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DEFECT OPERATORS AND FREDHOLMNESS FOR TOEPLITZ PAIRS WITH INNER SYMBOLS

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Dedicated to Professor Sun Shunhua on his 70th birthday

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ABSTRACT. In this paper, we study defect operators and Fredholmness for Toeplitz pairs with inner symbols on both the Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$. The defect operator theory is closely related to function theory on the unit disk and the bidisk. Fredholmness of Toeplitz pairs on $H^2(\mathbb{D}^2)$ with rational inner symbols is completely characterized. We also establish an index formula for a general isometric pair.

KEYWORDS: *Defect operator, Toeplitz pair, Fredholm index, polydisc, Hardy space.*

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INTRODUCTION

Given a tuple of commuting operators $T = (T_1, \dots, T_n)$, acting on a Hilbert space H, using Douglas-Paulsen's Hilbert module language [5], we endow H with a $\mathbb{C}[z_1, \dots, z_n]$-module structure by

$$p \cdot x = p(T_1, \dots, T_n)x, \quad p \in \mathbb{C}[z_1, \dots, z_n], x \in H.$$

In this paper, we will mainly be concerned with isometric Hilbert modules on the polynomial ring $\mathbb{C}[z_1, \dots, z_n]$. Say a Hilbert module H to be isometric if the canonical operators $T_1, \dots, T_n$ acting on H are all isometries. A standard example is the Hardy module $H^2(\mathbb{D}^n)$, where $\mathbb{D}^n$ is the unit polydisk and the module action is multiplications by coordinate functions. Of course, all submodules of the Hardy module $H^2(\mathbb{D}^n)$ are isometric. To investigate Hardy submodules of $H^2(\mathbb{D}^2)$, Yang has developed some general techniques, see [8], [14], [17], [15], [18], [19], [16], [13] and references therein.

We organize the present paper as follows. In Section 1, we give some basic properties of *defect operators* of isometric pairs. In particular, we establish an index formula for a general isometric pair in this section. This formula is as follows: if an isometric pair $T = (T_1, T_2)$ is Fredholm, then its Fredholm index reads as follows:

$$\operatorname{Ind} T = \dim \Delta_T^{(-1)} - \dim \Delta_T^{(1)},$$

where $\Delta_T^{(\mu)}$ denotes the eigenspace of the defect operator Δ_T corresponding to the eigenvalue μ. Using a general theory for isometric pairs developed in Section 1, we study compactness of the defect operators of Toeplitz pairs with inner symbols on both the Hardy spaces $H^2(\mathbb{D})$ and $H^2(\mathbb{D}^2)$ in Section 2. In the case of $H^2(\mathbb{D})$, compactness of defect operators is closely related to Hankel operator theory, and in the case of $H^2(\mathbb{D}^2)$, compactness of defect operators appears only in the trivial case.

A result, due to Theorem 5.2.5(b) of [10], shows that an inner function η in $A(\mathbb{D}^2)$ must be rational, and it has the form $\eta = \frac{p}{q}$ with $Z(q) \cap \overline{\mathbb{D}}^2 = \emptyset$. For a pair $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$ of inner functions with $Z(q_i) \cap \overline{\mathbb{D}}^2 = \emptyset$ for $i = 1, 2$, it is proved in Section 3 that the pair $(T_{\frac{p_1}{q_1}}, T_{\frac{p_2}{q_2}})$ on $H^2(\mathbb{D}^2)$ is Fredholm if and only if

$$Z(p_1) \cap Z(p_2) \cap \partial\mathbb{D}^2 = \emptyset.$$

1. DEFECT OPERATOR FOR ISOMETRIC PAIRS

In papers [7], [8], Guo and Yang studied defect operators of Hardy submodules over the polydisk. Recently, some interested results are obtained by Yang in [18], [19].

For a submodule M of $H^2(\mathbb{D}^d)$, let R_{z_i} denote the restriction of the multiplication operator M_{z_i} to M for $i = 1, \dots, d$. Then $R = (R_{z_1}, \dots, R_{z_d})$ is an isometric tuple acting on M. The defect operator Δ_M of the submodule M is defined by

$$\Delta_M = \sum_{(0,\dots,0) \leqslant \alpha \leqslant (1,\dots,1)} (-1)^{|\alpha|} R^\alpha R^{*\alpha},$$

where $R^\alpha = R_{z_1}^{\alpha_1} \cdots R_{z_d}^{\alpha_d}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ of nonnegative integers. As is shown in [7], [8], [18], [19], this operator carries the key information of the Hardy submodule. Motivated by this observation, as done for Hardy submodules, we define the defect operator Δ_T for an isometric tuple $T = (T_1, \dots, T_d)$ acting on H as:

$$\Delta_T = \sum_{(0,\dots,0) \leqslant \alpha \leqslant (1,\dots,1)} (-1)^{|\alpha|} T^\alpha T^{*\alpha}.$$

In this paper, we are mainly concerned with isometric pairs $T = (T_1, T_2)$ acting on H; then by the definition,

$$\Delta_T = I - T_1 T_1^* - T_2 T_2^* + T_1 T_2 T_1^* T_2^*,$$

and hence as done by Yang for Hardy submodules over the bidisk [18]

$$\Delta_T = [T_1^*, T_1][T_2^*, T_2] + T_1 [T_1^*, T_2] T_2^*, \tag{1.1}$$

where $[T_1^*, T_2] = T_1^* T_2 - T_2 T_1^*$. Since Δ_T is selfadjoint, this means

$$\Delta_T = [T_2^*, T_2][T_1^*, T_1] + T_2[T_1^*, T_2]^* T_1^*,$$

and it follows that

$$\Delta_T^2 = [T_1^*, T_1][T_2^*, T_2][T_1^*, T_1] + T_1[T_1^*, T_2][T_1^*, T_2]^* T_1^*.$$

Notice that T_1, T_2 are isometries, and

$$[T_1^*, T_1][T_2^*, T_2][T_1^*, T_1] = ([T_1^*, T_1][T_2^*, T_2])([T_1^*, T_1][T_2^*, T_2])^*$$

and

$$T_1[T_1^*, T_2][T_1^*, T_2]^* T_1^* = (T_1[T_1^*, T_2])(T_1[T_1^*, T_2])^*.$$

We therefore have the following.

PROPOSITION 1.1. *For any isometric pair $T = (T_1, T_2)$ on H,*
(i) *Δ_T is finite rank if and only if both $[T_1^*, T_1][T_2^*, T_2]$ and $[T_1^*, T_2]$ are finite rank.*
(ii) *Δ_T is compact if and only if $[T_1^*, T_1][T_2^*, T_2]$ and $[T_1^*, T_2]$ are both compact.*

Let $T = (T_1, T_2)$ be a commuting operator pair on H; the Koszul complex associated with T [11] is defined by

$$0 \to H \overset{d_1}{\to} H \oplus H \overset{d_2}{\to} H \to 0,$$

where the boundary operators d_1, d_2 are given by

$$d_1(\xi) = (-T_2\xi, T_1\xi),\ d_2(\xi_1, \xi_2) = T_1\xi_1 + T_2\xi_2, \quad \text{for } \xi, \xi_1, \xi_2 \in H.$$

Obviously, $d_2 d_1 = 0$. The pair (T_1, T_2) is called Fredholm [4] if

$$\mathcal{H}_0 = \ker(d_1),\ \mathcal{H}_1 = \frac{\ker(d_2)}{\operatorname{Ran}(d_1)},\ \mathcal{H}_2 = \frac{H}{\operatorname{Ran}(d_2)},$$

are all of finite dimension, and in this case, the Fredholm index of T is defined by

$$\operatorname{Ind} T = -\dim \mathcal{H}_0 + \dim \mathcal{H}_1 - \dim \mathcal{H}_2.$$

To establish the Fredholm index of an isometric pair, we need some preliminaries. For an isometric pair $T = (T_1, T_2)$ on H, it is easy to see

$$-1 \leqslant \Delta_T \leqslant 1.$$

Given $-1 \leqslant \mu \leqslant 1$, write $\Delta_T^{(\mu)}$ for $\ker(\mu I - \Delta_H)$. Then we have the following proposition.

