we have* $V^l_{i,\nu_1\cdots\nu_k}(k) = \frac{1}{N}$. *In particular, we have* $u^l_{\mu_1\cdots\mu_l} = \frac{1}{N^l}$.

Proof. By the previous lemma, one directly has $V^l_{i,\nu_1\cdots\nu_k}(k) = \frac{1}{N}$. For $l = 1$, it follows that $\frac{1}{N} = V^1_{i,\nu_1\cdots\nu_k} = N^k u^{k+1}_{i\nu_1\cdots\nu_k}$ for all $i, \nu_1, \dots, \nu_k \in \{1, \dots, N\}$. ∎

Therefore a KMS state φ satisfies the equality

$$
\varphi(E_{\mu_1\cdots\mu_l}) = \frac{1}{N^l} \quad \text{for } \mu_1, \dots, \mu_l \in \{1, \dots, N\}.
$$

Conversely we see

PROPOSITION 4.7. *If a state φ on the C^*-algebra $A_{\mathfrak{L}^{Ch(D_N)}}$ generated by the projections $E_{\mu_1\cdots\mu_l}, \mu_1,\ldots,\mu_l \in \{1,\ldots,N\}$ satisfies the condition*

$$\varphi(E_{\mu_1\cdots\mu_l}) = \frac{1}{N^l} \quad \text{for } \mu_1,\ldots,\mu_l \in \{1,\ldots,N\},$$

then φ can be uniquely extended on $O_{\mathfrak{L}^{Ch(D_N)}}$ to a KMS *state at inverse temperature* $\log(N+1)$.

Proof. By the relation (1.2), we have

$$\sum_{j=1}^{N} T_j^* E_{\mu_1\cdots\mu_l} T_j = T_{\mu_1}^* T_{\mu_1} E_{\mu_2\cdots\mu_l} T_{\mu_1} T_{\mu_1}^* = E_{\mu_2\cdots\mu_l}$$

so that

$$\begin{aligned}\varphi\Big(\sum_{j=1}^{N}(S_j^* E_{\mu_1\cdots\mu_l} S_j + T_j^* E_{\mu_1\cdots\mu_l} T_j)\Big) &= \sum_{j=1}^{N} \varphi(E_{j\mu_1\cdots\mu_l}) + \varphi(E_{\mu_2\cdots\mu_l}) \\ &= N \times \frac{1}{N^{k+1}} + \frac{1}{N^{k-1}} = (N+1)\varphi(E_{\mu_1\cdots\mu_l}).\end{aligned}$$

Hence φ satisfies (4.1) and it can be extended to a KMS state on $O_{\mathfrak{L}^{Ch(D_N)}}$. The uniqueness of the extension to a KMS state on $O_{\mathfrak{L}^{Ch(D_N)}}$ comes from a general theory of [26], [13] or a similar result to [22]. ∎

Therefore we conclude Theorem 1.2. We finally remark that the value $\log(N+1)$ is the topological entropy of the subshift D_N. Corresponding result for Cuntz-Krieger algebras has been shown in [4].

5. TRACIAL STATE ON THE CANONICAL AF ALGEBRA

We denote by $F_{\mathfrak{L}^{Ch(D_N)}}$ the fixed point algebra ${O_{\mathfrak{L}^{Ch(D_N)}}}^{\alpha^{D_N}}$ of $O_{\mathfrak{L}^{Ch(D_N)}}$ under the gauge action α^{D_N}. By [16], one knows that the algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ is an AF algebra. The restriction of a KMS state to the subalgebra $F_{\mathfrak{L}^{Ch(D_N)}}$ yields a tracial state on it. In this section we will prove that tracial state on $F_{\mathfrak{L}^{Ch(D_N)}}$ is unique. Its proof needs some combinatorial properties of the generators $S_i, T_i, i = 1,\ldots,N$. By using the uniqueness of tracial state on $F_{\mathfrak{L}^{Ch(D_N)}}$, one can determine the type of the von Neumann algebra $M = \pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$ generated by the GNS-representation π_φ of $O_{\mathfrak{L}^{Ch(D_N)}}$ by the KMS state φ. As a consequence we will show that the algebra M is a factor of type $\mathrm{III}_{1/(N+1)}$.

