J. OPERATOR THEORY
58:2(2007), 269–310

MARTINGALES, ENDOMORPHISMS, AND COVARIANT SYSTEMS OF OPERATORS IN HILBERT SPACE

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Dedicated to the memory of J.L. Doob

Communicated by Şerban Strătilă

ABSTRACT. In the theory of wavelets, in the study of subshifts, in the analysis of Julia sets of rational maps of a complex variable, and, more generally, in the study of dynamical systems, we are faced with the problem of building a unitary operator from a mapping r in a compact metric space X. The space X may be a torus, or the state space of subshift dynamical systems, or a Julia set.

While our motivation derives from some wavelet problems, we have in mind other applications as well; and the issues involving covariant operator systems may be of independent interest.

KEYWORDS: *Wavelet, Julia set, subshift, Cuntz algebra, iterated function system (IFS), Perron-Frobenius-Ruelle operator, multiresolution, martingale, scaling function, transition probability.*

MSC (2000): 47C15, 33C50, 42C15, 42C40, 46E22, 47B32, 60J15, 60G42.

1. INTRODUCTION

In this paper, we aim at combining and using ideas from one area of mathematics (operator theory and traditional analysis) in a different area (martingale theory from probability). We have in mind applications to both wavelets and symbolic dynamics. So our paper is interdisciplinary: results in one area often benefit the other. In fact, the benefits go both ways.

Our construction is based on a closer examination of an eigenvalue problem for a transition operator, also called a Perron-Frobenius-Ruelle operator.

Under suitable conditions on the given filter functions, our construction takes place in the Hilbert space $L^2(\mathbb{R}^d)$. In a variety of examples, for example for frequency localized wavelets, more general filter functions are called for. This then entails basis constructions in Hilbert spaces of L^2-martingales. These martingale Hilbert spaces consist of L^2-functions on certain projective limit spaces X_∞ built on a given mapping $r : X \to X$ which is onto, and finite-to-one. We study function theory on X_∞ in a suitable general framework, as suggested by applications; and we develop our theory in the context of Hilbert space and operator theory.

We hope that these perhaps unexpected links between more traditional and narrowly defined fields will inspire further research. Since we wish to reach several audiences, we have included here a few more details than is perhaps standard in more specialized papers. The general question we address already has a number of incarnations in the literature, but they have so far not been unified. Here are two such examples which capture the essence of our focus. (a) Extension of non-invertible endomorphisms in one space X to automorphisms in a bigger space naturally containing X. (b) Some non-invertible operator S (contractive or isometric) in a fixed Hilbert space $\mathcal{H}$ is given. It is assumed that S is contractive and that it satisfies a certain covariance condition specified by a system of operators in $\mathcal{H}$. The question is then to extend S to a unitary operator U in a bigger Hilbert space which naturally contains $\mathcal{H}$, such that U satisfies a covariance condition arising by dilation from the initially given system on $\mathcal{H}$.

The dilation idea in operator theory is fundamental; i.e., the idea of extending (or dilating) an operator system on a fixed Hilbert space H_0 to a bigger ambient Hilbert space H in such a way as to get orthogonality relations in the dilated space H; see for example [35] and Remark 3.3 below. In an operator algebraic framework such an extension is of course encoded by Stinespring's theorem [39]. Our present setting is motivated by this, but goes beyond it in a number of ways, as we show in Sections 5–8 below.

Our basic viewpoint may be understood from the example of wavelets: A crucial strength of wavelet bases is their algorithmic and computational features. What this means in terms of the two Hilbert spaces are three things: First we must have a concrete function representation of the dilated space H; and secondly we aim for recursive and matrix based algorithms, much like the familiar case of Gram-Schmidt algorithms which lets us compute orthonormal bases, or frames (see e.g., [5]) in the dilated space H. Thirdly, we reverse the traditional point of view. Hence, the dilation idea is turned around: Starting with H, we wish to select a subspace H_0 which is computationally much more feasible. This idea is motivated by image processing where such a selected subspace H_0 corresponds to a chosen resolution, and where "resolution" is to be understood in the sense of optics; see e.g., [25] and [27]. The selection of subspace H_0 is made in such a way as to yield recursive algorithms to be used in computation of orthonormal bases, or frames in H, but starting with data from H_0.

Examples of (a) occur in thermodynamics, such as it is presented in its rigorous form by David Ruelle in [36] and [37]. Both (a) and (b) are present in the approach to wavelets that goes under the name multiresolution analysis (MRA) [14]. In this case, we can take X to be $\mathbb{R}/\mathbb{Z}$, or equivalently the circle, or the unit-interval $[0, 1)$, and the extension of X can be taken to be the real line $\mathbb{R}$ (see [14]), or it may be a suitable solenoid over X; see, e.g., [11] and [10]. In this case, the endomorphism in X is multiplication by 2 modulo the integers $\mathbb{Z}$, and the extension to $\mathbb{R}$ is simply $x \to 2x$. The more traditional settings for (b) are scattering theory [29] or the theory of extensions, or unitary dilations of operators in Hilbert space, as presented for example in [28], [9], and in the references given there.

Specifically, we study the problem of inducing operators on Hilbert space from non-invertible transformations on compact metric spaces. The operators, or representations must satisfy relations which mirror properties of the given point transformations.

While our setup allows a rather general formulation in the context of C^*-algebras, we will emphasize the case of induction from an abelian C^*-algebra. Hence, we will stress the special case when X is a given compact metric space, and $r : X \to X$ is a finite-to-one mapping of X onto X. Several of our results are in the measurable category; and in particular we are not assuming continuity of r, or any contractivity properties.

1.1. WAVELETS. Our results will apply to wavelets. In the theory of multiresolution wavelets, the problem is to construct a special basis in the Hilbert space $L^2(\mathbb{R}^d)$ from a set of numbers a_n, $n \in \mathbb{Z}^d$.

The starting point is the scaling identity

$$\varphi(t) = N^{1/2} \sum_{n\in\mathbb{Z}^d} a_n \varphi(At - n), \quad (t \in \mathbb{R}^d), \tag{1.1}$$

where A is a d by d matrix over $\mathbb{Z}$, with eigenvalues $|\lambda| > 1$, and $N = |\det A|$, and where φ is a function in $L^2(\mathbb{R}^d)$.

The first problem is to determine when (1.1) has a solution in $L^2(\mathbb{R}^d)$, and to establish how these solutions (*scaling functions*) depend on the coefficients a_n.

When the Fourier transform is applied, we get the equivalent formulation,

$$\widehat{\varphi}(x) = N^{-1/2} m_0({A^{\mathrm{tr}}}^{-1} x)\widehat{\varphi}({A^{\mathrm{tr}}}^{-1} x), \tag{1.2}$$

where $\widehat{\varphi}$ denotes the Fourier transform,

$$\widehat{\varphi}(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi x\cdot t} \varphi(t)\,\mathrm{d}t$$

and where now m_0 is a function on the torus

$$\mathbb{T}^d = \{z = (z_1, \dots, z_d) \in \mathbb{C}^d : |z_j| = 1, 1 \leqslant j \leqslant d\} = \mathbb{R}^d/\mathbb{Z}^d,$$

i.e.,

$$m_0(z) = \sum_{n\in\mathbb{Z}^d} a_n z^n = \sum_{n\in\mathbb{Z}^d} a_n \mathrm{e}^{-\mathrm{i}2\pi n\cdot x}.$$

The duality between the compact group $\mathbb{T}^d$ and the lattice $\mathbb{Z}^d$ is given by

$$\langle\, z \mid n \,\rangle = z^n = z_1^{n_1} \cdots z_d^{n_d}, \quad (z = (z_1, \dots, z_d), n = (n_1, \dots, n_d)).$$

In this case, matrix multiplication $x \mapsto Ax$ on $\mathbb{R}^d$ passes to the quotient $\mathbb{R}^d/\mathbb{Z}^d$, and we get an N-to-one mapping $x \mapsto Ax \mod \mathbb{Z}^d$, which we denote by r_A.

The function m_0 is called a low pass filter, and it is chosen such that the operator $S = S_{m_0}$ given by

$$(Sf)(z) = m_0(z)f(Az)$$

is an isometry on $H_0 = L^2(\mathbb{T}^d, \text{Haar measure})$. Moreover, $L^\infty(\mathbb{T}^d)$ acts as multiplication operators on H_0. If $g \in L^\infty(\mathbb{T})$

$$(M(g)f)(z) = g(z)f(z)$$

and

$$SM(g) = M(g(A\cdot))S. \tag{1.3}$$

A main problem is the extension of this covariance relation (1.3) to a bigger Hilbert space $H_0 \to H_{\text{ext}}$, $S \to S_{\text{ext}}$, such that S_{ext} is unitary in H_{ext}. We now sketch briefly this extension in some concrete cases of interest.

In Section 5, we construct a sequence of measures $\omega_0, \omega_1, \dots$ on $\mathbb{T}^d$ such that $L^2(\mathbb{T}^d, \omega_0) \simeq H_0$, and such that there are natural isometric embeddings

$$L^2(\mathbb{T}^d, \omega_n) \hookrightarrow L^2(\mathbb{T}^d, \omega_{n+1}), \quad f \mapsto f \circ r_A. \tag{1.4}$$

The limit in (1.4) defines a *martingale Hilbert space* $\mathcal{H}$ in such a way that the norm of the L^2-martingale f is

$$\|f\|^2 = \lim_{n\to\infty} \|P_n f\|^2_{L^2(\mathbb{T}^d, \omega_n)}.$$

We also state a pointwise a.e. convergence result (Section 6). If $\Psi : L^2(\mathbb{T}^d, \omega_n) \to L^2(\mathbb{R}^d)$ is defined by

$$\Psi : f_n \mapsto f_n(A^{-n}x)\widehat{\varphi}(x),$$

then Ψ is an isometry of $L^2(\mathbb{T}^d, \omega_n)$ into $L^2(\mathbb{R}^d)$.

Specifically

$$\int_{\mathbb{T}^d} |f_n|^2 \, d\omega_n = \int_{\mathbb{R}^d} |f_n(A^{-n}x)\widehat{\varphi}(x)|^2 \, dx. \tag{1.5}$$

As a result we have induced a system

$$(r_A, \mathbb{T}^d) \to (S_{m_0}, L^2(\mathbb{T}^d)) \to (U_A, L^2(\mathbb{R}^d)),$$

where

$$(U_A\xi)(x) = N^{1/2}f(Ax), \quad (f \in L^2(\mathbb{R}^d)) \tag{1.6}$$

U_A unitary; the system is determined by the given filter function m_0. It can be checked (see details in Section 6) that Ψ is an isometry, and that

$$U_A M(g) = M(g(A\cdot))U_A$$

holds on $L^2(\mathbb{R}^d)$. Moreover Ψ maps onto $L^2(\mathbb{R}^d)$ if the function m_0 doesn't vanish on a subset of positive measure.

In the case of wavelets, we ask for a wavelet basis in $L^2(\mathbb{R}^d)$ which is consistent with a suitable resolution subspace in $L^2(\mathbb{R}^d)$. Whether the basis is orthonormal, or just a Parseval frame, it may be constructed from a system of subband filters m_i, say with N frequency bands. These filters m_i may be realized as functions on $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, the d-torus. Typically the scaling operation is specified by a given expansive integral d by d matrix A.

Let $N := |\det A|$. Pass A to the quotient $X = \mathbb{R}^d/\mathbb{Z}^d$, and we get a mapping r of X onto X such that $\# r^{-1}(x) = N$ for all x in X, and the N branches of the inverse are strictly contractive in $X = \mathbb{R}^d/\mathbb{Z}^d$ if the eigenvalues of A satisfy $|\lambda| > 1$.

The subband filters m_i are defined in terms of this map, r_A, and the problem is now to realize the wavelet data in the Hilbert space $L^2(\mathbb{R}^d)$ in such a way that $r = r_A : X \to X$ induces the unitary scaling operator $f \mapsto N^{1/2} f(Ax)$ in $L^2(\mathbb{R}^d)$, see (1.6).

1.2. EXAMPLES (JULIA SETS, SUBSHIFTS). In this paper we will show that this extension from spaces X, with a finite-to-one mapping $r : X \to X$, to operator systems may be done quite generally, to apply to the case when X is a Julia set for a fixed rational function of a complex variable, i.e., $r(z) = \frac{p_1(z)}{p_2(z)}$, with p_1, p_2 polynomials, $z \in \mathbb{C}$ and $N = \max(\deg p_1, \deg p_2)$. Then $r : X(r) \to X(r)$ is N-to-1 except at the singular points of r. Here $X(r)$ denotes the Julia set of r. It also applies to shift invariant spaces $X(A)$ when A is a 0-1 matrix, and

$$X(A) = \left\{ (x_i) \in \prod_{\mathbb{N}} \{1, \dots, N\} : A(x_i, x_{i+1}) = 1 \right\}$$

and

$$r_A(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

is the familiar subshift. Note that $r_A : X(A) \to X(A)$ is onto if and only if every column in A contains at least one entry 1.

1.3. MARTINGALES. Part of the motivation for our paper derives from the papers by Richard Gundy [23], [24], [22], [21]. The second named author also acknowledges enlightening discussions with R. Gundy. The fundamental idea in these papers by Gundy et al is that multiresolutions should be understood as martingales in the sense of Doob [15],[16],[17] and Neveu [34]. And moreover that this is a natural viewpoint.

One substantial advantage of this viewpoint is that we are then able to handle the construction of wavelets from subband filters that are only assumed measurable, i.e., filters that fail to satisfy the regularity conditions that are traditionally imposed in wavelet analysis.

A second advantage is that the martingale approach applies to a number of wavelet-like constructions completely outside the traditional scope of wavelet analysis in the Hilbert space $L^2(\mathbb{R}^d)$. But more importantly, the martingale tools apply even when the operation of scaling doesn't take place in $\mathbb{R}^d$ at all, but rather in a compact Julia set from complex dynamics; or the scaling operation may be one of the shift in the subshift dynamics that is understood from that thermodynamical formalism of David Ruelle [36].

1.4. THE GENERAL THEORY. In each of the examples, we are faced with a given space X, and a finite-to-one mapping $r : X \to X$. The space X is equipped with a suitable family of measures μ_h, and the L^∞-functions on X act by multiplication on the corresponding L^2-spaces, $L^2(X, \mu_h)$. It is easy to see that there are L^2-isometries which intertwine the multiplication operators $M(g)$ and $M(g \circ r)$, as g ranges over $L^\infty(X)$. We have

$$\begin{array}{ccc} L^2(X,\mu_h) & \xrightarrow{S} & L^2(X,\mu_h) \\ \downarrow & & \downarrow \\ \mathcal{H}_{\text{ext}} & \xrightarrow{U} & \mathcal{H}_{\text{ext}} \end{array} \tag{1.7}$$

where the vertical maps are given by inclusions. Specifically,

$$SM(g) = M(g \circ r)S, \quad \text{and} \quad UM(g)U^{-1} = M(g \circ r). \tag{1.8}$$

But for spectral theoretic calculations, we need to have representations of $M(g)$ and $M(g \circ r)$ unitarily equivalent. That is true in traditional wavelet applications, but the unitary operator U in (1.8) is not acting on $L^2(X, \mu_h)$. Rather, the unitary U is acting by matrix scaling on a different Hilbert space, namely $L^2(\mathbb{R}^d, \text{Lebesgue measure})$,

$$U_A f(t) = |\det A|^{1/2} f(At), \quad (t \in \mathbb{R}^d, f \in L^2(\mathbb{R}^d).$$

In the other applications, Julia set, and shift-spaces, we aim for a similar construction. But in these other cases, it is not at all clear what the Hilbert space corresponding to $L^2(\mathbb{R}^d)$, and the corresponding unitary matrix scaling operator, should be.