PROPOSITION 1.2. *For an isometric pair $T = (T_1, T_2)$ on H,*
(i) $\Delta_T^{(1)} = \ker T_1^* \cap \ker T_2^* = H \ominus (T_1 H + T_2 H)$.
(ii) $\Delta_T^{(-1)} = T_1(H \ominus T_2 H) \cap T_2(H \ominus T_1 H)$.

Proof. (i) $\ker T_1^* \cap \ker T_2^* \subseteq \Delta_T^{(1)}$ is obvious. And now we begin to check the other direction. For $\xi \in \Delta_T^{(1)}$,

$$0 = (I - \Delta_T)\xi = T_1 T_1^* \xi + T_2(I - T_1 T_1^*) T_2^* \xi = T_2 T_2^* \xi + T_1 (I - T_2 T_2^*) T_1^* \xi.$$

Since $(I - T_1 T_1^*)$ and $(I - T_2 T_2^*)$ are both projections, it implies that

$$0 = \langle (I - \Delta_T)\xi, \xi \rangle = \|T_1^*\xi\|^2 + \|(I - T_1T_1^*)T_2^*\xi\|^2 = \|T_2^*\xi\|^2 + \|(I - T_2T_2^*)T_1^*\xi\|^2.$$

Thus $\xi \in \ker T_1^* \cap \ker T_2^*$.

(ii) Since $(I - T_1 T_1^*)$ and $(I - T_2 T_2^*)$ are both projections, we have

$$\begin{aligned} T_1(H \ominus T_2 H) \cap T_2(H \ominus T_1 H) &= \mathrm{Ran}(T_1(I - T_2T_2^*)) \cap \mathrm{Ran}(T_2(I - T_1T_1^*)) \\ &= \mathrm{Ran}(T_1(I - T_2T_2^*)T_1^*) \cap \mathrm{Ran}(T_2(I - T_1T_1^*)T_2^*). \end{aligned}$$

Since $T_1(I - T_2T_2^*)T_1^*$ and $T_2(I - T_1T_1^*)T_2^*$ are projections, for $\xi \in \Delta_T^{(-1)}$ we have

$$\begin{aligned} -\|\xi\|^2 = \langle -\xi, \xi \rangle = \langle \Delta_T \xi, \xi \rangle &= \|(I - T_1T_1^*)\xi\|^2 - \|T_2(I - T_1T_1^*)T_2^*\xi\|^2 \\ &= \|(I - T_2T_2^*)\xi\|^2 - \|T_1(I - T_2T_2^*)T_1^*\xi\|^2. \end{aligned}$$

Since $\|T_2(I - T_1T_1^*)T_2^*\xi\| \leqslant \|\xi\|$ and $\|T_1(I - T_2T_2^*)T_1^*\xi\| \leqslant \|\xi\|$, we have $\|\xi\| = \|T_1(I - T_2T_2^*)T_1^*\xi\| = \|T_2(I - T_1T_1^*)T_2^*\xi\|$. And then

$$\xi = T_1(I - T_2T_2^*)T_1^*\xi = T_2(I - T_1T_1^*)T_2^*\xi.$$

Thus $\xi \in \mathrm{Ran}(T_1(I - T_2T_2^*)T_1^*) \cap \mathrm{Ran}(T_2(I - T_1T_1^*)T_2^*)$.

On the other hand, $\forall \xi \in \mathrm{Ran}(T_1(1 - T_2T_2^*)) \cap \mathrm{Ran}(T_2(I - T_1T_1^*))$, there exists $\eta \in H$ such that $\xi = T_1(I - T_2T_2^*)\eta$, i.e. $T_1\eta = \xi + T_1T_2T_2^*\eta$. Since $\xi \in \mathrm{Ran}(T_2)$, $T_1\eta \in \mathrm{Ran}(T_2)$, and hence $T_1\eta = T_2T_2^*T_1\eta$. It follows that

$$\begin{aligned} \Delta_T \xi &= (I - T_1T_1^* - T_2T_2^* + T_1T_2T_1^*T_2^*)\xi = -T_2T_2^*T_1(I - T_2T_2^*)\eta \\ &= -T_1\eta + T_1T_2T_2^*\eta = -\xi. \end{aligned}$$

This means that $\mathrm{Ran}(T_1(1 - T_2T_2^*)) \cap \mathrm{Ran}(T_2(I - T_1T_1^*)) \subseteq \Delta_T^{(-1)}$. ∎

THEOREM 1.3. *Given an isometric pair $T = (T_1, T_2)$ on H,*

(i) *if the defect operator Δ_T is compact, then $T = (T_1, T_2)$ is Fredholm;*

(ii) *if $T = (T_1, T_2)$ is Fredholm, then the Fredholm index is given by*

$$\mathrm{Ind}\, T = \dim \Delta_T^{(-1)} - \dim \Delta_T^{(1)}.$$

Proof. (i) By Curto [4], T is Fredholm if and only if $\widehat{T}$ is Fredholm, where

$$\widehat{T} = \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix},$$

and in this case the Fredholm index $\mathrm{Ind}\, T = \mathrm{Ind}\, \widehat{T}$. Hence it is enough to prove that $\widehat{T}$ is Fredholm. Since T is a pair of isometries, a simple computation gives that

$$\widehat{T}^*\widehat{T} = \begin{pmatrix} I + T_2T_2^* & [T_1^*, T_2] \\ [T_1^*, T_2]^* & I + T_1T_1^* \end{pmatrix},$$

and

$$\widehat{T}\widehat{T}^* = \begin{pmatrix} T_1T_1^* + T_2T_2^* & 0 \\ 0 & 2 \end{pmatrix}. \tag{1.2}$$

From the fact that $I + T_1T_2T_1^*T_2^*$ is invertible, and Δ_T is compact, we see that

$$T_1T_1^* + T_2T_2^* = I + T_1T_2T_1^*T_2^* - \Delta_T$$

is Fredholm. Moreover, by Proposition 1.1(ii), $[T_1^*, T_2]$ is compact. This implies that both $\widehat{T}^*\widehat{T}$ and $\widehat{T}\widehat{T}^*$ are Fredholm, and hence $\widehat{T}$ is Fredholm.

(ii) Now assume that $T = (T_1, T_2)$ is Fredholm, and notice that T_1, T_2 are isometries. We have

$$\begin{aligned}\ker \widehat{T}^* &= \Big\{(\xi_1, \xi_2) : \begin{pmatrix} T_1^* & -T_2 \\ T_2^* & T_1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0\Big\} \\ &= \{(\xi_1, \xi_2) : T_1^*\xi_1 - T_2\xi_2 = 0, T_2^*\xi_1 + T_1\xi_2 = 0\} = \{(\xi_1, 0) : \xi_1 \in \ker T_1^* \cap \ker T_2^*\}.\end{aligned}$$

Proposition 1.2 implies $\ker \widehat{T}^* = \{(\xi_1, 0) : \xi_1 \in \Delta_T^{(1)}\}$. Below, we claim that $\dim \ker \widehat{T} = \dim \Delta_T^{(-1)}$. In fact,

$$\begin{aligned}\ker \widehat{T} &= \Big\{(\xi_1, \xi_2) : \begin{pmatrix} T_1 & T_2 \\ -T_2^* & T_1^* \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0\Big\} \\ &= \{(\xi_1, \xi_2) : T_1\xi_1 + T_2\xi_2 = 0, -T_2^*\xi_1 + T_1^*\xi_2 = 0\} \\ &= \{(\xi_1, \xi_2) : T_1\xi_1 + T_2\xi_2 = 0, T_2^*\xi_1 = 0, T_1^*\xi_2 = 0\}.\end{aligned}$$