PROPOSITION 5.1. *The AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ is simple.*

Proof. We note that the Bratteli diagram of the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ is given by the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ [16]. By Proposition 3.1, the λ-graph system

$\mathfrak{L}^{Ch(D_N)}$ is λ-irreducible so that there is no proper hereditary subset. Hence the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ is simple. ∎

Let us consider the inverse monoid $\mathbb{M}_{1,N}$ generated by $N+1$ elements $\alpha_0, \beta_1, \dots, \beta_N$ with relations: $\alpha_0\beta_i = \mathbf{1}$ for $i = 1, \dots, N$. The set of the elements $\alpha_0, \beta_1, \dots, \beta_N$ is denoted by $\Sigma_{1,N}$. Then a word $\gamma_1 \cdots \gamma_n$ of $\Sigma_{1,N}$ is called *acceptable* if $\gamma_1 \cdots \gamma_n = \mathbf{1}$. It is clear that if a word $\gamma_1 \cdots \gamma_n$ is acceptable, then $n = 2k$ for some $k \in \mathbb{N}$ and $\gamma_1 = \alpha_0, \gamma_n = \beta_i$ for some $i = 1, \dots, N$. Let $L_{1,N}(2k)$ be the set of all acceptable words of length $2k$. Recall that $\mathbb{D}_1$ is the Dyck inverse monoid generated by 2 elements α, β that satisfy $\alpha\beta = \mathbf{1}$. The acceptable words of $\mathbb{D}_1$ are similarly defined to those of $\mathbb{M}_{1,N}$. It is well-known that the cardinal number of the set of all acceptable words of $\mathbb{D}_1$ of length $2k$ is the k-th Catalan number $C_k = \frac{1}{k+1}\binom{2k}{k}(= \frac{2k!}{k!(k+1)!})$, where C_0 is defined to be 1. Therefore we have

$$|L_{1,N}(2k)| = N^k C_k. \tag{5.1}$$

Let $S_i, T_i, i = 1, \dots, N$ be the generators of $O_{\mathfrak{L}^{Ch(D_N)}}$ as in Theorem 1.1. We put for $\gamma \in \Sigma_{1,N}$

$$\widetilde{T}_\gamma = \begin{cases} T_0 & \text{if } \gamma = \alpha_0, \\ T_j & \text{if } \gamma = \beta_j, \end{cases}$$

where $T_0 = \sum_{j=1}^{N} S_j$.

LEMMA 5.2. *For $i = 1, \dots, N$ we have:*

(i) $S_i^* S_i \geqslant T_i T_i^*$.

(ii) *For an acceptable word $\gamma_1 \cdots \gamma_{2k} \in L_{1,N}(2k)$, one has*

$$S_i^* S_i \geqslant \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i T_i^* \widetilde{T}_{\gamma_{2k}}^* \cdots \widetilde{T}_{\gamma_1}^*. \tag{5.2}$$

Proof. (i) The inequality (i) is clear from (3.6).

(ii) As $\gamma_1 \cdots \gamma_{2k} \in L_{1,N}(2k)$ is acceptable, k symbols of $\{\gamma_1, \dots, \gamma_{2k}\}$ are α_0 and the other k symbols are of $\Sigma^+ = \{\beta_1, \dots, \beta_N\}$. Let $\beta_{i_1}, \dots, \beta_{i_k}$ be the symbols in Σ^+ that appear in $\gamma_1 \cdots \gamma_{2k}$. Since $S_i T_j = 0$ for $i \neq j$ by Lemma 3.6, one has $T_0 T_j = S_j T_j$. The acceptable word $\gamma_1 \cdots \gamma_{2k}$ uniquely determines an admissible word $\mu(\gamma) = (\mu(\gamma)_1, \dots, \mu(\gamma)_{2k}) \in B_{2k}(D_N)$ of the Dyck shift D_N such that the corresponding element of the Dyck inverse monoid $\mathbb{D}_N$ is the unit $\mathbf{1}$. In the λ-graph system $\mathfrak{L}^{Ch(D_N)}$, every vertex v_j^{2k+2} in V_{2k+2} has a unique in-coming path labeled $\mu(\gamma)\beta_i$. All the paths labeled $\mu(\gamma)\beta_i$ start at the unique vertex in V_1 corresponding to the word β_i. An edge in $E_{0,1}$ labeled α_i is unique and it terminates at the vertex corresponding to the word β_i. This means that the following inequality holds

$$s_{\alpha_i}^* s_{\alpha_i} \geqslant s_{\mu(\gamma)\beta_i} s_{\mu(\gamma)\beta_i}^*$$

in the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$. As $s_{\alpha_i} = S_i$ and $s_{\mu(\gamma)\beta_i} = \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i$ one concludes the desired inequality. ∎

The inequality (i) in Lemma 5.2 is interpreted to be the inequality (5.3) for $k = 0$.