We provide two answers to this question, one at an abstract level, and a second one which is a concrete function representation; Sections 4 and 5. At the abstract level, we show that the construction may be accomplished in Hilbert spaces which serve as unitary dilations of the initial structure, see (1.7). In the concrete, we show that the extended Hilbert spaces may be taken as Hilbert spaces of L^2-martingales on X. In fact, we present these as Hilbert spaces of L^2-functions built from a projective limit

$$X \xleftarrow{r} X \xleftarrow{r} X \leftarrow \cdots \leftarrow X_\infty.$$

This is analogous to the distinction between an abstract spectral theorem on the one hand, and a concrete spectral representation, on the other. To know details about multiplicities, and multiplicity functions (Section 4), we need the latter.

Our concrete version of the dilation Hilbert space $\mathcal{H}_{\text{ext}}$ from (1.7) is then

$$\mathcal{H}_{\text{ext}} \simeq L^2(X_\infty, \widehat{\mu}_h)$$

for a suitable measure $\widehat{\mu}_h$ on X_∞.

2. FUNCTIONS AND MEASURES ON X

Consider

(•) X a compact metric space;

(•) $\mathfrak{B} = \mathfrak{B}(X)$ a Borel sigma-algebra of subsets of X;

(•) $r : X \to X$ an onto, measurable map such that $\#r^{-1}(x) < \infty$ for all $x \in X$;

(•) $W : X \to [0, \infty)$;

(•) μ a positive Borel measure on X.

2.1. TRANSFORMATIONS OF FUNCTIONS AND MEASURES. (•) Let $g \in L^\infty(X)$. Then the following is the multiplication operator on $L^\infty(X)$ or on $L^2(X, \mu)$:

$$M(g)f = gf. \tag{2.1}$$

(•) Composition:

$$S_0 f = f \circ r, \text{ or } (S_0 f)(x) = f(r(x)), \quad (x \in X). \tag{2.2}$$

(•) If $m_0 \in L^\infty(X)$, we set

$$S_{m_0} = M(m_0) S_0,$$

or equivalently

$$(S_{m_0} f)(x) = m_0(x) f(r(x)), \quad (x \in X, f \in L^\infty(X)). \tag{2.3}$$

(•) $r^{-1}(E) := \{x \in X : r(x) \in E\}$ for $E \in \mathfrak{B}(X)$. Then

$$\mu \circ r^{-1}(E) = \mu(r^{-1}(E)), \quad (E \in \mathfrak{B}(X)).$$

2.2. PROPERTIES OF MEASURES μ ON X. DEFINITIONS. (i) Invariance:

$$\mu \circ r^{-1} = \mu. \tag{2.4}$$

(ii) Strong invariance:

$$\int_X f(x)\, \mathrm{d}\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y)\, \mathrm{d}\mu, \quad (f \in L^\infty(X)). \tag{2.5}$$

(iii) For $W : X \to [0, \infty)$, we have

$$(R_W f)(x) = \sum_{r(y)=x} W(y) f(y). \tag{2.6}$$

If $m_0 \in L^\infty(X, \mu)$ is complex valued, we use the notation $R_{m_0} := R_W$ where $W(x) = |m_0(x)|^2 / \#r^{-1}(r(x))$.

(a) A function $h : X \to [0, \infty)$ is said to be an eigenfunction for R_W if

$$R_W h = h. \tag{2.7}$$

(b) A Borel measure ν on X is said to be a left-eigenfunction for R_W if

$$\nu R_W = \nu, \tag{2.8}$$

or equivalently

$$\int_X R_W f \,\mathrm{d}\nu = \int_X f \,\mathrm{d}\nu, \quad \text{for all } f \in L^\infty(X).$$

LEMMA 2.1. (i) *For measures μ on X we have the implication* (2.5) $\Rightarrow$ (2.4)*, but not conversely.*

(ii) *If W is given and if ν and h satisfy* (2.8) *and* (2.7) *respectively, then the following satisfies* (2.4)*:*

$$\mathrm{d}\mu := h \,\mathrm{d}\nu. \tag{2.9}$$

(iii) *If μ satisfies* (2.5)*, and $m_0 \in L^\infty(X)$, then S_{m_0} is an isometry in $L^2(X, h\,\mathrm{d}\mu)$ if and only if*

$$R_{m_0} h = h.$$

Proof. (i) Suppose μ satisfies (2.5). Let $f \in L^\infty(X)$. Then

$$\int_X f \circ r \,\mathrm{d}\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(r(y)) \,\mathrm{d}\mu(x) = \int_X f \,\mathrm{d}\mu.$$

(ii) Let W, ν and h be as in the statement of part (ii) of the lemma. Then

$$\int_X f \circ r \,\mathrm{d}\mu = \int_X f \circ r\, h \,\mathrm{d}\nu = \int_X R_W(f \circ r\, h) \,\mathrm{d}\nu = \int_X f R_W h \,\mathrm{d}\nu = \int_X f h \,\mathrm{d}\nu = \int_X f \,\mathrm{d}\mu,$$

which is the desired conclusion (2.4). It follows in particular that (2.5) is strictly stronger than (2.4).

(iii) For $f \in L^\infty(X)$, we have

$$\begin{aligned} \|S_{m_0} f\|^2_{L^2(X, h\,\mathrm{d}\mu)} &= \int_X |m_0(x) f(rx)|^2 h(x) \,\mathrm{d}\mu \\ &= \int_X |f(x)|^2 \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 h(y) \,\mathrm{d}\mu(x) \\ &= \int_X |f(x)|^2 R_{m_0} h(x) \,\mathrm{d}\mu(x) = \int_X |f|^2 h \,\mathrm{d}\mu = \|f\|^2_{L^2(X, h\,\mathrm{d}\mu)} \end{aligned}$$

if and only if $R_{m_0} h = h$ and (iii) follows. ∎

We will use standard facts from measure theory: for example, we may identify positive Borel measures on X with positive linear functionals on $C(X)$ via

$$\Lambda_\omega(f) = \int_X f \,\mathrm{d}\omega.$$

In fact, we will identify Λ_ω and ω. For two measures μ and ν on X, we will use the notation $\mu \prec \nu$ to denote absolute continuity. For example $\mu \prec \nu$ holds in (2.9).

2.3. EXAMPLES. We illustrate the definitions:

EXAMPLE 2.2. Let $X = [0,1] = \mathbb{R}/\mathbb{Z}$. Fix $N \in \mathbb{Z}_+$, $N > 1$. Let

$$r(x) = Nx \mod 1.$$

Invariance:

$$\int_0^1 f(Nx)\,\mathrm{d}\mu(x) = \int_0^1 f(x)\,\mathrm{d}\mu(x), \quad (f \in L^\infty(\mathbb{R}/\mathbb{Z})). \tag{2.10}$$

Strong invariance:

$$\frac{1}{N}\int_0^1 \sum_{k=0}^{N-1} f(\frac{x+k}{N})\,\mathrm{d}\mu(x) = \int_0^1 f(x)\,\mathrm{d}\mu(x). \tag{2.11}$$

The Lebesgue measure $\mu = \lambda$ is the unique probability measure on $[0,1] = \mathbb{R}/\mathbb{Z}$ which satisfies (2.11).

Examples of measures μ on $\mathbb{R}/\mathbb{Z}$ which satisfy (2.10) but not (2.11) are

(•) $\mu = \delta_0$, the Dirac mass at $x = 0$;

(•) $\mu = \mu_{\mathbf{C}}$, the Cantor middle-third measure on $[0,1]$ (see [19]), i.e., $\mu_{\mathbf{C}}$ is determined by

(-) $\frac{1}{2}\int \big(f(\frac{x}{3}) + f(\frac{x+2}{3})\big)\,\mathrm{d}\mu_{\mathbf{C}}(x) = \int f(x)\,\mathrm{d}\mu_{\mathbf{C}}(x)$,

(-) $\mu_{\mathbf{C}}([0,1]) = 1$,

(-) $\mu_{\mathbf{C}}$ is supported on the middle-third Cantor set.

EXAMPLE 2.3. Let $X = [0,1) = \mathbb{R}/\mathbb{Z}$, λ the Lebesgue measure, $X_{\mathbf{C}}$ the middle-third Cantor set, $\mu_{\mathbf{C}}$ the Cantor measure. Then $r : X \to X$, $r(x) = 3x \mod 1$, $r_{\mathbf{C}} = r_{X_{\mathbf{C}}} : X_{\mathbf{C}} \to X_{\mathbf{C}}$.

Consider the following properties for a Borel probability measure μ on $\mathbb{R}$:

$$\int f\,\mathrm{d}\mu = \frac{1}{3}\int \Big(f(\frac{x}{3}) + f(\frac{x+1}{3}) + f(\frac{x+2}{3})\Big)\,\mathrm{d}\mu(x); \tag{2.12}$$

$$\int f\,\mathrm{d}\mu = \frac{1}{2}\int \Big(f(\frac{x}{3}) + f(\frac{x+2}{3})\Big)\,\mathrm{d}\mu(x). \tag{2.13}$$

Then (2.12) has a unique solution $\mu = \lambda$. Moreover (2.13) has a unique solution, $\mu = \mu_{\mathbf{C}}$, and $\mu_{\mathbf{C}}$ is supported on the Cantor set $X_{\mathbf{C}}$.

Let $\mathbb{R}/\mathbb{Z} = [0,1)$. Then $\#r^{-1}(x) = 3$ for all $x \in [0,1)$. If $x = \frac{x_1}{3} + \frac{x_2}{3^2} + \cdots$, $x_i \in \{0,1,2\}$, is the representation of x in base 3, then $r(x) \sim (x_2, x_3, \ldots)$, and $r^{-1}(x) = \{(0, x_1, x_2, \ldots), (1, x_1, x_2, \ldots), (2, x_1, x_2, \ldots)\}$.

On the Cantor set $\#r_{\mathbf{C}}^{-1}(x) = 2$ for all $x \in X_{\mathbf{C}}$. If $x = \frac{x_1}{3} + \frac{x_2}{3^2} + \cdots$, $x_i \in \{0,2\}$ is the usual representation of $X_{\mathbf{C}}$ in base 3, then

$$r_{\mathbf{C}}(x) = (x_2, x_3, \ldots) \quad \text{and} \quad X_{\mathbf{C}} \simeq \prod_{\mathbb{N}} \{0,2\}.$$

In the representation $\prod_{\mathbb{N}} \mathbb{Z}_3$ of $X = [0,1)$, $\mu = \lambda$ is the product (Bernoulli) measure with weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

In the representation $\prod_{\mathbb{N}}\{0,2\}$ of $X_{\mathbf{C}}$, $\mu_{\mathbf{C}}$ is the product (Bernoulli) measure with weights $(\frac{1}{2}, \frac{1}{2})$.

EXAMPLE 2.4. Let $N \in \mathbb{Z}_+$, $N \geqslant 2$ and let $A = (a_{ij})_{i,j=1}^N$ be an N by N matrix with all $a_{ij} \in \{0,1\}$. Set

$$X(A) := \left\{(x_i) \in \prod_{\mathbb{N}}\{1,\dots,N\} : A(x_i, x_{i+1}) = 1\right\}$$

and let $r = r_A$ be the restriction of the shift to $X(A)$, i.e.,

$$r_A(x_1, x_2, \dots) = (x_2, x_3, \dots), \quad (x = (x_1, x_2, \dots) \in X(A)).$$

LEMMA 2.5. *Let A be as above. Then*

$$\#r_A^{-1}(x) = \#\{y \in \{1,\dots,N\} : A(y, x_1) = 1\}.$$

It follows that $r_A : X(A) \to X(A)$ is onto if and only if A is *irreducible*, i.e., if and only if for all $j \in \mathbb{Z}_N$, there exists an $i \in \mathbb{Z}_N$ such that $A(i,j) = 1$. Suppose in addition that A is *aperiodic*, i.e., there exists $p \in \mathbb{Z}_+$ such that $A^p > 0$ on $\mathbb{Z}_N \times \mathbb{Z}_N$. We have the following lemma:

LEMMA 2.6 (D. Ruelle, [36], [6]). *Let A be irreducible and aperiodic and let $\phi \in C(X(A))$ be given. Assume that ϕ is a Lipschitz function.*

(i) *Set*

$$(R_\phi f)(x) = \sum_{r_A(y)=x} \mathrm{e}^{\phi(y)} f(y), \quad \textit{for } f \in C(X(A)).$$

Then there exist $\lambda_0 > 0$,

$$\lambda_0 = \sup\{|\lambda| : \lambda \in \operatorname{spec}(R_\phi)\},$$

$h \in C(X(A))$ strictly positive and ν a Borel measure on $X(A)$ such that

$$R_\phi h = \lambda_0 h, \quad \nu R_\phi = \lambda_0 \nu,$$

and $\nu(h) = 1$. The data is unique.

(ii) *In particular, setting*

$$(R_0 f)(x) = \frac{1}{\#r_A^{-1}(x)} \sum_{r_A(y)=x} f(y),$$

we may take $\lambda_0 = 1$, $h = 1$ and $\nu =: \mu_A$, where μ_A is a probability measure on $X(A)$ satisfying the strong invariance property

$$\int_{X(A)} f \, \mathrm{d}\mu_A = \int_{X(A)} \frac{1}{\#r_A^{-1}(x)} \sum_{r_A(y)=x} f(y) \, \mathrm{d}\mu_A(x), \quad (f \in L^\infty(X(A)).$$

3. POSITIVE DEFINITE FUNCTIONS AND DILATIONS

We now recall a result relating operator systems to positive definite functions. The idea dates back to Kolmogorov, but has been used recently in for example [20] and [18] (see also [2]).

DEFINITION 3.1. A map $K : X \times X \to \mathbb{C}$ is called *positive definite* if, for any $x_1, \dots, x_n \in X$ and any $\xi_1, \dots, \xi_n \in \mathbb{C}$,

$$\sum_{i,j=1}^{n} K(x_i, x_j)\overline{\xi}_i \xi_j \geqslant 0.$$

THEOREM 3.2 (Kolmogorov-Aronszajn). *Let $K : X \times X \to \mathbb{C}$ be positive definite. Then there exist a Hilbert space and a map $v : X \to H$ such that*

$$\overline{\text{span}}\{v(x) : x \in X\} = H, \quad \textit{and} \quad \langle\, v(x) \mid v(y)\, \rangle = K(x, y), \quad (x, y \in X).$$

Moreover H and v are unique up to isomorphism.

Proof. We sketch the idea of the proof. Take H to be the completion of the space

$$\{f : X \to \mathbb{C} : f \text{ has finite support}\}$$

with respect to the scalar product

$$\langle\, f \mid g\, \rangle = \sum_{x,y \in X} \overline{f(x)} K(x, y) g(y).$$

Then define $v(x) := \delta_x$. ∎

REMARK 3.3. Theorem 3.2 has a long history in operator theory. The version above is purely geometric, but as noted, for example in [35] and [8], it is possible to take the Hilbert space H in the theorem of the form $L^2(\Omega, \mathfrak{B}, \mu)$ where $(\Omega, \mathfrak{B}, \mu)$ is a probability space; i.e., $\mathfrak{B}$ is a sigma-algebra on some measure space Ω, μ a measure defined on $\mathfrak{B}$, $\mu(\Omega) = 1$. In that case, $v(x, \cdot)$ is a stochastic process. As is well known, it is even possible to make this choice such that the process is Gaussian. Examples of this include Brownian motion, and fractional Brownian motion, see also [31], [1], [25]; and [32] for a more operator theoretic approach.