Since $T_1 \oplus T_2 : H \oplus H \to H \oplus H$ is an isometry, this means that $\ker \widehat{T}$ and $T_1 \oplus T_2(\ker \widehat{T})$ have the same dimension. Notice that

$$\begin{aligned}T_1 \oplus T_2(\ker \widehat{T}) &= \{(T_1\xi_1, T_2\xi_2) : T_1\xi_1 + T_2\xi_2 = 0, T_2^*\xi_1 = 0, T_1^*\xi_2 = 0\} \\ &= \{(T_1\xi_1, -T_1\xi_1) : \xi_1 \in H\} \cap \{(T_2\xi_2, -T_2\xi_2) : \xi_2 \in H\} \\ &\qquad \cap \{(T_1\xi_1, T_2\xi_2) : T_1^*\xi_2 = 0, T_2^*\xi_1 = 0\} \\ &= \{(f, -f) : f \in T_1(\ker T_2^*) \cap T_2(\ker T_1^*)\}.\end{aligned}$$

By Proposition 1.2, we have

$$\dim \ker \widehat{T} = \dim\{(f, -f) : f \in \Delta_T^{(-1)}\} = \dim \Delta_T^{(-1)},$$

the claim follows. Hence,

$$\operatorname{Ind} T = \dim \ker \widehat{T} - \dim \ker \widehat{T}^* = \dim \Delta_T^{(-1)} - \dim \Delta_T^{(1)}. \quad ∎$$

REMARK 1.4. For an isometric pair $T = (T_1, T_2)$ on H, from the proof of Theorem 1.3(ii), we see:

(i) $\mathcal{H}_0 = 0$;

(ii) $\dim \mathcal{H}_1 = \dim \ker \widehat{T} = \dim \Delta_T^{(-1)} = \dim [T_1(H \ominus T_2H) \cap T_2(H \ominus T_1H)]$;

(iii) $\dim \mathcal{H}_2 = \dim \ker \widehat{T}^* = \dim \Delta_T^{(1)} = \dim H \ominus (T_1H + T_2H)$.

Thus, the isometric pair $T = (T_1, T_2)$ is Fredholm if and only if both $T_1(H \ominus T_2H) \cap T_2(H \ominus T_1H)$ and $\frac{H}{T_1H+T_2H}$ are of finite dimension.

The following proposition shows that if T_1 and T_2^* are essentially commutative, then the converse of Theorem 1.3(i) is also true.

PROPOSITION 1.5. *Given an isometric pair $T = (T_1, T_2)$ on H satisfying that $T_1T_2^* - T_2^*T_1$ is compact, if $T = (T_1, T_2)$ is Fredholm, then Δ_T is compact.*

Proof. If $T = (T_1, T_2)$ is Fredholm, then $\widehat{T}$ is Fredholm, and hence by (1.2), $T_1T_1^* + T_2T_2^*$ is Fredholm. By the equality

$$(T_1T_1^* + T_2T_2^*)\Delta_T = T_2[T_1, T_2^*]T_1^* + T_1[T_1, T_2^*]^*T_2^*,$$

the defect operator Δ_T is compact. ∎

The following is an example showing that the converse of Theorem 1.3(i) is not true in general.

EXAMPLE 1.6. For the isometric pair $S = (T_{z_1}, T_{\eta_a})$ on the Hardy space $H^2(\mathbb{D}^2)$, where $\eta_a(z) = \frac{z_1 - a}{1 - \bar{a}z_1}$, $0 < |a| < 1$, Theorem 3.2 in Section 3 shows that $S = (T_{z_1}, T_{\eta_a})$ is Fredholm. However, the defect operator Δ associated with S is not compact. The defect operator Δ of S is given by

$$\Delta = I - T_{z_1}T_{z_1}^* - T_{\eta_a}T_{\eta_a}^* + T_{z_1}T_{\eta_a}T_{z_1}^*T_{\eta_a}^*.$$

For $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, let K_λ denote the reproducing kernel of $H^2(\mathbb{D}^2)$ at λ, and $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$ be the normalized reproducing kernel at λ. It is easy to verify that $k_\lambda \xrightarrow{w} 0$ as $\lambda \to \partial\mathbb{D}^2$. Considering the equality

$$\langle \Delta k_\lambda, k_\lambda \rangle = (1 - |\lambda_1|^2)\Big(1 - \Big|\frac{\lambda_1 - a}{1 - \bar{a}\lambda_1}\Big|^2\Big),$$

and taking $\lambda_1 = 0$, then the above $= 1 - |a|^2 \neq 0$. This shows that Δ is not compact.

2. DEFECT OPERATORS ON THE HARDY SPACES

In this section, we will see that properties of defect operators on the Hardy spaces are closely related to function theory on the Hardy spaces. For $n \geqslant 1$, let P be the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. The Toeplitz operator $T_f : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n)$ with symbol $f \in L^\infty(\mathbb{T}^n)$ is defined by $T_f(h) = P(fh)$ for all $h \in H^2(\mathbb{D}^n)$. The Hankel operator $H_f : H^2(\mathbb{D}^n) \to L^2(\mathbb{T}^n) \ominus H^2(\mathbb{D}^n)$ with symbol f is defined as $H_f h = (I - P)(fh)$ for all $h \in H^2(\mathbb{D}^n)$. For $f, g \in L^\infty(\mathbb{T}^n)$, Toeplitz and Hankel operators are connected by the following formula

$$T_{fg} - T_fT_g = H_{\bar{f}}^*H_g.$$

Given two inner functions η_1 and η_2 on $\mathbb{D}^n$, then $T = (T_{\eta_1}, T_{\eta_2})$ is an isometric pair on the Hardy space $H^2(\mathbb{D}^n)$. The defect operator Δ_T on $H^2(\mathbb{D}^n)$ associated with T is defined by

$$\Delta_T = I - T_{\eta_1}T_{\eta_1}^* - T_{\eta_2}T_{\eta_2}^* + T_{\eta_1\eta_2}T_{\eta_1\eta_2}^*.$$

For $\lambda \in \mathbb{D}^n$, let K_λ denote the reproducing kernel of $H^2(\mathbb{D}^n)$ at λ, and $k_\lambda = \frac{K_\lambda}{\|K_\lambda\|}$ be the normalized reproducing kernel at λ. Considering the equality

$$\langle \Delta_T k_\lambda, k_\lambda \rangle = (1 - |\eta_1(\lambda)|^2)(1 - |\eta_2(\lambda)|^2),$$

one sees that $\Delta_T = 0$ only if either η_1 or η_2 is a constant.

Below, we consider defect operators on the Hardy space $H^2(\mathbb{D})$.

THEOREM 2.1. *Given two inner functions η_1 and η_2 on $\mathbb{D}$, and let Δ_T be the defect operator on $H^2(\mathbb{D})$ defined by the isometric pair $T = (T_{\eta_1}, T_{\eta_2})$. Then*

(i) *Δ_T is of finite rank if and only if either η_1 or η_2 is a finite Blaschke product.*

(ii) *Δ_T is compact if and only if $H^\infty[\overline{\eta}_1] \cap H^\infty[\overline{\eta}_2] \subseteq H^\infty + C(\mathbb{T})$, where $H^\infty[f]$ denotes the closed subalgebra of L^∞ generated by H^∞ and f.*

Proof. (i) Suppose first that η_1 or η_2 is a finite Blaschcke product. Without loss of generality, assume $\eta_1(z) = B(z)$ is a finite Blaschke product. Then $I - T_{\eta_1}T_{\eta_1}^*$ is finite rank. Since

$$\Delta_T = I - T_{\eta_1}T_{\eta_1}^* - T_{\eta_2}(I - T_{\eta_1}T_{\eta_1}^*)T_{\eta_2}^*,$$

this shows that Δ_T is of finite rank. Conversely, if Δ_T is of finite rank, then by Proposition 1.1, the operator

$$[T_{\eta_1}^*, T_{\eta_2}] = T_{\overline{\eta}_1}T_{\eta_2} - T_{\eta_2}T_{\overline{\eta}_1} = H_{\overline{\eta}_2}^* H_{\overline{\eta}_1}$$

has finite rank. By Theorem 4 of [1], either $H_{\overline{\eta}_1}$ or $H_{\overline{\eta}_2}$ is of finite rank. We may assume that $H_{\overline{\eta}_1}$ is of finite rank. By Beurling theorem [2], it is easy to prove that η_1 is a finite Blaschke product.

