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A CLASSIFICATION THEOREM FOR DIRECT LIMITS OF EXTENSIONS OF CIRCLE ALGEBRAS BY PURELY INFINITE C^*-ALGEBRAS

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ABSTRACT. We give a classification theorem for a class of C^*-algebras which are direct limits of finite direct sums of $\mathcal{E}_0$-algebras. The invariant consists of the following: (1) the set of Murray-von Neumann equivalence classes of projections; (2) the set of homotopy classes of hyponormal partial isometries; (3) a map d; and (4) total K-theory.

KEYWORDS: *Classification, nuclear C^*-algebras, extensions.*

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INTRODUCTION

The program of classifying all nuclear C^*-algebras was initiated by George Elliott. His work involving AF-algebras has provided a foundation for other classification theorems. Elliott [8] proved that the class of all AF-algebras are classified by their dimension groups. In 1993, Elliott [9] showed that the class of all AT-algebras with real rank zero are classified by their K_1 group and their ordered K_0 group. This remarkable result lead to many classification results for separable nuclear C^*-algebras. See [11] for references. Most of these results involved simple C^*-algebras with stable rank one or purely infinite simple C^*-algebras.

In 1997, Lin and Su [20] classified a class of (not necessarily simple) separable nuclear C^*-algebras with real rank zero that can be expressed as a direct limit of generalized Toeplitz algebras. The invariant $V_*(A)$ used by Lin and Su consists of the following three objects: (1) $V(A)$, the set of Murray-von Neumann equivalence classes of projections; (2) $k(A)_+$, certain equivalence classes of hyponormal partial isometries; and (3) a map $d : k(A)_+ \to V(A)$. Lin and Su used V_* to classify the class of all unital separable C^*-algebras with real rank zero that are direct limit of generalized Toeplitz algebras.

V_* was then used by Lin [17] to classify another class of (not necessarily simple) nuclear separable C^*-algebras. Its basic building blocks consist of C^*-algebras that are finite direct sums of corners of unital essential extensions of $M_k(C(S^1))$ by a stable Cuntz algebra $\mathcal{O}_m \otimes \mathcal{K}$. Lin proved that all unital C^*-algebras with real rank zero that are direct limit of these building blocks are classified by V_*.

In this paper, we consider C^*-algebras that are direct limits of finite direct sums of corners of unital essential extensions of $M_k(C(S^1))$ by a separable nuclear purely infinite simple C^*-algebra I satisfying the Universal Coefficient Theorem (UCT). A C^*-algebra of this form will be called an $\mathcal{AE}$-algebra. We will classify a subclass of the class of all unital $\mathcal{AE}$-algebras with real rank zero. A C^*-algebra in this subclass will be called an $\mathcal{AE}_0$-algebra. We will show that the class of all unital $\mathcal{AE}_0$-algebra with real rank zero are classified by V_* and total K-theory. In addition, we will show that unital AT-algebras with real rank zero and the C^*-algebras classified in [17] are $\mathcal{AE}_0$-algebras. Also, a unital separable nuclear purely infinite simple C^*-algebra with torsion free K_1 and satisfying the UCT is an $\mathcal{AE}_0$-algebra.

The paper is organized as follows. In Section 1, we give some definitions and basic properties. In Section 2, we give perturbation lemmas that are used throughout the paper. In Section 3, we introduce the invariant. In Section 4 and Section 5, we prove a uniqueness theorem and an existence theorem. In Section 6, we prove our main result.

1. DEFINITIONS AND BASIC PROPERTIES

1.1. PRELIMINARIES. (i) Let X be a compact subset of S^1. Then the identity z on X will be called the standard unitary generator.

(ii) Let C be a C^*-algebra. If $C \subset A$ and $1_C = 1_A$, then C is called a unital C^*-subalgebra.

(iii) $\mathrm{Hom}(A, B)$ will denote the set of all $*$-homomorphisms from A to B.

(iv) We write $p \sim q$ if p is Murray-von Neumann equivalent to q.

(v) An extension $0 \to B \xrightarrow{\mathrm{i}} E \xrightarrow{\pi} A \to 0$ of C^*-algebras is said to be *essential* if for every nonzero ideal J of E we have $J \cap \mathrm{i}(B) \neq 0$. If E is a unital essential extension of A by B, then E can be identified as a unital C^*-subalgebra of $\mathcal{M}(B)$. Note that an essential extension can be described by its Busby invariant $\tau_E : A \to \mathcal{M}(B)/B$.

(vi) Suppose E is an extension of A by B. Then the exponential map and the index map associated to E will be denoted by δ_0^E and δ_1^E respectively.

(vii) Let $\mathcal{N}$ be the bootstrap category of [26] and let $\mathcal{P}$ be the class of all $I \in \mathcal{N}$ such that I is a non-unital separable nuclear purely infinite simple C^*-algebra with $K_*(I)$ finitely generated. Let $\mathcal{P}_1$ be the class of all unital separable nuclear

purely infinite simple C^*-algebras in $\mathcal{N}$.
(viii) All semigroups have identities and homomorphisms preserve identities.

1.2. $\mathcal{E}$-ALGEBRAS.

DEFINITION 1.1. Suppose $E = pM_n(E')p$ for some $n \in \mathbb{Z}_{>0}$ and for some projection $p \in M_n(E')$. Suppose E' is a unital essential extension of $C(S^1)$ by a C^*-algebra $I \in \mathcal{P}$. Let π be the quotient map induced by the extension E'. If $\pi(p) = 1_{M_n(C(S^1))}$ or $p \in M_n(I)$, then E is said to be an *$\mathcal{E}$-algebra*.

NOTATION 1.2. Let E be an $\mathcal{E}$-algebra. If E is not a purely infinite simple C^*-algebra, then E is a unital essential extension of $M_n(C(S^1))$ by I, for some $I \in \mathcal{P}$, $n \in \mathbb{Z}_{>0}$. Denote I by $I(E)$ and $M_n(C(S^1))$ by $Q(E)$. If E is in $\mathcal{P}_1$, then set $I(E) = E$ and $Q(E) = 0$. If $E = \bigoplus_{i=1}^{k} E_i$, where each E_i is an $\mathcal{E}$-algebra, then set $I(E) = \bigoplus_{i=1}^{k} I(E_i)$ and $Q(E) = \bigoplus_{i=1}^{k} Q(E_i)$. Note that E is in $\mathcal{N}$.

DEFINITION 1.3. Let $E = pM_n(E')p$ be an $\mathcal{E}$-algebra. Then E is called an *$\mathcal{E}_0$-algebra* if one of the following hold:
(i) $K_1(I(E')) = 0$;
(ii) $K_1(I(E'))$ is a nonzero torsion free group and $\ker \delta_1^{E'} \neq \{0\}$.

PROPOSITION 1.4. *Suppose E_1 and E_2 are finite direct sums of $\mathcal{E}$-algebras. Let $i_k : I(E_k) \to E_k$ be the inclusion map and let $\pi_k : E_k \to Q(E_k)$ be the quotient map. If $\varphi \in \mathrm{Hom}(E_1, E_2)$, then there exist unique $I(\varphi) \in \mathrm{Hom}(I(E_1), I(E_2))$ and $Q(\varphi) \in \mathrm{Hom}(Q(E_1), Q(E_2))$ such that $i_2 \circ I(\varphi) = \varphi \circ i_1$ and $Q(\varphi) \circ \pi_1 = \pi_2 \circ \varphi$.*

Proof. Since $I(E_1)$ is a finite direct sum of purely infinite simple C^*-algebras and $Q(E_2)$ is a finite C^*-algebra, we have $\pi_2 \circ \varphi \circ i_1 = 0$. Hence, there exists $I(\varphi) \in \mathrm{Hom}(I(E_1), I(E_2))$ such that $i_2 \circ I(\varphi) = \varphi \circ i_1$. Thus, $Q(\varphi)$ exists. ∎

REMARK 1.5. Suppose E_1 is an $\mathcal{E}$-algebra and $\ker \varphi \neq \{0\}$. Then $\varphi \circ i_1 = 0$. Hence, there exists $\varphi_{Q_1} \in \mathrm{Hom}(Q(E_1), E_2)$ such that $\varphi_{Q_1} \circ \pi_1 = \varphi$ and $\pi_2 \circ \varphi_{Q_1} = Q(\varphi)$.

DEFINITION 1.6. If $E = \varinjlim(E_n, \varphi_{n,n+1})$, where E_n is a direct sum of $\mathcal{E}$-algebras, then E is called an *$\mathcal{AE}$-algebra*. If each E_n is a direct sum of $\mathcal{E}_0$-algebras, then E is called an *$\mathcal{AE}_0$-algebra*.

Let $E = \varinjlim(E_n, \varphi_{n,n+1})$ be an $\mathcal{AE}$-algebra. Set $I(E) = \varinjlim(I(E_n), I(\varphi_{n,n+1}))$ and set $Q(E) = \varinjlim(Q(E_n), Q(\varphi_{n,n+1}))$. Then E is an extension of $Q(E)$ by $I(E)$.

PROPOSITION 1.7. *Proposition* 1.4 *is still true when E_1 and E_2 are $\mathcal{AE}$-algebras.*

Proof. Note that $Q(E_1)$ and $Q(E_2)$ are finite C^*-algebras and $I(E_1), I(E_2)$ are direct limits of finite direct sums of purely infinite simple C^*-algebras. ∎

PROPOSITION 1.8. *Suppose E is an $\mathcal{E}$-algebra such that $Q(E) \cong M_k(C(S^1))$. If e is a projection in E not in $I(E)$, then every projection $p \in M_n(Q(eEe))$ lifts to a projection in $M_n(eEe)$ for all $n \in \mathbb{Z}_{>0}$. Consequently, $\delta_0^{eEe} = 0$.*

Proof. By Theorem 1.2 in [30], $RR(I(E)) = 0$. The proposition now follows from Lemma 2.5, Lemma 2.8, and Remark 2.9 in [29]. ∎

PROPOSITION 1.9. *Let $0 \to B \to E \to A \to 0$ be a unital extension. Suppose*
(i) *every hereditary C^*-subalgebra of B has an approximate identity consisting of projections;*
(ii) *every projection in A lifts to a projection in E.*
Suppose C is a unital C^*-subalgebra of A such that $C \cong M_n(\mathbb{C})$. Let $\{\overline{e}_{ij}\}_{i,j=1}^n$ be a system of matrix units for C. Then, there exists a system of matrix units $\{e_{ij}\}_{i,j=1}^n$ in E such that $1_E - \sum_{i=1}^n e_{ii} \in B$.

Effros [7] proved the above proposition when B is an AF-algebra. The key components of the proof are (i) and (ii).

COROLLARY 1.10. *Suppose E is an $\mathcal{E}$-algebra such that $Q(E) \cong M_l(C(S^1))$. Let $e \in E$ be a projection not in $I(E)$. Suppose $\{q_{ij}\}_{i,j=1}^n$ is a system of matrix units for $M_n(\mathbb{C}) \subset Q(eEe) \cong M_n(C(S^1))$. Then there exists a system of matrix units $\{e_{ij}\}_{i,j=1}^n$ in eEe such that $\pi(e_{ij}) = q_{ij}$ and $e - \sum_{i=1}^n e_{ii} \in I(eEe)$.*

Proof. By Theorem 1.2 in [30] and Corollary 2.6 in [2], $I(eEe)$ satisfies (i) in Proposition 1.9. By Proposition 1.8, eEe satisfies (ii) in Proposition 1.9. ∎

PROPOSITION 1.11. *Let A be a unital nuclear C^*-algebra in $\mathcal{N}$ and let $I \in \mathcal{P}$. Let τ be in $\mathrm{Hom}(A, \mathcal{M}(I)/I)$ such that τ is injective and unital. Then, for any strongly unital trivial extension τ_0 (i.e. there exists $\sigma_0 \in \mathrm{Hom}(A, \mathcal{M}(I))$ such that $\pi \circ \sigma_0 = \tau_0$ and $\sigma_0(1_A) = 1_{\mathcal{M}(I)}$), there exists $u \in U(\mathcal{M}(I))$ such that $\mathrm{Ad}(\pi(u)) \circ \tau = \tau \oplus \tau_0$.*

Proof. Let $s_1, s_2 \in \mathcal{M}(I)$ be isometries such that $1_{\mathcal{M}(I)} = s_1 s_1^* + s_2 s_2^*$. Then $\tau \oplus \tau_0$ can be written as $\pi(s_1)\tau(a)\pi(s_1^*) + \pi(s_2)\tau_0(a)\pi(s_2^*)$. Let $E_1 = \pi^{-1}(\tau(A))$. Define $\eta : E_1 \to \mathcal{M}(I)$ by $\eta(x) = s_1 x s_1^* + s_2(\sigma_0 \circ \pi(x))s_2^*$ for all $x \in E_1$. Since $s_2(\sigma_0 \circ \pi(\cdot))s_2^*|_I = 0$, by Theorem 8.3.1, pp. 125 in [25] ([13]), there exists $u \in U(\mathcal{M}(I))$ such that $\eta(x) - u^* x u \in I$ for all $x \in E_1$. Hence $\pi(u^*)\tau(a)\pi(u) = (\tau \oplus \tau_0)(a)$. ∎

COROLLARY 1.12. *Let E and E' be two $\mathcal{E}$-algebras such that $Q = Q(E) \cong Q(E')$ and $I = I(E) \cong I(E')$. If $[\tau_E] = [\tau_{E'}]$ in $\mathrm{Ext}(Q, I)$, then $E \cong E'$.*

The next lemma in well-known and we state it without proof.

LEMMA 1.13. *Let $I \in \mathcal{P}$. Let E be a unital C^*-subalgebra of $\mathcal{M}(I)$ which contains I as an essential ideal. Then:*

(i) *Every unitary in E/I lifts to a non-unitary isometry in E.*
(ii) *$u \in U(E/I)$ can be lifted to $v \in U(E)$ if and only if $\delta_1^E([u]) = 0$.*
Consequently, if E is an $\mathcal{E}$-algebra and $Q(E) = C(S^1)$, then E has a splitting if and only if $[\tau_E] = 0$ in $\mathrm{Ext}(C(S^1), I(E))$.

DEFINITION 1.14. Suppose $I \in \mathcal{P}$. A unitary $U \in \mathcal{M}(I)$ is *quasi-diagonal* if for every $\varepsilon > 0$, there exist a sequence $\{e_k\}_{k=1}^{\infty}$ of mutually orthogonal projections and a dense sequence $\{\lambda_k\}_{k=1}^{\infty}$ in $\mathrm{sp}(U)$ such that:
(i) $\left\{f_n = \sum_{k=1}^{n} e_k\right\}_{n=1}^{\infty}$ is an approximate identity of I consisting of projections;
(ii) $U - \sum_{k=1}^{\infty} \lambda_k e_k \in I$; and
(iii) $\left\|U - \sum_{k=1}^{\infty} \lambda_k e_k\right\| < \varepsilon$;
where the sums converge in the strict topology.

PROPOSITION 1.15. *Suppose $I \in \mathcal{P}$. Suppose E is a unital C^*-subalgebra of $\mathcal{M}(I)$ that contains I as an essential ideal. Let $\pi : E \to E/I$ be the quotient map. If $U \in U(E)$ such that $\mathrm{sp}(U) = \mathrm{sp}(\pi(U))$, then U is quasi-diagonal. Consequently, if E is an $\mathcal{E}$-algebra such that E is a trivial extension, then E is a quasi-diagonal extension and every $v \in U(E)$ is quasi-diagonal.*

Proof. Choose a sequence $\{\lambda_k\}_{k=1}^{\infty}$ in $\mathrm{sp}(U)$ such that for all $n \in \mathbb{Z}_{>0}$, the sequence $\{\lambda_k\}_{k=n}^{\infty}$ is dense in $\mathrm{sp}(U)$. Let $\{p_k\}_{k=1}^{\infty}$ be a sequence of mutually orthogonal projections in I such that $\left\{q_n = \sum_{k=1}^{n} p_k\right\}_{n=1}^{\infty}$ is an approximate identity for I. Set $U' = \sum_{k=1}^{\infty} \lambda_k p_k$, where the series converges in the strict topology. Then $\mathrm{sp}(\pi(U)) = \mathrm{sp}(\pi(U')) = X \subset S^1$. Define $\sigma_1, \sigma_2 : C(X) \to \mathcal{M}(I)$ by $\sigma_1(z) = U$ and $\sigma_2(z) = U'$. Let $E_1 = C^*(U, I)$ and let $E_2 = C^*(U', I)$. Let $s_1, s_2 \in \mathcal{M}(I)$ be isometries such that $1_{\mathcal{M}(I)} = s_1 s_1^* + s_2 s_2^*$. Define $\eta_1 : E_1 \to \mathcal{M}(I)$ by $\eta_1(x) = s_1 x s_1^* + s_2(\sigma_2 \circ \pi(x)) s_2^*$ for all $x \in E_1$ and define $\eta_1 : E_2 \to \mathcal{M}(I)$ by $\eta_2(a) = s_1(\sigma_1 \circ \pi(a)) s_1^* + s_2 a s_2^*$ for all $a \in E_2$. Note that $\eta_1(U) = \eta_2(U')$.