For the purpose of the present discussion, it will be enough to know the Hilbert space H abstractly, but in the main part of our paper (Sections 5–8), the particular function representation will be of significance. To see this, take for example the case of the more familiar wavelet construction from Example 1.1 above. In the present framework, the space X is then the d-torus $\mathbb{T}^d$, while the ambient dilation Hilbert space H is $L^2(\mathbb{R}^d)$. Since wavelet bases must be realized in the ambient Hilbert space, it is significant to have much more detail than is encoded in the purely geometric data of Theorem 3.2. Even when comparing with the function theoretic version of [35], the wavelet example illustrates that it is significant to go beyond probability spaces.

One of our aims is to offer a framework for more general wavelet bases, including state spaces in symbolic dynamics and Julia sets (such as [12].) A main reason for the usefulness of wavelet bases is their computational features. As is well known [38], there are many function theoretic orthonormal bases (ONB), or Parseval frames in analysis where the basis coefficients do not lend themselves to practical algorithmic schemes. If for example we are in $L^2(\mathbb{R}^d)$, then the computation of each basis coefficients typically involves a separate integration over $\mathbb{R}^d$; not at all a computationally attractive proposition.

What our present approach does is that it selects a subspace of the ambient Hilbert space which is computationally much more feasible. As stressed in [5] and [27], such a selection corresponds to a choice of resolution, a notion from optics; and one dictated in turn by applications. In the present setup, the chosen resolution corresponds to an initial space, which in this context may be encoded by X from Theorem 3.2 above. As we will see later, there are ways to do this such that the computation of basis coefficients becomes algorithmic. We will talk about wavelet bases in this much more general context, even though wavelets are traditionally considered only in $L^2(\mathbb{R}^d)$. With good choices, we find that computation of the corresponding basis coefficients may be carried with a certain recursive algorithm involving only matrix iteration; much like in the familiar case of Gram-Schmidt algorithms.

THEOREM 3.4. *Let K be a positive definite map on a set X. Let $s : X \to X$ be a map that is compatible with K in the sense that*

$$K(s(x), s(y)) = K(x, y), \quad (x, y \in X). \tag{3.1}$$

Then there exist a Hilbert space H, a map $v : X \to H$ and a unitary operator U on H such that

$$\langle\, v(x) \mid v(y)\, \rangle = K(x, y), \quad (x, y \in X); \tag{3.2}$$

$$\overline{\text{span}}\{U^{-n}(v(x)) : x \in X, n \geqslant 0\} = H; \tag{3.3}$$

$$Uv(x) = v(s(x)), \quad (x \in X). \tag{3.4}$$

Moreover, this is unique up to an intertwining isomorphism.

Proof. Let $\widetilde{X} := X \times \mathbb{Z}$. Define $\widetilde{K} : \widetilde{X} \times \widetilde{X} \to \mathbb{C}$ by

$$\widetilde{K}((x, n), (y, m)) = K(s^{n+M}(x), s^{m+M}(y)), \quad (x, y \in X, n, m \in \mathbb{Z}),$$

where $M \geqslant \max\{-m, -n\}$.

The compatibility condition (3.1) implies that the definition does not depend on the choice of M. We check that $\widetilde{K}$ is positive definite. Take $(x_i, n_i) \in \widetilde{X}$ and $\xi_i \in \mathbb{C}$. Then, for M big enough we have:

$$\sum_{i,j} \widetilde{K}((x_i, n_i), (x_j, n_j))\overline{\xi}_i\xi_j = \sum_{i,j} K(s^{M+n_i}(x_i), s^{M+n_j}(x_j))\overline{\xi}_i\xi_j \geqslant 0.$$

Using now the Kolmogorov construction (see Theorem 3.2), there exists a Hilbert space H and a map $\widetilde{v} : \widetilde{X} \to H$ such that

$$\langle\, \widetilde{v}(x,n) \mid \widetilde{v}(y,m)\,\rangle = \widetilde{K}((x,n),(y,m)), \quad ((x,m),(y,n) \in \widetilde{X});$$
$$\overline{\text{span}}\{\widetilde{v}(x,m) : (x,m) \in \widetilde{X}\} = H.$$

Define v by

$$v(x) = \widetilde{v}(x,0), \quad (x \in X).$$

Then (3.2) is satisfied. Define

$$U\widetilde{v}(x,n) = \widetilde{v}(x,n+1), \quad ((x_n) \in \widetilde{X}).$$

U is well defined and an isometry because, for M sufficiently big,

$$\begin{aligned}\langle\, \widetilde{v}(x,n+1) \mid \widetilde{v}(y,m+1)\,\rangle &= K(s^{M+n+1}(x), s^{M+m+1}(y))\\ &= K(s^{M+n}(x), s^{M+m}(y)) = \langle\, \widetilde{v}(x,n) \mid \widetilde{v}(y,m)\,\rangle.\end{aligned}$$

U has dense range so U is unitary. Also (3.3) is immediate (we need only $n \geqslant 0$ because $U^n(v(x)) = v(s^n(x))$, for $n \geqslant 0$, will follow form (3.4)).

For (3.4) we compute

$$\begin{aligned}\langle\, Uv(x) \mid \widetilde{v}(y,n)\,\rangle &= \widetilde{K}((x,1),(y,n)) = K(s^{M+1}(x), s^{M+n}(y))\\ &= K(s^M(s(x)), s^{M+n}(y)) = \langle\, v(s(x)) \mid \widetilde{v}(y,n)\,\rangle.\end{aligned}$$

For uniqueness, if H', v', U' satisfy the same conditions, then the formula $W(U^n v(x)) = U'^n v'(x)$ defines an intertwining isomorphism. ∎

THEOREM 3.5. *Let $\mathcal{A}$ be a unital C^*-algebra, α an endomorphism on $\mathcal{A}$, μ a state on $\mathcal{A}$ and, $m_0 \in \mathcal{A}$, such that*

$$\mu(m_0^* \alpha(f) m_0) = \mu(f), \quad (f \in \mathcal{A}). \tag{3.5}$$

Then there exist a Hilbert space H, a representation π of $\mathcal{A}$ on H, U a unitary on H, and a vector $\varphi \in \mathcal{A}$, with the following properties:

$$U\pi(f)U^* = \pi(\alpha(f)), \quad (f \in \mathcal{A}); \tag{3.6}$$

$$\langle\, \varphi \mid \pi(f)\varphi\,\rangle = \mu(f), \quad (f \in \mathcal{A}); \tag{3.7}$$

$$U\varphi = \pi(\alpha(1)m_0)\varphi; \tag{3.8}$$

$$\overline{\text{span}}\{U^{-n}\pi(f)\varphi : n \geqslant 0, f \in \mathcal{A}\} = H. \tag{3.9}$$

Moreover, this is unique up to an intertwining isomorphism.

We call (H, U, π, φ) the covariant system associated to μ and m_0.

Proof. Define K and s by

$$K(x,y) = \mu(x^* y), \quad s(x) = \alpha(x)m_0, \quad (x,y \in \mathcal{A}).$$

K is positive definite and compatible with s so, with Theorem 3.4, there exists a Hilbert space H, a map v from $\mathcal{A}$ to H, and a unitary U with the mentioned properties.

Define $\varphi = v(1)$, then

$$\pi(f)(U^{-n}v(x)) = U^{-n}v(\alpha^n(f)x), \quad (f, x \in \mathcal{A}, n \geqslant 0).$$

Some straightforward computations show that π is a well defined representation of $\mathcal{A}$ that satisfies all requirements. ∎

COROLLARY 3.6. *Let X be a measure space, $r : X \to X$ a measurable, onto map and μ a probability measure on X such that*

$$\int_X f \, \mathrm{d}\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y) \, \mathrm{d}\mu(x). \tag{3.10}$$

Let $h \in L^1(X)$, $h \geqslant 0$ such that

$$\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 h(y) = h(x), \quad (x \in X).$$

Then there exist (uniquely up to isomorphisms) a Hilbert space H, a unitary U, a representation π of $L^\infty(X)$ and a vector $\varphi \in H$ such that:

$$U\pi(f)U^{-1} = \pi(f \circ r), \quad (f \in L^\infty(X));$$

$$\langle \varphi \mid \pi(f)\varphi \rangle = \int_X fh \, \mathrm{d}\mu, \quad (f \in L^\infty(X));$$

$$U\varphi = \pi(m_0)\varphi;$$

$$\overline{\mathrm{span}}\{U^{-n}\pi(f)\varphi : n \geqslant 0, f \in L^\infty(X)\} = H.$$

We call (H, U, π, φ) the covariant system associated to m_0 and h.

Proof. Take $\mu(f) = \int_X fh \, \mathrm{d}\mu$, $\alpha(f) = f \circ r$; and use Theorem 3.5. ∎

We regard Theorem 3.5 as a dilation result. In this context we have a second closely related result:

THEOREM 3.7. (i) *Let H be a Hilbert space, S an isometry on H. Then there exist a Hilbert space $\widehat{H}$ containing H and a unitary $\widehat{S}$ on $\widehat{H}$ such that:*

$$\widehat{S}|_H = S, \tag{3.11}$$

$$\overline{\bigcup_{n \geqslant 0} \widehat{S}^{-n} H} = \widehat{H}. \tag{3.12}$$

Moreover these are unique up to an intertwining isomorphism.

(ii) *If $\mathcal{A}$ is a C^*-algebra, α is an endomorphism on $\mathcal{A}$ and π is a representation of $\mathcal{A}$ on H such that*

$$S\pi(g) = \pi(\alpha(g))S, \quad (g \in \mathcal{A}); \tag{3.13}$$

then there exists a unique representation $\widehat{\pi}$ on $\widehat{H}$ such that:

$$\widehat{\pi}(g)|_H = \pi(g), \quad (g \in \mathcal{A}); \tag{3.14}$$

$$\widehat{S}\widehat{\pi}(g) = \widehat{\pi}(\alpha(g))\widehat{S}, \quad (g \in \mathcal{A}). \tag{3.15}$$

Proof. (i) Consider the set of symbols

$$\mathcal{H}_{\mathrm{sym}} := \Big\{ \sum_{j\in\mathbb{Z}} S^j\xi_j : \xi_j \in H, \xi_j = 0 \text{ except for finiteley many } j\text{'s}\Big\}.$$

Define the scalar product

$$\Big\langle \sum_{i\in\mathbb{Z}} S^i\xi_i \,\Big|\, \sum_{j\in\mathbb{Z}} S^j\eta_j \Big\rangle = \sum_{i,j\in\mathbb{Z}} \langle\, S^{i+m}\xi_i \mid S^{j+m}\eta_j \,\rangle, \tag{3.16}$$

where m is chosen sufficiently large, such that $i+m, j+m \geqslant 0$ for all $i, j \in \mathbb{Z}$ with $\xi_i \neq 0, \eta_j \neq 0$.

Since S is an isometry this definition does not depend on the choice of m. We denote the completion of $\mathcal{H}_{\mathrm{sym}}$ with this scalar product by $\widehat{H}$. H can be isometrically identified with a subspace of $\widehat{H}$ by

$$\xi \mapsto \sum_{i\in\mathbb{Z}} S^i\xi_i, \quad \text{where } \xi_i = \begin{cases} 0 & \text{if } \; i \neq 0, \\ \xi & \text{if } \; i = 0. \end{cases}$$

Define

$$\widehat{S}\Big(\sum_{i\in\mathbb{Z}} S^i\xi_i\Big) = \sum_{i\in\mathbb{Z}} S^{\mathrm{i}+1}\xi_i.$$

In the definition of $\widehat{H}$, we use (3.16) as an inner product, and we set

$$\widehat{H} = \Big(\mathcal{H}_{\mathrm{sym}} / \Big\{ \sum_j S^j\xi_j : \sum_{i,j} \langle\, S^{i+m}\xi_i \mid S^{j+m}\xi_j \,\rangle = 0 \Big\}\Big)^{\wedge}$$

where $\wedge$ stands for completion.

Since $\xi = S^{-1}(S\xi)$ in $\mathcal{H}_{\mathrm{sym}}$, for $\xi \in H$, we get natural isometric embeddings as follows, see (3.12),

$$H \subset \widehat{S}^{-1}H \subset \widehat{S}^{-2}H \subset \cdots \subset \widehat{S}^{-n}H \subset \widehat{S}^{-n-1}H \subset \cdots.$$

It can be checked that $\widehat{H}$ and $\widehat{S}$ satisfy the requirements.

(ii) We know that the spaces

$$\{\widehat{S}^{-n}\xi : n \geqslant 0, \xi \in H\}$$

span a dense subspace of $\widehat{H}$. Define

$$\widehat{\pi}(g)(\widehat{S}^{-n}\xi) = \widehat{S}^{-n}\pi(\alpha^n(g))\xi, \quad (g \in \mathcal{A}, n \geqslant 0, \xi \in H).$$

We check only that $\widehat{\pi}(g)$ is a well defined, bounded operator, the rest of our claims follow from some elementary computations. Take m large:

$$\left\|\widehat{\pi}(g)\Big(\sum_i \widehat{S}^{-n_i}\xi_i\Big)\right\|^2 = \left\|\sum_i \widehat{S}^{-n_i}\pi(\alpha^{n_i}(g))\xi_i\right\|^2 = \left\|\sum_i \widehat{S}^{m-n_i}\pi(\alpha^{n_i}(g))\xi_i\right\|^2$$
$$= \left\|\sum_i \pi(\alpha^m(g))\widehat{S}^{m-n_i}\xi_i\right\|^2 \leqslant \|g\|^2\left\|\sum_i \widehat{S}^{-n_i}\xi_i\right\|^2. \quad \blacksquare$$

EXAMPLE 3.8. This example is from [3], and it illustrates the conclusions in Theorem 3.7.

Consider

1. $H = l^2(\mathbb{N}_0)$.
2. $S(c_0, c_1, \dots) = (c_1, c_2, \dots)$, the unilateral shift.
3. $\delta_k(j) = \delta_{k,j} =$ Kronecker delta, for $k, j \in \mathbb{N}_0$.
4. $\pi(g_k)\delta_j := \exp(i2\pi k 2^{-j})\delta_j, \quad j \in \mathbb{N}_0, k \in \mathbb{Z}[\frac{1}{2}]$.

When Theorem 3.7 is applied we get:

1′. The dilation Hilbert space $\widehat{H}$ is $l^2(\mathbb{Z})$.
2′. $\widehat{S}$ is the bilateral shift on $l^2(\mathbb{Z})$ i.e., $\widehat{S}\delta_j = \delta_{j-1}$ for $j \in \mathbb{Z}$.
3′. Same as in 3. but for $k, j \in \mathbb{Z}$.
4′. The operator $\widehat{\pi}(g_k)$ is given by the same formula 4., but for $j \in \mathbb{Z}$.

The commutation relation (3.15) now takes the form

$$\widehat{S}\widehat{\pi}(g_k) = \widehat{\pi}(g_{2^{-1}k})\widehat{S} \text{ on } l^2(\mathbb{Z}), \quad \text{for } k \in \mathbb{Z}[\frac{1}{2}]; \tag{3.17}$$

$$\widehat{S}^{-n}H = \overline{\mathrm{span}}\{\delta_{-n}, \delta_{-n+1}, \delta_{-n+2}, \dots\} \subset \widehat{H}. \tag{3.18}$$

3.1. OPERATOR VALUED FILTERS. In this subsection we study the multiplicity configurations of the representations π from above. Our first result shows that the two functions m_0, and h in Section 2.1 may be operator valued. The explicit multiplicity functions are then calculated in the next section.