(ii) Suppose that Δ_T is compact. Then Proposition 1.1 implies that

$$[T_{\eta_1}^*, T_{\eta_2}] = H_{\overline{\eta}_2}^* H_{\overline{\eta}_1}$$

is compact. By Theorem 1 of [12], we have

$$H^\infty[\overline{\eta}_1] \cap H^\infty[\overline{\eta}_2] \subseteq H^\infty + C(\mathbb{T}).$$

Conversely, if $H^\infty[\overline{\eta}_1] \cap H^\infty[\overline{\eta}_2] \subseteq H^\infty + C(\mathbb{T})$, then by Theorem 1 of [1],

$$[T_{\eta_1}^*, T_{\eta_2}] = H_{\overline{\eta}_2}^* H_{\overline{\eta}_1}$$

is compact. Applying Lemma 1 of [1] shows that $H_{\overline{\eta}_2}H_{\overline{\eta}_1}^*$ is compact, and hence $H_{\overline{\eta}_1}H_{\overline{\eta}_2}^*$ is compact. This implies that

$$[T_{\eta_1}^*, T_{\eta_1}][T_{\eta_2}^*, T_{\eta_2}] = H_{\overline{\eta}_1}^* H_{\overline{\eta}_1} H_{\overline{\eta}_2}^* H_{\overline{\eta}_2},$$

is compact. Applying Proposition 1.1 shows that Δ_T is compact. ∎

Now, we consider defect operators on the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk.

THEOREM 2.2. *Consider two inner functions η_1 and η_2 on $\mathbb{D}^2$, and let Δ_T be the defect operator on $H^2(\mathbb{D}^2)$ defined by the isometric pair $T = (T_{\eta_1}, T_{\eta_2})$. If $\Delta_T \neq 0$, then the following statements are equivalent:*

(i) *Δ_T is compact.*

(ii) *Δ_T is finite rank.*

(iii) *Δ_T is a finite rank projection.*

(iv) *There exist finite Blaschke products $B_1(z)$ and $B_2(w)$ such that $\eta_1 = B_1(z)$, $\eta_2 = B_2(w)$ or $\eta_1 = B_2(w)$, $\eta_2 = B_1(z)$.*

Proof. (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are obvious.

(iv)$\Rightarrow$(iii). Without loss of generality, assume that $\eta_1 = B_1(z)$ and $\eta_2 = B_2(w)$. Since $T_z T_w^* = T_w^* T_z$, this implies

$$T_{\eta_1} T_{\eta_2}^* = T_{\eta_2}^* T_{\eta_1}.$$

Using (1.1), we have

$$\Delta_T = I - T_{\eta_1} T_{\eta_1}^* - T_{\eta_2} T_{\eta_2}^* + T_{\eta_1 \eta_2} T_{\eta_1 \eta_2}^* = [T_{\eta_1}^*, T_{\eta_1}][T_{\eta_2}^*, T_{\eta_2}].$$

It is easy to see that Δ_T is a projection. By Proposition 1.2(i), we have

$$\operatorname{Ran}(\Delta_T) = \Delta_T^{(1)} = \ker T_{\eta_1}^* \cap \ker T_{\eta_2}^* = H^2(\mathbb{D}^2) \ominus [B_1(z) H^2(\mathbb{D}^2) + B_2(w) H^2(\mathbb{D}^2)].$$

By Lemma 2.2.9 of [3], the above equality implies that Δ_T has finite rank, and it follows that Δ_T is a finite rank projection.

(i)$\Rightarrow$(iv). At first, we will prove that both η_1 and η_2 are functions of one variable. To show this, it is easy to see that the normalized reproducing kernel $\{k_\lambda\}$ converge weakly to 0 as $\lambda \to \partial\mathbb{D}^2$. Since Δ_T is compact, this means

$$\lim_{\lambda \to \partial\mathbb{D}^2} \langle \Delta_T k_\lambda, k_\lambda \rangle = \lim_{\lambda \to \partial\mathbb{D}^2} (1 - |\eta_1(\lambda)|^2)(1 - |\eta_2(\lambda)|^2) = 0. \tag{2.1}$$

Now, take $\lambda_1 = 0$, then for all $\zeta \in \mathbb{T}$,

$$\lim_{w \to \zeta} (1 - |\eta_1(0, w)|^2)(1 - |\eta_2(0, w)|^2) = 0.$$

Set

$$\mathbb{S}_1 = \{\zeta : \zeta \in \mathbb{T}, \lim_{w \to \zeta} |\eta_1(0, w)| = 1\}, \quad \mathbb{S}_2 = \{\zeta : \zeta \in \mathbb{T}, \lim_{w \to \zeta} |\eta_2(0, w)| = 1\}.$$

Then $m(\mathbb{S}_1 \cup \mathbb{S}_2) = 1$, here $m = \frac{\mathrm{d}\theta}{2\pi}$. *Now, we may assume* $m(\mathbb{S}_1) > 0$. Write η_1 as

$$\eta_1(z, w) = \sum_{j=0}^{\infty} z^j f_j(w),$$

where $f_j \in H^2(\mathbb{D})$. This expansion implies that $f_j(w) = \int_{\mathbb{T}} \eta_1(z, w) \bar{z}^j \mathrm{d}m(z)$, and hence

$$|f_j(w)| \leqslant \int_{\mathbb{T}} |\eta_1(z, w) \bar{z}^j| \mathrm{d}m(z) \leqslant 1. \tag{2.2}$$

Set

$$\mathbb{S} = \bigcap_j \{\zeta : \zeta \in \mathbb{T},\ f_j \text{ has nontangential limit at } \zeta\}.$$

Then $m(\mathbb{S}) = 1$, and hence $m(\mathbb{S}_1 \cap \mathbb{S}) > 0$. For any $\zeta \in \mathbb{S}_1 \cap \mathbb{S}$, using the estimate (2.2) and Lebesgue Dominated Convergence theorem,

$$\eta_1(z, \zeta) = \lim_{w \to \zeta} \Big(\sum_{j=0}^{\infty} z^j f_j(w) \Big) = \sum_{j=0}^{\infty} z^j f_j(\zeta)$$

is analytic in $\mathbb{D}$. Since $\zeta \in \mathbb{S}_1$, we have $|\eta_1(0, \zeta)| = 1$. The maximum modulus theorem implies that $\eta_1(z, \zeta)$ is a constant of modulo 1. So $f_j(\zeta) = 0$ for $j = 1, 2, \ldots$. This shows that f_j vanishes on $\mathbb{S}_1 \cap \mathbb{S}$ for $j = 1, 2, \ldots$. Since $m(\mathbb{S}_1 \cap \mathbb{S}) > 0$, this ensures $f_j = 0$ for $j = 1, 2, \ldots$. So, $\eta_1(z, w) = f_0(w)$. Since $\Delta_T \neq 0$, $|\eta_1(z, 0)| = |f_0(0)| < 1$. Let λ_2 be 0 in the formula (2.1), for any $\zeta \in \mathbb{T}$, we have

$$\lim_{\lambda_1 \to \zeta} (1 - |\eta_2(\lambda_1, 0)|) = 0.$$

The same discussions give $\eta_2(z, w) = g(z)$.

Without loss of generality, we assume that $\eta_1(z, w) = \varphi(z)$ and $\eta_2(z, w) = \psi(w)$. We will show that both φ and ψ are finite Blaschke products. By the formula (2.1),

$$\lim_{(\lambda_1, \lambda_2) \to \partial \mathbb{D}^2} (1 - |\varphi(\lambda_1)|^2)(1 - |\psi(\lambda_2)|^2) = 0.$$

The above implies that

$$\lim_{\lambda_1 \to \mathbb{T}} |\varphi(\lambda_1)| = 1, \quad \lim_{\lambda_2 \to \mathbb{T}} |\psi(\lambda_2)| = 1.$$

From the above equalities, it is not difficult to prove that both φ and ψ are continuous inner functions on $\overline{\mathbb{D}}$, and hence they are finite Blaschke products. ∎

Let both $\eta(z)$ and $\zeta(w)$ be nontrivial inner functions. Considering the isometric pair $T = (T_{\eta(z)}, T_{\zeta(w)})$ on $H^2(\mathbb{D}^2)$, the next theorem shows that T is Fredholm only if both η and ζ are finite Blaschke products. In Section 3, we will consider Fredholmness of Toeplitz pairs with rational inner symbols.