Recall that by Proposition 4.1, the C^*-algebra $C^*(\widetilde{T}_{\alpha_0}, \widetilde{T}_{\beta_1}, \dots, \widetilde{T}_{\beta_N})$ generated by $\widetilde{T}_{\alpha_0} = T_0, \widetilde{T}_{\beta_i} = T_i, i = 1, \dots, N$ is canonically isomorphic to the Cuntz algebra O_{N+1}.

LEMMA 5.3. *For $i = 1, \dots, N$ and distinct acceptable words $\gamma_1 \cdots \gamma_{2k} \in L_{1,N}(2k)$ and $\delta_1 \cdots \delta_{2n} \in L_{1,N}(2n)$, the projections*

$$\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i T_i^* \widetilde{T}_{\gamma_{2k}}^* \cdots \widetilde{T}_{\gamma_1}^* \quad \text{and} \quad \widetilde{T}_{\delta_1} \cdots \widetilde{T}_{\delta_{2n}} T_i T_i^* \widetilde{T}_{\delta_{2n}}^* \cdots \widetilde{T}_{\delta_1}^*$$

are orthogonal.

Proof. If $n = k$, the assertion is immediate. Suppose $k > n$. If the projection $\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i T_i^* \widetilde{T}_{\gamma_{2k}}^* \cdots \widetilde{T}_{\gamma_1}^*$ is not orthogonal to $\widetilde{T}_{\delta_1} \cdots \widetilde{T}_{\delta_{2n}} T_i T_i^* \widetilde{T}_{\delta_{2n}}^* \cdots \widetilde{T}_{\delta_1}^*$, one sees $\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2n+1}} = \widetilde{T}_{\delta_1} \cdots \widetilde{T}_{\delta_{2n}} T_i$ so that $\gamma_j = \delta_j$ for $j = 1, \dots, 2n$ and $\gamma_{2n+1} = \beta_i$. Since both $\gamma_1 \cdots \gamma_{2k}$ and $\delta_1 \cdots \delta_{2n}$ are acceptable, one has that $\gamma_{2n+1} \cdots \gamma_{2k}$ is acceptable. Hence γ_{2n+1} must be α_0, that is a contradiction. ∎

In what follows, let ϕ be a tracial state on the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$. We will prove that ϕ coincides with the restriction of the KMS state φ to the subalgebra $F_{\mathfrak{L}^{Ch(D_N)}}$.

LEMMA 5.4. *For an acceptable word $\gamma_1 \cdots \gamma_{2k} \in L_{1,N}(2k)$, one has*

$$\phi(\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i T_i^* \widetilde{T}_{\gamma_{2k}}^* \cdots \widetilde{T}_{\gamma_1}^*) = \frac{1}{(N+1)^{2k+1}}.$$

Proof. Since the C^*-algebra $C^*(\widetilde{T}_{\alpha_0}, \widetilde{T}_{\beta_1}, \dots, \widetilde{T}_{\beta_N})$ is canonically isomorphic to the Cuntz algebra O_{N+1}, the projection $\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k}} T_i T_i^* \widetilde{T}_{\gamma_{2k}}^* \cdots \widetilde{T}_{\gamma_1}^*$ is a minimal projection in the matrix algebra $M_{(N+1)^{2k+1}}(\mathbb{C})$ of size $(N+1)^{2k+1}$. As the state ϕ gives rise to a unique tracial state on the matrix algebra, the desired equality holds. ∎

Now we refer a combinatorial property of the Catalan numbers $C_k, k \in \mathbb{Z}_+$. It is well-known that the equality

$$\sum_{k=0}^{\infty} C_k x^k = \frac{1}{2x}(1 - \sqrt{1-4x}).$$

holds. Hence we have

LEMMA 5.5. $\sum_{k=0}^{\infty} \frac{N^k}{(N+1)^{2k+1}} C_k = \frac{1}{N}$.

We then have

PROPOSITION 5.6. $\phi(S_i^* S_i) = \frac{1}{N}$.