By Theorem 8.3.1, pp. 125 in [25] ([13]), there exist unitaries $V, W \in \mathcal{M}(I)$ such that $\|\eta_1(U) - VUV^*\| < \varepsilon/2$, $\|\eta_2(U') - WU'W^*\| < \varepsilon/2$, $\eta_1(U) - VUV^* \in I$, and $\eta_2(U') - WU'W^* \in I$. Hence, we have that $\|U - V^*WU'W^*V\| < \varepsilon$ and $U - V^*WU'W^*V \in I$. Set $e_k = V^*Wp_kW^*V$.

The last statement follows from Lemma 1.13 and the above result. ∎

LEMMA 1.16. *Suppose E is an $\mathcal{E}$-algebra such that $Q(E) = C(S^1)$. Let $\pi : E \to Q(E)$ be the quotient map. Then $K_1(E) \cong K_1(I(E)) \oplus \mathrm{ran}(\pi_{*,1})$, where $\mathrm{ran}(\pi_{*,1})$ is generated by $[\pi(u)]$ for some $u \in U(E)$.*

Proof. The lemma follows from the exactness of the six-term exact sequence in K-theory, $\delta_0^E = 0$ (Proposition 1.8), and $K_1(C(S^1)) = \langle [z] \rangle$. ∎

PROPOSITION 1.17. *Let E be an $\mathcal{E}$-algebra and let p be a nonzero projection in E. Then pEp is an $\mathcal{E}$-algebra. Moreover, if E is an $\mathcal{E}_0$-algebra, then pEp is an $\mathcal{E}_0$-algebra.*

Proof. If $p \in I(E)$, then pEp is in $\mathcal{P}_1$ such that $K_*(pEp) \cong K_*(pI(E)p)$. Hence, pEp is an $\mathcal{E}$-algebra. Suppose $p \notin I(E)$. Let $\{\overline{e}_{ij}\}_{i,j=1}^n$ be system of matrix units of $M_n(C(S^1)) \cong Q(E)$. By Corollary 1.10, there exists a system of matrix units $\{e_{ij}\}_{i,j=1}^n \subset pEp$ such that $\pi(e_{ij}) = \overline{e}_{ij}$ and $p - \sum_{i=1}^{n} e_{ii} \in pI(E)p$. Note that $e_{11}pI(E)pe_{11} \in \mathcal{P}$. Hence, $e_{11}pEpe_{11}$ is an $\mathcal{E}$-algebra with $Q(e_{11}pEpe_{11}) \cong C(S^1)$. Then pEp is isomorphic to $qM_n(E')q$ and $qM_n(E')q$ is an $\mathcal{E}$-algebra, where $I' = e_{11}pI(E)pe_{11}$, $E' = e_{11}pEpe_{11}$, and $q = \sum_{i=1}^{n} e_{ii}$. Note that $I(E') \cong I(pEp)$.

The last statement is clear from the above arguments. ∎

2. PERTURBATION LEMMAS

We now give several lemmas that will play an important role in many of the proofs in the sequel. Let A be a unital C^*-algebra. Denote the unit of $M_m(A)$ by 1_m. If $m = 0$, then $1_0 = 0$. The unitization of A will be denoted by $\widetilde{A}$.

LEMMA 2.1. *Let $A \in \mathcal{P}_1$ and let $\varepsilon > 0$. Suppose $u_1, u_2 \in U(A)$ such that $[u_1] = [u_2]$ in $K_1(A)$ and $\sup\{\mathrm{dist}(\lambda, \mathrm{sp}(u_i)) : \lambda \in S^1\} < \varepsilon$. Then, there exists $W \in U(A)$ such that $\|W^* u_1 W - u_2\| < 3\varepsilon$.*

Proof. The lemma follows from Lemma 2.2 in [17] when $[u_1] = [u_2] = 0$ and from Theorem 3 in [10] when $[u_1] \neq 0$. ∎

If E is a Hilbert A-module, let $\mathcal{L}_A(E)$ be the set of all module homomorphisms $T : E \to E$ for which T has an adjoint T^*. It is a well-known fact that $\mathcal{L}_A(E)$ is a C^*-algebra with respect to the operator norm. Let $x, y \in E$. Define $\theta_{x,y} \in \mathcal{L}_A(E)$ by $\theta_{x,y}(z) = x\langle y, z\rangle$. Let $\mathcal{K}_A(E)$ be the C^*-subalgebra of $\mathcal{L}_A(E)$ generated by the collection $\{\theta_{x,y}\}_{x,y\in E}$. Then $\mathcal{K}_A(E)$ is an ideal of $\mathcal{L}_A(E)$.

DEFINITION 2.2. Let $A^{(n)} = \bigoplus_{k=1}^{n} A$ be the Hilbert A-module of orthogonal direct sum of n copies of A. H_A will denote the following Hilbert A-module:

$$\Big\{\{a_n\} : a_n \in A \text{ and } \Big\{\sum_{k=1}^{n} a_k^* a_k\Big\}_{n=1}^{\infty} \text{ converges in norm as } n \to \infty\Big\}.$$

PROPOSITION 2.3 (See Proposition 15.2.12 in [28]). *$\mathcal{L}_A(A^{(n)}) \cong M_n(A) \cong \mathcal{K}_A(A^{(n)})$, $\mathcal{K}_A(H_A) \cong A \otimes \mathcal{K}$, and $\mathcal{L}_A(H_A) \cong \mathcal{M}(A \otimes \mathcal{K})$ for any C^*-algebra A.*

We will use the above identifications throughout.

Let

$$s_n = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \in \mathcal{L}_A(A^{(n)}).$$

Suppose A is a unital C^*-algebra. Let $e_i = (0, \dots, 0, 1_A, 0, \dots, 0)$ be the element in $A^{(n)}$ that has 1_A in the i^{th} coordinate. Then $s_n(e_i) = e_{i+1}$ for all $i = 1, 2, \dots, n-1$ and $s_n(e_n) = e_1$. Note that s_n is a unitary in $\mathcal{L}_A(A^{(n)})$. Define the *standard unilateral shift* S in $\mathcal{L}_A(H_A)$ by $S(\{a_n\}_{n=1}^{\infty}) = \{0, a_1, a_2, \dots\}$.

LEMMA 2.4. *Suppose that A is a unital C^*-algebra. For any $u \in U(M_m(A))$ and for any integer $n > m$, set $w_n = \begin{pmatrix} u & 0 \\ 0 & 1_{n-m} \end{pmatrix} s_n$. Then*

$$\sup\{\operatorname{dist}(\lambda, \operatorname{sp}(w_n)) \,:\, \lambda \in S^1\} < \frac{\pi}{n} + 2\Big(\frac{m}{n}\Big)^{1/2}.$$

The above lemma was proved by Lin (see Lemma 2.4 in [17]) for the case $K_1(A) = 0$. By the proof, we see that the assumption $K_1(A) = 0$ may be omitted.

LEMMA 2.5. *Let $A \in \mathcal{P}_1$. Then, for any $\varepsilon > 0$ and for any $k \in \mathbb{Z}_{>0}$, there exists $N > k$ such that for all $n \geqslant N$ and for any $u, v \in U(M_k(A))$ with $[u] = [v]$ in $K_1(A)$, there exists $W \in U(M_n(A))$ such that*

$$\|W \operatorname{diag}(u, 1_{n-k}) s_n W^* - \operatorname{diag}(v, 1_{n-k}) s_n\| < \varepsilon.$$

Proof. The lemma follows from Lemma 2.4 and Lemma 2.1. ∎

LEMMA 2.6 (Rørdam). *Let A be a unital C^*-algebra. Let $l, k \in \mathbb{Z}_{>0}$ and let $m > k + l$. Suppose $u \in U(M_k(A))$, $v \in U(A)$, and $w_1 \in U(M_m(A))$. Suppose that*

$$\|w_1 \operatorname{diag}(1_l, v, 1_{m-l-1}) s_m w_1^* - \operatorname{diag}(1_l, u, 1_{m-k-l}) s_m\| \leqslant \frac{7 - 2\pi}{l}.$$

Then, there exists $w \in U(M_{2m}(A))$ such that $w p_{2m} = p_{2m} w = p_{2m}$ and

$$\|w \operatorname{diag}(1_l, v, 1_{2m-l-1}) s_{2m} w^* - \operatorname{diag}(1_l, u, 1_{2m-k-l}) s_{2m}\| \leqslant \frac{7}{l},$$

where $p_{2m} = \operatorname{diag}(0, 0, \dots, 0, 1) \in M_{2m}(A)$.

Proof. Let $n = 2m$. Define $X \in M_n(\mathbb{C}) \subset M_n(A) \cong \mathcal{L}_A(A^{(n)})$ as follows: $Xe_i = e_{2i-1}$ and $Xe_{i+m} = e_{2i}$ for $1 \leqslant i \leqslant m$, where $\{e_i\}_{i=1}^n$ is the standard orthonormal basis for $\mathbb{C}^n$. A direct computation shows that

$$X s_n X^* = \begin{pmatrix} 0 & 0 & \dots & 0 & v_1 \\ 1_2 & 0 & \dots & 0 & 0 \\ 0 & 1_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1_2 & 0 \end{pmatrix} \quad \text{and} \quad X \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} X^* = \begin{pmatrix} 0 & 0 & \dots & 0 & 1_2 \\ 1_2 & 0 & \dots & 0 & 0 \\ 0 & 1_2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1_2 & 0 \end{pmatrix},$$

where $v_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that the eigenvalues of v_1 are 1 and -1. Hence, there exists a unitary v_2 such that $v_2^l = v_1$ and $\|v_2 - 1_2\| \leqslant \pi/l$.

Set $z = \operatorname{diag}(v_1, v_2^{l-1}, v_2^{l-2}, \dots, v_2, 1_2, \dots, 1_2) \in M_n(\mathbb{C})$. Then

$$z^*Xs_nX^*z = \begin{pmatrix} 0 & 0 & & & & 0 & 1_2 \\ v_2 & 0 & & & & 0 & 0 \\ & v_2 & & & & 0 & 0 \\ & & \ddots & & & \vdots & \vdots \\ & & & 1_2 & & 0 & 0 \\ & & & & \ddots & \vdots & \vdots \\ 0 & & & & & 1_2 & 0 \end{pmatrix} \quad \text{and} \quad \left\| X^*z^*Xs_nX^*zX - \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \right\| \leqslant \frac{\pi}{l}.$$

Let $z = \{a(i,j)\}_{i,j=1}^n$ and let $X^*zX = \{b(i,j)\}_{i,j=1}^n$. Then

$$\begin{aligned} b(l+i, l+j) &= a(2l+2i-1, 2l+2j-1) && \text{if} \quad i \neq j, \\ b(i, m+l+j) &= a(2i-1, 2(l+j)) = 0 && \text{if} \quad 0 < i \leqslant l \text{ and } j > 0, \text{ and} \\ b(m+l+i, j) &= a(2(l+i), 2j-1) = 0 && \text{if} \quad i > 0 \text{ and } 0 < j \leqslant l. \end{aligned}$$

Therefore

$$X^*zX = \begin{pmatrix} B & 0 & C & 0 \\ 0 & 1_{m-l} & 0 & 0 \\ D & 0 & 1_l & 0 \\ 0 & 0 & 0 & 1_{m-l} \end{pmatrix} \quad \text{and} \quad X^*z^*X = \begin{pmatrix} B^* & 0 & D^* & 0 \\ 0 & 1_{m-l} & 0 & 0 \\ C^* & 0 & 1_l & 0 \\ 0 & 0 & 0 & 1_{m-l} \end{pmatrix},$$

where $B, C,$ and D are in $M_l(\mathbb{C})$. Hence, $\operatorname{diag}(1_l, u, 1_{n-k-l})$ and $\operatorname{diag}(1_l, v, 1_{n-l-1})$ commute with X^*zX and X^*z^*X. Let $w = X^*zX \operatorname{diag}(w_1, 1_m) X^*z^*X$. Then, we have

$$\begin{aligned} &\|w \operatorname{diag}(1_l, v, 1_{n-l-1}) s_n w^* - \operatorname{diag}(1_l, u, 1_{n-k-l}) s_n\| \\ &\leqslant \frac{\pi}{l} + \|X^*zX \operatorname{diag}(w_1, 1_m) \operatorname{diag}(1_l, v, 1_{n-l-1}) \operatorname{diag}(s_m, s_m) \operatorname{diag}(w_1^*, 1_m) X^*z^*X \\ &\qquad - \operatorname{diag}(1_l, u, 1_{n-k-l}) s_n\| \\ &\leqslant \frac{\pi}{l} + \frac{7-2\pi}{l} + \|X^*zX \operatorname{diag}(1_l, u, 1_{n-k-l}) \operatorname{diag}(s_m, s_m) X^*zX - \operatorname{diag}(1_l, u, 1_{n-k-l}) s_n\| \\ &\leqslant \frac{\pi}{l} + \frac{7-2\pi}{l} + \frac{\pi}{l} = \frac{7}{l}. \end{aligned}$$

Finally, $wp_n = p_n$ since $X^*zXe_n = X^*ze_n = X^*e_n = e_n$. ∎

LEMMA 2.7. *Let $B \in \mathcal{P}_1$ and set $A = B \otimes \mathcal{K}$. Let E be the unital C^*-subalgebra of $\mathcal{M}(A)$ generated by S and A, where S is the standard unilateral shift in $\mathcal{M}(A) \cong \mathcal{L}_B(H_B)$. Let $\pi : E \to E/A$ be the quotient map. Suppose T is an isometry in E such that $\pi(T) = \pi(S)$ and $[1 - TT^*] = [1 - SS^*]$ in $K_0(A)$. Then, for any $\varepsilon > 0$, there exists $w \in U(\widetilde{A})$ such that $\|w^*Tw - S\| < \varepsilon$.*

Proof. Let $p_1 = 1 - SS^* = 1_B$ and $q_1 = 1 - TT^*$. Since $[p_1] = [q_1]$ in $K_0(A)$, by Lemma 1.1 in [31], there exists a unitary $w_{00} \in \widetilde{A} \subset E$ such that $w_{00}^* p_1 w_{00} = q_1$.

By replacing T with $w_{00}Tw_{00}^*$, we may assume $p_1 = q_1$. Let $u = p_1 + TS^*$. Then $u \in \widetilde{A}$ and $uS = T$.

Let $p_i = p_1 \otimes e_{ii}$ and let $E_k = \sum_{i=1}^{k} p_i$, where $\{e_{ij}\}_{i,j=1}^{\infty}$ is the standard system of matrix units for $\mathcal{K}$. Choose $m' \in \mathbb{Z}_{>0}$ such that $7/m' < \varepsilon$. Since $\{E_n\}_{n=1}^{\infty}$ forms an approximate identity for A consisting of projections, there exist $v \in U(\widetilde{A})$ and a positive integer $L > m'$ such that (1) $p_i v = v p_i$ for $i = 2, \dots, k$; (2) $p_1 v = v p_1 = p_1$; (3) $v = vE_L + 1 - E_L$; and (4) $\|v - u\| < \varepsilon/2$. Suppose there exists $w \in U(\widetilde{A})$ such that $\|w^*Sw - vS\| < \varepsilon/2$. Then, $\|w^*Sw - T\| \leqslant \|w^*Sw - vS\| + \|vS - T\| < \varepsilon$. Hence, by replacing T by vS, we may assume $u = v$.

Let $\{e_i\}_{i=1}^{\infty} = \{(1, 0, \dots), (0, 1, 0, \dots), \dots\}$ be the standard orthonormal basis for H_B. Define a scalar matrix $w_0 \in \mathcal{L}_B(H_B)$ as follows:

$$\begin{aligned} w_0(e_i) &= e_{L+i+1} && \text{for} \quad i = 1, 2, \dots, L+1, \\ w_0(e_{L+i+1}) &= e_i && \text{for} \quad i = 1, 2, \dots, L+1, \\ w_0(e_{2L+2+j}) &= e_{2L+2+j} && \text{for} \quad j = 1, 2, \dots. \end{aligned}$$

Note that $w_0 \in U(\widetilde{A})$ and $w_0^*Sw_0E_L = SE_L$. Let $u' = w_0^*uw_0$. Then $u' = E_L + (E_{2L+2} - E_L)u'(E_{2L+2} - E_L) + 1 - E_{2L+2}$.