THEOREM 2.3. *Let $\eta(z), \zeta(w)$ be nontrivial inner functions on $\mathbb{D}$, then the pair $(T_{\eta(z)}, T_{\zeta(w)})$, acting on $H^2(\mathbb{D}^2)$, is Fredholm if and only if they are all finite Blaschke products.*

Proof. The "if" part comes from Theorem 1.3 and Theorem 2.2. Now let us see the "only if" part. It is easy to verify that $\eta(z)H^2(\mathbb{D}^2) + \zeta(w)H^2(\mathbb{D}^2)$ is closed. Hence by Remark 1.4, $\eta H^2(\mathbb{D}^2) + \zeta H^2(\mathbb{D}^2)$ is of finite codimension. Suppose η is not a finite Blaschke product. Consider M_η as an operator on $H^2(\mathbb{D})$, and then $\ker M_\eta^*$ is of infinite dimension. Let $\{e_n(z)\}_{n=1}^{\infty}$ be an orthogonal basis for $\ker M_\eta^*$. Since ζ is nontrivial, this means that $\ker M_\zeta{}^*$ is not zero, where M_ζ is viewed as an operator on $H^2(\mathbb{D})$. Let $e_0(w) \in \ker M_\zeta^*$ and $e_0(w) \neq 0$. Then $\{e_n(z)e_0(w)\}_{n=1}^\infty$ is an orthogonal set in $H^2(\mathbb{D}^2)$. Moreover, for nonnegative integers i, j we have

$$\langle e_n(z)e_0(w), \eta(z)z^i w^j\rangle = \langle e_n(z), \eta(z)z^i\rangle\langle e_0(w), w^j\rangle = \langle M_\eta^* e_n, z^i\rangle\langle e_0(w), w^j\rangle = 0,$$

and

$$\langle e_n(z)e_0(w), \zeta(w)z^i w^j\rangle = \langle e_n(z), z^i\rangle\langle e_0(w), \zeta(w)w^j\rangle = \langle e_n(z), z^i\rangle\langle M_\zeta^* e_0, w^j\rangle = 0.$$

The above reasoning shows $e_n(z)e_0(w) \perp (\eta H^2(\mathbb{D}^2) + \zeta H^2(\mathbb{D}^2))$ for $n = 1, 2, \ldots$, and hence $\eta H^2(\mathbb{D}^2) + \zeta H^2(\mathbb{D}^2)$ is infinite codimensional. This contradiction says that $\eta(z)$ is a finite Blaschke product. The same reasoning shows that $\zeta(w)$ is a finite Blaschke product. ∎

3. TOEPLITZ PAIRS ON THE HARDY SPACE $H^2(\mathbb{D}^2)$

For an inner function φ on the unit disk $\mathbb{D}$, the Toeplitz operator T_φ on the Hardy space $H^2(\mathbb{D})$ is Fredholm if and only if φ is a finite Blaschke product.

Motivated by this result, we naturally have the following question.

Question. Can one completely characterize the Fredholmness for Toeplitz tuples with inner symbols on the Hardy space $H^2(\mathbb{D}^n)$ for $n > 1$?

In this section, we will give a complete characterization for the Fredholmness of Toeplitz pairs on $H^2(\mathbb{D}^2)$ with inner symbols in the bidisk algebra $A(\mathbb{D}^2)$. At first, we mention the following lemma which comes from [10]. This lemma describes inner functions in the polydisk algebra $A(\mathbb{D}^n)$.

LEMMA 3.1. *Let f be an inner function on $\mathbb{D}^n$, and $f \in A(\mathbb{D}^n)$. Then $f = \frac{p}{q}$ for some polynomials p, q, and q has no zero on $\overline{\mathbb{D}}^n$.*

The following is the main result in this section.

THEOREM 3.2. *Let $\eta_1 = \frac{p_1}{q_1}$, $\eta_2 = \frac{p_2}{q_2}$ be two inner functions in $A(\mathbb{D}^2)$. Then the Toeplitz pair $T = (T_{\eta_1}, T_{\eta_2})$ on $H^2(\mathbb{D}^2)$ is Fredholm if and only if*

$$Z(p_1) \cap Z(p_2) \cap \partial\mathbb{D}^2 = \emptyset,$$

and in this case, the Fredholm index of the Toeplitz pair is given by

$$\operatorname{Ind}(T_{\eta_1}, T_{\eta_2}) = -\frac{\dim\ H^2(\mathbb{D}^2)}{[\eta_1 H^2(\mathbb{D}^2) + \eta_2 H^2(\mathbb{D}^2)]}.$$

Before proving this theorem, let us see two examples.

EXAMPLE 3.3. Let $\varphi_i(z)$, $\psi_i(w)$ be finite Blaschke products for $i = 1, 2$, and

$$\varphi_i(z, w) = \prod_{j=1}^{n_i} \frac{z - a_{ij}}{1 - \bar{a}_{ij} z}, \quad \psi_i(z, w) = \prod_{j=1}^{m_i} \frac{w - b_{ij}}{1 - \bar{b}_{ij} w}$$

where a_{ij}, $b_{ij} \in \mathbb{D}$. So by Theorem 3.2, $(T_{\varphi_1\psi_1}, T_{\varphi_2\psi_2})$ is Fredholm if and only if

$$\{a_{1j}\}_{j=1}^{n_1} \cap \{a_{2j}\}_{j=1}^{n_2} = \emptyset, \quad \{b_{1j}\}_{j=1}^{m_1} \cap \{b_{2j}\}_{j=1}^{m_2} = \emptyset.$$

When the above conditions are satisfied, the Fredholm index of the pair is

$$\operatorname{Ind}(T_{\varphi_1\psi_1}, T_{\varphi_2\psi_2}) = -n_1 n_2 m_1 m_2.$$

EXAMPLE 3.4. Let $\varphi(z, w) = \frac{z-\frac{1}{2}}{1-\frac{1}{2}z}$, $\psi(z, w) = \frac{zw-\frac{1}{2}}{1-\frac{1}{2}zw}$, then

$$Z(z-\frac{1}{2}) \cap Z(zw-\frac{1}{2}) = \{(\frac{1}{2}, 1)\} \in \partial\mathbb{D}^2.$$

Theorem 3.2 implies $(T_{\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}}, T_{\frac{zw-\frac{1}{2}}{1-\frac{1}{2}zw}})$ is not Fredholm.

The proof of Theorem 3.2 comes from the next Theorem 3.5 whose proof is long. We will place the proof of Theorem 3.5 to the later part of this section.

THEOREM 3.5. *Let $p_1, p_2 \in \mathbb{C}[z, w]$, and $r = GCD(p_1, p_2)$. Set $p = r\widetilde{p}_1$ and $p_2 = r\widetilde{p}_2$, then $p_1H^2(\mathbb{D}^2) + p_2H^2(\mathbb{D}^2)$ is closed if and only if*

$$Z(r) \cap \mathbb{T}^2 = \emptyset, \quad \textit{and} \quad Z(\widetilde{p}_1) \cap Z(\widetilde{p}_2) \cap \partial\mathbb{D}^2 = \emptyset.$$

Proof of Theorem 3.2. Suppose first $Z(p_1) \cap Z(p_2) \cap \partial\mathbb{D}^2 = \emptyset$. Combining Lemma 3.1 and Theorem 3.5 implies that $\eta_1 H^2(\mathbb{D}^2) + \eta_2 H^2(\mathbb{D}^2) = p_1H^2(\mathbb{D}^2) + p_2H^2(\mathbb{D}^2)$ is closed. Let $r = GCD(p_1, p_2)$, and $p_1 = r\,\widetilde{p}_1$, $p_2 = r\,\widetilde{p}_2$. Since $Z(r) \cap \partial\mathbb{D}^2 = \emptyset$, this implies that $Z(r) \cap \overline{\mathbb{D}}^2 = \emptyset$. Therefore, we have

$$p_1H^2(\mathbb{D}^2) + p_2H^2(\mathbb{D}^2) = \widetilde{p}_1H^2(\mathbb{D}^2) + \widetilde{p}_2H^2(\mathbb{D}^2).$$