Proof. We first note the equality $\phi(T_iT_i^*) = \frac{1}{N+1}$. By Lemma 5.2, Lemma 5.3, Lemma 5.4 and Lemma 5.5 with (5.1), we have

$$\phi(S_i^*S_i) \geqslant \sum_{k=0}^{\infty} \frac{|L_{1,N}(2k)|}{(N+1)^{2k+1}} = \frac{1}{N} \quad \text{for } i = 1, \dots, N.$$

The relation (1.1) implies $\phi(S_i^*S_i) = \frac{1}{N}$. ∎

We will next show that $\phi(S_{\mu_l\cdots\mu_1}^* S_{\mu_l\cdots\mu_1}) = \frac{1}{N^l}$ for all $l \in \mathbb{N}$.

In the Dyck inverse monoid $\mathbb{D}_1$ generated by 2 elements α, β satisfying $\alpha\beta = \mathbf{1}$, we set for $k \geqslant l-1$

$$C_k^{(l-1)} = |\{(\delta_1, \dots, \delta_{2k}) : \delta_i \in \{\alpha, \beta\}, \delta_1 \cdots \delta_{2k} = \mathbf{1}, \delta_1 = \cdots = \delta_{l-1} = \alpha\}|,$$
$$C_k^{(0)} = C_k.$$

Hence we have $C_k^{(1)} = C_k$.

LEMMA 5.7. $C_k^{(l-1)} = C_k^{(l-2)} - C_{k-1}^{(l-3)}$.

Proof. We have

$$C_k^{(l-1)} = C_k^{(l-2)} - |\{(\delta_1, \dots, \delta_{2k}) \in C_k^{(l-2)} : \delta_{l-1} = \beta\}|.$$

By deleting $\delta_{l-2}\delta_{l-1}$ in the set $\{(\delta_1, \dots, \delta_{2k}) \in C_k^{(l-2)} : \delta_{l-1} = \beta\}$, we see

$$|\{(\delta_1, \dots, \delta_{2k}) \in C_k^{(l-2)} : \delta_{l-1} = \beta\}| = C_{k-1}^{(l-3)}$$

so that the desired equality holds. ∎

Let $\mathbb{M}_{N+1,N}$ be the inverse monoid generated by $\alpha_0, \alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N$ with relations

$$\alpha_0\beta_i = \alpha_i\beta_i = \mathbf{1} \quad \text{for } i = 1, \dots, N \quad \text{and} \quad \alpha_i\beta_j = 0 \quad \text{for } i \neq j.$$

In the inverse monoid $\mathbb{M}_{N+1,N}$, we put for $k \geqslant l-1$ and $\mu_1, \dots, \mu_{l-1} \in \{1, \dots, N\}$, $L_N(2k; \mu_{l-1} \cdots \mu_1) = \{(\gamma_1, \dots, \gamma_{2k-(l-1)}) \in \Sigma_{1,N}^{2k-(l-1)} : \alpha_{\mu_{l-1}} \cdots \alpha_{\mu_1}\gamma_1 \cdots \gamma_{2k-(l-1)} = \mathbf{1}\}$. It is easy to see that

$$|L_N(2k; \mu_{l-1} \cdots \mu_1)| = C_k^{(l-1)} N^{k-(l-1)}. \tag{5.3}$$

Similarly to the previous discussions we have

LEMMA 5.8. *For $k \geqslant l-1$, $\mu_1, \mu_2, \dots, \mu_l \in \{1, \dots, N\}$ and $(\gamma_1, \dots, \gamma_{2k-(l-1)}) \in L_N(2k; \mu_{l-1} \cdots \mu_1)$ we have:*

(i) $S_{\mu_1}^* \cdots S_{\mu_l}^* S_{\mu_l} \cdots S_{\mu_1} \geqslant \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} T_{\mu_l}^* \widetilde{T}_{\gamma_{2k-(l-1)}}^* \cdots \widetilde{T}_{\gamma_1}^*$.

(ii) *Let $(\gamma'_1, \dots, \gamma'_{2k'-(l-1)}) \in L_N(2k'; \mu_{l-1} \cdots \mu_1)$ be an acceptable word such that $(\gamma_1, \dots, \gamma_{2k-(l-1)}) \neq (\gamma'_1, \dots, \gamma'_{2k'-(l-1)})$. Then the projections*

$$\widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} T_{\mu_l}^* \widetilde{T}_{\gamma_{2k-(l-1)}}^* \cdots \widetilde{T}_{\gamma_1}^* \text{ and } \widetilde{T}_{\gamma'_1} \cdots \widetilde{T}_{\gamma'_{2k'-(l-1)}} T_{\mu_l} T_{\mu_l}^* \widetilde{T}_{\gamma'_{2k'-(l-1)}}^* \cdots \widetilde{T}_{\gamma'_1}^*$$

are orthogonal.