Set $u'' = p_1 + u'w_0^*Sw_0S^*$. It is easy to check that $u''S = w_0^*Tw_0$. Clearly, $u''p_1 = p_1 = p_1u''$. Note, for $1 < i \leqslant L$, $u'w_0^*Sw_0S^*(e_i) = u'w_0^*Sw_0(e_{i-1}) = u'w_0^*S(e_{L+i}) = u'w_0^*(e_{L+i+1}) = u'(e_i) = e_i$. Also, for all $i > 2L + 4$

$$u'w_0^*Sw_0S^*(e_i) = u'w_0^*Sw_0(e_{i-1}) = u'w_0^*S(e_{i-1}) = u'w_0^*(e_i) = u'(e_i) = e_i.$$

Therefore, $u'' = E_L + (E_{2L+4} - E_L)u''(E_{2L+4} - E_L) + (1 - E_{2L+4})$. So, we may assume that $u = u''$.

Let $u_1 = (E_{2L+4} - E_L)u''(E_{2L+4} - E_L)$. Note that we may identify u_1 as an element in $U(M_{L+4}(B))$. By Lemma 2.5, there exist $m > 2L + 4$, a unitary $w_1 \in M_m(B)$, and a unitary $v \in B = p_{L+1}Ap_{L+1}$ such that

$$\|w_1(E_L + v + E_m - E_{L+1})s_m w_1^* - (E_L + u_1 + E_m - E_{2L+4})s_m\| \leqslant \frac{7 - 2\pi}{L}.$$

By Lemma 2.6, there exists $w_2 \in U(M_{2m}(B))$ such that

$$\|w_2(E_L + v + E_{2m} - E_{L+1})s_{2m} w_2^* - (E_L + u_1 + E_{2m} - E_{2L+4})s_{2m}\| \leqslant \frac{7}{L}$$

and $w_2 p_{2m} = p_{2m} w_2 = p_{2m}$. Set $w_3 = w_2 + 1 - E_{2m}$. Then $wp_{2m} = p_{2m}$. Thus

$$w_3(E_L + v + 1 - E_{L+1})Sw_3^*(1 - E_{2m-1}) = (E_L + v + 1 - E_{L+1})S(1 - E_{2m-1}).$$

Note that $w_3^*E_{2m-1} = E_{2m-1}w_3^*E_{2m-1}$ and $(E_L + v + E_{2m} - E_{L+1})s_{2m}E_{2m-1} = (E_L + v + 1 - E_{L+1})SE_{2m-1}$. So, we have $(E_L + v + E_{2m} - E_{L+1})s_{2m}w_3^*E_{2m-1} = (E_L + v + 1 - E_{L+1})Sw_3^*E_{2m-1}$.

Let $w = w_3 \operatorname{diag}(v^*, \dots, v^*, 1, \dots)$, where v^* is repeated $L-1$ times. Then $w \in U(\widetilde{A})$ such that

$$\begin{aligned}\|wSw^* - uS\| &= \|(w_3(E_L + v + 1 - E_{L+1})Sw_3^* - uS)E_{2m-1}\| \\ &\leqslant \|w_2(E_L + v + E_{2m} - E_{L+1})s_{2m}w_2^* - (E_L + u_1 + E_{2m} - E_{L+4})s_{2m}\| \\ &\leqslant \frac{7}{L} < \varepsilon. \quad \blacksquare\end{aligned}$$

LEMMA 2.8. *Let A be a separable purely infinite simple C^*-algebra and let s, t be two non-unitary isometries in A (in $\widetilde{A}$ if A does not have a unit). Then, for any $\varepsilon > 0$, there exists $W \in U(A)$* (or $U(\widetilde{A})$) *such that $\|W^* s W - t\| < \varepsilon$.*

Proof. Suppose A is unital. Then the conclusion follows from Lemma 2.9 in [17]. Suppose A is non-unital. Since A is a separable C^*-algebra with real rank zero, there exist a projection $e \in A$ and non-unitary isometries $s', t' \in eAe$ such that $\|s' + 1 - e - s\| < \varepsilon/3$ and $\|t' + 1 - e - t\| < \varepsilon/3$. By the unital case, there exists $w \in U(eAe)$ such that $\|w^* s' w - t'\| < \varepsilon/3$. Let $W = w + 1 - e \in U(\widetilde{A})$. Then $\|W^* s W - t\| < \varepsilon$. ∎

LEMMA 2.9. *Let E be an $\mathcal{E}$-algebra. Then, for any $\varepsilon > 0$ and for any $U_1, U_2 \in U(E)$ satisfying* $\operatorname{sp}(U_1) = \operatorname{sp}(U_1) = S^1$, *$\pi(U_1) = \pi(U_2)$, and $[U_1] = [U_2]$ in $K_1(E)$, there exists $W \in U(\widetilde{I(E)})$ such that $\|W^* U_1 W - U_2\| < \varepsilon$.*

Proof. By Proposition 1.15, we may assume $U_i = \sum_{k=1}^{\infty} \lambda_k^{(i)} e_k^{(i)}$, where the sum converges in the strict topology. Using a similar argument as in the proof of Proposition 2.10 in [17], we get $W \in U(E)$ such that $\|W^* U_1 W - U_2\| < \varepsilon$. ∎

LEMMA 2.10. *Let $I \in \mathcal{P}$. Let E be a unital C^*-subalgebra of $\mathcal{M}(I)$ which contains I as an essential ideal. Suppose S_1, S_2 are two non-unitary isometries such that $\pi(S_1) = \pi(S_2)$ and $[1 - S_1 S_1^*] = [1 - S_2 S_2^*] = 0$ in $K_0(I)$. Then, for any $\varepsilon > 0$, there exists a unitary $W \in \widetilde{I}$ such that $\|W^* S_1 W - S_2\| < \varepsilon$.*

Proof. By Lemma 1.13, there exists $U \in U(E)$ and $\pi(U) = \pi(S_2)$. By Proposition 1.15, we may assume that there exists a sequence of mutually orthogonal projections $\{e_n\}_{n=1}^{\infty}$ in I such that $U = \sum_{n=1}^{\infty} U_n$, where $U_n \in U(e_n I e_n)$ and the sum converges in the strict topology.

Let $T = \sum_{n=2}^{\infty} U_n + v$, where the sum converges in the strict topology and $v \in e_1 I e_1$ such that $v^* v = e_1$ and $v v^* \neq e_1$. Note that $T^* T = 1$, $T T^* \neq 1$, and $\pi(T) = \pi(S_1)$. Arguing as in the proof of Lemma 2.11 in [17], there exists $W_1 \in U(\widetilde{I})$ such that $\|W_1^* S_1 W_1 - T\| < \varepsilon/2$. Since $\pi(T) = \pi(S_2)$, we again get $W_2 \in U(\widetilde{I})$ such that $\|W_2^* S_2 W_2 - T\| < \varepsilon/2$. Let $W = W_1 W_2^*$. Then $\|W^* S_1 W - S_2\| < \varepsilon$. ∎

3. THE INVARIANT

DEFINITION 3.1. Let A be a C^*-algebra. Let $V(A)$ be the set of all Murray-von Neumann equivalence classes of projections in matrices (of all sizes) over A. If addition on $V(A)$ is given by $[p] + [q] = [p \oplus q]$ for all $[p], [q] \in V(A)$, then $V(A)$ is an abelian semigroup. If $\varphi \in \mathrm{Hom}(A, B)$, then $V(\varphi)$ will denote the induced homomorphism.

LEMMA 3.2 (Proposition 5.5.5 in [1]). *Let A be a C^*-algebra such that A has an approximate identity consisting of projections. Then the map from the Grothendieck group of $V(A)$ to $K_0(A)$ is an isomorphism.*

If A is a non-unital separable purely infinite simple C^*-algebra, then A has an approximate identity consisting of projections. Throughout the rest of this section, $[p]_0$ will denote the image of $[p]$ in $K_0(A)$.

LEMMA 3.3. *Let E be an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$. Then for every projection $p \in M_n(E)$ not in $M_n(I(E))$, there exist a projection $e(p) \in I(E)$ and a positive integer $n(p) \leqslant n$ such that $p \sim 1_{n(p)} \oplus e(p)$.*

Proof. Since $Q(E) = C(S^1)$, there exists a projection $q \in M_n(C(S^1))$ such that $\pi(q) = 1_m$ and $p \sim q$. So, we may assume $\pi(p) = 1_m = \pi(1_m)$. Since $I(E)$ has an approximate identity consisting of projections, there exists a partial isometry $v \in M_n(E)$ such that $v^*v \leqslant 1_m$, $vv^* \leqslant p$, and $\pi(v) = 1_m$. Let e be a nonzero projection in $I(E)$ such that $[e]_0 = 0$ in $K_0(I(E))$. By Proposition 1.5 in [4], there exists $e' \in I(E)$ such that $(1_m - v^*v) \oplus e' \sim e$. Hence, $p \sim v^*v \oplus (p - vv^*) \sim 1_m \oplus e' \oplus (p - vv^*)$. ∎

NOTATION 3.4. (i) Denote the disjoint union of X_1 and X_2 by $X_1 \sqcup X_2$.

(ii) Suppose E is an $\mathcal{E}$-algebra such that $Q(E) \cong M_n(C(S^1))$. If $x \in K_0(I(E))$, then denote the image of x in $K_0(I(E))/\operatorname{ran}\delta_1^E$ by $\overline{x}$.

THEOREM 3.5. (i) *Let E be a separable purely infinite simple C^*-algebra. Consider the abelian semigroup $\{0\} \sqcup K_0(E)$, where $0 + x = x$ for all $x \in K_0(E)$. Then $V(E) \cong \{0\} \sqcup K_0(E)$, where the isomorphism sends $[0]$ to 0 and sends $[p]$ to $[p]_0$ for every nonzero projection p.*

(ii) *Let E be an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$. Consider the abelian semigroup $V = \{0\} \sqcup K_0(I(E)) \sqcup (\mathbb{Z}_{>0} \oplus (K_0(I(E))/\operatorname{ran}\delta_1^E))$, where addition is defined as follows: Addition in $K_0(I(E))$ and in $\mathbb{Z}_{>0} \oplus (K_0(I(E))/\operatorname{ran}\delta_1^E)$ are the usual addition in those semigroups. If $x \in K_0(I(E))$ and $(z_1, \overline{z}_2) \in \mathbb{Z}_{>0} \oplus (K_0(I(E))/\operatorname{ran}\delta_1^E)$, then $x + (z_1, \overline{z}_2) = (z_1, \overline{z}_2 + \overline{x})$. Suppose $\alpha : V(E) \to V$ is defined by $\alpha([p]) = 0$ if $p = 0$; $\alpha([p]) = [p]_0$ if $p \in M_n(I(E)) \setminus \{0\}$; and $\alpha([p]) = (n(p), \overline{[e(p)]_0})$ otherwise, where $p \sim 1_{n(p)} \oplus e(p)$ is the decomposition given in Lemma 3.3. Then α is a well-defined isomorphism. Moreover, the natural map from $V(I(E))$ to $V(E)$ is injective and α sends its image onto $\{0\} \sqcup K_0(I(E))$.*

(iii) *Let E be a trivial extension with $Q(E) = C(S^1)$. Then α in* (ii) *gives an isomorphism from $V(E)$ onto $\{0\} \sqcup K_0(I(E)) \oplus \mathbb{Z}_{\geqslant 0}$. Moreover, the natural map from $V(I(E))$ to $V(E)$ is injective and α sends its image onto $\{0\} \sqcup K_0(I(E))$.*

(iv) *Let E be an $\mathcal{E}$-algebra. Then the map x to* $\operatorname{diag}(x, 0)$ *induces an isomorphism from $V(E)$ onto $V(M_n(E))$. If p is a projection in $M_n(E)$ not in $M_n(I(E))$, then the inclusion from $pM_n(E)p$ to $M_n(E)$ induces an isomorphism from $V(pM_n(E)p)$ onto $V(M_n(E))$. Hence, $V(E)$ is a finitely generated abelian semigroup.*

Proof. (i) is a consequence of Theorem 1.4 in [4] and Lemma 3.2.

It is easy to check that if $p \in I(E)$, $q \in E$, and $p \sim q$, then $q \in I(E)$. Therefore,

$$\begin{aligned} V(E) &= V(I(E)) \sqcup \{[p] \ : \ p \in M_n(E) \setminus M_n(I(E)) \text{ for some } n\} \\ &\cong \{0\} \sqcup K_0(I(E)) \sqcup \{[p] \ : \ p \in M_n(E) \setminus M_n(I(E)) \text{ for some } n\}. \end{aligned}$$

(ii) and (iii) now follows from the exactness of the six-term exact sequence in K-theory and Lemma 3.3. The last statement of the theorem is clear. ∎

Let A be a C^*-algebra. An element $s \in A$ is called *hyponormal* if $s^*s \geqslant ss^*$. For a unital C^*-algebra A, let $S_n(A)$ be the set of all nonzero hyponormal partial isometries in $M_n(A)$. Let $S(A) = \bigcup_{n=1}^{\infty} S_n(A)$, where we embed $S_n(A)$ into $S_{n+1}(A)$ by sending s to $\operatorname{diag}(s, 1)$.

DEFINITION 3.6. For $v_1, v_2 \in S_n(A)$, we write $v_1 \simeq v_2$ if and only if there exists $m \in \mathbb{Z}_{\geqslant 0}$ such that $(1_n - v_1^*v_1 + v_1) \oplus 1_m$ is homotopic to $(1_n - v_2^*v_2 + v_2) \oplus 1_m$ in $S_{n+m}(A)$. Set $k(A)_+ = S(A)/\simeq$. Let $[s]_k$ denote the equivalence class represented by s. If addition is defined by $[u]_k + [v]_k = [u \oplus v]_k$, then $k(A)_+$ becomes an abelian semigroup. If A is a non-unital C^*-algebra, then $k(A)_+ = k(\widetilde{A})_+$. If $\varphi \in \operatorname{Hom}(A, B)$, then $k(\varphi)_+$ will denote the induced homomorphism.

LEMMA 3.7. *Let A be a unital C^*-algebra. Then the following hold:*

(i) *For all projections $p, q \in M_n(A)$, we have $[p]_k = [q]_k$.*

(ii) *If s is an isometry and u is a unitary, then $[su]_k = [s]_k + [u]_k = [us]_k$.*

(iii) *$k(A)_+ = \{[s]_k \ : \ s$ is an isometry in $M_n(A)$ for some $n \in \mathbb{Z}_{>0}\}$.*

(iv) *$k(B(\ell^2))_+$ is isomorphic to $\mathbb{Z}_{\geqslant 0} \sqcup \{\mathcal{S}\}$ where addition in $\mathbb{Z}_{\geqslant 0}$ is the usual one and if $x \in \mathbb{Z}_{\geqslant 0}$, then $x + \mathcal{S} = \mathcal{S}$.*

Proof. Everything is easy to check. We only prove (iv) to familiarize the reader with some techniques that will be used later. Let s and t be isometries in $B(\ell^2)$. Note that $[s]_k = [t]_k$ implies that $1_{B(\ell^2)} - ss^* \sim 1_{B(\ell^2)} - tt^*$. Suppose $1_{B(\ell^2)} - ss^* \sim 1_{B(\ell^2)} - tt^*$. Then there exists a unitary u in $B(\ell^2)$ such that $u(1 - ss^*)u^* = 1_{B(\ell^2)} - tt^*$. Let $w = (usu^*)^*t$. It is easy to check that w is a unitary and $usu^*w = t$. Note that there exists a norm continuous path of unitaries w_t in $B(\ell^2)$ such that $w_0 = 1_{B(\ell^2)}$ and $w_1 = w$. Thus, $[s]_k = [usu^*]_k = [usu^*w]_k = [t]_k$.

Note that $1_{B(\ell^2)} - ss^* \sim 1_{B(\ell^2)} - tt^*$ if $\pi(s)$ and $\pi(t)$ are not unitaries in $B(\ell^2)/\mathcal{K}$. Hence, the map that sends $[s]_k$ to $[1_{B(\ell^2)} - ss^*]$ is an isomorphism. ∎

LEMMA 3.8. *Let A be a unital C^*-algebra. Suppose that s and v are isometries in A such that $\|s - v\| < 2/(4\sqrt{2}+1)$. Then $[s]_k = [v]_k$ in $k(A)_+$.*

Proof. Suppose s or v is in $U(A)$. Then, $s, v \in U(A)$ and $s^*v \in U_0(A)$ since $\|s - v\| < 2/(4\sqrt{2}+1) < 2$. Thus, $[v]_k = [s]_k$ in $k(A)_+$. Suppose s and v are non-unitary isometries. Note that $\|(1 - ss^*) - (1 - vv^*)\| < 4/(4\sqrt{2}+1)$. Hence, there exists $w \in U(A)$ such that $\|1 - w\| < 4\sqrt{2}/(4\sqrt{2}+1)$, $w^*(1 - vv^*)w = 1 - ss^*$, and $\|w^*vw - s\| < 2$. Let $x = w^*vw$. Then x is an isometry with $1 - xx^* = 1 - ss^*$. Also, $u = x^*s \in U(A)$, $xu = s$, and $\|1 - u\| < 2$. Hence, by Lemma 3.7, $[s]_k = [w^*vw]_k = [v]_k$ in $k(A)_+$. ∎

LEMMA 3.9. *Let A be a unital C^*-algebra. If $[u]_k = [s]_k$, where $u \in U(A)$ and s is an isometry in A, then $s \in U(A)$. If $\iota : K_1(A) \to k(A)_+$ be the natural map, then*

$$k(A)_+ = \iota(K_1(A)) \sqcup \{[s]_k \;:\; s \text{ non-unitary isometry in } S(A)\}.$$

Moreover, ι is injective and ι is an isomorphism whenever A has cancellation.