Since $GCD(\widetilde{p}_1, \widetilde{p}_2) = 1$, $\widetilde{p}_1\mathbb{C}[z, w] + \widetilde{p}_2\mathbb{C}[z, w]$ is a finite codimensional ideal by [14], and hence by Theorem 2.2.3 of [3],

$$\widetilde{p}_1H^2(\mathbb{D}^2) + \widetilde{p}_2H^2(\mathbb{D}^2) = \overline{\widetilde{p}_1\mathbb{C}[z, w] + \widetilde{p}_2\mathbb{C}[z, w]}$$

is of finite codimension. Thus $\mathcal{H}_2$ is finite dimensional. Since (T_{η_1}, T_{η_2}) is an isometric pair, let us consider

$$\begin{aligned}
\Delta^{(-1)} &= T_{\eta_1}[H^2(\mathbb{D}^2) \ominus T_{\eta_2}H^2(\mathbb{D}^2)] \bigcap T_{\eta_2}[H^2(\mathbb{D}^2) \ominus T_{\eta_1}H^2(\mathbb{D}^2)] \\
&= [\eta_1H^2(\mathbb{D}^2) \ominus \eta_1\eta_2H^2(\mathbb{D}^2)] \bigcap [\eta_2H^2(\mathbb{D}^2) \ominus \eta_2\eta_1H^2(\mathbb{D}^2)] \\
&= [\eta_1H^2(\mathbb{D}^2) \cap \eta_2H^2(\mathbb{D}^2)] \ominus \eta_1\eta_2H^2(\mathbb{D}^2) \\
&= [p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2)] \ominus p_1p_2H^2(\mathbb{D}^2).
\end{aligned}$$

Below, we claim $p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2) = p_1p_2H^2(\mathbb{D}^2)$. Clearly, $p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2) \supseteq p_1p_2H^2(\mathbb{D}^2)$. It is enough to show

$$p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2) \subseteq p_1p_2H^2(\mathbb{D}^2).$$

For $f \in p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2)$, there are g and h in $H^2(\mathbb{D}^2)$ such that $f = p_1g = p_2h$, and hence $\frac{h}{p_1} = \frac{g}{p_2}$ on $\mathbb{D}^2 \setminus Z(p_1) \cup Z(p_2)$. Set

$$\phi = \frac{h}{p_1} = \frac{g}{p_2}.$$

Then ϕ can be analytically extended to $\mathbb{D}^2 \setminus Z(p_1) \cap Z(p_2)$. Since

$$Z(p_1) \cap Z(p_2) \cap \partial\mathbb{D}^2 = \emptyset,$$

this means that $Z(p_1) \cap Z(p_2) \cap \mathbb{D}^2$ is a finite set, and hence ϕ can be analytically extended to $\mathbb{D}^2$ by Hartogs' theorem. Furthermore, we have $\phi \in H^2(\mathbb{D}^2)$. The reasoning is as follows: since the set $Z(p_1) \cap Z(p_2) \cap \mathbb{D}^2$ is finite, there exists $0 < s < 1$ and $\varepsilon > 0$ such that $|p_1(r\xi)|^2 + |p_1(r\xi)|^2 > \varepsilon$ for all $\xi \in \mathbb{T}^2$ if $r > s$. Therefore, if $r > s$, we have

$$|\phi(r\xi)|^2 = \frac{|h(r\xi)|^2}{|p_1(r\xi)|^2} = \frac{|g(r\xi)|^2}{|p_2(r\xi)|^2} = \frac{|h(r\xi)|^2 + |g(r\xi)|^2}{|p_1(r\xi)|^2 + |p_2(r\xi)|^2} < \frac{|h(r\xi)|^2 + |g(r\xi)|^2}{\varepsilon}, \quad \xi \in \mathbb{T}^2.$$

The above reasoning insures $\phi \in H^2(\mathbb{D}^2)$, and it follows that $h = p_1\phi, g = p_2\phi$ and $f = p_1p_2\phi$. This says $f \in p_1p_2H^2(\mathbb{D}^2)$, and hence

$$p_1H^2(\mathbb{D}^2) \cap p_2H^2(\mathbb{D}^2) = p_1p_2H^2(\mathbb{D}^2).$$

We conclude $\Delta^{(-1)} = 0$. By Remark 1.4(ii), we see $\mathcal{H}_1 = 0$. For an isometric pair, $\mathcal{H}_0 = 0$ is obvious. So, (T_{η_1}, T_{η_2}) is Fredholm. Conversely, suppose that (T_{η_1}, T_{η_2}) is Fredholm. By (1.2), $T_{\eta_1}T_{\eta_1}^* + T_{\eta_2}T_{\eta_2}^*$ is Fredholm, and hence there exists a positive invertible operator X and a compact operator K such that

$$T_{\eta_1}T_{\eta_1}^* + T_{\eta_2}T_{\eta_2}^* = X + K.$$

This gives

$$\|T_{\eta_1}^*k_\lambda\|^2 + \|T_{\eta_2}^*k_\lambda\|^2 = |\eta_1(\lambda)|^2 + |\eta_2(\lambda)|^2 = \langle Xk_\lambda, k_\lambda\rangle + \langle Kk_\lambda, k_\lambda\rangle.$$

Considering that X is positive and invertible, and K is compact, and noticing that $k_\lambda \xrightarrow{w} 0$ as $\lambda \to \partial\mathbb{D}^2$, there exists a positive constant c such that

$$\Big|\frac{p_1(\xi)}{q_1(\xi)}\Big|^2 + \Big|\frac{p_2(\xi)}{q_2(\xi)}\Big|^2 \geqslant c$$

for all $\xi \in \partial\mathbb{D}^2$. This gives

$$Z(p_1) \cap Z(p_2) \cap \partial\mathbb{D}^2 = \emptyset.$$

If $T = (T_{\eta_1}, T_{\eta_2})$ is Fredholm, from the above reasoning, both $\mathcal{H}_0$ and $\mathcal{H}_1$ are 0, and hence we have

$$\operatorname{Ind}(T_{\eta_1}, T_{\eta_2}) = -\dim \mathcal{H}_2 = -\operatorname{codim}(\eta_1H^2(\mathbb{D}^2) + \eta_2H^2(\mathbb{D}^2)). \quad \blacksquare$$

Now we come back to the proof of Theorem 3.5. Since the polynomial ring $\mathbb{C}[z, w]$ is Noetherian, each ideal I is finitely generated, that is, there exist finitely many $p_1, \dots, p_l$ in I such that $I = p_1\mathbb{C}[z, w] + \cdots + p_l\mathbb{C}[z, w]$. Such a tuple $\{p_1, \dots, p_l\}$ is called a set of generators of I. Below, we will use $I\,H^2(\mathbb{D}^2)$ to denote $p_1H^2(\mathbb{D}^2) + \cdots + p_lH^2(\mathbb{D}^2)$. It is easy to see that this expression is independent on the choice of sets of generators of I.

The following lemma is used to prove the sufficiency of Theorem 3.5.

LEMMA 3.6. *Let M be a finite codimensional submodule of $H^2(\mathbb{D}^2)$. Set $Q = M \cap \mathbb{C}[z, w]$, then $M = QH^2(\mathbb{D}^2)$.*

Proof. One may give the proof of Lemma 3.6 by using Putinar's method as in [9]. Here, we present a proof to the reader by induction on $\operatorname{card}(Z(M))$, the cardinality of $Z(M)$. Since M is of finite codimension, by Theorem 2.2.3 of [3], $Q = M \cap \mathbb{C}[z, w]$ is an ideal of $\mathbb{C}[z, w]$ with the same codimension as M, $M = [Q]$, and $Z(M) = Z(Q) \subset \mathbb{D}^2$ is a finite set, where $[Q]$ denotes $\overline{QH^2(\mathbb{D}^2)}$.