COROLLARY 3.9. *Let X, r, and μ be as in Corollary 3.6. Let I be a finite or countable set. Suppose $H : X \to \mathcal{B}(l^2(I))$ has the property that $H(x) \geqslant 0$ for almost every $x \in X$, and $H_{ij} \in L^1(X)$ for all $i, j \in I$. Let $M_0 : X \to \mathcal{B}(l^2(I))$ such that $x \mapsto \|M_0(x)\|$ is essentially bounded. Assume in addition that*

$$\frac{1}{\#r^{-1}(x)}\sum_{r(y)=x} M_0^*(y)H(y)M_0(y) = H(x), \quad \textit{for a.e. } x \in X. \tag{3.19}$$

Then there exist a Hilbert space $\widehat{K}$, a unitary operator $\widehat{U}$ on $\widehat{K}$, a representation $\widehat{\pi}$ of $L^\infty(X)$ on $\widehat{K}$, and a family of vectors $(\varphi_i) \in \widehat{K}$, such that:

$$\widehat{U}\widehat{\pi}(g)\widehat{U}^{-1} = \widehat{\pi}(g \circ r), \quad (g \in L^\infty(X));$$

$$\widehat{U}\varphi_i = \sum_{j\in I}\widehat{\pi}((M_0)_{ji})\varphi_j, \quad (i \in I);$$

$$\langle \varphi_i \mid \widehat{\pi}(f)\varphi_j \rangle = \int_X f H_{ij} \, \mathrm{d}\mu, \quad (i,j \in I, f \in L^\infty(X));$$

$$\overline{\operatorname{span}}\{\widehat{\pi}(f)\varphi_i : n \geqslant 0, f \in L^\infty(X), i \in I\} = \widehat{K}.$$

These are unique up to an intertwining unitary isomorphism. (All functions are assumed weakly measurable in the sense that $x \mapsto \langle \xi \mid F(x)\eta \rangle$ *is measurable for all* $\xi, \eta \in l^2(I)$*.)*

Proof. Consider the Hilbert space

$$K := \Big\{ f : X \to \mathbb{C}^I : f \text{ is measurable}, \int_X \langle f(x) \mid H(x)f(x) \rangle \, \mathrm{d}\mu(x) < \infty \Big\}.$$

Define S on K by

$$(Sf)(x) = M_0(x)(f(r(x))), \quad (x \in X, f \in K).$$

We check that S is an isometry. For $f, g \in K$:

$$\begin{aligned}
\langle Sg \mid Sf \rangle &= \int_X \langle M_0(x)g(r(x)) \mid H(x)M_0(x)f(r(x)) \rangle \, \mathrm{d}\mu(x) \\
&= \int_X \langle g(r(x)) \mid M_0(x)^* H(x) M_0(x) f(r(x)) \rangle \, \mathrm{d}\mu(x) \\
&= \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} \langle g(x) \mid M_0(y)^* H(y) M_0(y) f(x) \rangle \, \mathrm{d}\mu(x) \\
&= \int_X \langle g(x) \mid H(x) f(x) \rangle \, \mathrm{d}\mu(x) = \langle g \mid f \rangle,
\end{aligned}$$

where we used (3.19) in the last step. The converse implication holds as well, i.e., if S is an isometry then (3.19) is satisfied. Define now

$$(\pi(g)f)(x) = g(x)f(x), \quad (x \in X, g \in L^\infty(X), f \in K).$$

π defines a representation of $L^\infty(X)$ on K. Moreover, the covariance relation is satisfied:

$$S\pi(g) = \pi(g \circ r)S.$$

Then we use Theorem 3.7 to obtain a Hilbert space $\widehat{K}$ containing K, a unitary $\widehat{U} := \widehat{S}$ on $\widehat{K}$, and a representation $\widehat{\pi}$ on $\widehat{K}$ that dilate S and π.

Define $\varphi_i \in K \subset \widehat{K}$,

$$\varphi_i(x) := \delta_i, \text{ for all } x \in X, \quad (i \in I).$$

We have that

$$\langle \varphi_i \mid \widehat{\pi}(f)\varphi_j \rangle = \int_X \langle \delta_i \mid H(x)(f(x)\delta_j) \rangle \, \mathrm{d}\mu(x) = \int_X f(x) H_{ij}(x) \, \mathrm{d}\mu(x),$$

$$(\widehat{U}\varphi_i)(x) = (S\varphi_i)(x) = M_0(x)\delta_i = ((M_0)_{ji}(x))_{j\in I} = \Big(\sum_{j\in I}\widehat{\pi}((M_0)_{ji})\varphi_j\Big)(x).$$

Also it is clear that

$$\overline{\text{span}}\{\widehat{\pi}(f)\varphi_i : f \in L^\infty(X), i \in I\} = K.$$

These relations, together with Theorem 3.7, prove our assertions. ∎

4. MULTIPLICITY THEORY

One of the tools from operator theory which has been especially useful in the analysis of wavelets is multiplicity theory for abelian C^*-algebras $\mathcal{A}$.

We first recall a few well known facts, see e.g., [33]. By Gelfand's theorem, every abelian C^*-algebra with unit is $C(X)$ for a compact Hausdorff space X; and every representation of $\mathcal{A}$ is the orthogonal sum of cyclic representations. While the cardinality of the set of cyclic components in this decomposition is an invariant, the explicit determination of the cyclic components is problematic, as the construction depends on Zorn's lemma. So for this reason, it is desirable to turn the abstract spectral theorem for representations into a concrete one. In the concrete spectral representation, $C(X)$ is represented as an algebra of multiplication operators on a suitable L^2-space; as opposed to merely an abstract Hilbert space. When we further restrict attention to normal representations of $\mathcal{A}$, we will be working with the algebra $L^\infty(X)$ defined relative to the Borel sigma-algebra of subsets in X.

With this, we are able to compute a concrete spectral representation, and thereby to strengthen the conclusion from Theorem 3.7.

Our L^2-space which carries the representation may be realized concretely when the additional structure from Section 2.1 is introduced, i.e., is added to the assumptions in Theorem 3.7. Hence, we will work with the given finite-to-one mapping $r : X \to X$, and the measure μ from before. Recall from Section 2 that μ is assumed strongly r-invariant.

Theorem 3.7 provides an abstract unitary dilation of a given covariant system involving a representation π and a fixed isometry S on a Hilbert space H. In the present section, we specialize the representation π in Theorem 3.7 to the algebra $\mathcal{A} = L^\infty(X)$, and $\alpha : \mathcal{A} \to \mathcal{A}$, is $\alpha(g) := g \circ r$. While our conclusion from Theorem 3.7 still offers a unitary dilation U in an abstract Hilbert space $\widehat{H}$, we are now able to show that $\widehat{H}$ has a concrete spectral representation. Since $\widehat{H}$ is the closure of an ascending union of resolution subspaces defined from U, the question arises as to how the multiplicities of the restricted representations of the resolution subspaces in $\widehat{H}$ are related to one-another.

The answer to this is known in the case of wavelets, see e.g., [4]. In this section we show that there is a version of the Baggett et al multiplicity formula in the much more general setting of Theorem 3.7. In particular, we get the multiplicity formula in the applications where X is a Julia set, or a state space of sub-shift dynamical system. As we noted in Section 2 above, each of these examples carries a natural mapping r, and a strongly r-invariant measure μ.

Consider X a measure space, $r : X \to X$ an onto, measurable map such that $\#r^{-1}(x) < \infty$ for all $x \in X$. Let μ be a measure on X such that

$$\int_X f \, \mathrm{d}\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y) \, \mathrm{d}\mu(x), \quad (f \in L^\infty(X)). \tag{4.1}$$

Suppose now that H is a Hilbert space with an isometry S on it and with a normal representation π of $L^\infty(X)$ on H that satisfies the covariance relation

$$S\pi(g) = \pi(g \circ r)S, \quad (g \in L^\infty(X)). \tag{4.2}$$

Theorem 3.7 shows that there exists a Hilbert space $\widehat{H}$ containing H, a unitary $\widehat{S}$ on $\widehat{H}$ and a representation $\widehat{\pi}$ of $L^\infty(X)$ on $\widehat{H}$ such that:

$(V_n := \widehat{S}^{-n}(H))_n$ form an increasing sequence of subspaces with dense union,

and $\quad \widehat{S}|_H = S, \quad \widehat{\pi}|_H = \pi, \quad \widehat{S}\widehat{\pi}(g) = \widehat{\pi}(g \circ r)\widehat{S}.$

THEOREM 4.1. (i) $V_1 = \widehat{S}^{-1}(H)$ *is invariant for the representation* $\widehat{\pi}$*. The multiplicity functions of the representation* $\widehat{\pi}$ *on* V_1*, and on* $V_0 = H$*, are related by*

$$m_{V_1}(x) = \sum_{r(y)=x} m_{V_0}(y), \quad (x \in X). \tag{4.3}$$

(ii) *If* $W_0 := V_1 \ominus V_0 = \widehat{S}^{-1}H \ominus H$*, then*

$$m_{V_0}(x) + m_{W_0}(x) = \sum_{r(y)=x} m_{V_0}(y), \quad (x \in X). \tag{4.4}$$

Proof. Note that $\widehat{S}$ maps V_1 to V_0, and the covariance relation implies that the representation $\widehat{\pi}$ on V_1 is isomorphic to the representation $\pi^r : g \mapsto \pi(g \circ r)$ on V_0. Therefore we have to compute the multiplicity of the latter, which we denote by $m^r_{V_0}$.

By the spectral theorem there exists a unitary isomorphism $J : H(= V_0) \to L^2(X, m_{V_0}, \mu)$, where, for a *multiplicity function* $m : X \to \{0, 1, \ldots, \infty\}$, we use the notation:

$$L^2(X, m, \mu) := \Big\{ f : X \to \bigcup_{x \in X} \mathbb{C}^{m(x)} : f(x) \in \mathbb{C}^{m(x)}, \int_X \|f(x)\|^2 \, \mathrm{d}\mu(x) < \infty \Big\}.$$

In addition J intertwines π with the representation of $L^\infty(X)$ by multiplication operators, i.e.,

$$(J\pi(g)J^{-1}(f))(x) = g(x)f(x) \quad (g \in L^\infty(X), f \in L^2(X, m_{V_0}, \mu), x \in X).$$

REMARK 4.2. Here we are identifying H with $L^2(X, m_{V_0}, \mu)$ via the *spectral representation*. We recall the details of this representation $H \ni f \mapsto \widetilde{f} \in L^2(X, m_{V_0}, \mu)$.

Recall that any normal representation $\pi \in \mathrm{Rep}(L^\infty(X), H)$ is the orthogonal sum

$$H = \sum_{k \in C}{}^{\oplus} [\pi(L^\infty(X))k], \tag{4.5}$$

where the set C of vectors $k \in H$ is chosen such that:

($\bullet$) $\|k\| = 1$,

$$\langle\, k \mid \pi(g)k \,\rangle = \int_X g(x) v_k(x)^2 \,\mathrm{d}\mu(x), \quad \text{for all } k \in C; \tag{4.6}$$

($\bullet$) $\langle\, k' \mid \pi(g)k \,\rangle = 0, \quad g \in L^\infty(X), k, k' \in C, k \neq k'$; orthogonality.

The formula (4.5) is obtained by a use of Zorn's lemma. Here, v_k^2 is the Radon-Nikodym derivative of $\langle\, k \mid \pi(\cdot)k \,\rangle$ with respect to μ, and we use that π is assumed normal.

For $f \in H$, set

$$f = \sum_{k \in C}{}^{\oplus} \pi(g_k)k, \quad g_k \in L^\infty(X) \quad \text{and} \quad \widetilde{f} = \sum_{k \in C}{}^{\oplus} g_k v_k \in L^2_\mu(X, l^2(C)).$$

Then $Wf = \widetilde{f}$ is the desired spectral transform, i.e.,

$$W \text{ is unitary}, \quad W\pi(g) = M(g)W, \quad \text{and} \quad \|\widetilde{f}(x)\|^2 = \sum_{k \in C} |g_k(x) v_k(x)|^2.$$

Indeed, we have

$$\begin{aligned} \int_X \|\widetilde{f}(x)\|^2 \,\mathrm{d}\mu(x) &= \int_X \sum_{k \in C} |g_k(x)|^2 v_k(x)^2 \,\mathrm{d}\mu(x) = \sum_{k \in C} \int_X |g_k|^2 v_k^2 \,\mathrm{d}\mu \\ &= \sum_{k \in C} \langle\, k \mid \pi(|g_k|^2)k \,\rangle = \sum_{k \in C} \|\pi(g_k)k\|^2 = \Big\| \sum_{k \in C}{}^{\oplus} \pi(g_k)k \Big\|_H^2 = \|f\|_H^2. \end{aligned}$$

It follows in particular that the multiplicity function $m(x) = m_H(x)$ is

$$m(x) = \#\{k \in C : v_k(x) \neq 0\}.$$

Setting

$$X_i := \{x \in X : m(x) \geqslant i\}, \quad (i \geqslant 1),$$

we see that

$$H \simeq \sum{}^{\oplus} L^2(X_i, \mu) \simeq L^2(X, m, \mu),$$

and the isomorphism intertwines $\pi(g)$ with multiplication operators.

Returning to the proof of the theorem, we have to find the similar form for the representation π^r. Let

$$\widetilde{m}(x) := \sum_{r(y) = x} m_{V_0}(y), \quad (x \in X). \tag{4.7}$$

Define the following unitary isomorphism:

$$L : L^2(X, m_{V_0}, \mu) \to L^2(X, \widetilde{m}, \mu), \quad (L\xi)(x) = \frac{1}{\sqrt{\#r^{-1}(x)}}(\xi(y))_{r(y)=x}.$$

(Note that the dimensions of the vectors match because of (4.7)). This operator L is unitary. For $\xi \in L^2(X, m_{V_0}, \mu)$, we have

$$\begin{aligned}\|L\xi\|^2_{L^2(X,m_{V_0},\mu)} &= \int_X \|L\xi(x)\|^2 \,\mathrm{d}\mu(x) = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} \|\xi(y)\|^2 \,\mathrm{d}\mu(x)\\ &= \int_X \|\xi(x)\|^2 \,\mathrm{d}\mu(x).\end{aligned}$$

And L intertwines the representations. Indeed, for $g \in L^\infty(X)$,

$$L(g \circ r\, \xi)(x) = (g(r(y))\xi(y))_{r(y)=x} = g(x)L(\xi)(x).$$

Therefore, the multiplicity of the representation $\pi^r : g \mapsto \pi(g \circ r)$ on V_0 is $\widetilde{m}$, and this proves (i).

(ii) follows from (i).