Proof. The first part of the lemma follows from Lemma 3.8. If A has cancellation, then every isometry is a unitary. Hence, by Lemma 3.7, ι is surjective. ∎

LEMMA 3.10. *Let E be an $\mathcal{E}$-algebra such that $Q(E) = C(S^1)$. Then there exist a projection $e \in I(E)$, a $*$-isomorphism $I(\eta) : I(E) \to eI(E)e \otimes \mathcal{K}$, a unital and injective $\eta \in \operatorname{Hom}(E, \mathcal{M}(eI(E)e \otimes \mathcal{K}))$, and $y \in E$ such that $\pi(y) = z$, $\eta(y)$ is the standard unilateral shift for $eI(E)e \otimes \mathcal{K} = I$, and the following diagram commutes:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(E) & \longrightarrow & E & \longrightarrow & Q(E) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I(\eta)} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle Q(\eta)} & & \\
0 & \longrightarrow & I & \longrightarrow & \mathcal{M}(I) & \longrightarrow & \mathcal{M}(I)/I & \longrightarrow & 0
\end{array}$$

Proof. Let T be a non-unitary isometry such that $\pi(T) = z$. Set $e' = 1 - TT^*$. By Corollary 2.6 in [2] and Theorem 3.2(i) in [30], there is a $*$-isomorphism $\gamma : I(E) \to e'I(E)e' \otimes \mathcal{K}$. Hence, γ extends to a $*$-isomorphism γ from $\mathcal{M}(I(E))$ onto $\mathcal{M}(e'I(E)e' \otimes \mathcal{K})$. Set $\eta' = \gamma|_E$. Let S' be the standard unilateral shift for $e'I(E)e' \otimes \mathcal{K} = I'$. Let τ and τ' be the Busby invariants for the extensions $C^*(\eta'(T), I')$ and $C^*(S', I')$ respectively. Since $[1 - S'(S')^*] = [e'] = \delta_1^E([z])$, we have $[\tau] = [\tau']$ in $\operatorname{Ext}(C(S^1), I(E))$. By Proposition 1.11, there exists a unitary $W \in \mathcal{M}(e'I(E)e' \otimes \mathcal{K})$ such that $\pi(WS'W^*) = z$. Set $S = WS'W^*$, $e = We'W^*$, and $\eta = \operatorname{Ad}(W) \circ \eta'$. ∎

LEMMA 3.11. *Let E be an $\mathcal{E}$-algebra such that $Q(E) \cong M_n(C(S^1))$. Suppose S and T are two non-unitary isometries. Then $[\pi(S)] = [\pi(T)]$ in $K_1(Q(E))$ if and only if $[S]_k = [T]_k$ in $k(E)_+$.*

Proof. If $[S]_k = [T]_k$, then by Lemma 3.9, $[\pi(S)] = [\pi(T)]$. Suppose $[\pi(S)] = [\pi(T)]$. Then there exists $w \in U_0(E)$ such that $\pi(S) = \pi(wT)$. By Lemma 3.7, $[wT]_k = [T]_k$. Hence, we may assume $\pi(S) = \pi(T)$.

Let $0 < \varepsilon < 2/(4\sqrt{2}+1)$. Suppose $[1 - SS^*]_0 \neq 0$. Then $C^*(S, I(E))$ is a unital essential extension of $C(S^1)$ by $I(E)$. By Lemma 3.10, $\pi(T) = \pi(S) = \pi(S_1)$, where S_1 is the standard unilateral shift of $eI(E)e \otimes \mathcal{K} \cong I(E)$. Applying Lemma 2.7 to T and S_1, then to S and S_1, there exists $v \in U(\widetilde{I(E)})$ such that $\|v^*Sv - T\| < \varepsilon$. Suppose $[1 - SS^*]_0 = 0$. Then by Lemma 2.10, there exists $v \in U(\widetilde{I(E)})$ such that $\|v^*Sv - T\| < \varepsilon$. Hence, by Lemma 3.8 and Lemma 3.7, $[S]_k = [T]_k$. ∎

Note that in the proof of the above lemma we proved the following.

PROPOSITION 3.12. *Let E be an $\mathcal{E}$-algebra such that $Q(E) \cong M_n(C(S^1))$. Let $\varepsilon > 0$. If S and T are two non-unitary isometries in E such that $\pi(S) = \pi(T)$, then there exists $w \in U(\widetilde{I(E')})$ such that $\|w^*Sw - T\| < \varepsilon$.*

THEOREM 3.13. (i) *Let E be a purely infinite simple C^*-algebra. Consider the abelian semigroup $K_1(E) \sqcup \{\mathcal{S}\}$, where addition is defined as follows: If $x \in K_1(E)$, then $x + n\mathcal{S} = \mathcal{S}$ for all $n \in \mathbb{Z}$. Addition in $K_1(E)$ is the usual one. Define $\alpha : k(E)_+ \to K_1(E) \sqcup \{\mathcal{S}\}$ by $\alpha([s]_k) = [s]$ if s is a unitary and $\alpha([s]_k) = \mathcal{S}$ otherwise. Then α is a well-defined isomorphism.*

(ii) *Let E be an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$ and E is a trivial extension. Consider the abelian semigroup $G = \{(x, y) : x \in k(I(E))_+ \text{ and } y \in \mathbb{Z}\}$ with coordinate-wise addition. Then $k(E)_+$ is isomorphic to G via some isomorphism α. Moreover, the map from $k(I(E))_+$ to $k(E)_+$ is injective and α sends its image onto the sub-semigroup $\{(x, 0) : x \in k(I(E))_+\}$ of G.*

(iii) *Let E be an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$ and E is a non-trivial extension. Consider the abelian semigroup $K_1(E) \sqcup \mathbb{Z}$ with addition defined as follows: if $x \in K_1(E)$ and $n \in \mathbb{Z}$, then $x + n = m + n$ where $m = \pi_{*,1}(x) \in K_1(Q(E)) \cong \mathbb{Z}$. Addition in $K_1(E)$ and in $\mathbb{Z}$ are the usual addition in those semigroups. (Note that the zero in $K_1(E)$ is the zero in $K_1(E) \sqcup \mathbb{Z}$ but the zero in $\mathbb{Z}$ is not the zero in $K_1(E) \sqcup \mathbb{Z}$.) Then there exists an isomorphism α from $k(E)_+$ onto $K_1(E) \sqcup \mathbb{Z}$. Moreover, α sends the image of the natural injective map from $k(I(E))_+$ to $k(E)_+$ onto $K_1(I(E)) \sqcup \{0\}$.*

(iv) *Let E be an $\mathcal{E}$-algebra. Then the map x to* $\operatorname{diag}(x, 1)$ *induces an isomorphism from $k(E)_+$ onto $k(M_n(E))_+$. If p is a projection in $M_n(E)$ not in $M_n(I(E))$, then the inclusion from $pM_n(E)p$ to $M_n(E)$ induces an isomorphism from $k(pM_n(E)p)_+$ onto $k(M_n(E))_+$. Hence, $k(E)_+$ is a finitely generated abelian semigroup.*

Proof. (i) follows from Lemma 2.8, Lemma 3.8, and Lemma 3.7.

(ii) Let s be a non-unitary isometry in $M_n(E)$. Since E is a trivial extension and $\pi(s)$ is a unitary in $M_n(C(S^1))$, there exists $u \in U(M_n(E))$ such that $\pi(s) = \pi(u)$. Hence, su^* is a non-unitary isometry in $M_n(\widetilde{I(E)})$. So, $[s]_k = [su^*]_k + [u]_k$ in $k(E)_+$. Note that this decomposition is unique.

Let β be the isomorphism from $K_1(E)$ onto $K_1(I(E)) \oplus \mathbb{Z}$. Define α from $k(E)_+$ to $k(I(E))_+ \oplus \mathbb{Z}$ by $\alpha([w]_k) = \beta([u])$ if w is a unitary. If w is a non-unitary isometry in $M_n(E)$, then by the above observation $[w]_k = [s]_k + [u]_k$ for some $u \in$

$U(M_n(E))$ and non-unitary isometry $s \in M_n(\widetilde{I(E)})$. Define $\alpha([w]) = ([s]_k, [u])$, where $[s]_k$ is now considered as an element in $k(I(E))_+$. Then α is a well-defined isomorphism with the desired property.

(iii) By Lemma 3.11, $\{[s]_k : s \text{ is a non-unitary isometry in } S(E)\} \cong \mathbb{Z}$, where the isomorphism is induced by π. Define $\alpha : k(E)_+ \to K_1(E) \sqcup \mathbb{Z}$ by $\alpha([s]_k) = [s]_k$ if s is a unitary and $\alpha([s]_k) = [\pi(s)]$ otherwise. Then α is a well-defined isomorphism with the desired property.

The last statement of the theorem is clear. ∎

COROLLARY 3.14. *Let E be an $\mathcal{E}$-algebra such that $Q(E) \cong M_n(C(S^1))$. If $k(E)$ denotes the Grothendieck group of $k(E)_+$, then $\pi : E \to Q(E)$ induces an isomorphism from $k(E)$ onto $K_1(Q(E))$.*

Proof. By Lemma 3.9, $k(Q(E))_+ \cong \mathbb{Z}$ and by Theorem 3.13, $k(E) \cong \mathbb{Z}$. It is now easy to see that π induces an isomorphism from $k(E)$ onto $K_1(Q(E))$. ∎

DEFINITION 3.15. Let A be a C^*-algebra and let

$$\overline{V}(A) = \{([u^*u], [u]_k) : u \in S(A)\} \subset V(A) \oplus k(A)_+.$$

Define $d_A : k(A)_+ \to V(A)$ by $d_A([u]_k) = [u^*u - uu^*]$. For convenience, we will sometimes denote d_A by just d. A homomorphism $\alpha : \overline{V}(A) \to \overline{V}(B)$ consists of two homomorphisms $\alpha_v : V(A) \to V(B)$ and $\alpha_k : k(A)_+ \to k(B)_+$ such that if $\alpha([s]_k) = [v]_k$, then $\alpha_v([s^*s]) = [v^*v]$.

LEMMA 3.16. *Let A be a C^*-algebra. Define $\Theta_v^A : V(A) \to \overline{V}(A)$ by $\Theta_v^A([p]) = ([p], 0)$ and define $\Theta_k^A : k(A)_+ \to \overline{V}(A)$ by $\Theta_k^A([s]_k) = ([s^*s], [s]_k)$. Then Θ_v^A and Θ_k^A are injective homomorphisms. Using Θ_v^A and Θ_k^A, we may identify $V(A)$ and $k(A)_+$ as subsemigroups of $\overline{V}(A)$.*

The proof of the above lemma is easy and we leave it for the reader.

DEFINITION 3.17. Define $V_*(A)$ to be the set of triples

$$\{([u^*u], [u]_k, d_A([u]_k)) : u \in S(A)\} \subset V(A) \oplus k(A)_+ \oplus V(A).$$

Let A and B be C^*-algebras. A homomorphism $\eta : V_*(A) \to V_*(B)$ is a homomorphism $\eta : \overline{V}(A) \to \overline{V}(B)$ such that $\eta_v \circ d_A = d_B \circ \eta_k$. If $\varphi \in \mathrm{Hom}(A, B)$, then denote the induced homomorphism on $V_*(A)$ by $V_*(\varphi)$.

LEMMA 3.18. *Let E and E' be two finite direct sums of $\mathcal{E}$-algebras.*

(i) *Suppose $\alpha : \overline{V}(E) \to \overline{V}(E')$ is a homomorphism. Then, α_v maps $V(I(E))$ to $V(I(E'))$ and α_k maps $k(I(E))_+$ to $k(I(E'))_+$.*

(ii) *If $\eta : V_*(E) \to V_*(E')$ is a homomorphism, then η induces a homomorphism from $V_*(I(E))$ to $V_*(I(E'))$. Also, if $\iota : K_1(E) \to k(E)_+$ and $\iota' : K_1(E') \to k(E')_+$ are the injective homomorphisms given in Lemma 3.9, then $\eta_k : k(E)_+ \to k(E')_+$ maps $\iota(K_1(E))$ to $\iota'(K_1(E'))$.*

Proof. (i) Using the identifications in Theorem 3.5 and Theorem 3.13 and all semigroup homomorphisms are assumed to preserve identities, one easily checks that α_v maps $V(I(E))$ to $V(I(E'))$ and α_k maps $k(I(E))_+$ to $k(I(E'))_+$.

(ii) The first part of (ii) follows from (i). Let $\pi : E \to Q(E)$ and $\pi' : E' \to Q(E')$ be the quotient maps. Suppose $u \in U(M_n(E))$ and $\eta_k([u]_k) = [s]_k$ for some isometry $s \in M_m(E')$. Then $0 \sim 1_m - ss^*$. Hence, $ss^* = 1_m$. Thus, η_k maps $\iota(K_1(E))$ to $\iota'(K_1(E'))$. ∎

Let E_1 and E_2 be finite direct sums of $\mathcal{E}$-algebras. Then E_i is an extension of $Q(E_i)$ by $I(E_i)$. By Proposition 1.8, $\delta_0^{E_i} = 0$. So, the six-term exact sequence in K-theory associated to E_i has the form

$$0 \to K_1(I(E_i)) \to K_1(E_i) \to K_1(Q(E_i)) \to K_0(I(E_i)) \to K_0(E_i) \to K_0(Q(E_i)) \to 0.$$

Denote this exact sequence by $\mathbf{K}(E_i)$. A map from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$ consists of six group homomorphisms $\alpha = \{\alpha_i\}_{i=1}^6$ making the obvious diagram commute.

Let E be a unital extension of $Q(E)$ by $I(E)$. Set

$$\Gamma = \{(x, y) \,:\, x \in K_0(Q(E))_+, y \in K_1(Q(E));\ \text{if } x = 0, \text{then } y = 0\}.$$

Let $K_*(Q(E))$ denote the graded group $K_0(Q(E)) \oplus K_1(Q(E))$ with the partial order generated by Γ. By Proposition 1.4, every $\varphi \in \mathrm{Hom}(E_1, E_2)$ between two direct sums of $\mathcal{E}$-algebras induces a map $\{\alpha_i\}_{i=1}^6$ from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$ such that $\alpha_6 \oplus \alpha_3$ preserves the order.

Using a similar method as in Section 1.16 in [17] we get the following.

PROPOSITION 3.19. *Let E_1 and E_2 be two finite direct sums of $\mathcal{E}$-algebras. Suppose $\eta : V_*(E_1) \to V_*(E_2)$ is a homomorphism. Then η induces a map $\{\alpha_i\}_{i=1}^6$ from $\mathbf{K}(E_1)$ to $\mathbf{K}(E_2)$ such that $\alpha_6 \oplus \alpha_3$ preserves the order.*

Let E be an $\mathcal{AE}$-algebra. It is easy to check that $\delta_0^E = 0$. Hence, if E and E' are two $\mathcal{AE}$-algebras, then a map from $\mathbf{K}(E)$ to $\mathbf{K}(E')$ is defined exactly the same way as for $\mathcal{E}$-algebras. Note that by Proposition 1.7, every $\varphi \in \mathrm{Hom}(E, E')$ between two $\mathcal{AE}$-algebras induces a map from $\mathbf{K}(E)$ to $\mathbf{K}(E')$.

PROPOSITION 3.20. *Let $E = \varinjlim(E_i, \varphi_{i,i+1})$ and $E' = \varinjlim(E'_i, \varphi'_{i,i+1})$ be unital $\mathcal{AE}$-algebras. Let $\alpha : V_*(E) \to V_*(E')$ be a homomorphism.*

(i) *α induces the following commutative diagram for some increasing sequence of natural numbers $\{m_k\}_{k=1}^\infty$:*

$$\begin{array}{ccccccc}
V_*(E_1) & \longrightarrow & V_*(E_2) & \longrightarrow & \cdots & \longrightarrow & V_*(E) \\
\downarrow{\scriptstyle \alpha^{(1)}} & & \downarrow{\scriptstyle \alpha^{(2)}} & & & & \downarrow{\scriptstyle \alpha} \\
V_*(E'_{m_1}) & \longrightarrow & V_*(E'_{m_2}) & \longrightarrow & \cdots & \longrightarrow & V_*(E')
\end{array}$$

(ii) *There exists a unique map $\{\alpha_i\}_{i=1}^6$ from $\mathbf{K}(E)$ to $\mathbf{K}(E')$ induced by α.*

Furthermore, the map $\alpha_6 \oplus \alpha_3$ from $K_*(Q(E))$ to $K_*(Q(E'))$ preserves the order. If $\alpha_v([1_E]) = [1_{E'}]$, then $\alpha^{(i)}$ may be chosen such that $\alpha_v^{(i)}([1_{E_i}]) = [1_{E'_{m_i}}]$ for all i.