(iii) $\phi(\widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l}T^*_{\mu_l}\widetilde{T}^*_{\gamma_{2k-(l-1)}}\cdots\widetilde{T}^*_{\gamma_1}) = \frac{1}{(N+1)^{2k-(l-1)+1}}$.

LEMMA 5.9. $\sum_{k=l-1}^{\infty}\frac{1}{(N+1)^{2k-l+2}}C_k^{(l-1)}N^{k-(l-1)} = \frac{1}{N^l}$.

Proof. As $C_k^{(0)} = C_k^{(1)} = C_k$, the desired equalities hold for $l = 1, 2$. Suppose that the desired equalities hold for all l less than m. By Lemma 5.7 one has

$$\begin{aligned}
&\sum_{k=m-1}^{\infty}\frac{1}{(N+1)^{2k-m+2}}C_k^{(m-1)}N^{k-(m-1)}\\
&=\sum_{k=m-1}^{\infty}\frac{1}{(N+1)^{2k-m+2}}C_k^{(m-2)}N^{k-(m-1)}-\sum_{k=m-1}^{\infty}\frac{1}{(N+1)^{2k-m+2}}C_{k-1}^{(m-3)}N^{k-(m-1)}.
\end{aligned}$$

The first summand above goes to

$$\begin{aligned}
&\Big\{\sum_{k=m-2}^{\infty}\frac{1}{(N+1)^{2k-(m-1)+2}}C_k^{(m-2)}N^{k-(m-2)}-\frac{1}{(N+1)^{2(m-2)-(m-1)+2}}C_{m-2}^{(m-2)}\Big\}\frac{N+1}{N}\\
&=\Big(\frac{1}{N^{m-1}}-\frac{1}{(N+1)^{m-1}}\Big)\frac{N+1}{N}.
\end{aligned}$$

The second summand above goes to

$$\begin{aligned}
&\sum_{k-1=m-2}^{\infty}\frac{1}{(N+1)^{2(k-1)-m+4}}C_{k-1}^{(m-3)}N^{k-1-(m-2)}\\
&=\Big\{\sum_{h=m-3}^{\infty}\frac{1}{(N+1)^{2h-\{(m-3)+1\}+2}}C_h^{(m-3)}N^{h-(m-3)}-\frac{1}{(N+1)^{2(m-3)-(m-2)+2}}C_{m-3}^{(m-3)}\Big\}\frac{1}{N}\\
&=\Big(\frac{1}{N^{m-2}}-\frac{1}{(N+1)^{m-2}}\Big)\frac{1}{N}.
\end{aligned}$$

Hence we see that the desired equality holds for $l = m$. ∎

Therefore we have

PROPOSITION 5.10. $\phi(S^*_{\mu_1}\cdots S^*_{\mu_l}S_{\mu_l}\cdots S_{\mu_1}) = \frac{1}{N^l}$.

Proof. By Lemma 5.8 with (5.4), one has

$$\phi(S^*_{\mu_1}\cdots S^*_{\mu_l}S_{\mu_l}\cdots S_{\mu_1}) \geqslant \sum_{k=l-1}^{\infty}\frac{1}{(N+1)^{2k-l+2}}C_k^{(l-1)}N^{k-(l-1)} = \frac{1}{N^l}.$$

As the relation (1.1) implies the equality

$$\sum_{\mu_1,\dots,\mu_l=1}^{N}\phi(S^*_{\mu_1}\cdots S^*_{\mu_l}S_{\mu_l}\cdots S_{\mu_1}) = 1,$$

one gets the desired equality. ∎

For $\eta_1, \dots, \eta_k \in \Sigma = \{\alpha_1, \dots, \alpha_N, \beta_1, \dots, \beta_N\}$, let $s_{\eta_i}, \eta_i \in \Sigma$ be the operator defined in Proposition 3.8. We will next prove the equality

$$\phi(s_{\eta_1} \cdots s_{\eta_k} S^*_{\mu_1} \cdots S^*_{\mu_l} S_{\mu_l} \cdots S_{\mu_1} s^*_{\eta_k} \cdots s^*_{\eta_1}) = \frac{1}{(N+1)^k N^l}$$

for $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$. We first note the equality

$$S_n T_n = T_0 T_n \quad \text{for } n = 1, \dots, N$$

because $S_j T_n = 0$ for $j \neq n$ by Lemma 3.5. For $\eta_i \in \Sigma$, we put

$$\overline{T}_{\eta_i} = \begin{cases} T_0 & \text{if } \eta_i \in \{\alpha_1, \dots, \alpha_N\}, \\ T_j & \text{if } \eta_i = \beta_j. \end{cases}$$