First assume $\operatorname{card}(Z(M)) = 1$ and $Z(M) = \{0\}$. Then as done in Lemma 2.5.1 of [3], there is a finite dimensional space R of polynomials such that

$$H^2(\mathbb{D}^2) = M \oplus R.$$

Let $s = \max\{\deg(q) : q \in R\}$, where $\deg(q)$ is homogeneous degree of polynomial q. Since R is finite dimensional, $s < \infty$. Let $M_s = z^{s+1}H^2(\mathbb{D}^2) + w^{s+1}H^2(\mathbb{D}^2)$, then $M_s \perp R$ and hence $M_s \subseteq M$. Clearly, M_s is closed and finite codimensional with the codimension $(s+1)^2$. Moreover, $M_s^\perp$ is a polynomial space. So $M \ominus M_s \subseteq M_s^\perp$ is also a finite dimensional space of polynomials. Let $\{e_1, \dots, e_t\}$ be a base for $M \ominus M_s$. We have

$$M = M_s + (M \ominus M_s) = z^{s+1}H^2(\mathbb{D}^2) + w^{s+1}H^2(\mathbb{D}^2) + e_1H^2(\mathbb{D}^2) + \cdots + e_tH^2(\mathbb{D}^2).$$

Let $\{p_1, \dots, p_l\}$ be a set of generators of Q. Then we have

$$\begin{aligned} Q = M \cap \mathbb{C}[z, w] &= p_1\mathbb{C}[z, w] + \cdots + p_l\mathbb{C}[z, w] \\ &\supseteq z^{s+1}\mathbb{C}[z, w] + w^{s+1}\mathbb{C}[z, w] + e_1\mathbb{C}[z, w] + \cdots + e_t\mathbb{C}[z, w]. \end{aligned}$$

This implies $\{z^{s+1}, w^{s+1}, e_1, \dots, e_t\} \subseteq Q$, and hence

$$\begin{aligned} M &= z^{s+1}H^2(\mathbb{D}^2) + w^{s+1}H^2(\mathbb{D}^2) + e_1H^2(\mathbb{D}^2) + \cdots + e_tH^2(\mathbb{D}^2) \\ &\subseteq p_1H^2(\mathbb{D}^2) + \cdots + p_lH^2(\mathbb{D}^2) \subseteq M. \end{aligned}$$

It follows that

$$M = p_1H^2(\mathbb{D}^2) + \cdots + p_lH^2(\mathbb{D}^2) = QH^2(\mathbb{D}^2).$$

Now assume $Z(M) = \{\lambda = (\lambda_1, \lambda_2)\}$. Set

$$M(\lambda) = M \circ \varphi_\lambda = \{f \circ \varphi_\lambda : \forall f \in M\}$$

where $\varphi_\lambda(z,w) = (\frac{z+\lambda_1}{1+\overline{\lambda}_1 z}, \frac{w+\lambda_2}{1+\overline{\lambda}_2 w})$. It is not difficult to see that $M(\lambda)$ is a submodule with the same codimension as M. Noticing $Z(M(\lambda)) = \{0\}$, and letting

$$Q(\lambda) = M(\lambda) \cap \mathbb{C}[z,w] = r_1\mathbb{C}[z,w] + \cdots + r_k\mathbb{C}[z,w],$$

the above reasoning implies that

$$M(\lambda) = r_1 H^2(\mathbb{D}^2) + \cdots + r_k H^2(\mathbb{D}^2).$$

For a polynomial $r(z,w) = \sum a_{ij} z^i w^j$,

$$r \circ \varphi_\lambda = \sum a_{ij} \frac{(z+\lambda_1)^i (w+\lambda_2)^j}{(1+\overline{\lambda}_1 z)^i (1+\overline{\lambda}_2 w)^j} = \frac{\widetilde{r}}{(1+\overline{\lambda}_1 z)^m (1+\overline{\lambda}_2 w)^d},$$

where m, d are enough large such that $\widetilde{r}$ is a polynomial. Since both $\frac{1}{(1+\overline{\lambda}_1 z)^m}$ and $\frac{1}{(1+\overline{\lambda}_2 w)^d}$ are invertible in H^∞, we have

$$\begin{aligned} M = M(\lambda) \circ \varphi_\lambda &= r_1 \circ \varphi_\lambda H^2(\mathbb{D}^2) + \cdots + r_k \circ \varphi_\lambda H^2(\mathbb{D}^2) \\ &= \widetilde{r}_1 H^2(\mathbb{D}^2) + \cdots + \widetilde{r}_k H^2(\mathbb{D}^2) = QH^2(\mathbb{D}^2). \end{aligned}$$

Now assume that the lemma is true for $\operatorname{card}(Z(M)) \leqslant k$. We will prove it is true in the case $\operatorname{card}(Z(M)) = k+1$.

Let $Z(M) = \{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}\}$. By Corollary 2.2.6(2) of [3], M can be decomposed as

$$M = M_1 \cap M_2$$

such that $Z(M_1) = \{\lambda_1, \ldots, \lambda_k\}$ and $Z(M_2) = \{\lambda_{k+1}\}$.

By the assumption,

$$M_1 = Q_1 H^2(\mathbb{D}^2) \quad \text{and} \quad M_2 = Q_2 H^2(\mathbb{D}^2),$$

where

$$Q_1 = M_1 \cap \mathbb{C}[z,w] \quad \text{and} \quad Q_2 = M_2 \cap \mathbb{C}[z,w].$$

Let $\{g_i\}_{i=1}^s$ be a set of generators of Q_1, and $\{q_j\}_{j=1}^t$ a set of generators of Q_2. Set $Q = M \cap \mathbb{C}[z,w]$. Then clearly, $Q_1 Q_2 \subseteq Q$. It is enough to prove $M \subseteq QH^2(\mathbb{D}^2)$. For $f \in M$, since $M \subseteq M_1 = Q_1 H^2(\mathbb{D}^2)$, $f \in Q_1 H^2(\mathbb{D}^2)$, and hence f has the form $f = \sum_{i=1}^s g_i h_i$ where $h_i \in H^2(\mathbb{D}^2)$. Since Q_2 is a finite codimensional ideal of $\mathbb{C}[z,w]$, $\mathbb{C}[z,w] = Q_2 \dotplus R$ for some finite dimensional subspace R of polynomials. Thus, $H^2(\mathbb{D}^2) = \overline{\mathbb{C}[z,w]} = \overline{Q}_2 + R = Q_2 H^2(\mathbb{D}^2) + R$. Therefore, each h_i can be expressed as $h_i = \sum_{j=1}^t q_j f_{ij} + r_i$, here $f_{ij} \in H^2(\mathbb{D}^2)$ and $r_i \in R$. So, we have

$$f = \sum_{i=1}^s g_i \Big(\sum_{j=1}^t q_j f_{ij} + r_i \Big) = \sum_{i,j} g_i q_j f_{ij} + \sum_{i=1}^s g_i r_i.$$

Because both f and $\sum_{i,j} g_i q_j f_{ij}$ are in M, the polynomial $\sum_{i=1}^{s} g_i r_i$ is in M, and hence it is in Q. Since all $g_i q_j$ are in Q, the above reasoning gives $M \subseteq Q\,H^2(\mathbb{D}^2)$. Clearly, $Q\,H^2(\mathbb{D}^2) \subseteq M$. So, we have $M = QH^2(\mathbb{D}^2)$. ∎