Conclusions. By definition, if $k \in C$,

$$\langle\, k \mid \pi(g)k\,\rangle = \int_X g(x)v_k(x)^2 \,\mathrm{d}\mu(x), \quad \text{and}$$

$$\langle\, k \mid \pi^r(g)k\,\rangle = \int_X g(r(x))v_k(x)^2 \,\mathrm{d}\mu(x) = \int_X g(x)\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} v_k(x)^2 \,\mathrm{d}\mu(x);$$

and so

$$m^r(x) = \#\Big\{k \in C : \sum_{r(y)=x} v_k(y)^2 > 0\Big\} = \sum_{r(y)=x} \#\{k \in C : v_k(y)^2 > 0\} = \sum_{r(y)=x} m(y).$$

Let $C^m(x) := \{k \in C : v_k(x) \neq 0\}$. Then we showed that

$$C^m(x) = \bigcup_{y \in X, r(y)=x} C^m(y)$$

and that $C^m(y) \cap C^m(y') = \emptyset$ when $y \neq y'$ and $r(y) = r(y') = x$. Setting $\mathcal{H}(x) = l^2(C^m(x))$, we have

$$\mathcal{H}(x) = l^2(C^m(x)) = \sum_{r(y)=x}{}^{\oplus} l^2(C^m(y)) = \sum_{r(y)=x}{}^{\oplus} \mathcal{H}(y). \quad \blacksquare$$

REMARK 4.3. There are many representations $(\pi, U, \widehat{H})$ for which

$$U\pi(g)U^{-1} = \pi(g \circ r), \quad (g \in C(X)),$$

holds; but for which the spectral measures of π are not absolutely continuous; i.e., the measure

$$g \mapsto \langle \widehat{h} \mid \pi(g)\widehat{h} \rangle = \int_X g(x)\,\mathrm{d}\mu_{\widehat{h}}(x)$$

is singular with respect to the Julia-measure μ for some $\widehat{h} \in \widehat{H}$. But for the purpose of wavelet analysis, it is necessary to restrict our attention to normal representations π.

5. PROJECTIVE LIMITS

We work in either the category of measure spaces or topological spaces.

DEFINITION 5.1. Let $r : X \to X$ be onto, and assume that $\#r^{-1}(x) < \infty$ for all $x \in X$. We define the *projective limit* of the system:

$$X \xleftarrow{r} X \xleftarrow{r} X \xleftarrow{r} \cdots \leftarrow X_\infty \tag{5.1}$$

as

$$X_\infty := \{\widehat{x} = (x_0, x_1, \dots) : r(x_{n+1}) = x_n, \text{ for all } n \geqslant 0\}.$$

Let $\theta_n : X_\infty \to X$ be the projection onto the n-th component:

$$\theta_n(x_0, x_1, \dots) = x_n, \quad ((x_0, x_1, \dots) \in X_\infty).$$

Taking inverse images of sets in X through these projections, we obtain a sigma algebra on X_∞, or a topology on X_∞. We have an induced mapping $\widehat{r} : X_\infty \to X_\infty$ defined by

$$\widehat{r}(\widehat{x}) = (r(x_0), x_0, x_1, \dots), \quad \text{and with inverse } \widehat{r}^{-1}(\widehat{x}) = (x_1, x_2, \dots). \tag{5.2}$$

so $\widehat{r}$ is an automorphism, i.e., $\widehat{r} \circ \widehat{r}^{-1} = \mathrm{id}_{X_\infty}$ and $\widehat{r}^{-1} \circ \widehat{r} = \mathrm{id}_{X_\infty}$.

Note that

$$\theta_n \circ \widehat{r} = r \circ \theta_n = \theta_{n-1};$$

$$\begin{tikzcd} X_\infty \arrow[r, "\theta_n"] \arrow[dr, "\theta_{n-1}"'] & X \arrow[d, "r"] \\ & X \end{tikzcd} \quad \begin{tikzcd} X_\infty \arrow[r, "\widehat{r}"] \arrow[dr, "\theta_{n-1}"'] & X_\infty \arrow[d, "\theta_n"] \\ & X \end{tikzcd} \quad \begin{tikzcd} X_\infty \arrow[r, "\widehat{r}"] \arrow[d, "\theta_n"] & X_\infty \arrow[d, "\theta_n"] \\ X \arrow[r, "r"'] & X. \end{tikzcd}$$

Consider a probability measure μ on X that satisfies

$$\int_X f\,\mathrm{d}\mu = \int_X \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} f(y)\,\mathrm{d}\mu(x). \tag{5.3}$$

It is known that such measures μ on X exist for a general class of systems $r : X \to X$. The measure μ is said to be *strongly r-invariant*. We have already discussed some examples in Section 2 above.

If $X = X(A)$ is the state space of a sub-shift, we saw that $\mu = \mu_A$ may be constructed as an application of Ruelle's theorem (see Lemma 2.6). If $X = \text{Julia}(r)$ is the Julia set of some rational mapping, then it is also known [7], [30] that a strongly r-invariant measure μ on $X = \text{Julia}(r)$ exists.

For $m_0 \in L^\infty(X)$, define

$$(R\xi)(x) = \frac{1}{\# r^{-1}(x)} \sum_{r(y)=x} |m_0(y)|^2 \xi(y), \quad (\xi \in L^1(X)). \tag{5.4}$$

The next two theorems (Theorem 5.3–5.4) are key to our dilation theory. The dilations which we construct take place at three levels as follows:

(•) Dynamical systems:

$$(X, r, \mu) \text{ endomorphism } \to (X_\infty, \widehat{r}, \widehat{\mu}), \text{ automorphism.}$$

(•) Hilbert spaces:

$$L_2(X, h\,\mathrm{d}\mu) \to (R_{m_0} h = h) \to L^2(X_\infty, \widehat{\mu}).$$

(•) Operators:

$$\begin{aligned} &S_{m_0} \text{ isometry } \to U \text{ unitary (if } m_0 \text{ is non-singular);} \\ &M(g) \text{ multiplication operator } \to M_\infty(g). \end{aligned}$$

DEFINITION 5.2. A function m_0 on a measure space is called *singular* if $m_0 = 0$ on a set of positive measure.

In general, the operators S_{m_0} on $H_0 = L^2(X, h\,\mathrm{d}\mu)$, and U on $L^2(X_\infty, \widehat{\mu})$, may be given only by abstract Hilbert space axioms; but in our *martingale representation*, we get the following two concrete formulas:

$$\begin{aligned} (S_{m_0}\xi)(x) &= m_0(x)\xi(r(x)), && (x \in X, \xi \in H_0); \\ (Uf)(\widehat{x}) &= m_0(x_0) f(\widehat{r}(\widehat{x})), && (\widehat{x} \in X_\infty, f \in L^2(X_\infty, \widehat{\mu})). \end{aligned}$$

THEOREM 5.3. *If $h \in L^1(X)$, $h \geqslant 0$ and $Rh = h$, then there exists a unique measure $\widehat{\mu}$ on X_∞ such that*

$$\widehat{\mu} \circ \theta_n^{-1} = \omega_n, \quad (n \geqslant 0),$$

where

$$\omega_n(f) = \int_X R^n(fh)\,\mathrm{d}\mu, \quad (f \in L^\infty(X)). \tag{5.5}$$

Proof. It is enough to check that the measures ω_n and ω_{n+1} are *compatible*, i.e., we have to check if

$$\omega_{n+1}(f \circ r) = \omega_n(f), \quad (f \in L^\infty(X)).$$

But

$$R^{n+1}(f \circ r\, h) = R^n(R(f \circ r\, h)) = R^n(fRh) = R^n(fh). \quad \blacksquare$$

Note that we can identify functions on X with functions on X_∞ by

$$f(x_0, x_1, \dots) = f(x_0), \quad (f : X \to \mathbb{C}).$$

THEOREM 5.4.

$$\frac{\mathrm{d}(\widehat{\mu} \circ \widehat{r}^{-1})}{\mathrm{d}\widehat{\mu}} = |m_0|^2. \tag{5.6}$$

Proof. Equation (5.6) can be rewritten as

$$\int_{X_\infty} |m_0|^2 f \circ \widehat{r}\, \mathrm{d}\widehat{\mu} = \int_{X_\infty} f\, \mathrm{d}\widehat{\mu}, \quad (f \in L^\infty(\widehat{\mu})).$$

By the uniqueness of $\widehat{\mu}$, it is enough to check that

$$\int_{X_\infty} |m_0|^2(x_0)(f \circ \theta_n) \circ \widehat{r}(\widehat{x})\, \mathrm{d}\widehat{\mu}(\widehat{x}) = \omega_n(f), \quad (f \in L^\infty(X)),$$

or, equivalently (since $\theta_n \widehat{r} = r\theta_n$ and $x_0 = r^n(x_n)$):

$$\omega_n(|m_0|^2 \circ r^n\, f \circ r) = \omega_n(f). \tag{5.7}$$

We can compute:

$$\begin{aligned}
\int_X R^n(|m_0|^2 \circ r^n f \circ r\, h)\, \mathrm{d}\mu &= \int_X |m_0|^2 R^n(f \circ r\, h)\, \mathrm{d}\mu = \int_X |m_0|^2 R^{n-1}(fRh)\, \mathrm{d}\mu \\
&= \int_X |m_0|^2 R^{n-1}(fh)\, \mathrm{d}\mu = \int_X R(R^{n-1}(fh))\, \mathrm{d}\mu,
\end{aligned}$$

and we used (5.3) for the last equality. This proves (5.7) and the theorem. $\blacksquare$

THEOREM 5.5. *Suppose m_0 is non-singular, i.e., it does not vanish on a set of positive measure. Define U on $L^2(X_\infty, \widehat{\mu})$ by:*

$$\begin{aligned}
Uf &= m_0 f \circ \widehat{r}, && (f \in L^2(X_\infty, \widehat{\mu})), \\
\pi(g)f &= gf, && (g \in L^\infty(X), f \in L^2(X_\infty, \widehat{\mu})), \\
\varphi &= 1.
\end{aligned}$$

Then $(L^2(X_\infty, \widehat{\mu}), U, \pi, \varphi)$ is the covariant system associated to m_0 and h as in Corollary 3.6. *Moreover, if $M_g f = gf$ for $g \in L^\infty(X_\infty, \widehat{\mu})$ and $f \in L^2(X_\infty, \widehat{\mu})$, then*

$$UM_gU^{-1} = M_{g \circ \widehat{r}}.$$

Proof. Theorem 5.4 shows that U is isometric. Since m_0 is non-singular, the same theorem can be used to deduce that

$$U^* f = \frac{1}{m_0 \circ \widehat{r}^{-1}} f \circ \widehat{r}^{-1}$$

is a well defined inverse for U.

The covariance relation follows by a direct computation. Also we obtain

$$U^{-n}\pi(g)U^n f = g \circ \widehat{r}^{-n} f, \quad (g \in L^\infty(X), f \in L^2(X_\infty, \widehat{\mu})),$$

which shows that φ is cyclic.

The other requirements of Corollary 3.6 are easily obtained by computation. ∎

REMARK 5.6. When m_0 is singular U is just an isometry (not onto). However, we still have many of the relations: the covariance relation becomes

$$U\pi(f) = \pi(f \circ r)U, \quad (f \in L^\infty(X)),$$

the scaling equation remains true,

$$U\varphi = \pi(m_0)\varphi, \tag{5.8}$$

and the correlation function of φ is h:

$$\langle\, \varphi \mid \pi(f)\varphi \,\rangle = \int_X fh \, \mathrm{d}\mu, \quad (f \in L^\infty(X)).$$

We further note that equation (5.8) is an abstract version of the scaling identity from wavelet theory. In Section 1 we recalled the scaling equation in its two equivalent forms, the additive version (1.1), and its multiplicative version (1.2). The two versions are equivalent via the Fourier transform.

6. MARTINGALES

We give now a different representation of the construction of the covariant system associated to m_0 and h given in Theorem 5.5.

Let

$$H_n := \{f \in L^2(X_\infty, \widehat{\mu}) : f = \xi \circ \theta_n, \xi \in L^2(X, \omega_n)\}.$$

Then H_n form an increasing sequence of closed subspaces which have dense union.

We can identify the functions in H_n with functions in $L^2(X, \omega_n)$, by

$$i_n(\xi) = \xi \circ \theta_n, \quad (\xi \in L^2(X, \omega_n)).$$

The definition of $\widehat{\mu}$ makes i_n an isomorphism between H_n and $L^2(X, \omega_n)$.

Define

$$\mathcal{H} := \left\{(\xi_0, \xi_1, \ldots) : \xi_n \in L^2(X, \omega_n), R(\xi_{n+1}h) = \xi_n h, \sup_n \int_X R^n(|\xi_n|^2 h) \, \mathrm{d}\mu < \infty\right\},$$

with the scalar product

$$\langle\, (\xi_0, \xi_1, \ldots) \mid (\eta_0, \eta_1, \ldots) \,\rangle = \lim_{n\to\infty} \int_X R^n(\overline{\xi}_n \eta_n h) \, \mathrm{d}\mu.$$

THEOREM 6.1. *The map $\Phi : L^2(X_\infty, \widehat{\mu}) \to \mathcal{H}$ defined by*

$$\Phi(f) = (i_n^{-1}(P_n f))_{n\geqslant 0},$$

where P_n is the projection onto H_n, is an isomorphism. Then:

$$\Phi U \Phi^{-1}(\xi_n)_{n\geqslant 0} = (m_0 \circ r^n\, \xi_{n+1})_{n\geqslant 0}, \quad \Phi\pi(g)\Phi^{-1}(\xi_n)_{n\geqslant 0} = (g \circ r^n\, \xi_n)_{n\geqslant 0},$$

$$\Phi\varphi = (1, 1, \ldots).$$

Proof. Let $\xi_n := i_n^{-1}(P_n f)$. We check that $R(\xi_{n+1}h) = \xi_n h$. For this it is enough to see that the projection of $\xi_{n+1} \circ \theta_{n+1}$ onto H_n is $\left(\frac{R(\xi_{n+1}h)}{h}\right) \circ \theta_n$. We compute the scalar products with $g \circ \theta_n \in H_n$:

$$\langle\, \xi_{n+1} \circ \theta_{n+1} \mid g \circ \theta_n \,\rangle = \int_{X_\infty} \overline{\xi}_{n+1} \circ \theta_{n+1} g \circ r \circ \theta_{n+1} \,\mathrm{d}\widehat{\mu} = \int_X R^{n+1}(\overline{\xi}_{n+1} g \circ rh) \,\mathrm{d}\mu$$

$$= \int_X R^n\Big(g \frac{R(\overline{\xi}_{n+1}h)}{h} h\Big) \,\mathrm{d}\mu = \Big\langle\, \frac{R(\xi_{n+1}h)}{h} \circ \theta_n \,\Big|\, g \circ \theta_n \,\Big\rangle.$$

Since the union of (H_n) is dense, $P_n f$ converges to f. As each i_n is isometric,

$$\langle\, f \mid g \,\rangle = \lim_{n\to\infty} \langle\, P_n f \mid P_n g \,\rangle = \lim_{n\to\infty} \langle\, \Phi(f)_n \mid \Phi(g)_n \,\rangle_{L^2(X,\omega_n)} = \langle\, \Phi(f) \mid \Phi(g) \,\rangle.$$

Now we check that Φ is onto. Take $(\xi_n)_{n\geqslant 0} \in \mathcal{H}$. Then define

$$f_n := \xi_n \circ \theta_n = i_n^{-1}(\xi_n).$$

The previous computation shows that

$$P_n f_{n+1} = f_n.$$

Also

$$\sup_n \|f_n\|^2 = \sup_n \int_X R^n(|\xi_n|^2 h) \,\mathrm{d}\mu < \infty.$$

But then, by a standard Hilbert space argument, f_n is a Cauchy sequence which converges to some

$$f = \lim_{n\to\infty} f_n = f_0 + \sum_{k=0}^{\infty} (f_{k+1} - f_k) \in L^2(X_\infty, \mu)$$

with $P_n f = f_n$ for all $n \geqslant 0$, and we conclude that $\Phi(f) = (\xi_n)_{n\geqslant 0}$.