LEMMA 5.11. *For $\mu_1, \dots, \mu_l \in \{1, \dots, N\}$, $\gamma_1 \cdots \gamma_{2k-(l-1)} \in L_N(2k; \mu_{l-1} \cdots \mu_1)$ and $\eta_1 \cdots \eta_m \in B_m(D_N)$ with $m \leqslant l$, suppose $s_{\eta_1} \cdots s_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} \neq 0$. Then we have*

$$s_{\eta_1} \cdots s_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = \overline{T}_{\eta_1} \cdots \overline{T}_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l}. \tag{5.4}$$

Proof. By applying the relation $\alpha_0 \beta_i = \mathbf{1}$ for $i = 1, \dots, N$, we may write the word $\gamma_1 \cdots \gamma_{2k-(l-1)}$ as the reduced word such as $\beta_{h_1} \cdots \beta_{h_{l-1}}$. We will prove the assertion by induction on m. For $m = 1$, if $\eta_m = \beta_j$ for some $j = 1, \dots, N$, then $s_{\beta_j} = T_j$ so that (5.5) hods. If $\eta_m = \alpha_j$ for some $j = 1, \dots, N$, the condition $s_{\alpha_j} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} \neq 0$ implies $j = h_1$ and

$$s_{\alpha_r} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = 0 \quad \text{for all } r \neq h_1,$$

because $\alpha_r \gamma_1 \cdots \gamma_{2k-(l-1)}$ is not admissible in D_N. Since $T_0 = \sum_{r=1}^{N} s_{\alpha_r}$, one has $s_{\alpha_r} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = T_0 \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l}$, so that (5.5) holds. Suppose next that the desired equality holds for all m' less than m. Since

$$s_{\eta_2} \cdots s_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} \neq 0$$

one has by hypothesis of induction

$$s_{\eta_1} s_{\eta_2} \cdots s_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = s_{\eta_1} \overline{T}_{\eta_2} \cdots \overline{T}_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l}.$$

If $\eta_1 = \beta_j$ for some $j = 1, \dots, N$, then $s_{\eta_1} = T_j$ and hence (5.5) holds. Suppose next $\eta_1 = \alpha_j$ for some $j = 1, \dots, N$. We note that $\overline{T}_{\eta_2} \cdots \overline{T}_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} \neq 0$ and $m \leqslant l - 1$. Since the reduced form of $\gamma_1 \cdots \gamma_{2k-(l-1)}$ is $\beta_{h_1} \cdots \beta_{h_{l-1}}$, the reduced form of $\eta_2 \cdots \eta_m \gamma_1 \cdots \gamma_{2k-(l-1)}$ must be of the form $\beta_{q_1} \cdots \beta_{q_t}$. Hence j must be q_1 and

$$s_{\alpha_p} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = 0 \quad \text{for all } p \neq q_1,$$

because $\alpha_p \gamma_1 \cdots \gamma_{2k-(l-1)}$ is not admissible in D_N. Hence we have

$$s_{\alpha_j} \overline{T}_{\eta_2} \cdots \overline{T}_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l} = T_0 \overline{T}_{\eta_2} \cdots \overline{T}_{\eta_m} \widetilde{T}_{\gamma_1} \cdots \widetilde{T}_{\gamma_{2k-(l-1)}} T_{\mu_l}$$

so that (5.5) holds. Therefore the equality (5.5) holds for all $m \leqslant l$. ∎

Keeping the notations above we have

LEMMA 5.12. $\phi(\overline{T}_{\eta_1}\cdots\overline{T}_{\eta_m}\widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l}T^*_{\mu_l}\widetilde{T}^*_{\gamma_{2k-(l-1)}}\cdots\widetilde{T}^*_{\gamma_1}\overline{T}^*_{\eta_m}\cdots\overline{T}^*_{\eta_1}) = \frac{1}{(N+1)^{2k-l+2+m}}$.