Proof. Let $[0]$ denote the identity of $V(E)$, $V(E')$, $V(E_n)$, and $V(E'_n)$. Note that $V_*(E) = \varinjlim(V_*(E_i), V_*(\varphi_{i,i+1}))$ and $V_*(E') = \varinjlim(V_*(E'_i), V_*(\varphi'_{i,i+1}))$. Denote the maps from $k(E_n)_+$ to $V(E_n)$, from $k(E'_n)_+$ to $V(E'_n)$, from $k(E)_+$ to $V(E)$, and from $k(E')_+$ to $V(E')$ by d_n, d'_n, d, and d' respectively. We will show that there exists $m_1 \in \mathbb{Z}_{>0}$ and a homomorphism $\alpha^{(1)} : V_*(E_1) \to V_*(E_{m_1})$ such that $\alpha \circ V_*(\varphi_{1,\infty}) = V_*(\varphi'_{m_1,\infty}) \circ \alpha^{(1)}$. Note that we may assume E_1 has only one summand and E_1 is not a unital purely infinite simple C^*-algebra.

Case 1: Suppose E_1 is a non-trivial extension. Then $k(E_1)_+ \cong K_1(E_1) \sqcup \mathbb{Z}$ and $V(E_1) \cong \{[0]\} \sqcup K_0(I(E_1)) \sqcup ((K_0(I(E_1)) / \operatorname{ran} \delta_1^{E_1}) \oplus \mathbb{Z}_{>0})$. Let s_1, s_2, and s_3 be non-unitary isometries in E_1 such that $[s_1]_k = 1$, $[s_2]_k = -1$, and $[s_3]_k = 1 - 1 = 0$.

Suppose $\alpha_v \circ V(\varphi_{1,\infty})|_{V(I(E_1))} = [0]$. For each $n \in \mathbb{Z}_{>0}$, choose $x_n \in V(E'_n)$ such that $V(\varphi'_{n,\infty})(x_n) = \alpha_v \circ V(\varphi_{1,\infty})((0,1))$. Define $\alpha_{1,n}$ from $V(E_1)$ to $V(E_n)$ by $\alpha_{1,n}|_{V(I(E_1))} = [0]$ and $\alpha_{1,n}((a,k)) = kx_n$. It is clear that $\alpha_{1,n}$ is a homomorphism such that $\alpha_v \circ V(\varphi_{1,\infty}) = V(\varphi'_{n,\infty}) \circ \alpha_{1,n}$.

Since $K_1(E_1)$ is a finitely generated abelian group, there exists $n_2 \in \mathbb{Z}_{>0}$ such that for all $n \geqslant n_2$, there exists a homomorphism $\gamma_{1,n} : K_1(E_1) \to K_1(E'_n)$ with $\alpha_k \circ k(\varphi_{1,\infty})_+|_{K_1(E_1)} = (\varphi'_{n,\infty})_{*,1} \circ \gamma_{1,n}$. Note that $\operatorname{ran}(\alpha_k \circ k(\varphi_{1,\infty})_+)$ is a subset of $K_1(E')$ since $\alpha_v \circ V(\varphi_{1,\infty})|_{V(I(E_1))} = [0]$. Let $(\alpha_k \circ k(\varphi_{1,\infty})_+)([s_1]_k) = [t'_1]_k$ and $(\alpha_k \circ k(\varphi_{1,\infty})_+)([s_2]_k) = [t'_2]_k$, where $t'_1, t'_2 \in U(M_n(E'))$. Note that $[t'_1]_k + [t'_2]_k = 0$ in $K_1(E')$. Therefore, there exists $n_2 \in \mathbb{Z}_{>0}$ such that for all $n \geqslant n_2$, there exist $y_{n,i} \in K_1(E'_n)$ with $(\alpha_k \circ k(\varphi_{1,\infty})_+)([s_i]_k) = k(\varphi'_{n,\infty})_+(y_{n,i})$ for $i = 1, 2$ and $y_{n,1} + y_{n,2} = 0$ in $K_1(E'_n)$. Define $\beta_{1,n} : k(E_1)_+ \to k(E'_n)_+$ by $\beta_{1,n}|_{K_1(E_1)} = \gamma_{1,n}$ and $\beta_{1,n}(\ell_1[s_1]_k + \ell_2[s_2]_k) = \ell_1 y_{n,1} + \ell_2 y_{n,2}$ for all $\ell_1, \ell_2 \in \mathbb{Z}_{>0}$.

Choose $m_1 = \max\{n_1, n_2\}$. Set $\alpha^{(1)} = (\alpha_{1,m_1}, \beta_{1,m_1})$. Then $\alpha^{(1)} : \overline{V}(E_1) \to \overline{V}(E'_{m_1})$ is a homomorphism. Note that $(d'_{m_1} \circ \beta_{1,m_1})([s_i]_k) = (\alpha_{1,m_1} \circ d_1)([s_i]_k) = [0]$. Therefore, $\alpha^{(1)}$ is the desired homomorphism.

Suppose $\alpha_v \circ V(\varphi_{1,\infty})|_{V(I(E_1))} \neq [0]$. It is easy to see that there exists $n_1 \in \mathbb{Z}_{>0}$ such that for all $n \geqslant n_1$, there exists a homomorphism $\alpha_1^{I,n} : V_*(I(E_1)) \to V_*(I(E'_n))$ with $V_*(I(\varphi'_{n,\infty})) \circ \alpha_1^{I,n} = \alpha|_{V_*(I(E_1))} \circ V_*(I(\varphi_{1,\infty}))$. By Proposition 3.19, $\alpha_1^{I,n}$ induces a homomorphism $\lambda_{i,n} : K_i(I(E_1)) \to K_i(I(E'_n))$ for $i = 0, 1$. Also, $\alpha_1^{I,n}$ induces two homomorphisms $\alpha_1^{I,n,v} : V(I(E_1)) \to V(I(E'_n))$ and $\alpha_1^{I,n,k} : k(I(E_1))_+ \to k(I(E'_n))_+$. Since $\operatorname{ran}(\alpha_1^{I,n,v} \circ d_1)$ is finitely generated, we may assume that $\operatorname{ran}(\alpha_1^{I,n,v} \circ d_1)$ is a subset of $\operatorname{ran} d'_n$ for all $n \geqslant n_1$. Since $V(E_1)$ and $k(E_1)_+$ are finitely generated, there exists $n_2 \geqslant n_1$ such that $\operatorname{ran}(\alpha \circ V_*(\varphi_{1,\infty}))$ is a subset of $\operatorname{ran}(V_*(\varphi'_{n,\infty}))$ for all $n \geqslant n_2$. Note that $\operatorname{ran}(\alpha_1^{I,n,v} \circ d_1)$ is a subset of $\operatorname{ran} d'_n$ for all $n \geqslant n_2$. Therefore, for all $n \geqslant n_2$, the homomorphism $\lambda_{0,n}$ induces a homomorphism $\widetilde{\lambda}_{0,n} : K_0(I(E_1)) / \operatorname{ran} \delta_1^{E_1} \to K_0(I(E'_n)) / \operatorname{ran} \delta_1^{E'_n}$ such that the diagram commutes

$$\begin{array}{ccccc} K_0(I(E_1)) & \xrightarrow{\lambda_{0,n}} & K_0(I(E_n')) & \longrightarrow & K_0(I(E_n'))/\operatorname{ran}\delta_1^{E'_{m_1}} \\ \downarrow & & & \nearrow_{\widetilde{\lambda}_{0,n}} & \\ K_0(I(E_1))/\operatorname{ran}\delta_1^{E_1} & & & & \end{array}$$

For all $n \geqslant n_2$, choose $x_n \in V(E_n')$ such that $\alpha_v \circ V(\varphi_{1,\infty})((0,1)) = V(\varphi'_{n,\infty})(x_n)$. Define $\alpha_{1,v,n} : V(E_1) \to V(E_n')$ by $\alpha_{1,v,n}|_{V(I(E_1))} = \alpha_1^{I,n,v}$ and $\alpha_{1,v,n}((a,k)) = \widetilde{\lambda}_{0,n}(a) + kx_n$ for $k \in \mathbb{Z}_{>0}$. Then, $\alpha_{1,v,n}$ is a homomorphism such that $V(\varphi'_{n,\infty}) \circ \alpha_{1,v,n} = \alpha_v \circ V(\varphi_{1,\infty})$.

Since $K_1(E_1)$ is finitely generated, there exists $n_3 \geqslant n_2$ such that for all $n \geqslant n_3$ we have a homomorphism $\beta_{1,n} : K_1(E_1) \to K_1(E_n')$ with $\alpha_k \circ k(\varphi_{1,\infty})_+|_{K_1(E_1)} = (\varphi'_{n,\infty})_{*,1} \circ \beta_{1,n}$ and $\beta_{1,n}|_{K_1(I(E_1))} = \lambda_{1,n}$. Also, there exists $n_4 \geqslant n_3$ such that for all $n \geqslant n_4$, there exist $y_{1,n}, y_{2,n} \in k(E_n')_+$ with $(\alpha_k \circ k(\varphi_{1,\infty})_+)([s_i]_k) = k(\varphi'_{n,\infty}(y_{i,n}))_+$ and $y_{1,n} + y_{2,n}$ is the identity of the subsemigroup $k(E_n')_+ \setminus K_1(E_n')$.

Define $\alpha_{1,k,n} : k(E_1)_+ \to k(E_n')_+$ by $\alpha_{1,k}|_{K_1(E_1)} = \beta_{1,n}$ and $\alpha_{1,k,n}(\ell_1[s_1]_k + \ell_2[s_2]_k) = \ell_1 y_{1,n} + \ell_2 y_{2,n}$ for all $\ell_1, \ell_2 \in \mathbb{Z}_{>0}$. Then $\alpha_{1,k,n}$ is a homomorphism such that $k(\varphi'_{n,\infty})_+ \circ \alpha_{1,k,n} = k(\varphi_{1,\infty})_+ \circ \alpha_k$. Note that there exists $m_1 \geqslant n_4$ such that $(V(\varphi'_{n_4,m_1}) \circ \alpha_{1,v,n_4} \circ d_1)([s_i]_k) = (V(\varphi'_{n_4,m_1}) \circ d'_{n_4})(y_{i,n_4})$. Hence, $\alpha^{(1)} = (V(\varphi'_{n_4,m_1}) \circ \alpha_{1,v,n_4}, k(\varphi'_{n_4,m_1})_+ \circ \beta_{1,n_4})$ is the desired homomorphism.

Case 2: Suppose E_1 is a trivial extension. Then $k(E_1)_+ \cong k(I(E_1))_+ \oplus \mathbb{Z}$ and $V(E_1) \cong \{[0]\} \sqcup (K_0(I(E_1)) \oplus \mathbb{Z}_{\geqslant 0})$. This case is proved in a similar fashion as in Case 1 but it is easier.

Next, starting with E_2, there exist $m_2' \geqslant m_1$ and a homomorphism $\beta^{(2)} : V_*(E_2) \to V_*(E'_{m_2'})$ such that $\alpha \circ V_*(\varphi_{2,\infty}) = V_*(\varphi'_{m_2',\infty}) \circ \beta^{(2)}$. Hence, there exists $m_2 > m_2'$ such that $V_*(\varphi'_{m_2',m_2}) \circ V_*(\varphi'_{m_1,m_2'}) \circ \alpha^{(1)} = V_*(\varphi'_{m_2',m_2}) \circ \beta^{(2)} \circ V_*(\varphi_{1,2})$. Let $\alpha^{(2)} = V_*(\varphi'_{m_2',m_2}) \circ \beta^{(2)}$. Then, the following diagram commutes:

$$\begin{array}{ccccc} V_*(E_1) & \xrightarrow{V_*(\varphi_{1,2})} & V_*(E_2) & \xrightarrow{V_*(\varphi_{2,\infty})} & V_*(E) \\ \downarrow{\scriptstyle \alpha^{(1)}} & & \downarrow{\scriptstyle \alpha^{(2)}} & & \downarrow{\scriptstyle \alpha} \\ V_*(E'_{m_1}) & \xrightarrow[V_*(\varphi'_{m_1,m_2})]{} & V_*(E'_{m_2}) & \xrightarrow[V_*(\varphi'_{m_2,\infty})]{} & V_*(E') \end{array}$$

Continuing this process, we get the desired result. ∎

Let C_n be the mapping cone of the degree n map $\theta_n : C_0((0,1)) \to C_0((0,1))$. Then $C_n \in \mathcal{N}$, $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$, and $K_1(C_n) = 0$. The *total K-theory* of A is defined to be $\underline{K}(A) = \bigoplus_{n=0}^{\infty} K_*(A; \mathbb{Z}/n\mathbb{Z})$, where $K_*(A;\mathbb{Z}/n\mathbb{Z}) = K_*(A \otimes C_n)$ for $n \geqslant 2$, $K_*(A;\mathbb{Z}/0\mathbb{Z}) = K_*(A)$, and $K_*(A;\mathbb{Z}/1\mathbb{Z}) = 0$. It is a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_{\geqslant 0}$ graded group.

Let Λ denote the category of Bockstein maps. Denote the group of all $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_{\geqslant 0}$ graded group homomorphisms which are Λ-linear by $\mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$. See Section 4 in [5] for more details.

Consider the following extension of C^*-algebras $0 \to B \xrightarrow{\varphi} E \xrightarrow{\psi} A \to 0$. Let $\varphi_n = \varphi \otimes \mathrm{id}_{C_n}$ and let $\psi_n = \varphi \otimes \mathrm{id}_{C_n}$. Since C_n is nuclear, we have the following six-term exact sequence:

$$\begin{array}{ccccc} K_0(B;\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{(\varphi_n)_{*,0}} & K_0(E;\mathbb{Z}/n\mathbb{Z}) & \xrightarrow{(\psi_n)_{*,0}} & K_0(A;\mathbb{Z}/n\mathbb{Z}) \\ {\scriptstyle \delta_1^{E\otimes C_n}} \uparrow & & & & \downarrow {\scriptstyle \delta_0^{E\otimes C_n}} \\ K_1(A;\mathbb{Z}/n\mathbb{Z}) & \xleftarrow[(\psi_n)_{*,1}]{} & K_1(E;\mathbb{Z}/n\mathbb{Z}) & \xleftarrow[(\varphi_n)_{*,1}]{} & K_1(B;\mathbb{Z}/n\mathbb{Z}) \end{array}$$

If E is an $\mathcal{E}$-algebra or an $\mathcal{AE}$-algebra, then set

$$\widetilde{K}(E) = \left(\underline{K}(B),\ \underline{K}(E),\ \underline{K}(A),\ \bigoplus_{n=0}^{\infty}((\varphi_n)_{*,0} \oplus (\varphi_n)_{*,1}),\ \bigoplus_{n=0}^{\infty}((\psi_n)_{*} \oplus (\psi_n)_{*,1}),\ \delta_1^E\right).$$

DEFINITION 3.21. Let E and E' be two finite direct sums of $\mathcal{E}$-algebras or E and E' are $\mathcal{AE}$-algebras. Then a homomorphism $\eta : \widetilde{K}(E) \to \widetilde{K}(E')$ is a system of Λ-linear maps,

$$\eta_1 : \underline{K}(I(E)) \to \underline{K}(I(E')), \quad \eta_2 : \underline{K}(E) \to \underline{K}(E'), \quad \text{and } \ \eta_3 : \underline{K}(Q(E)) \to \underline{K}(Q(E'))$$

making the obvious diagrams commute.

The invariant used to classify all unital $\mathcal{AE}_0$-algebras with real rank zero is $(V_*(E), \widetilde{K}(E))$. A homomorphism $\eta : (V_*(E_1), \widetilde{K}(E_1)) \to (V_*(E_2), \widetilde{K}(E_2))$, where each E_i is a finite direct sum of $\mathcal{E}$-algebras, is a system of two homomorphisms $\eta_1 : V_*(E_1) \to V_*(E_2)$ and $\eta_2 : \widetilde{K}(E_1) \to \widetilde{K}(E_2)$ such that the following diagrams commute:

$$\begin{array}{ccc} \underline{K}(I(E_1)) & \longrightarrow & K_*(I(E_1)) \\ {\scriptstyle \eta_2} \downarrow & & \downarrow {\scriptstyle (\alpha_1,\alpha_4)} \\ \underline{K}(I(E_2)) & \longrightarrow & K_*(I(E_2)) \end{array} \qquad \begin{array}{ccc} \underline{K}(E_1) & \longrightarrow & K_*(E_1) \\ {\scriptstyle \eta_2} \downarrow & & \downarrow {\scriptstyle (\alpha_2,\alpha_5)} \\ \underline{K}(E_2) & \longrightarrow & K_*(E_2) \end{array} \qquad \begin{array}{ccc} \underline{K}(Q(E_1)) & \longrightarrow & K_*(Q(E_1)) \\ {\scriptstyle \eta_2} \downarrow & & \downarrow {\scriptstyle (\alpha_3,\alpha_6)} \\ \underline{K}(Q(E_2)) & \longrightarrow & K_*(Q(E_2)) \end{array},$$

where the horizontal maps are the projection maps and $\{\alpha_i\}_{i=1}^{6}$ are the unique maps (in Proposition 3.19) induced by η_1. If E and E' are two unital $\mathcal{AE}$-algebras, then a homomorphism from $(V_*(E), \widetilde{K}(E))$ to $(V_*(E'), \widetilde{K}(E'))$ is defined similarly but now we use Proposition 3.20 to get the unique maps $\{\alpha_i\}_{i=1}^{6}$. By Proposition 1.4 and by Proposition 1.7, if $\varphi \in \mathrm{Hom}(E, E')$, E and E' are $\mathcal{E}$-algebras or $\mathcal{AE}$-algebras, then φ induces a homomorphism $[\varphi] : (V_*(E), \widetilde{K}(E)) \to (V_*(E), \widetilde{K}(E))$.