The form of $\Phi U \Phi^{-1}$ and $\Phi\pi(g)\Phi^{-1}$ can be obtained from the next lemma (using the fact that $P_n U f = U P_{n+1}$). ∎

LEMMA 6.2. *The following diagram is commutative, where $\alpha(\xi) = \xi \circ r$:*

$$\begin{array}{ccc} L^2(X, \omega_n) & \xrightarrow{\alpha} & L^2(X, \omega_{n+1}) \\ \downarrow i_n & & \downarrow i_{n+1} \\ H_n & \hookrightarrow & H_{n+1} \end{array}.$$

If $\xi\circ\theta_{n+k}\in H_{n+k}$, *then*

$$P_n(\xi\circ\theta_{n+k})=\frac{R^k(\xi h)}{h}\circ\theta_n; \tag{6.1}$$

$$U^*f=\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\frac{1}{m_0\circ\widehat{r}^{-1}}f\circ\widehat{r}^{-1},\quad (f\in L^2(X_\infty,\widehat{\mu})); \tag{6.2}$$

$$UP_{n+1}U^*=P_n,\quad (n\geqslant 0). \tag{6.3}$$

Proof. For $\xi\in L^2(X,\omega_n)$, $\xi\circ\theta_n=\xi\circ r\circ\theta_{n+1}=i_{n+1}(\alpha(\xi))$, thus the diagram commutes.

We have to check that, for all $\eta\in L^2(X,\omega_n)$ we have

$$\langle\,\xi\circ\theta_{n+k}\mid\eta\circ\theta_n\,\rangle=\Big\langle\,\frac{R^k(\xi h)}{h}\circ\theta_n\;\Big|\;\eta\circ\theta_n\,\Big\rangle.$$

But

$$\begin{aligned}\langle\,\xi\circ\theta_{n+k}\mid\eta\circ\theta_n\,\rangle&=\int_X R^{n+k}(\overline{\xi}\eta\circ r^k\,h)\,\mathrm{d}\mu=\int_X R^n\Big(\frac{R^k(\overline{\xi}h)}{h}\eta h\Big)\,\mathrm{d}\mu\\&=\Big\langle\,\frac{R^k(\xi h)}{h}\circ\theta_n\;\Big|\;\eta\circ\theta_n\,\Big\rangle.\end{aligned}$$

Equation (6.2) can be proved by a direct computation.

Since (H_n) are dense in $L^2(X_\infty,\widehat{\mu})$, we can check (6.3) on H_{n+k}. Take $\xi\circ\theta_{n+k}\in H_{n+k}$, then

$$\begin{aligned}UP_{n+1}U^*(\xi\circ\theta_{n+k})&=UP_{n+1}\Big(\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\frac{1}{m_0\circ\widehat{r}^{-1}}\xi\circ\theta_{n+k}\circ\widehat{r}^{-1}\Big)\\&=UP_{n+1}\Big(\Big(\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\circ r^{n+k+1}\frac{1}{m_0\circ r^{n+k}}\xi\Big)\circ\theta_{n+k+1}\Big)\\&=U\Big(\Big(\frac{R^k\Big(\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\circ r^{n+k+1}\frac{1}{m_0\circ r^{n+k}}\xi h\Big)}{h}\Big)\circ\theta_{n+1}\Big)\\&=U\Big(\Big(\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\circ r^{n+1}\frac{1}{m_0\circ r^n}\frac{R^k(\xi h)}{h}\Big)\circ\theta_{n+1}\Big)\\&=m_0\chi_{\{m_0\circ\widehat{r}^{-1}\neq 0\}}\circ r\frac{1}{m_0}\frac{R^k(\xi h)}{h}\circ\theta_n=P_n(\xi\circ\theta_{n+k}).\qquad\blacksquare\end{aligned}$$

As a consequence of Lemma 6.2 we also have:

PROPOSITION 6.3. *The identification of functions in* $L^2(X,\omega_n)$ *with martingales is given by*

$$\Phi(i_n(\xi))=\Big(\frac{R^n(\xi h)}{h},\dots,\frac{R(\xi h)}{h},\xi,\xi\circ r,\xi\circ r^2,\dots\Big),\quad(\xi\in L^2(X,\omega_n),n\geqslant 0). \tag{6.4}$$

The condition that m_0 be non-singular is essential if one wants U to be unitary. We illustrate this by an example.

EXAMPLE 6.4 (Shannon's wavelet). Let $\mathbb{R}/\mathbb{Z} \simeq [-\frac{1}{2}, \frac{1}{2})$. By this we mean that functions on $[-\frac{1}{2}, \frac{1}{2})$ are viewed also as functions on $\mathbb{R}$ via periodic extension, i.e., $f(x+n) = f(x)$ if $x \in [-\frac{1}{2}, \frac{1}{2})$ and $n \in \mathbb{Z}$.

Set

$$m_0(x) = \sqrt{2}\chi_{[-\frac{1}{4},\frac{1}{4})}(x).$$

Then

$$\widehat{\varphi}(x) = \prod_{k=1}^{\infty} \frac{1}{\sqrt{2}} m_0(\frac{x}{2^k}) = \chi_{[-\frac{1}{4},\frac{1}{4})}(\frac{x}{2}) = \chi_{[-\frac{1}{2},\frac{1}{2})}(x), \tag{6.5}$$

and

$$\varphi(t) = \frac{\sin \pi t}{\pi t}.$$

For functions in $L^1(\mathbb{R}/\mathbb{Z})$, the Ruelle operator R_{m_0} is

$$\begin{aligned}(R_{m_0}f)(x) &= \chi_{[-\frac{1}{4},\frac{1}{4})}(\frac{x}{2})f(\frac{x}{2}) + \chi_{[-\frac{1}{4},\frac{1}{4})}(\frac{x+1}{2})f(\frac{x+1}{2}) = \chi_{[-\frac{1}{4},\frac{1}{4})}(\frac{x}{2})f(\frac{x}{2}) \\ &= \chi_{[-\frac{1}{2},\frac{1}{2})}(x)f(\frac{x}{2}) = f(\frac{x}{2}), \quad \text{for } x \in [-\frac{1}{2}, \frac{1}{2}).\end{aligned}$$

Hence $R_{m_0}1 = 1$. Note from (6.5) that $\widehat{\varphi}(x+n) = 0$ if $n \in \mathbb{Z} \setminus \{0\}$.

Let $\xi \in L^2(\mathbb{R}/\mathbb{Z})$. Then we get

$$\begin{aligned}\int_{X_\infty} |\xi \circ \theta_n|^2 \, \mathrm{d}\widehat{\mu} = \int_X |\xi|^2 \, \mathrm{d}\omega_n &= \int_{-1/2}^{1/2} R^n(|\xi|^2)(x) \, \mathrm{d}x \\ &= \int_{-1/2}^{1/2} |\xi(2^{-n}x)|^2 \, \mathrm{d}x = 2^n \int_{-1/2^{n+1}}^{1/2^{n+1}} |\xi(x)|^2 \, \mathrm{d}x.\end{aligned}$$

But then $L^2(X, \omega_n) = L^2([-\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}), 2^n \, \mathrm{d}x)$ and we see that the map

$$\alpha : L^2(X, \omega_n) \to L^2(X, \omega_{n+1}), \quad \alpha(\xi) = \xi(2\cdot)$$

is an isometry (Lemma 6.2) which is also surjective with inverse $\xi \mapsto \xi(\frac{x}{2})$.

With Lemma 6.2, we get that the inclusion of H_n in H_{n+1} is in fact an identity, therefore

$$L^2(X_\infty, \widehat{\mu}) = H_0 = L^2([-\frac{1}{2}, \frac{1}{2}), \mathrm{d}x).$$

When m_0 is non-singular, then Theorem 5.5 shows us that the covariant system $(L^2(X_\infty, \widehat{\mu}), U, \pi, \varphi)$ has U unitary so, by uniqueness, it is isomorphic to the one constructed via the Kolmogorov theorem in Corollary 3.6, which we denote by $(\widetilde{H}, \widetilde{U}, \widetilde{\pi}, \widetilde{\varphi})$.

The next theorem shows that even when m_0 is singular, the covariant system $(L^2(X_\infty, \widehat{\mu}), U, \pi, \varphi)$ can be embedded in the $(\widetilde{H}, \widetilde{U}, \widetilde{\pi}, \widetilde{\varphi})$.

THEOREM 6.5. *There exists a unique isometry $\Psi : L^2(X_\infty, \widehat{\mu}) \to \widetilde{H}$ such that*

$$\Psi(\xi \circ \theta_n) = \widetilde{U}^{-n}\widetilde{\pi}(\xi)\widetilde{U}^n\widetilde{\varphi}, \quad (\xi \in L^\infty(X, \mu)).$$

Ψ intertwines the two systems, i.e.,

$$\Psi U = \widetilde{U}\Psi, \quad \Psi\pi(g) = \widetilde{\pi}(g)\Psi, \text{ for } g \in L^\infty(X, \mu), \quad \Psi\varphi = \widetilde{\varphi}.$$

Proof. Let $j_n : H_n \to \widetilde{H}$ be defined on a dense subspace by

$$j_n(\xi \circ \theta_n) = \widetilde{U}^{-n}\widetilde{\pi}(\xi)\widetilde{U}^n\widetilde{\varphi}, \quad (\xi \in L^\infty(X, \mu)).$$

Then j_n is a well defined isometry because

$$\|\xi \circ \theta_n\|^2_{L^2(\widehat{\mu})} = \int_X R^n(|\xi|^2 h)\,\mathrm{d}\mu = \int_X |m_0^{(n)}|^2|\xi|^2\,\mathrm{d}\mu = \|\widetilde{U}^{-n}\widetilde{\pi}(\xi)\widetilde{U}^n\widetilde{\varphi}\|^2,$$

where

$$m_0^{(n)} := m_0 \cdot m_0 \circ r \circ \cdots \circ m_0 \circ r^{n-1}.$$

Also note that

$$j_{n+1}(\xi \circ \theta_n) = j_n(\xi \circ r \circ \theta_{n+1}) = \widetilde{U}^{-n-1}\widetilde{\pi}(\xi \circ r)\widetilde{U}^{n+1}\widetilde{\varphi} = \widetilde{U}^{-n}\widetilde{\pi}(\xi)\widetilde{U}^n\widetilde{\varphi},$$

so we can construct Ψ on $L^2(X_\infty, \widehat{\mu})$ such that it agrees with j_n on H_n.

Next, we check the intertwining properties; it is enough to verify them on H_n:

$$\begin{aligned}
\widetilde{U}\Psi(\xi \circ \theta_n) &= \widetilde{U}\widetilde{U}^{-n}\widetilde{\pi}(\xi)\widetilde{U}^n\widetilde{\varphi} = \widetilde{U}^{-n+1}\widetilde{\pi}(\xi \circ r)\widetilde{U}^{n-1}\widetilde{U}\widetilde{\varphi} \\
&= \widetilde{U}^{-n+1}\widetilde{\pi}(\xi)\widetilde{U}^{n-1}\widetilde{\pi}(m_0)\widetilde{\varphi}, \\
\Psi U(\xi \circ \theta_n) &= \Psi(m_0\xi \circ \theta_n \circ \widehat{r}) = \Psi((m_0 \circ r^{n-1}\xi) \circ \theta_{n-1}) \\
&= \widetilde{U}^{-n+1}\widetilde{\pi}(m_0 \circ r^{n-1}\xi)\widetilde{U}^{n-1}\widetilde{\varphi} = \widetilde{U}^{-n+1}\widetilde{\pi}(\xi)\widetilde{U}^{n-1}\widetilde{\pi}(m_0)\widetilde{\varphi}.
\end{aligned}$$

The other intertwining relations can be checked by some similar computations. ∎

6.1. CONDITIONAL EXPECTATIONS. We can consider the σ-algebras

$$\mathfrak{B}_n := \theta_n^{-1}(\mathfrak{B}),$$

$\mathfrak{B}$ being the σ-algebra of Borel subsets in X. Note that $\theta_n^{-1}(E) = \theta_{n+1}^{-1}(r^{-1}(E))$. If follows that

$$\mathfrak{B}_0 \subset \mathfrak{B}_1 \subset \cdots \subset \mathfrak{B}_n \subset \mathfrak{B}_{n+1} \subset \cdots.$$

We set $\mathfrak{B}_\infty = \bigcup_{n \geqslant 0} \mathfrak{B}_n$ which is a sigma-algebra on X_∞.

The functions on X_∞ which are $\mathfrak{B}_n$ measurable are the functions which depend only on $x_0, \ldots, x_n$. H_n consists of function in $L^2(X_\infty, \mathfrak{B}_n, \widehat{\mu})$. Also we can regard $L^\infty(X_\infty, \mathfrak{B}_n, \widehat{\mu})$ as an increasing sequence of subalgebras of $L^\infty(X_\infty, \widehat{\mu})$. The map

$$i_n : L^\infty(X, \omega_n) \to L^\infty(X_\infty, \mathfrak{B}_n, \widehat{\mu})$$

is an isomorphism.

An application of the Radon-Nikodym theorem shows that there exists a unique *conditional expectation* $E_n : L^1(X_\infty, \widehat{\mu}) \to L^1(X_\infty, \mathfrak{B}_n, \widehat{\mu})$ determined by the relation

$$\int_{X_\infty} E_n(f) g \, \mathrm{d}\widehat{\mu} = \int_{X_\infty} f g \, \mathrm{d}\widehat{\mu}, \quad (g \in L^\infty(X_\infty, \mathfrak{B}_n, \widehat{\mu})). \tag{6.6}$$

We enumerate the properties of these conditional expectations.

PROPOSITION 6.6. *We have:*

$$E_n(fg) = f E_n(g), \quad (f \in L^\infty(X_\infty, \mathfrak{B}_n, \widehat{\mu}), g \in L^1(X_\infty, \widehat{\mu})); \tag{6.7}$$

$$E_n(f) \geqslant 0, \text{ if } f \geqslant 0; \tag{6.8}$$

$$E_m E_n = E_n E_m = E_n, \text{ if } m \geqslant n; \tag{6.9}$$

$$\int_{X_\infty} E_n(f) \, \mathrm{d}\widehat{\mu} = \int_{X_\infty} f \, \mathrm{d}\widehat{\mu}; \tag{6.10}$$

$$E_n(f) = P_n(f), \text{ if } f \in L^2(X_\infty, \widehat{\mu}). \tag{6.11}$$

DEFINITION 6.7. A sequence $(f_n)_{n \geqslant 0}$ of measurable functions on X_∞ is said to be a *martingale* if

$$E_n f_{n+1} = f_n, \quad (n \geqslant 0),$$

where E_n is a family of conditional expectations as in Proposition 6.6.

PROPOSITION 6.8. *If* $\xi \in L^1(X, \omega_{n+k})$ *then*

$$E_n(\xi \circ \theta_{n+k}) = \frac{R^k(\xi h)}{h} \circ \theta_n. \tag{6.12}$$

Proof. If $\xi \in L^2(X, \omega_n)$, the formula follows from Lemma 6.2. The rest follows by approximation. ∎

Proposition 6.8 offers a direct link between the operator powers R^k and the conditional expectations E_n. It shows in particular how our martingale construction depends on the Ruelle operator R. For a sequence $(\xi_n)_{n \geqslant 0}$ of measurable functions on X, $(\xi_n \circ \theta_n)_{n \geqslant 0}$ is a martingale if and only if

$$R(\xi_{n+1} h) = \xi_n h, \quad (n \geqslant 0).$$

A direct application of Doob's theorem (Theorem IV-1-2, in [34]) gives the following:

PROPOSITION 6.9. *If* $\xi_n \in L^1(X, \omega_n)$ *is a sequence of functions with the property that*

$$R(\xi_{n+1} h) = \xi_n h, \quad (n \geqslant 0),$$

then the sequence $\xi_n \circ \theta_n$ *converges* $\widehat{\mu}$*-almost everywhere.*

Then Proposition IV-2-3 from [34], translates into

PROPOSITION 6.10. *Suppose $\xi_n \in L^1(X, \omega_n)$ is a sequence with the property that*

$$R(\xi_{n+1}h) = \xi_n h, \quad (n \geqslant 0).$$

The following conditions are equivalent:

(i) *The sequence $\xi_n \circ \theta_n$ converges in $L^1(X_\infty, \widehat{\mu})$.*

(ii) $\sup_n \int_X R^n(|\xi|h)\, \mathrm{d}\mu < \infty$ *and the a.e. limit $\xi_\infty = \lim_n \xi_n \circ \theta_n$ satisfies $\xi_n \circ \theta_n = E_n(\xi_\infty)$.*

(iii) *There exists a function $\xi \in L^1(X_\infty, \widehat{\mu})$ such that $\xi_n \circ \theta_n = E_n(\xi)$ for all n.*

(iv) *The sequence $\xi_n \circ \theta_n$ satisfies the uniform integrability condition:*

$$\sup_n \int_X R^n(\chi_{\{|\xi_n|>a\}}\xi_n h)\, \mathrm{d}\mu \downarrow 0 \text{ as } a \uparrow \infty.$$

If one of the conditions is satisfied, the martingale $(\xi_n)_n$ is called regular.