Proof. The operators $\overline{T}_{\eta_i}$ and $\widetilde{T}_{\gamma_j}$ are T_n for $n = 0,1,\ldots,N$. Since T_n are isometries satisfying $\sum_{j=0}^{N} T_jT_j^* = 1$, one gets the desired equality. ∎

PROPOSITION 5.13. *Suppose* $s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \neq 0$. *Then*

$$\phi(s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1}) = \frac{1}{(N+1)^m N^l}.$$

Proof. By Lemma 5.8 (i), one has for $k \geqslant l-1$,

$$E_{\mu_1\cdots\mu_l} \geqslant \widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l}T^*_{\mu_l}\widetilde{T}^*_{\gamma_{2k-(l-1)}}\cdots\widetilde{T}^*_{\gamma_1}$$

for all $\gamma_1\cdots\gamma_{2k-(l-1)} \in L_N(2k;\mu_{l-1}\cdots\mu_1)$ so that $s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \geqslant s_{\eta_1}\cdots s_{\eta_m}\widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l}T^*_{\mu_l}\widetilde{T}^*_{\gamma_{2k-(l-1)}}\cdots\widetilde{T}^*_{\gamma_1}s^*_{\eta_m}\cdots s^*_{\eta_1}$. Since the λ-graph system $\mathfrak{L}^{Ch(D_N)}$ is predecessor-separated, the projection $E_{\mu_1\cdots\mu_l}$ is a minimal projection in the commutative C^*-algebra generated by all projections of the form $s^*_{\zeta_m}\cdots s^*_{\zeta_1}s_{\zeta_1}\cdots s_{\zeta_m}$ for all $\zeta_1,\ldots,\zeta_m \in \Sigma, m \leqslant l$. The condition $s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \neq 0$ implies the inequality

$$s^*_{\eta_m}\cdots s^*_{\eta_1} \geqslant E_{\mu_1\cdots\mu_l}$$

so that $s_{\eta_1}\cdots s_{\eta_m}\widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l} \neq 0$. Hence by Lemma 5.11 one sees $s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \geqslant \overline{T}_{\eta_1}\cdots\overline{T}_{\eta_m}\widetilde{T}_{\gamma_1}\cdots\widetilde{T}_{\gamma_{2k-(l-1)}}T_{\mu_l}T^*_{\mu_l}\widetilde{T}^*_{\gamma_{2k-(l-1)}}\cdots\widetilde{T}^*_{\gamma_1}\overline{T}^*_{\eta_m}\cdots\overline{T}^*_{\eta_1}$. By (5.4), Lemma 5.8 (ii), Lemma 5.9 and Lemma 5.12, one gets

$$\begin{aligned}\phi(s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1}) &\geqslant \sum_{k=l-1}^{\infty}\frac{1}{(N+1)^{2k-l+2+m}}C_k^{(l-1)}N^{k-(l-1)}\\ &= \frac{1}{(N+1)^m}\sum_{k=l-1}^{\infty}\frac{1}{(N+1)^{2k-l+2}}C_k^{(l-1)}N^{k-(l-1)}\\ &= \frac{1}{N^l(N+1)^m}.\end{aligned}$$

Since each vertex $(\beta_{\mu_1}\cdots\beta_{\mu_l}) \in V_l$ corresponding to the projection $E_{\mu_1\cdots\mu_l}$ has N incoming edges labeled $\beta_j, j = 1,\ldots,N$ and one incoming edge labeled α_{μ_1}, the total number

$$|\{\delta_1\cdots\delta_m \in B_m(D_N) : s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \neq 0\}|$$

is $(N+1)^m$. By the equality

$$\sum_{\eta_1,\ldots,\eta_m\in B_m(D_N)}\ \sum_{\mu_1,\ldots,\mu_l=1,\ldots,N} s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} = 1,$$

we obtain

$$\phi(s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1}) = \frac{1}{N^l(N+1)^m}. \quad \blacksquare$$

Therefore we conclude

THEOREM 5.14. *The AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ has a unique tracial state. The admitted tracial state ϕ is the restriction of the* KMS *state φ on $O_{\mathfrak{L}^{Ch(D_N)}}$ to $F_{\mathfrak{L}^{Ch(D_N)}}$.*

Proof. By Proposition 5.10, the equality $\phi(E_{\mu_1\cdots\mu_l}) = \frac{1}{N^l}$ holds for $\mu_1,\dots,\mu_l \in \{1,\dots,N\}$. For $\eta_1,\dots,\eta_m \in \Sigma$, if $s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \neq 0$ we see that the equality $s^*_{\eta_m}\cdots s^*_{\eta_1}s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l} = E_{\mu_1\cdots\mu_l}$ holds so that we have

$$\phi(s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1}) = \frac{1}{(N+1)^m}\phi(s^*_{\eta_m}\cdots s^*_{\eta_1}s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}).$$