4. THE UNIQUENESS THEOREM

4.1. AUTOMORPHISMS.

DEFINITION 4.1. Let E be a C^*-algebra. Then $\mathrm{Aut}(E)$ will denote the group of all $*$-automorphism of E. The topology on $\mathrm{Aut}(E)$ will be the norm topology, i.e. $\|\alpha\| = \sup\limits_{\|a\| \leqslant 1} \|\alpha(a)\|$. Let $\mathrm{Aut}_0(E)$ denote the set of all $\alpha \in \mathrm{Aut}(E)$ that are in the same path component as id_E with the norm topology.

THEOREM 4.2 (Theorem 3.2 in [18]). *Let E be a C^*-algebra. Then every $\alpha \in \mathrm{Aut}_0(E) \subset \mathrm{Aut}(E)$ is approximately inner.*

Let E be a separable C^*-algebra. Then the above theorem follows from the following facts:

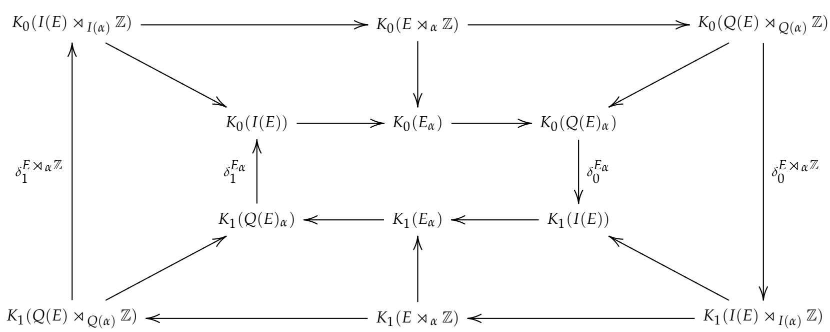
(1) By Corollary 8.7.8 in [21], every $\alpha \in \mathrm{Aut}_0(E)$ is a product of derivable $*$-automorphisms; and

(2) By Lemma 8.6.12 in [21], every derivation is approximately inner. This observation was made by Lin in [15]. Lin then showed in [18] that the general case can be reduced to the separable case.

COROLLARY 4.3. *Suppose $\alpha, \beta \in \mathrm{Aut}(E)$. Suppose that $\beta^{-1} \circ \alpha \in \mathrm{Aut}_0(E)$ and β is approximately inner. Then α is approximately inner.*

DEFINITION 4.4. Let E be a unital essential extension of A by $I \in \mathcal{P}$. Note that we may assume E is a unital C^*-subalgebra of $\mathcal{M}(I)$. Suppose $U \in \mathcal{M}(I)$ is a unitary such that $Ux - xU \in I$ for all $x \in E$. Define E_α to be the C^*-subalgebra of $\mathcal{M}(I)$ generated by E and U. Define A_α to be the C^*-subalgebra of $\mathcal{M}(I)/I$ generated by A and $\pi(U)$. Note that $\alpha = \mathrm{Ad}(U) \in \mathrm{Aut}(E)$ such that $\pi \circ \alpha = \pi$.

LEMMA 4.5. *Let E be an $\mathcal{E}$-algebra such that $Q(E) = C(S^1)$ and let U be a unitary in $\mathcal{M}(I(E))$ such that $Ux - xU \in I(E)$ for all $x \in E$. Set $\alpha = \mathrm{Ad}(U)$. Let $h : E \rtimes_\alpha \mathbb{Z} \to E_\alpha$ be the canonical surjective map. Then h gives the following commutative diagram:*



Proof. Note that h sends $I(E) \rtimes_{I(\alpha)} \mathbb{Z}$ to $I(E)$. Hence, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(E) \rtimes_{I(\alpha)} \mathbb{Z} & \longrightarrow & E \rtimes_\alpha \mathbb{Z} & \longrightarrow & Q(E) \rtimes_{Q(\alpha)} \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I(h)} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle Q(h)} & & \\
0 & \longrightarrow & I(E) & \longrightarrow & E_\alpha & \longrightarrow & Q(E)_\alpha & \longrightarrow & 0
\end{array}$$

∎

Set $I(E)^\alpha = I(E) \rtimes_{I(\alpha)} \mathbb{Z}$, $E^\alpha = E \rtimes_{\alpha\mathbb{Z}}$, and $Q(E)^\alpha = Q(E) \rtimes_{Q(\alpha)} \mathbb{Z}$.

$$\begin{array}{ccccccccccc}
\longrightarrow & K_i(I(E)) & \longrightarrow & K_i(E) & \longrightarrow & K_i(Q(E)) & \xrightarrow{\delta_i^E} & K_{i+1}(I(E)) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_i(I(E)^\alpha) & \longrightarrow & K_i(E^\alpha) & \longrightarrow & K_i(Q(E)^\alpha) & \xrightarrow{\delta_i^{E^\alpha}} & K_{i+1}(I(E)^\alpha) & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & K_{i+1}(I(E)) & \longrightarrow & K_{i+1}(E) & \longrightarrow & K_{i+1}(Q(E)) & \xrightarrow{\delta_{i+1}^E} & K_{i+2}(I(E)) & \longrightarrow
\end{array}$$

FIGURE 1. K-theory of the Crossed Product

LEMMA 4.6. *Let E be an $\mathcal{E}$-algebra and let $\alpha \in \mathrm{Aut}(E)$. Then Figure 1 is a commutative diagram.*

Proof. Use the fact that the Pimsner-Voiculescu exact sequence is natural. ∎

LEMMA 4.7. *Let E be an $\mathcal{E}_0$-algebra and let $U \in \mathcal{M}(I(E))$ be a unitary such that $Ux - xU \in I(E)$ for all $x \in E$. Set $\alpha = \mathrm{Ad}(U)$. Suppose $Q(E)_\alpha \cong Q(E) \otimes C(S^1)$ and $\alpha_{*,i} = (\mathrm{id}_E)_{*,i}$ on $K_i(E)$. Then $\delta_0^{E_\alpha} = 0$.*

Proof. Note that the homomorphism from $K_0(Q(E) \rtimes_\alpha \mathbb{Z})$ to $K_0(Q(E)_\alpha)$ in Lemma 4.5 is an isomorphism. If $K_1(I(E)) = 0$, then it is clear that $\delta_0^{E_\alpha} = 0$. Suppose $K_1(I(E))$ is torsion free and $\ker \delta_1^E \neq \{0\}$. Hence, $\operatorname{ran} \delta_1^E$ is a torsion group. Note that $\delta_0^E = 0$. By performing a diagram chase in Figure 1, we see that $\operatorname{ran} \delta_0^{E \rtimes_\alpha \mathbb{Z}}$ is a torsion group. Hence, by Lemma 4.5, $\operatorname{ran} \delta_0^{E_\alpha}$ is a torsion group. Since $K_1(I(E))$ is torsion free, $\delta_0^{E_\alpha} = 0$. ∎

LEMMA 4.8. *Let E be an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$. Let $U \in \mathcal{M}(I(E))$ be a unitary such that $Ux - xU \in I(E)$ for all $x \in E$. Set $\alpha = \mathrm{Ad}(U)$. Then there exists a norm continuous path $V_t \in U(\mathcal{M}(I(E)))$ such that:*

(i) *$V_t x - x V_t \in I(E)$ for all $x \in E$ and for all $t \in [0, 1]$;*
(ii) *$V_0 = U$; and*
(iii) *$Q(E)_{\beta_1} \cong Q(E) \otimes C(S^1)$, where $\beta_1 = \mathrm{Ad}(V_1)$.*

Proof. Let $[x, y]$ denote the element $xy - yx$. Note that $Q(E)_\alpha \cong C(X)$, where X is a compact subset of $S^1 \times S^1$. Let $\{\xi_n\}_{n=1}^\infty$ be a subset of X such that for all $k \in \mathbb{Z}_{>0}$, $\{\xi_n\}_{n=k}^\infty$ is dense in X and let $\{e_n\}_{n=1}^\infty$ be an approximate identity of $I(E)$ consisting of projections. For all $f \in C(X)$, let $\sigma_0(f) = \sum_{n=1}^{\infty} f(\xi_n)(e_n - e_{n-1})$, where the sum converges in the strict topology. By Proposition 3.2 in [16], there exists an abelian AF-algebra $B \subset \mathcal{M}(I(E))/I(E)$ such that $\operatorname{ran}(\tau_0) \subset B$, where $\tau_0 = \pi \circ \sigma_0$. Hence, there exists a self-adjoint element $h_1 \in \mathcal{M}(I(E))/I(E)$ such that $\tau_0(\pi(U)) = \exp(\mathrm{i}h_1)$ and $[h_1, \tau_0(b)] = 0$. Let $\tau = \tau_\alpha \oplus \tau_0$. By Proposition 1.11, there exists a unitary $Z \in \mathcal{M}(I(E))$ such that $\tau_\alpha = \operatorname{Ad}(\pi(Z))(\tau_\alpha \oplus \tau_0)$. Let $e_1 = \operatorname{Ad}(\pi(Z))(\tau_\alpha(1) \oplus 0)$ and let $e_2 = \operatorname{Ad}(\pi(Z))(0 \oplus \tau_0(1))$. Note that e_1 lifts to a projection $P_1 \in \mathcal{M}(I(E))$. Set $P_2 = 1 - P_1$. Then $P_1 + P_2 = 1$ and $\pi(P_i) = e_i$.

Let $\tau_2 : Q(E) \otimes C([0, 1]) \to e_2\mathcal{M}(I(E))/I(E)e_2$ be a strongly unital essential trivial extension. Let g be a self-adjoint element in $C([0, 1])$ such that $\operatorname{sp}(g) = [0, 2\pi]$. Then the C^*-algebra C which is generated by $Q(E) \otimes 1$ and $\exp(\mathrm{i}(1 \otimes g))$ is isomorphic to $Q(E) \otimes C(S^1)$. Let $h_2 = \tau_2(1 \otimes g)$. If $\tau_2' = \tau_2|_C$, then τ_2' is a strongly unital trivial essential extension such that $[h_2, \tau_2'(x \otimes 1)] = 0$ for all $x \in Q(E)$. Note that by Proposition 1.11, $\operatorname{Ad}(\pi(Z)) \circ \tau_0|_{Q(E)}$ is unitarily equivalent to $\tau_2'|_{Q(E)\otimes 1}$. Hence, by conjugating τ_2' by the image of a unitary in $P_2\mathcal{M}(I(E))P_2$, we may assume $\operatorname{Ad}(\pi(Z)) \circ \tau_0|_{Q(E)} = \tau_2'|_{Q(E)\otimes 1}$. Let $v_t = \operatorname{Ad}(\pi(Z))(\pi(U) \oplus \exp(\mathrm{i}(1-t)h_1))(e_1 + \exp(\mathrm{i}th_2))$. Then $v_0 = \pi(U)$ and $C^*(v_1, \tau_2'(Q(E) \otimes 1)) \cong Q(E) \otimes C(S^1)$. Hence, there exists a norm continuous path of unitaries $V_t \in \mathcal{M}(I(E))$ such that $V_t x - x V_t \in I(E)$ for all $x \in E$, $V_0 = U$, and $\pi(V_1) = v_1$. ∎

THEOREM 4.9. *Let E be an $\mathcal{E}_0$-algebra such that $Q(E) = C(S^1)$. Let $U \in \mathcal{M}(I(E))$ be a unitary such that $Ux - xU \in I(E)$ for all $x \in E$. Let $\alpha = \operatorname{Ad}(U)$. If $\alpha_{*,i} = (\mathrm{id}_E)_{*,i}$ on $K_i(E)$ for $i = 0, 1$, then α is approximately inner.*

Proof. By Lemma 4.8 and Corollary 4.3, we may assume $Q(E)_\alpha = Q(E) \otimes C(S^1) = C(S^1 \times S^1)$. Let $u = \pi(U)$. Note that $\operatorname{sp}(u) = S^1$. Let $H = (1/2)(U + U^*)$ and let $W = \exp(\mathrm{i}H)$. Since $\operatorname{sp}(U) = S^1$, we have $\operatorname{sp}(H) = [-\pi, \pi]$. Define $\beta = \operatorname{Ad}(W)$. Since H is in the C^*-algebra generated by U, we have $Wx - xW \in I(E)$ for all $x \in E$. Let $\beta_t = \operatorname{Ad}(\exp(\mathrm{i}H(1-t)))$. Then $\beta_t \in \operatorname{Aut}(E)$ such that $\beta_0 = \beta$, $\beta_1 = \mathrm{id}_E$, and $Q(\beta_t) = \mathrm{id}_{Q(E)}$.

Apply Lemma 4.8 to get a norm continuous path of unitaries V_t in $\mathcal{M}(I(E))$ such that $V_0 = W$, $V_t x - x V_t \in I(E)$ for all $x \in E$, and $Q(E)_{\sigma_1} = C(S^1 \times S^1)$ where $\sigma_1 = \operatorname{Ad}(V_1)$. Let τ_α and τ_{σ_1} be the Busby invariants associated to the extensions E_α and E_{σ_1} respectively.

Since $I(E) \in \mathcal{P}$, there exists an isomorphism λ_i from $K_i(\mathcal{M}(I(E))/I(E))$ to $K_{1-i}(I(E))$ such that $\delta_i^{E_\alpha} = \lambda_i \circ (\tau_\alpha)_{*,i}$. By Lemma 4.7, $\delta_0^{E_\alpha} = 0$. Hence, $(\tau_\alpha)_{*,0} = 0$. It is clear that $(\tau_{\sigma_1})_{*,0} = 0$.

It is easy to check that the homomorphism $(\tau_\alpha)_{*,1} : K_1(Q(E)_\alpha) \to K_0(I(E))$ is completely determined by $(\tau_\alpha)_{*,1}([z]) = (\lambda_1^{-1} \circ \delta_1^E)([z])$ and $(\tau_\alpha)_{*,1}([\pi(U)]) =$

0 and the homomorphism $(\tau_{\sigma_1})_{*,1}$ is completely determined by $(\tau_{\sigma_1})_{*,1}([z]) = (\lambda_1^{-1} \circ \delta_1^E)([z])$ and $(\tau_{\sigma_1})_{*,1}([\pi(V_1)]) = 0$. Hence, $(\tau_\alpha)_{*,1} = (\tau_{\sigma_1})_{*,1}$ on $K_1(C(S^1 \times S^1))$.

By Proposition 1.11, $\mathrm{Ad}(\pi(Z)) \circ \tau_{\sigma_1} = \tau_\alpha$ for some unitary Z in $\mathcal{M}(I(E))$. So, $Z^* V_1 Z - U \in I(E)$. Let $V = [Z^* V_1 Z]^* U$. Then $\pi(V) = 1$. Hence, α is approximately inner since $\alpha = \mathrm{Ad}(Z^* V_1 Z) \circ \mathrm{Ad}(V)$ and $\mathrm{Ad}(Z^* V_1 Z) \in \mathrm{Aut}_0(E)$. ∎

LEMMA 4.10. *Suppose E_1 and E_2 are $\mathcal{E}_0$-algebras. Suppose E_1 is a unital C^*-subalgebra of E_2, $Q(E_1) = C(S^1)$, and $I(E_1)$ is a nonzero hereditary C^*-subalgebra of $I(E_2)$. Let $v \in \mathcal{M}(I(E_1))$ be a unitary such that $vx - xv \in I(E_1)$ for all $x \in E_1$. Set $\alpha = \mathrm{Ad}(v)$.*

Suppose $\alpha_{*,1}(n\xi) = n\xi$ in $K_1(E_2)$ for some $n \geqslant 1$, where ξ is the generator of the copy of $\mathrm{ran}(\pi_{*,1})$ *in $K_1(E_1) \cong K_1(I(E_1)) \oplus \mathrm{ran}(\pi_{*,1})$. Then $\alpha_{*,i} = (\mathrm{id}_{E_1})_{*,i}$ on $K_i(E_1)$ for $i = 0, 1$.*

Proof. Let $\iota_i : I(E_i) \to E_i$ be the inclusion map and let $\pi_i : E_i \to Q(E_i)$ be the quotient map. Let $j : E_1 \to E_2$ be the inclusion map and let $\xi = [w]$ for some $w \in U(E_1)$. It is easy to check that $\alpha_{*,0} = (\mathrm{id}_{E_1})_{*,0}$.