The proof of the sufficiency of Theorem 3.5. Let $r = GCD(p_1, p_2)$, and set $p_1 = r\widetilde{p}_1$, $p_2 = r\widetilde{p}_2$. Since $Z(r) \cap \mathbb{T}^2 = \emptyset$, r is bounded below on $\mathbb{T}^2$, and hence $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed if and only if $\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2)$ is closed. Hence we may assume $GCD(p_1, p_2) = 1$. By [14], $I = p_1\mathbb{C}[z, w] + p_2\mathbb{C}[z, w]$ is a finite codimensional ideal, and hence $Z(I) = Z(p_1) \cap Z(p_2)$ is a finite set. Write $Z(I) = \Lambda_1 \cup \Lambda_2$, where $\Lambda_1 \subset \mathbb{D}^2$ and $\Lambda_2 \subset \mathbb{C}^2 \backslash \overline{\mathbb{D}}^2$. Then I can be decomposed as $I = I_1 \cap I_2$ such that $Z(I_1) = \Lambda_1$ and $Z(I_2) = \Lambda_2$. Using the characteristic space theory for polynomials ([3], Chapter 2), we have

$$I = I_1 \cap I_2 = I_1 I_2.$$

Noticing that $Z(I_2) \cap \overline{\mathbb{D}}^2 = \emptyset$, this gives $I_2 H^2(\mathbb{D}^2) = H^2(\mathbb{D}^2)$, and hence

$$[I] = [I_1 I_2] = [I_1].$$

Since $Z(I_1) \subset \mathbb{D}^2$, by Theorem 2.2.8 of [3], $[I_1] \cap \mathbb{C}[z, w] = I_1$. Because $[I_1]$ is of finite codimension, by Lemma 3.6, $[I_1] = I_1 H^2(\mathbb{D}^2)$. Let $\{r_i\}_{i=1}^{s}$ and $\{q_j\}_{j=1}^{t}$ be sets of generators of I_1 and I_2, respectively. Then we have

$$\begin{aligned}[I_1] &= I_1 H^2(\mathbb{D}^2) = r_1 H^2(\mathbb{D}^2) + \cdots + r_s H^2(\mathbb{D}^2) = \sum_{i=1}^{s} r_i I_2 H^2(\mathbb{D}^2)\\ &= \sum_{i,j} r_i q_j H^2(\mathbb{D}^2) = I_1 I_2 H^2(\mathbb{D}^2) = I H^2(\mathbb{D}^2) = p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2).\end{aligned}$$

Thus $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed. ∎

The proof of the necessity of Theorem 3.5. We prove the necessity by two steps.
Step 1. $Z(r) \cap \mathbb{T}^2 = \emptyset$.

Set $r = GCD(p_1, p_2)$, and $p_1 = r\widetilde{p}_1$, $p_2 = r\widetilde{p}_2$. Since $GCD(\widetilde{p}_1, \widetilde{p}_2) = 1$, by [Ya1], $I = \widetilde{p}_1\mathbb{C}[z, w] + \widetilde{p}_2\mathbb{C}[z, w]$ is a finite codimensional ideal. So $\mathbb{C}[z, w] = I \dot{+} R$ where R is a finite dimensional space of polynomials, and hence $H^2(\mathbb{D}^2) = [I] \dot{+} R$.

Since $M = p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed, we have

$$\begin{aligned} M &= \overline{r(\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2))} \supseteq r\overline{(\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2))}\\ &\supseteq r(\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2)) = M.\end{aligned}$$

Thus,

$$rH^2(\mathbb{D}^2) = r[I] + rR = r\overline{(\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2))} + rR = M + rR.$$

Since R is a finite dimensional space, and M is closed, we see that $rH^2(\mathbb{D}^2)$ is closed. Therefore, the multiplication operator $M_r : H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ is injective, and it has the closed range. This implies that there is a positive constant C

such that
$$\|rf\|^2 \geqslant C\|f\|^2, \quad \forall f \in H^2(\mathbb{D}^2).$$
So, $\|r\,k_\lambda\|^2 \geqslant C$. Noticing that $\|r\,k_\lambda\|^2 = \int\limits_{\mathbb{T}^2} |r(\xi)|^2 P_\lambda(\xi) \mathrm{d}m_2$, where $P_\lambda(\xi)$ is the Poisson kernel at λ, this means that
$$\lim_{\lambda \to \varsigma} \|r\,k_\lambda\|^2 = |r(\varsigma)|^2, \quad \forall \varsigma \in \mathbb{T}^2,$$
and hence $|r(\varsigma)|^2 \geqslant C$ for all $\varsigma \in \mathbb{T}^2$.

Step 2. $Z(\widetilde{p}_1) \cap Z(\widetilde{p}_2) \cap \partial\mathbb{D}^2 = \varnothing$.

To prove this, we need the following lemma.

LEMMA 3.7. *Given $p_1, p_2 \in \mathbb{C}[z, w]$, $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed if and only if the operator $M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*$ has the closed range, and in this case*
$$\mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) = p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2).$$

Proof. On the one hand, if $\mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*)$ is closed, then
$$\begin{aligned}\mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) &= H^2(\mathbb{D}^2) \ominus \ker(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) \\ &= H^2(\mathbb{D}^2) \ominus (\ker M_{p_1}^* \cap \ker M_{p_2}^*) = \overline{p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)}.\end{aligned}$$
Since $\mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) \subseteq p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$, the above reasoning yields $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2) = \overline{p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)}$, and hence $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed.

On the other hand, if $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed, we define an operator $T: H^2(\mathbb{D}^2) \oplus H^2(\mathbb{D}^2) \to H^2(\mathbb{D}^2)$ by
$$T(h_1, h_2) = p_1 h_1 + p_2 h_2, \quad \forall\, h_1, h_2 \in H^2(\mathbb{D}^2).$$
It is easy to check that $T^* h = (M_{p_1}^* h, M_{p_2}^* h), \forall\, h \in H^2(\mathbb{D}^2)$. Considering the invertible operator $\widetilde{T} : H^2(\mathbb{D}^2) \oplus \frac{H^2(\mathbb{D}^2)}{\ker T} \to \mathrm{Ran}(T)$ induced by T, it is easy to see that for any $f \in \mathrm{Ran}(T)$, $\widetilde{T}\widetilde{T}^* f = TT^* f = (M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) f$. This means that
$$\begin{aligned}p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2) &= \mathrm{Ran}(T) = \mathrm{Ran}(\widetilde{T}) = \mathrm{Ran}(\widetilde{T}\widetilde{T}^*) = \mathrm{Ran}(TT^*) \\ &\subseteq \mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) \subseteq p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2).\end{aligned}$$
Hence,
$$\mathrm{Ran}(M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*) = p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$$
is closed. ∎

Step 2. By Step 1, r is bounded below on $\mathbb{T}^2$, hence $p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$ is closed if and only if $\widetilde{p}_1 H^2(\mathbb{D}^2) + \widetilde{p}_2 H^2(\mathbb{D}^2)$ is closed. So we can assume that $GCD(p_1, p_2) = 1$. Set $M = p_1 H^2(\mathbb{D}^2) + p_2 H^2(\mathbb{D}^2)$. Since $GCD(p_1, p_2) = 1$, by [14] we know that M is of finite codimension. Using Lemma 3.7, $M_{p_1} M_{p_1}^* + M_{p_2} M_{p_2}^*$ has closed range, and the codimension of the range is finite. This implies that

$M_{p_1}M_{p_1}^* + M_{p_2}M_{p_2}^*$ is Fredholm. From this fact, there exists a positive invertible operator X and a compact operator K such that

$$M_{p_1}M_{p_1}^* + M_{p_2}M_{p_2}^* = X + K.$$

Since the normalized reproducing kernel k_λ is convergent weakly to 0 as $\lambda \to \partial\mathbb{D}^2$, for any $\xi \in \partial\mathbb{D}^2$ we have

$$\begin{aligned} |p_1(\xi)|^2 + |p_2(\xi)|^2 &= \lim_{\lambda\to\xi}\langle (M_{p_1}M_{p_1}^* + M_{p_2}M_{p_2}^*)k_\lambda,\, k_\lambda\rangle \\ &= \lim_{\lambda\to\xi}\langle Xk_\lambda,\, k_\lambda\rangle + \lim_{\lambda\to\xi}\langle K k_\lambda, k_\lambda\rangle \geqslant \frac{1}{\|X^{-\frac{1}{2}}\|}. \end{aligned}$$

This insures that $Z(p_1)\cap Z(p_2)\cap\partial\mathbb{D}^2 = \emptyset$. ∎

REMARK 2. Let p_1 and p_2 be polynomials on $\mathbb{C}^2$, then the same reasoning as the above shows that the pair $(T_{p_1},\, T_{p_2})$ on $H^2(\mathbb{D}^2)$ is Fredholm if and only if

$$Z(p_1)\cap Z(p_2)\cap\partial\mathbb{D}^2 = \emptyset,$$

and in this case,

$$\mathrm{Ind}\,(T_{p_1},\, T_{p_2}) = -\frac{\dim\, H^2(\mathbb{D}^2)}{[p_1H^2(\mathbb{D}^2) + p_2H^2(\mathbb{D}^2)]}.$$

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