Hence the state ϕ satisfies the equalities:

$$\phi(s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1}) = \begin{cases} \frac{1}{(N+1)^m N^l} & \text{if } = s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} \neq 0, \\ 0 & \text{if } = s_{\eta_1}\cdots s_{\eta_m}E_{\mu_1\cdots\mu_l}s^*_{\eta_m}\cdots s^*_{\eta_1} = 0. \end{cases}$$

It coincides with the restriction of the unique KMS state φ to the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$. $\blacksquare$

REMARK. Theorem 1.2 is deduced from Theorem 5.14, because KMS state on the algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ is uniquely determined by a tracial state on the AF algebra satisfying (4.2).

By using uniqueness of tracial state on the AF algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ with Theorem 1.2, we may prove the following theorem.

THEOREM 5.15. *Let $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$ be the von Neumann algebra generated by the* GNS*-representation $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})$ of the algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ by the unique* KMS *state φ. Then $\pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$ is the injective factor of type* $\mathrm{III}_{1/(N+1)}$.

Proof. We put $M = \pi_\varphi(O_{\mathfrak{L}^{Ch(D_N)}})''$. As the KMS state φ on $O_{\mathfrak{L}^{Ch(D_N)}}$ is unique, the von Neumann algebra M is a factor. Since the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ is nuclear [16], it is an injective factor. The GNS representation π_φ is faithful so that we may regard $O_{\mathfrak{L}^{Ch(D_N)}}$ as a subalgebra of M. Similarly the von Neumann subalgebra $\pi_\varphi(F_{\mathfrak{L}^{Ch(D_N)}})''$ of M generated by the algebra $\pi_\varphi(F_{\mathfrak{L}^{Ch(D_N)}})$ is a factor because the C^*-algebra $F_{\mathfrak{L}^{Ch(D_N)}}$ has a unique tracial state by Theorem 5.14 and the tracial state is faithful.

Let σ be an action of $\mathbb{R}$ on the von Neumann algebra M defined by $\sigma_t(\pi_\varphi(a)) = \alpha_{-t\log(N+1)}(a)$, $a \in O_{\mathfrak{L}^{Ch(D_N)}}, t \in \mathbb{R}$. Since φ is a KMS state at inverse temperature $\log(N+1)$ for gauge action, it yields a KMS state at inverse temperature -1 for σ on $O_{\mathfrak{L}^{Ch(D_N)}}$. That is $\varphi(x\sigma_{-1}(y)) = \varphi(yx)$ for y an analytic elements of $(O_{\mathfrak{L}^{Ch(D_N)}}, \sigma, \mathbb{R})$ and $x \in O_{\mathfrak{L}^{Ch(D_N)}}$. Since φ is σ-invariant, the automorphisms $\sigma_t, t \in \mathbb{R}$ can be extended to automorphisms on the factor M, denoted by σ_t. This means that $\sigma_t = \sigma^\varphi_t$ the modular automorphisms of M. As the algebra $F_{\mathfrak{L}^{Ch(D_N)}}$

is realized as the fixed point algebra ${O_{\mathfrak{L}^{Ch(D_N)}}}^{\alpha^{DN}}$ of $O_{\mathfrak{L}^{Ch(D_N)}}$ under the gauge action α^{DN}, it is routine to check that the fixed point algebra M^σ coincides with $\pi_\varphi(F_{\mathfrak{L}^{Ch(D_N)}})''$. Since M^σ is a factor, the Connes spectrum $\Gamma(\sigma)$ coincides with the Arveson spectrum $Sp(\sigma)$. By a similar manner to the proof of Theorem 8 in [4] one knows that $Sp(\sigma) = \mathbb{Z}\log(N+1)$. This implies that the von Neumann algebra M is a factor of type III_λ, where $\lambda = \mathrm{e}^{-\log(N+1)} = \frac{1}{N+1}$. ∎

The above theorem means that the exponent of the topological entropy of the Dyck shift D_N appears in the type of the factor representation of the unique KMS state on the C^*-algebra $O_{\mathfrak{L}^{Ch(D_N)}}$ by the gauge action.

General construction of simple C^*-algebras of Dyck systems of topological Markov chains will be studied in a forthcoming paper [21].

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