Suppose $K_1(I(E_1)) = 0$. Then, $(\pi_1)_{*,1}$ is injective. Hence, $\alpha_{*,1} = (\mathrm{id}_{E_1})_{*,1}$ since $Q(\alpha)_{*,1} = \mathrm{id}_{K_1(Q(E_1))}$. Suppose $K_1(I(E_1))$ is torsion free and $\ker \delta_1^{E_1} \neq \{0\}$. Note that j induces the following commutative diagram such that the rows are exact sequences and $I(j)_{*,i}$ is an isomorphism for $i = 0, 1$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(I(E_1)) & \longrightarrow & K_1(E_1) & \longrightarrow & K_1(Q(E_1)) & \longrightarrow & K_0(I(E_1)) \\
 & & \downarrow{\scriptstyle I(j)_{*,1}} & & \downarrow{\scriptstyle j_{*,1}} & & \downarrow{\scriptstyle Q(j)_{*,1}} & & \downarrow{\scriptstyle I(j)_{*,0}} \\
0 & \longrightarrow & K_1(I(E_2)) & \longrightarrow & K_1(E_2) & \longrightarrow & K_1(Q(E_2)) & \longrightarrow & K_0(I(E_2))
\end{array}$$

Since $K_1(Q(E_2)) \cong \mathbb{Z} \cong K_1(Q(E'))$, the map $Q(j)_{*,1}$ is injective or the zero map.

Case 1. Suppose $[Q(j)(z)] = 0$ in $K_1(Q(E_2))$. Then $Q(j)_{*,1}$ is the zero map. Hence, $j_{*,1}(K_1(E_2)) \subset K_1(I(E_2))$. By performing a diagram chase in the above diagram, we see that $[w] = a_1 + a_2$, for some $a_1 \in K_1(I(E_1))$ and $a_2 \in \ker(j_{*,1})$. Since v is a unitary in $\mathcal{M}(I(E_1))$, we have $\alpha_{*,1}(a_1) = a_1$. Since $\pi_1 \circ \alpha = \pi_1$ and since $(\pi_1)_{*,1}|_{\ker(j_{*,1})}$ is injective, by a diagram chase, $\alpha_{*,1}(a_2) = a_2$. Therefore, $\alpha_{*,1} = (\mathrm{id}_{E_1})_{*,1}$ on $K_1(E_1)$.

Case 2. Suppose $[Q(j)(z)] \neq 0$. By the Five Lemma, $j_{*,1}$ is injective. Therefore, $[\alpha(w^n)] = [w^n] \neq 0$ in $K_1(E_1)$. Hence, by the exactness of the Pimsner-Voiculescu exact sequence, $[w^n]$ lifts to an element x in $K_0(E_1 \rtimes_\alpha \mathbb{Z})$.

Consider Figure 1, where E is replaced by E_1. Since $\pi_1 \circ \alpha = \pi_1$ and since the unitary group of $\mathcal{M}(I(E_1))$ is connected,

$$0 \to K_0(Q(E_1)) \to K_0(Q(E_1) \rtimes_{Q(\alpha)} \mathbb{Z}) \to K_1(Q(E_1)) \to 0 \quad \text{and}$$
$$0 \to K_1(I(E_1)) \to K_1(I(E_1) \rtimes_{I(\alpha)} \mathbb{Z}) \to K_0(I(E_1)) \to 0$$

are exact sequences. Note that $\delta_0^{E_1} = 0$, $K_1(I(E_1))$ is torsion free, and $\operatorname{ran}\delta_1^{E_1}$ is a torsion group. Hence, by an easy diagram chase in Figure 1, we see that $[w]$ lifts to an element in $K_0(E_1 \rtimes_\alpha \mathbb{Z})$. Hence, by the exactness of the Pimsner-Voiculescu exact sequence, $\alpha_{*,1} = (\mathrm{id}_{E_1})_{*,1}$ on $K_1(E_1)$. ∎

4.2. Uniqueness theorems.

DEFINITION 4.11 (Definition 1.4.11 in [12]). Let $A = C(S^1) \otimes B$, where B is a C^*-algebra and let $\varepsilon > 0$. We identify $C(S^1) \otimes B$ with $C(S^1, B)$. A finite subset $\mathcal{F} \subset A$ is *weakly approximately constant to within* ε if for any $t \in S^1$, there exists $U(t) \in U(A)$ such that $\|U(t)^* f(t) U(t) - f(1)\| < \varepsilon$ for all $f \in \mathcal{F}$.

Suppose C and D are unital C^*-algebras and $\varphi, \psi \in \mathrm{Hom}(C, D)$. Let $\mathcal{G} \subset C$. We say that φ and ψ are *approximately the same on* $\mathcal{G}$ *to within* $\varepsilon > 0$ if $\|\varphi(f) - \psi(f)\| < \varepsilon$ for all $f \in \mathcal{G}$.

DEFINITION 4.12. Let $\{e_{ij}\}_{i,j=1}^n$ be the standard system of matrix units in $M_n(\mathbb{C}) \subset M_n(C(S^1))$. Let z be the standard unitary generator of $C(S^1) = e_{11} M_n(C(S^1)) e_{11}$. Then $\{z\} \sqcup \{e_{ij}\}_{i,j=1}^n$ will be called the set of standard generators for $M_n(C(S^1))$.

THEOREM 4.13. *Let E be an $\mathcal{E}_0$-algebra. Let $\pi : E \to Q(E)$ denote the quotient map. Let $\mathcal{F} = \{z_i\}_{i=1}^n \subset E$ such that $\pi(\mathcal{F})$ contains the set of standard generators for $Q(E)$. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if E' is an $\mathcal{E}_0$-algebra and $\varphi_i \in \mathrm{Hom}(E, E')$ is unital and injective for $i = 1, 2$ with:*

(i) $[\varphi_1] = [\varphi_2]$ *on* $(V_*(E), \widetilde{K}(E))$ *and*

(ii) $Q(\varphi_1)$ *and* $Q(\varphi_2)$ *are approximately the same on* $\pi(\mathcal{F})$ *to within* δ,

then φ_2 and φ_1 are approximately unitarily equivalent on $\mathcal{F}$ to within ε.

Proof. Let $E = pM_l(E_1)p$ for some $p \in M_l(E)$ with $\pi(p) = 1_l$. If E is not isomorphic to $M_l(E_1)$, then define $\widetilde{\varphi}_i \in \mathrm{Hom}(E \otimes M_2, E' \otimes M_2)$ by $\widetilde{\varphi}_i = \varphi_i \otimes \mathrm{id}$. Since $\pi(p) = 1_l$, by Lemma 3.3, there exists a projection $e \in I(E)$ such that $(e \oplus 1_E) M_2(E)(e \oplus 1_E) \cong M_l(E_1)$. Since $[\varphi_1(e)] = [\varphi_2(e)]$ in $K_0(I(E'))$, by Lemma 1.1 in [31], there exists $W_1 \in U(\widetilde{I(E')})$ such that $W_1^* \varphi_1(e) W_1 = \varphi_2(e)$. Hence, we may assume $\varphi_1(e) = \varphi_2(e)$. Let $E_2 = (e \oplus 1_E) M_2(E)(e \oplus 1_E)$ and let $C = (\varphi_1(e) \oplus 1_{E'}) M_2(E')(\varphi_1(e) \oplus 1_{E'})$. Set $\psi_i = \widetilde{\varphi}_i|_{E_2}$. Then $\psi_i \in \mathrm{Hom}(E_2, C)$ is unital and injective. Note that $\psi_i(I(E_2)) \subset I(C)$ since $\varphi_i(I(E)) \subset I(E')$. Hence, ψ_i induces an element $Q(\psi_i)$ in $\mathrm{Hom}(Q(E_2), Q(C))$. Clearly, $Q(\varphi_i) = Q(\psi_i)$. Therefore, we can reduce the general case to the case $E = M_l(E_1)$ and hence to the case $l = 1$.

Suppose E is a non-trivial extension. By Lemma 3.10, there exists a non-unitary isometry $S_1 \in E$ such that E is generated by $(1 - S_1 S_1^*) I(E)(1 - S_1 S_1^*)$ and S_1. Let $q = 1 - S_1 S_1^*$. Then, we may assume $z_1 = S_1$ and $z_i = g_i \in qI(E)q$ for $i = 2, 3, \ldots, n$.

Recall that $K_1(E) \cong K_1(I(E)) \oplus \operatorname{ran}(\pi_{*,1})$. By Lemma 1.16, there exists $\xi \in U(E)$ such that $[\xi]$ generates the copy of $\operatorname{ran}(\pi_{*,1})$ in $K_1(E)$. Since E is generated by $\mathcal{F}$, there exists $0 < \delta' < \min\{1/2, \varepsilon/2\}$ such that if C is any unital C^*-algebra and γ and λ are unital $*$-homomorphisms from E to C such that $\|\gamma(x) - \lambda(x)\| < \delta'$ for all $x \in \mathcal{F}$, then $\|\gamma(\xi) - \lambda(\xi)\| < 1$.

Choose $0 < \rho < \delta'$ such that if C is a unital C^*-algebra, S_2 is an isometry in C, and $x \in C$ with $\|S_2 - x\| < \rho$, then x^*x is invertible in C. Also, ρ can be chosen such that $\|S_2 - x|x|^{-1}\| < \delta'/100$ and $\|S_2S_2^* - x|x|^{-2}x^*\| < \delta'/100$.

Let $0 < \delta < \rho$. Then, since $\|Q(\varphi_1)(z) - Q(\varphi_2)(z)\| < \delta$, there exists $a \in I(E')$ such that $\|\varphi_1(z_1) - \varphi_2(z_1) + a\| < \delta$. Let $z_1' = (\varphi_2(z_1) - a)|\varphi_2(z_1) - a|^{-1}$. Then $(z_1')^*z_1' = 1$ and $\|(1 - \varphi_1(z_1)\varphi_1(z_1)^*) - d\| < \delta'/100$, where $d = 1 - z_1'(z_1')^*$. Therefore, there exists $W' \in U(\widetilde{I(E')})$ such that $(W')^*(1 - \varphi_1(z_1)\varphi_1(z_1)^*)W' = d$ and $\|W' - 1\| < \delta'/50$. Hence, $\|(W')^*\varphi_1(z_1)W' - z_1'\| < \delta'/20$ and $\pi'(z_1') = \pi' \circ \varphi_2(z_1)$.

Set $S = z_1'$ and $T = \varphi_2(z_1)$. Since $[1 - SS^*] = [1 - TT^*]$ in $K_0(I(E'))$, by Lemma 1.1 in [31], there exists $W \in U(\widetilde{I(E')})$ such that $\mathrm{Ad}(W)(1 - SS^*) = 1 - TT^*$. Note that $1 - SS^* \neq 0$ and $1 - TT^* \neq 0$. By replacing $\mathrm{Ad}(W') \circ \varphi_1$ with $\mathrm{Ad}(WW') \circ \varphi_1$ and S with $\mathrm{Ad}(W)(S)$, we may assume that $1 - SS^* = 1 - TT^* = d$.

By Corollary 2.6 in [2] and Theorem 3.2(i) in [30], we may write $I(E') = d'I(E')d' \otimes \mathcal{K}$ with $[d] = [d']$ in $K_0(I(E'))$. By Lemma 1.1 in [31], there exists $V \in U(\widetilde{I(E)})$ such that $V^*dV = d'$. Suppose we have found $u \in U(E')$ such that $\|u^*V^*\varphi_1(x)Vu - V^*\varphi_2(x)V\| < \varepsilon$ for all $x \in \mathcal{F}$. Then $\|w^*\varphi_1(x)w - \varphi_2(x)\| < \varepsilon$ for all $x \in \mathcal{F}$, where $w = \mathrm{Ad}(V^*uV)$. Hence, we may assume $V = 1$ and $I(E') = dI(E')d \otimes \mathcal{K}$.

By Proposition 3.12, there exists $w_1 \in U(\widetilde{I(E')})$ such that $\|w_1^*Tw_1 - S\| < \delta/16$. We also have $\|w_1^*(1 - TT^*)w_1 - (1 - SS^*)\| < \delta/8$. Therefore, there exists $w_2' \in U(\widetilde{I(E')})$ such that $\|1 - w_2'\| < \delta/4$ and $(w_2')^*w_1^*(1 - TT^*)w_1w_2' = 1 - SS^*$. Hence, we may assume $w_1^*(1 - TT^*)w_1 = 1 - SS^*$ and $\|w_1^*Tw_1 - S\| < \delta/2$.

Note that $\mathrm{Ad}(w_1) \circ \varphi_2$ and $\mathrm{Ad}(W') \circ \varphi_1$ map $qI(E)q$ to $dI(E')d$ and $\mathrm{Ad}(w_1) \circ \varphi_2$ and $\mathrm{Ad}(W') \circ \varphi_1|_{qI(E)q}$ induce the same map on $\underline{K}(qI(E)q)$. Hence, by Theorem 4.10 in [18], there exists $u \in U(dI(E')d)$ such that

$$\|w_1^*\varphi_2(z_i)w_1 - u^*(W')^*\varphi_1(z_i)W'u\| < \frac{\delta}{2} \quad \text{for all } i \geqslant 2.$$

Let $X = \{S^m b(S^*)^n : b \in dI(E')d \text{ and } m, n \in \mathbb{Z}_{\geqslant 0}\}$ and let I be the closed linear span of the set X. Let E_2 be the C^*-algebra generated by I and S. Note that S is the standard unilateral shift of I and $1_{E_2} = 1_{E'}$. Hence, E_2 is a unital essential extension of $C(S^1)$ by $I = I(E_2)$. Let $w_2 = \sum_{n=0}^{\infty} S^n u(S^*)^n$, where the sum converges in the strict topology. Then w_2 is a unitary in $\mathcal{M}(I(E_2))$ and $w_2^*Sw_2 = S$. Hence, $\alpha = \mathrm{Ad}(w_2) \in \mathrm{Aut}(E_2)$ and

$$\|w_1^*\varphi_2(z_i)w_1 - w_2^*(W')^*\varphi_1(z_i)W'w_2\| < \frac{\delta}{2} < \delta'$$

for all $i = 2, \dots, n$ and

$$\|w_1^* \varphi_2(z_1) w_1 - w_2^* (W')^* \varphi_1(z_1) W' w_2\| < \frac{\delta}{16} + \frac{\delta'}{20} < \delta'.$$

Hence, by the choice of δ' and by Lemma 4.10, $(\mathrm{Ad}(v))_{*,i} = \mathrm{id}_{K_i(E_2)}$. Thus, by Theorem 4.9, there exists $w_3 \in U(E_2)$ such that $\|w_3^* x w_3 - w_2^* x w_2\| < \varepsilon/2$ for all $x \in \{S\} \cup \{(\mathrm{Ad}(W') \circ \varphi_1)(z_i) : i = 2, \dots, n\}$. Therefore,

$$\|w_1^* \varphi_2(z_i) w_1 - w_3^* (W')^* \varphi_1(z_i) W' w_3\| < \delta' + \frac{\varepsilon}{2} < \varepsilon$$

for all $i = 2, \dots, n$. Also, we have

$$\|w_1^* \varphi_2)(z_1) w_1 - w_3^* (W')^* \varphi_1(z_1) W' w_3\| < \frac{\delta}{16} + \frac{\varepsilon}{2} + \frac{\delta'}{20} < \varepsilon.$$

Now, suppose E is a trivial extension. Let $0 < \delta < \min\{1/8, \varepsilon/5\}$. By Proposition 1.15, E is generated by $I(E)$ and $U = \sum_{n=1}^{\infty} \lambda_n p_n$. Therefore, we may assume $z_1 = (1 - p_N) U (1 - p_N)$ and $z_2, \dots, z_n \in p_N I(E) p_N$, where p_N is a projection in $I(E)$. Since $V_*(I(\varphi_1)) = V_*(I(\varphi_2))$ on $V_*(I(E))$, by Lemma 1.1 in [31], there exists $W' \in U(\widetilde{I(E')})$ such that $(W')^* \varphi_1(p_N) W' = \varphi_2(p_N)$. Hence, we may assume $\varphi_1(p_N) = \varphi_2(p_N)$. Let $q = \varphi_1(1 - p_N)$.