Convergence in L^p is given by Proposition IV-2-7 in [34]:

PROPOSITION 6.11. *Let $p > 1$. Every martingale $(\xi_n)_n$ with $\xi_n \in L^p(X, \omega_n)$ and*

$$\sup_n \|\xi_n\|_p < \infty$$

is regular, and $\xi_n \circ \theta_n$ converges in $L^p(X_\infty, \widehat{\mu})$ to ξ_∞.

We have seen that functions f on X_∞ may be identified with sequences (ξ_n) of functions on X. When $r : X \to X$ is given, the induced mappings

$$\widehat{r} : X_\infty \to X_\infty, \quad \text{and } \widehat{r}^{-1} : X_\infty \to X_\infty \tag{6.13}$$

yield transformations of functions on X_∞ as follows $f \mapsto f \circ \widehat{r}$ and $f \mapsto f \circ \widehat{r}^{-1}$.

The 1-1 correspondence

$$f \text{ function on } X_\infty \leftrightarrow \xi_0, \xi_1, \dots \text{ functions on } X \tag{6.14}$$

is determined uniquely by

$$E_n(f) = \xi_n \circ \theta_n, \quad n = 0, 1, \dots. \tag{6.15}$$

When f and h are given, then the functions (ξ_n) in (6.14) must satisfy

$$R(\xi_{n+1}h) = \xi_n h, \quad (n \geqslant 0). \tag{6.16}$$

PROPOSITION 6.12. *Assume m_0 is non-singular. If f is a function on X_∞ and $f \leftrightarrow (\xi_n)$ as in* (6.14) *then*

$$f \circ \widehat{r} \leftrightarrow \xi_{n+1}, \tag{6.17}$$

$$f \circ \widehat{r}^{-1} \leftrightarrow \xi_{n-1}. \tag{6.18}$$

Specifically we have

$$E_n(f \circ \widehat{r}) = \xi_{n+1} \circ \theta_n, \tag{6.19}$$

$$E_n(f \circ \widehat{r}^{-1}) = \xi_{n-1} \circ \theta_n = \Big(\frac{R(\xi_n h)}{h}\Big) \circ \theta_n. \tag{6.20}$$

Or equivalently

$$f \circ \widehat{r} \leftrightarrow (\xi_1, \xi_2, \dots), \tag{6.21}$$

$$f \circ \widehat{r}^{-1} \leftrightarrow \Big(\frac{R(\xi_0 h)}{h}, \xi_0, \xi_1, \dots\Big). \tag{6.22}$$

Proof. Theorem 5.4 is used in both parts of the proof below. We have for $g : X \to \mathbb{C}$,

$$\begin{aligned}
\int_{X_\infty} E_n(f \circ \widehat{r})\, g \circ \theta_n \, d\widehat{\mu} &= \int_{X_\infty} f \circ \widehat{r}\, g \circ \theta_{n+1} \circ \widehat{r} \, d\widehat{\mu} = \int_{X_\infty} \frac{1}{|m_0|^2 \circ \widehat{r}^{-1}} f\, g \circ \theta_{n+1} \, d\widehat{\mu} \\
&= \int_{X_\infty} E_{n+1}(f) \Big(\frac{1}{|m_0|^2 \circ r^n} g\Big) \circ \theta_{n+1} \, d\widehat{\mu} \\
&= \int_{X_\infty} \xi_{n+1} \circ \theta_n \circ \widehat{r}^{-1} \Big(\frac{1}{|m_0|^2 \circ r^n} g\Big) \circ \theta_n \circ \widehat{r}^{-1} \, d\widehat{\mu} \\
&= \int_{X_\infty} |m_0|^2 \xi_{n+1} \circ \theta_n \frac{1}{|m_0|^2}\, g \circ \theta_n \, d\widehat{\mu} = \int_{X_\infty} \xi_{n+1} \circ \theta_n\, g \circ \theta_n \, d\widehat{\mu}.
\end{aligned}$$

Thus $E_n(f \circ \widehat{r}) = \xi_{n+1} \circ \theta_n$. Then

$$\begin{aligned}
\int_{X_\infty} E_n(f \circ \widehat{r}^{-1}) g_n \circ \theta_n \, d\widehat{\mu} &= \int_{X_\infty} f \circ \widehat{r}^{-1}\, g_n \circ \theta_{n-1} \circ \widehat{r}^{-1} \, d\widehat{\mu} = \int_{X_\infty} |m_0|^2 f\, g_n \circ \theta_{n-1} \, d\widehat{\mu} \\
&= \int_{X_\infty} E_{n-1}(f)(|m_0|^2 \circ r^{n-1}\, g) \circ \theta_{n-1} \, d\widehat{\mu} \\
&= \int_{X_\infty} \xi_{n-1} \circ \theta_n \circ \widehat{r}(|m_0|^2 \circ r^{n-1}\, g) \circ \theta_n \circ \widehat{r} \, d\widehat{\mu} \\
&= \int_{X_\infty} \frac{1}{|m_0|^2 \circ \widehat{r}^{-1}} \xi_{n-1} \circ \theta_n\, |m_0|^2 \circ \widehat{r}^{-1}\, g \circ \theta_n \, d\widehat{\mu} \\
&= \int_{X_\infty} \xi_{n-1} \circ \theta_n\, g \circ \theta_n \, d\widehat{\mu}
\end{aligned}$$

and this implies (6.20). ∎

7. INTERTWINING OPERATORS AND COCYCLES

In the paper [13], Dai and Larson showed that the familiar orthogonal wavelet systems have an attractive representation theoretic formulation. This formulation brings out the geometric properties of wavelet analysis especially nicely, and it led to the discovery of *wavelet sets*, i.e., singly generated wavelets in $L^2(\mathbb{R}^d)$, i.e., $\psi \in L^2(\mathbb{R}^d)$ such that $\widehat{\psi} = \chi_E$ for some $E \subset \mathbb{R}^d$, and

$$\{|\det A|^{j/2}\psi(A^j \cdot -k) : j \in \mathbb{Z}, k \in \mathbb{Z}^d\}$$

is an orthonormal basis.

The case when the initial resolution subspace for some wavelet construction is singly generated, the wavelet functions should be thought of as *wandering vectors*. If the scaling operation is realized as a unitary operator U in the Hilbert space $H := L^2(\mathbb{R}^d)$, then the notion of wandering refers to vectors, or subspaces, which are mapped into orthogonal vectors (respectively, subspaces) under powers of U. Since this approach yields wavelet bases derived directly from the initial data, i.e., from the wandering vectors, U, and the integral translations, the question of intertwining operators is a natural one. The initial data defines a representation ρ.

An operator in H which intertwines ρ with itself is said to be in the commutant of ρ; and Dai and Larson gave a formula for the commutant. They showed that the operators in the commutant are defined in a natural way from a class of invariant bounded measurable functions, called *wavelet multipliers*. This and other related results can be shown to generalize to the case of operators which intertwine two wavelet representations ρ and ρ'.

Since our present martingale construction is a generalization of the traditional wavelet resolutions, see [27], it is natural to ask for theorems which generalize the known theorems about wavelet functions. We prove in this section such a theorem, Theorem 7.2. The applications of this are manifold, and include the projective systems defined from Julia sets, and from the state space of a subshift in symbolic dynamics.

Our formula for the commutant in this general context of projective systems is shown to be related to the Perron-Frobenius-Ruelle operator in Corollary 7.3. This result implies in particular that the commutant is abelian; and it makes precise the way in which the representation ρ itself decomposes as a direct integral over the commutant.

Our proof of this corollary depends again on Doob's martingale convergence theorem, see (7.11) below, Section 6 above, and Chapter 2.7 of [27].

DEFINITION 7.1. If $m_0 \in L^\infty(X)$ and $h \in L^1(X)$, we call (m_0, h) a *Perron-Ruelle-Frobenius pair* if

$$R_{m_0}h = h.$$

THEOREM 7.2. *Let (m_0, h) and (m_0', h') be two Perron-Ruelle-Frobenius pairs with m_0, m_0' non-singular, and let $(L^2(X_\infty, \widehat{\mu}), U, \pi, \varphi)$, $(L^2(X_\infty, \widehat{\mu}'), U', \pi', \varphi')$ be the* *associated covariant systems. Let* $X_\infty = X_\infty^a \cup X_\infty^s$ *be the Jordan decomposition of* $\widehat{\mu}'$ *with respect to* $\widehat{\mu}$, $X_\infty^a \cap X_\infty^s = \emptyset$, *with* $\widehat{\mu}(X_\infty^s) = 0$ *and* $\widehat{\mu}'|_{X_\infty^a} \prec \widehat{\mu}$, *and denote by*

$$\Delta := \frac{\mathrm{d}\,\widehat{\mu}'|_{X_\infty^a}}{\mathrm{d}\widehat{\mu}}.$$

Then there is a 1-1 *correspondence between each two of the following sets of data:*

(i) *Operators* $A : L^2(X_\infty, \widehat{\mu}) \to L^2(X_\infty, \widehat{\mu}')$ *that intertwine the covariant system, i.e.,*

$$U'A = AU, \quad \text{and } \pi'(g)A = \pi(g)A, \text{ for } g \in L^\infty(X). \tag{7.1}$$

(ii) $\mathfrak{B}_\infty$*-measurable functions* $f : X \to \mathbb{C}$ *such that* $f|_{X_\infty^s} = 0$, $f\Delta^{1/2}$ *is* $\widehat{\mu}$*-bounded and*

$$m_0 f = m_0' f \circ \widehat{r}, \quad \widehat{\mu}'\text{-a.e.} \tag{7.2}$$

(iii) *Measurable functions* $h_0 : X \to \mathbb{C}$ *such that*

$$|h_0|^2 \leqslant chh' \ \mu\text{-a.e.}, \tag{7.3}$$

for some finite constant $c \geqslant 0$, *with*

$$\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} \overline{m_0'(y)} m_0(y) h_0(y) = h_0(x), \quad \text{for } \mu\text{-a.e. } x \in X. \tag{7.4}$$

From (i) *to* (ii) *the correspondence is given by*

$$A\xi = f\xi, \quad (\xi \in L^2(X_\infty, \widehat{\mu})). \tag{7.5}$$

From (ii) *to* (iii), *the correspondence is given by*

$$h_0 = E_0^{\widehat{\mu}'}(f)h' = E_0^{\widehat{\mu}}(f\Delta)h. \tag{7.6}$$

From (i) *to* (iii) *the correspondence is given by*

$$\langle\, \varphi' \mid A\pi(g)\varphi \,\rangle = \int_X g h_0 \,\mathrm{d}\mu, \quad (g \in L^\infty(X)). \tag{7.7}$$

Proof. Take A as in (i). Then for all $g \in L^\infty(X)$ and any $n \geqslant 0$ we have that

$$A(g \circ \widehat{r}^{-n}) = A(U^{-n}\pi(g)U^n)(1) = (U'^{-n}\pi'(g)U'^n)(A(1)) = g \circ \widehat{r}^{-n} \cdot (A(1)).$$

Denote by $f := A(1) \in L^2(X_\infty, \widehat{\mu}')$.

Since any $\mathfrak{B}_\infty$-measurable, bounded function $\xi : X_\infty \to \mathbb{C}$ can be pointwise $\widehat{\mu}$- and $\widehat{\mu}'$- approximated by functions of the form $g \circ \widehat{r}^{-n}$, we get that

$$A(\xi) = f\xi.$$

We have also that

$$\int_{X_\infty} |f|^2 |\xi|^2 \,\mathrm{d}\widehat{\mu}' \leqslant \|A\|^2 \int_{X_\infty} |\xi|^2 \,\mathrm{d}\widehat{\mu}$$

so

$$\int_{X_\infty^a} |f|^2 |\xi|^2 \Delta \,\mathrm{d}\widehat{\mu} + \int_{X_\infty^s} |f|^2 |\xi|^2 \,\mathrm{d}\widehat{\mu}' \leqslant \|A\|^2 \int_{X_\infty} |\xi|^2 \,\mathrm{d}\widehat{\mu}.$$

Taking $\xi = \chi_{X_\infty^s}$ we obtain that $f = 0$ $\widehat{\mu}'$-a.e. on X_∞^s; so we may take $f = 0$ on X_∞^s. Then we get also that $|f\Delta^{1/2}| \leqslant \|A\|$ $\widehat{\mu}$-a.e.

Then, again by approximation we obtain that

$$A\xi = f\xi, \quad \text{for } \xi \in L^2(X_\infty, \widehat{\mu}).$$

We have in addition the fact that $U'A = AU$, and this implies (7.2).

Conversely, the previous calculations show that any operator defined by (7.5) with f as in (ii), will be a bounded operator which intertwines the covariant systems.

Now take A as in (i) and consider the linear functional

$$g \in L^\infty(X) \mapsto \langle\, \varphi' \mid A\pi(g)\varphi \,\rangle.$$

This defines a measure on X which is absolutely continuous with respect to μ. Let h_0 be its Radon-Nikodym derivative. We have

$$\begin{aligned}\int_X gh_0 \,\mathrm{d}\mu &= \langle\, \varphi' \mid A\pi(g)\varphi \,\rangle = \langle\, U'\varphi' \mid U'A\pi(g)\varphi \,\rangle \\ &= \langle\, \pi'(m_0')\varphi' \mid A\pi(g\circ r)\pi(m_0)\varphi \,\rangle = \int_X \overline{m_0'} m_0 g\circ r\, h_0 \,\mathrm{d}\mu \\ &= \int_X g \frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} \overline{m_0'(y)} m_0(y) h_0(y) \,\mathrm{d}\mu(x).\end{aligned}$$

Thus $\frac{1}{\#r^{-1}(x)} \sum_{r(y)=x} \overline{m_0'(y)} m_0(y) h_0(y) = h_0(x)$ μ-a.e.

Next we check that $|h_0|^2 \leqslant \|A\|^2 hh'$ μ-a.e. By the Schwarz inequality, we have for all $f, g \in L^\infty(X)$,

$$|\langle\, \pi'(f)\varphi' \mid A\pi(g)\varphi \,\rangle|^2 \leqslant \|A\|^2 \|\pi'(f)\varphi'\|^2 \|\pi(g)\varphi\|^2,$$

which translates into

$$\left| \int_X \overline{f} g h_0 \,\mathrm{d}\mu \right|^2 \leqslant \|A\|^2 \int_X |g|^2 h' \,\mathrm{d}\mu \int_X |f|^2 h \,\mathrm{d}\mu. \tag{7.8}$$

If μ has some atoms then just take f and g to be the characteristic function of that atoms and this proves the inequality (7.3) for such points. The part of μ that does not have atoms is measure theoretically isomorphic to the unit interval with the Lebesgue measure. Then take x to be a Lebesgue differentiability point for h_0, h and h'. Take $f = g = \frac{1}{\mu(I)}\chi_I$ for some small interval centered at x. Letting I shrink to x and using Lebesgue's differentiability theorem, (7.8) implies (7.3).

For the converse, from (iii) to (i), let h_0 as in (iii), and define for $n \geqslant 0$ the sesquilinear form, B_n on $H_n' \times H_n$ (see Section 4): for $f, g \in L^\infty(X)$,

$$B_n(U'^{-n}\pi'(f)\varphi', U^{-n}\pi(g)\varphi) := \int_X \overline{f} g h_0 \,\mathrm{d}\mu.$$

An application of the Schwarz inequality and (7.3), shows that

$$|B_n(\xi,\eta)|^2 \leqslant c\|\xi\|^2\|\eta\|^2, \quad (\xi \in H'_n, \eta \in H_n).$$

The inclusion of H_n in H_{n+1} is given by

$$U^{-n}\pi(f)\varphi \mapsto U^{-n-1}\pi(f \circ r\, m_0)\varphi.$$

The forms B_n are compatible with these inclusion in the sense that

$$\begin{aligned}
&B_{n+1}(U'^{-n-1}\pi'(f \circ rm'_0)\varphi', U^{-n-1}\pi(g \circ rm_0)\varphi) \\
&\qquad = \int_X \overline{f \circ rm'_0}\, g \circ rm_0 h_0 \,\mathrm{d}\mu = \int_X \overline{f} g h_0 = B_n(U'^{-n}\pi'(f)\varphi', U^{-n}\pi(g)\varphi).
\end{aligned}$$

(We used (7.4) for the third equality.) Therefore the system $(B_n)_n$ extends to a sesquilinear map B on $H' \times H$ such that its restriction to $H'_n \times H_n$ is B_n, and B is bounded ($H = L^2(X_\infty, \widehat{\mu})$, $H' = L^2(X_\infty, \widehat{\mu}')$).) Then there exists a bounded operator $A : H \to H'$ such that

$$\langle\, \xi \mid A\eta \,\rangle = B(\xi,\eta), \quad (\xi \in H, \eta \in H').$$

We have to check that A is intertwining. But

$$\begin{aligned}
\langle\, U'^{-n}\pi'(f)\varphi' \mid AUU^{-n}\pi(g)\varphi \,\rangle &= B(U'^{-n}\pi'(f)\varphi', U^{-n}\pi(g \circ r\, m_0)\varphi) \\
&= \int_X \overline{f} g \circ r\, m_0 h_0 \,\mathrm{d}\mu \\
&= B(U'^{-n-1}\pi'(f)\varphi', U^{-n-1}\pi(g \circ r\, m_0)\varphi) \\
&= \langle\, U'^{-n-1}\pi'(f)\varphi' \mid AU^{-n-1}\pi(g \circ r\, m_0)\varphi \,\rangle \\
&= \langle\, U'^{-n}\pi'(f)\varphi' \mid U'AU^{-n}\pi(g)\varphi \,\rangle.
\end{aligned}$$

$$\begin{aligned}
\langle\, U'^{-n}\pi'(f)\varphi' \mid A\pi(k)U^{-n}\pi(g)\varphi \,\rangle &= B(U'^{-n}\pi'(f)\varphi', U^{-n}\pi(k \circ r^n\, g)\varphi) \\
&= \int_X \overline{f} k \circ r^n\, g h_0 \,\mathrm{d}\mu \\
&= B(U'^{-n}\pi'(\overline{k \circ r^n}\, f)\varphi', U^{-n}\pi(g)\varphi) \\
&= \langle\, U'^{-n}\pi'(\overline{k \circ r^n}\, f)\varphi' \mid AU^{-n}\pi(g)\varphi \,\rangle \\
&= \langle\, \pi'(\overline{k})U'^{-n}\pi'(f)\varphi' \mid AU^{-n}\pi(g)\varphi \,\rangle \\
&= \langle\, U'^{-n}\pi'(f)\varphi' \mid \pi'(k)AU^{-n}\pi(g)\varphi \,\rangle.
\end{aligned}$$

This shows that A is intertwining.

From (ii) to (iii), take f as in (ii). Then define the operator A as in (7.5). Using the previous correspondences we have that A is intertwining and there exists h_0 as in (iii), satisfying (7.7). We rewrite this in terms of f, and we have for all $g \in L^\infty(X)$:

$$\int_X E_0^{\widehat{\mu}'}(f) g h' \, \mathrm{d}\mu = \int_{X_\infty} f g \, \mathrm{d}\widehat{\mu}' = \langle \varphi' \mid A\pi(g)\varphi \rangle = \int_X g h_0 \, \mathrm{d}\mu.$$

Also we have the next which proves (7.6):

$$\int_{X_\infty} f g \, \mathrm{d}\widehat{\mu}' = \int_{X_\infty^a} f g \Delta \, \mathrm{d}\widehat{\mu} = \int_X E_0^{\widehat{\mu}}(f\Delta) g h \, \mathrm{d}\mu. \quad \blacksquare$$

COROLLARY 7.3. *Let (m_0, h) be a Perron-Ruelle-Frobenius pair with m_0 non-singular.*

(i) *For each operator A on $L^2(X_\infty, \widehat{\mu})$ which commutes with U and π, there exists a cocycle f, i.e., a bounded measurable function $f : X_\infty \to \mathbb{C}$ with $f = f \circ \widehat{r}$, $\widehat{\mu}$-a.e., such that*

$$A = M_f, \tag{7.9}$$

and, conversely each cocycle defines an operator in the commutant.

(ii) *For each measurable harmonic function $h_0 : X \to \mathbb{C}$, i.e., $R_{m_0} h_0 = h_0$, with $|h_0|^2 \leqslant ch^2$ for some $c \geqslant 0$, there exists a unique cocycle f such that*

$$h_0 = E_0(f) h, \tag{7.10}$$

and conversely, for each cocycle the function h_0 defined by (7.10) *is harmonic.*

(iii) *The correspondence $h_0 \to f$ in* (ii) *is given by*

$$f = \lim_{n \to \infty} \frac{h_0}{h} \circ \theta_n \tag{7.11}$$

where the limit is pointwise $\widehat{\mu}$-a.e., and in $L^p(X_\infty, \widehat{\mu})$ for all $1 \leqslant p < \infty$.

Proof. (i) and (ii) are direct consequences of Theorem 7.2. For (iii), we have that $f \in L^\infty(X_\infty, \widehat{\mu}) \subset L^p(X_\infty, \widehat{\mu})$. Using Proposition 6.12, we have that, since $f = f \circ \widehat{r}$, if $E_n(f) = \xi_n \circ \theta_n$, then $\xi_n = \xi_{n+1}$, for all $n \geqslant 0$. But from (7.10), we know that $\xi_0 = \frac{h_0}{h}$, so

$$E_n(f) = \frac{h_0}{h} \circ \theta_n.$$

(iii) follows now from Propositions 6.9, 6.10 and 6.11. $\blacksquare$

8. ITERATED FUNCTION SYSTEMS

In Section 6 we constructed our extension systems using martingales, and Doob's convergence theorem. We showed that our family of martingale Hilbert spaces may be realized as $L^2(X_\infty, \widehat{\mu})$, where both X_∞, and the associated measures $\widehat{\mu}$ on X_∞ are projective limits constructed directly from the following given data. Our construction starts with the following four: (1) a compact metric space

X, (2) a given mapping $r : X \to X$, (3) a strongly invariant measure μ on X, and (4) a function W on X which prescribes transition probabilities. From this, we construct our extension systems. In this section, we take a closer look at the measure $\widehat{\mu}$. We show that $\widehat{\mu}$ is in fact an average over an indexed family of measures P_x, x in X. Now P_x is constructed as a measure on a certain space of paths. The subscript x refers to the starting point of the paths, and P_x is defined on a sigma-algebra of subsets of path-space. (The reader is referred to [27] for additional details.) These are paths of a random walk, and the random walk is closely connected to the mathematics of the projective limit construction in Section 4. But the individual measures P_x carry more information than the averaged version μ from Section 4. As we show below, the construction of solutions to the canonical scaling identities in wavelet theory, and in dynamics, depend on the path space measures P_x. Our solutions will be infinite products, and the pointwise convergence of these infinite products depends directly on the analytic properties of the P_x's.

Let X be a metric space and $r : X \to X$ an N to 1 map. Denote by $\tau_k : X \to X$, $k \in \{1, \dots, N\}$, the branches of r, i.e., $r(\tau_k(x)) = x$ for $x \in X$, the sets $\tau_k(X)$ are disjoint and they cover X.

Let μ be a measure on X with the property

$$\mu = \frac{1}{N} \sum_{k=1}^{N} \mu \circ \tau_k^{-1}. \tag{8.1}$$

This can be rewritten as

$$\int_X f(x) \, d\mu(x) = \frac{1}{N} \sum_{k=1}^{N} \int_X f(\tau_k(x)) \, d\mu(x), \tag{8.2}$$

which is equivalent also to the strong invariance property.

Let $W, h \geqslant 0$ be two functions on X such that

$$\sum_{k=1}^{N} W(\tau_k(x)) h(\tau_k(x)) = h(x), \quad (x \in X). \tag{8.3}$$

Denote by

$$\Omega := \Omega_N := \prod_{\mathbb{N}} \{1, \dots, N\}.$$

Also we denote by

$$W^{(n)}(x) := W(x) W(r(x)) \cdots W(r^{n-1}(x)), \quad (x \in X).$$

PROPOSITION 8.1. *For every $x \in X$ there exists a positive Radon measure P_x on Ω such that, if f is a bounded measurable function on Ω which depends only on the first* *n coordinates* $\omega_1, \dots, \omega_n$, *then*

$$\int_\Omega f(\omega)\,\mathrm{d}P_x(\omega) = \sum_{\omega_1,\dots,\omega_n} W^{(n)}(\tau_{\omega_n}\tau_{\omega_{n-1}}\cdots\tau_{\omega_1}(x))h(\tau_{\omega_n}\tau_{\omega_{n-1}}\cdots\tau_{\omega_1}(x))f(\omega_1,\dots,\omega_n). \tag{8.4}$$

Proof. We check that P_x is well defined on functions which depend only on a finite number of coordinates. For this, take f measurable and bounded, depending only on $\omega_1,\dots,\omega_n$; and consider it as function which depends on the first $n+1$ coordinates. We have to check that the two formulas given by (8.4) yield the same result.

Consistency: As a function of the first $n+1$ coordinates, we have:

$$\begin{aligned}\int_X f(\omega)\,\mathrm{d}P_x(\omega) &= \sum_{\omega_1,\dots,\omega_{n+1}} W^{(n+1)}(\tau_{\omega_{n+1}}\cdots\tau_{\omega_1}(x))h(\tau_{\omega_{n+1}}\cdots\tau_{\omega_1}(x))f(\omega_1,\dots,\omega_{n+1})\\ &= \sum_{\omega_1,\dots,\omega_n} f(\omega_1,\dots,\omega_n)W^{(n)}(\tau_{\omega_n}\cdots\tau_{\omega_1}(x))\\ &\qquad\cdot\sum_{\omega_{n+1}} W(\tau_{\omega_{n+1}}\cdots\tau_{\omega_1}(x))h(\tau_{\omega_{n+1}}\cdots\tau_{\omega_1}(x))\\ &= \sum_{\omega_1,\dots,\omega_n} W^{(n)}(\tau_{\omega_n}\cdots\tau_{\omega_1}(x))f(\omega_1,\dots,\omega_n)h(\tau_{\omega_n}\cdots\tau_{\omega_1}(x)),\end{aligned}$$

so P_x is well defined. Using the Stone-Weierstrass and Riesz theorems, we obtain the desired measure. ∎

Consider now the space $X\times\Omega$. On this space we have the shift S:

$$S(x,\omega_1\cdots\omega_n\cdots) = (r(x),\omega_x\omega_1\cdots\omega_n\cdots),\quad (x\in X, (\omega_1\cdots\omega_n\cdots)\in\Omega), \tag{8.5}$$

where ω_x is defined by $x\in\tau_{\omega_x}(X)$. The inverse of the shift is given by the formula:

$$S^{-1}(x,\omega_1\cdots\omega_n\cdots) = (\tau_{\omega_1}(x),\omega_2\cdots\omega_n\cdots),\quad (x\in X, (\omega_1\cdots\omega_n\cdots)\in\Omega). \tag{8.6}$$

PROPOSITION 8.2. *Define the map* $\Psi : X_\infty \to X\times\Omega$ *by*

$$\Psi(x_0,x_1,\dots) = (x_0,\omega_1,\omega_2,\dots),\ \textit{where}\ x_n = \tau_{\omega_n}(x_{n-1}),\quad (n\geqslant 1).$$

Then Ψ *is a measurable bijection with inverse*

$$\Psi^{-1}(x,\omega_1,\omega_2,\dots) = (x,\tau_{\omega_1}(x),\tau_{\omega_2}\tau_{\omega_1}(x),\dots),\quad \textit{and}$$

$$\Psi\circ\widehat{r}\circ\Psi^{-1} = S. \tag{8.7}$$

Also

$$\int_{X_\infty} f\,\mathrm{d}\widehat{\mu} = \int_X\int_\Omega f\circ\Psi^{-1}(x,\omega)\,\mathrm{d}P_x(\omega)\,\mathrm{d}\mu(x),\quad (f\in L^1(X_\infty,\widehat{\mu})). \tag{8.8}$$

Proof. We know that $r(x_n) = x_{n-1}$ therefore $x_n = \tau_{\omega_n}(x_{n-1})$ for some $\omega_n \in \{1, \dots, N\}$. This correspondence defines Ψ and it is clear that the map is 1-1 and onto and the inverse has the given formula. A computation proves (8.7).

To check (8.8), it is enough to verify the conditions of Theorem 5.3. Take $\xi \in L^\infty(X)$, then $\xi \circ \theta_n \circ \Psi^{-1}$ depends only on x and $\omega_1, \dots, \omega_n$ so we have the next which proves (8.8):

$$\int_X \int_\Omega f \circ \theta_n \circ \Psi^{-1}(x, \omega)\, \mathrm{d}P_x(\omega)\, \mathrm{d}\mu(x)$$
$$= \int_X \sum_{\omega_1, \dots, \omega_n} W^{(n)}(\tau_{\omega_n}\tau_{\omega_{n-1}} \cdots \tau_{\omega_1}(x)) \cdot h(\tau_{\omega_n}\tau_{\omega_{n-1}} \cdots \tau_{\omega_1}(x))(f\theta_n\Psi^{-1})(x, \omega)\, \mathrm{d}\mu(x)$$
$$= \int_X R^n(fh)(x)\, \mathrm{d}\mu(x) = \int_{X_\infty} f \circ \theta_n\, \mathrm{d}\widehat{\mu}. \quad \blacksquare$$

Acknowledgements. We gratefully acknowledge discussions with the members of the group FRG/NSF; especially enlightening suggestions from Professors L. Baggett, D. Larson, and G. Olafsson.

In addition, this work was supported at the University of Iowa by a grant from the National Science Foundation (NSF-USA) under a Focused Research Program, DMS-0139473 (FRG), and at University of Central Florida by an NSF grant DMS-0704191.

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Received August 13, 2005.