Note that there exists a $b \in qI(E')q$ such that $\|\varphi_1(z_1) - \varphi_2(z_1) + b\| < \delta$. Since $0 < \delta < 1/2$, $|\varphi_2(z_1) - b|$ is invertible. Let $w = (\varphi_2(z_1) - b)|\varphi_2(z_1) - b|^{-1}$. Then w is a unitary in $qE'q$ and $\|\varphi_1(z_1) - w\| < 4\delta$. Note that $\pi'(w) = (\pi' \circ \varphi_2)(z_1)$ and $[w] = [\varphi_1(z_1)] = [\varphi_2(z_1)]$ in $K_1(qE'q)$. Therefore, by Lemma 2.9, there exists a unitary $U_1 \in \mathbb{C}q + qI(E')q$ such that $\|U_1^* w U_1 - \varphi_2(z_1)\| < \delta$. Hence, $\|U_1^* \varphi_1(z_1) U_1 - \varphi_2(z_1)\| < 5\delta < \varepsilon$. Since we have $[\varphi_1|_{p_N I(p)_N}] = [\varphi_2|_{p_N I(p)_N}]$ on $\underline{K}(p_N I(E) p_N)$, by Theorem 4.10 in [18], there exists $u_1 \in U((1-q)I(E')(1-q))$ such that $\|u_1^* \varphi_1(z_i) u_1 - \varphi_2(z_i)\| < \varepsilon$ for all $i = 2, \dots, n$. Let $W = U_1 + u_1$. Then $W \in U(E')$ such that $\|W^* \varphi_1(z_i) W - \varphi_2(z_i)\| < \varepsilon$ for all $i = 1, 2, \dots, n$.

By the proof, we see that δ is independent of φ_1, φ_2, and E'. ∎

THEOREM 4.14. *Let E be an $\mathcal{E}_0$-algebra such that $Q(E) \cong M_l(C(S^1))$ and let $\mathcal{F} \subset E$ be a finite subset of E such that $\pi(\mathcal{F})$ contains the standard generators for $Q(E)$. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that the following holds:*

Let E' be an $\mathcal{E}_0$-algebra. Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}(E, E')$ induce the same map on $(V_*(E), \widetilde{K}(E))$.

(i) *If φ_1 and φ_2 are injective and the maps $Q(\varphi_1)$ and $Q(\varphi_2)$ are approximately the same to within δ on $\pi(\mathcal{F})$, then φ_1 and φ_2 are approximately unitarily equivalent on $\mathcal{F}$ to within ε.*

(ii) *If φ_1 and φ_2 are not injective but $(\varphi_1)_Q$ and $(\varphi_2)_Q$ are injective and if the maps $Q(\varphi_1)$ and $Q(\varphi_2)$ are approximately the same on $\pi(\mathcal{F})$ to within δ, then φ_1 and φ_2 are approximately unitarily equivalent on $\mathcal{F}$ to within ε.*

Proof. It is easy to see that we may assume that φ_1 and φ_2 are unital.

(i) If E' is not a purely infinite simple C^*-algebra, then the conclusion follows from Theorem 4.13. If E' is a purely infinite simple C^*-algebra, then the conclusion follows from Theorem 6.7 in [19].

(ii) We will use the same notation as in the proof of Theorem 4.13. Since φ_i is not injective, there exists $(\varphi_i)_Q \in \mathrm{Hom}(Q(E), E')$ such that $\varphi_i = (\varphi_i)_Q \circ \pi$ and $\pi' \circ (\varphi_i)_Q = Q(\varphi_i)$. Note that $\mathrm{sp}(\varphi_1(z_1)) = \mathrm{sp}(\varphi_2(z_1)) = S^1$ since $(\varphi_1)_Q$ and $(\varphi_2)_Q$ are injective. By Lemma 2.9, there exists $U \in U(E')$ such that $\|U^* \varphi_2(z_1) U - \varphi_1(z_1)\| < \varepsilon$. ∎

THEOREM 4.15. *Let E and E' be $\mathcal{E}_0$-algebras such that $Q(E) \cong M_n(C(S^1))$, let $\varepsilon > 0$, and let $\mathcal{F} \subset E$ be a finite subset such that $\pi(\mathcal{F})$ is weakly approximately constant to within $\varepsilon/3$. Suppose $\varphi_1, \varphi_2 \in \mathrm{Hom}(E, E')$ induce the same map on $(V_*(E), \widetilde{K}(E))$. Then we have the following:*

(i) *Suppose $Q(E') \cong M_m(C(S^1))$. Let $\{f_{ij}\}_{i,j=1}^m$ be the standard system of matrix units for $M_m(\mathbb{C}) \subset M_m(C(S^1))$. If φ_1 and φ_2 are not injective, $(\varphi_1)_Q(z)$ and $(\varphi_2)_Q(z)$ have finite spectra in $f_{11}Q(E')f_{11} \cong C(S^1)$, then φ_1 and φ_2 are approximately unitarily equivalent on $\mathcal{F}$ to within $\varepsilon > 0$.*

(ii) *If $Q(\varphi_1)$ and $Q(\varphi_2)$ are zero, then φ_1 and φ_2 are approximately unitarily equivalent on $\mathcal{F}$ to within ε.*

Proof. (i) follows from the proof of Proposition 2.20(1) in [17].

(ii) Note that we may assume φ_1 and φ_2 are in $\mathrm{Hom}(E, pI(E')p)$ for some projection $p \in I(E')$ and $\varphi_1(1) = p = \varphi_2(1)$. If both φ_1 and φ_2 are injective, then the conclusion follows from Theorem 6.7 in [19]. If φ_1 and φ_2 are not injective, then $\varphi_i = (\varphi_i)_Q \circ \pi$. Since $V(\varphi_1) = V(\varphi_2)$ on $V(E)$, we may assume $(\varphi_1)_Q$ and $(\varphi_2)_Q$ both agree on $M_n(\mathbb{C}) \subset M_n(C(S^1)) \cong Q(E)$. Therefore, we may assume $n = 1$. Suppose $(\varphi_1)_Q$ and $(\varphi_2)_Q$ are injective. Then the conclusion follows from Lemma 2.1. Hence, we may assume $(\varphi_1)_Q$ is not injective. Therefore, $(\varphi_1)_Q(z)$ and $(\varphi_2)_Q(z)$ are in the same path component as p in $U(pI(E')p)$. By Corollary 2 in [22], there are $\xi_1, \ldots, \xi_l, \zeta_1, \ldots, \zeta_L \in S^1$ and mutually orthogonal projections $p_1, \ldots, p_l, q_1, \ldots, q_L \in pI(E')p$ such that

$$\left\|(\varphi_1)_Q(f) - \sum_{k=1}^{l} (\varphi_1)_Q(f(\xi_k)1)p_k\right\| < \frac{\varepsilon}{3} \quad \text{and} \quad \left\|(\varphi_2)_Q(f) - \sum_{j=1}^{L} (\varphi_2)_Q(f(\zeta_j)1)q_j\right\| < \frac{\varepsilon}{3}$$

for all $f \in \pi(\mathcal{F})$.

Set $(\varphi_1')_Q(g) = \sum_{k=1}^{l} (\varphi_1)_Q(g(\xi_k)1)p_k$ and $(\varphi_2')_Q(g) = \sum_{j=1}^{L} (\varphi_2)_Q(g(\zeta_j)1)q_j$ for all $g \in Q(E)$. Since $\pi(\mathcal{F})$ is weakly approximately constant to within $\varepsilon/3$, by part (i), $(\varphi_1')_Q$ and $(\varphi_2')_Q$ are approximately unitarily equivalent on $\pi(\mathcal{F})$ to within $\varepsilon/3$. Thus, φ_1 and φ_2 are approximately unitarily equivalent on $\mathcal{F}$ to within ε. ∎

5. THE EXISTENCE THEOREM

THEOREM 5.1. *Let E and E' be two finite direct sums of $\mathcal{E}$-algebras. Let $\alpha : V_*(E) \to V_*(E')$ be a homomorphism such that $\alpha_v([1_E]) = [P]$ for some projection $P \in E'$. Let $\{\alpha_i\}_{i=1}^6$ be the unique map from $\mathbf{K}(E)$ to $\mathbf{K}(E')$ induced by α. (See Proposition 3.19.) Suppose $\psi \in \operatorname{Hom}(Q(E), Q(E'))$ induces α_3 and α_6. Then there exists $\varphi \in \operatorname{Hom}(E, E')$ such that φ induces α, $\pi' \circ \varphi = \psi \circ \pi$, and $\varphi(1_E) = P$.*

Proof. By Corollary 2.6 in [2] and Theorem 1.2 in [30], $I(E) = qIq \otimes \mathcal{K}$, for some $I \in \mathcal{P}$ and for some projection $q \in I$. Let $\{e_{ij}\}_{i,j=1}^{\infty}$ be the standard system of matrix units for $\mathcal{K} \subset qIq \otimes \mathcal{K}$.

It is clear that we may assume E' has only one summand. Write $E = \bigoplus_{j=1}^{k} E_j$, where each E_j is an $\mathcal{E}$-algebra. Denote the unit of E_j by $1^{(j)}$. Write $\psi = \bigoplus_{j=1}^{k} \psi_j$, where each ψ_j is in $\operatorname{Hom}(Q(E_j), Q(E'))$. Let $e_j = \psi(1_{Q(E_j)})$. By Lemma 2.5 and Remark 2.9 in [29] and by Proposition 1.8, there exists a collection of mutually orthogonal projections $\{d_j\}_{j=1}^{k} \subset E'$ such that $\pi'(d_j) = e_j$ for all $j = 1, 2, \ldots, k$. We may assume that $P = d_1 + \cdots + d_k$ and $\alpha_v([1^{(j)}]) = [d_j]$. Thus, it is enough to show that for each j, there exists $\varphi_j \in \operatorname{Hom}(E_j, d_j E' d_j)$ that induces $\alpha|_{V_*(E_j)}$ such that $\pi' \circ \varphi = \psi_j \circ \pi$ and $\varphi_j(1_{E_j}) = d_j$. So, we may assume that E has only one summand. Moreover, since $d_j E' d_j$ is again an $\mathcal{E}$-algebra (Proposition 1.17), we may assume $P = 1_{E'}$.

Case 1. E' is a unital nuclear purely infinite simple C^*-algebra in $\mathcal{N}$. This case is an easy consequence of Theorem 6.7 in [19].

Case 2. Suppose E' is not a unital purely infinite simple C^*-algebra. Note that E can not be a unital purely infinite simple C^*-algebra. Let $E = qM_l(E_1)q$ and let $E' = PM_k(E_2)P$, where E_i is an $\mathcal{E}$-algebra with $Q(E_i) = C(S^1)$. We will show that we can reduce this general case to the case $l = 1$ and $q = 1$.

First assume that $P \neq 1_k$. Note that $e = 1_l - q$ is a projection in $I(E)$ such that $eq = 0$ and $e + q = 1_l$. Since $\alpha_v(V(I(E))) \subset V(I(E'))$, there exists a projection $e' \in (1_k - P)M_k(I(E_2))(1_k - P)$ such that $[e'] = \alpha([e])$. Let $E'' = (P + e')M_k(E_2)(P + e')$. Let π'' denote the quotient map from E'' onto $Q(E'')$ and let π' denote the quotient map from E' onto $Q(E')$.

Suppose we have found $\varphi' \in \operatorname{Hom}(M_l(E_1), E'')$ that induces α such that $\varphi'(1_l) = P + e'$ and $\pi' \circ \varphi' = \psi \circ \pi$. Let $P' = \varphi'(q)$. Then, there exists $U \in U(E'')$ such that $U^* P' U = P$. Denote the inclusion map from E' to E'' by j. It is easy to see that $Q(j)$ is an isomorphism. Choose u such that $Q(j)(u) = \pi''(U)$. Since $\delta_1^{E''}([\pi''(U)]) = 0$ and since $Q(j)_{*,1}$ is an isomorphism, we have $\delta_1^{E'}([u]) = 0$. By Lemma 1.13, there exists $W \in U(E')$ such that $\pi'(W) = u$. Let $W_0 = e' + W^*$ and let $V = UW_0$. Take $\varphi = \operatorname{Ad}(V) \circ \varphi'|_E$. Then φ induces α, $\pi' \circ \varphi = \psi \circ \pi$, and $\varphi(1_E) = P$. Hence, we may assume $E = M_l(E_1)$.

Assume that $P = 1_k$. Suppose we have found $\varphi' \in \mathrm{Hom}(M_l(E_1), M_k(E_2))$ such that φ' induces α, $\pi' \circ \varphi' = \psi \circ \pi$, and $\varphi'(1_l) = 1_k$. Note that $\varphi'(q) \sim 1_k$. Let $q' = \varphi'(q)$. Then $1_k - q' \in I(E')$ and $[1_k - q'] = 0$ in $K_0(I(E'))$. Since $q' \sim 1_k$, there exists v such that $v^*v = q'$ and $vv^* = 1_k$. Let $\varphi = \mathrm{Ad}(v) \circ \varphi'|_E$. It is easy to check that φ is the desired $*$-homomorphism.

We will now show that we may assume $E = E_1$. Let $\{p_{ij}\}_{i,j=1}^{l}$ be a system of matrix units for $M_l(\mathbb{C}) \subset M_l(E_1) = E$. Let $\overline{q}_{ij} = \psi(\pi(p_{ij}))$. By Corollary 1.10, there exists a system of matrix units $\{q'_{ij}\}_{i,j=1}^{l} \subset E'$ such that $\pi'(q'_{ij}) = \overline{q}_{ij}$ for all $i, j = 1, 2, \ldots, l$. Note that $\alpha_v([p_{ii}]) = \alpha_v([p_{11}])$, $[q'_{ii}] = [q'_{11}]$, and $\alpha_6([\pi(p_{ii})]) = [\overline{q}'_{11}]$. Since $\alpha_v([1_l]) = [1_{E'}]$, if $d = 1_{E'} - \sum_{i=1}^{l} q'_{ii} \neq 0$, then there are mutually orthogonal and mutually equivalent projections $a_1, a_2, \ldots, a_l \in dE'd$ such that $\sum_{i=1}^{l} a_i = d$. So, there exists a system of matrix units $\{a_{ij}\}_{i,j=1}^{l} \subset dE'd$ such that $a_{ii} = a_i$ for $i = 1, 2, \ldots, l$. Let $q_{ij} = q'_{ij} + a_{ij}$ for $i, j = 1, 2, \ldots, l$. Then $\pi'(q_{ij}) = \overline{q}_{ij}$ and $\alpha([p_{ii}]) = [q_{ii}]$. It is now clear that it is enough to show that there exists $\varphi \in \mathrm{Hom}(p_{11}Ep_{11}, q_{11}E'q_{11})$ such that φ induces α, $\pi' \circ \varphi = \psi \circ \pi$, and $\varphi(p_{11}) = q_{11}$. So, for the rest of the proof we will assume E is an $\mathcal{E}$-algebra with $Q(E) = C(S^1)$ and $\alpha([1_E]) = [1_{E'}]$.

Let W be an isometry in E' such that $\pi'(W) = \psi(z)$. Let $\underline{0}$ be the zero element in $K_0(I(E)) \subset V(I(E))$ which is represented by a nonzero projection. We break Case 2 into three sub-cases.

(i) Suppose $\alpha_v(V(I(E))) = \{0\}$. Then, there exists $\alpha' : V_*(Q(E)) \to V_*(E')$ such that $\alpha' \circ V_*(\pi) = \alpha$. Note that $k(Q(E))_+ \cong K_1(Q(E)) \cong \mathbb{Z}$ is generated by $[z]$. So, there exists $W \in U(E')$ such that $[W] = \alpha'([z])$ and $\pi'(W) = \psi(z)$. Let B be the C^*-subalgebra generated by W. Then there is $\varphi' \in \mathrm{Hom}(Q(E), B)$ such that $\varphi'(z) = W$ and $\varphi'(1_{Q(E)}) = 1_B$. Hence, $\varphi = \varphi' \circ \pi$ is the desired $*$-homomorphism.

(ii) Assume that E is a non-trivial extension and $\alpha_v(V(I(E))) \neq \{0\}$. Hence, $\alpha_v \circ d([S_1]) \neq 0$. By Lemma 1.13, we may assume W is a non-unitary isometry. We claim that there exists $\varphi' \in \mathrm{Hom}(E, E')$ such that $I(\varphi')$ induces $\alpha|_{V_*(I(E))}$, $\pi' \circ \varphi' = \psi \circ \pi$, and $\varphi'(1_E) = 1_{E'}$. Let f be a nonzero projection in $I(E')$ such that $\alpha_v([e_{11}]) = [f]$. By Corollary 2.6 in [2] and Theorem 1.2 in [30], we may assume $I(E') = fI(E')f \otimes \mathcal{K}$. Hence $I(E')$ has an approximate identity consisting of projections $\{p_i\}_{i=1}^{\infty}$ such that $f_i = p_i - p_{i-1}$ and $[f_i] = [f]$ for all $i \in \mathbb{Z}_{>1}$. Therefore, by Theorem 6.7 in [19], there exists $\varphi_1 \in \mathrm{Hom}(I(E), I(E'))$ such that φ_1 is injective, $\varphi_1(e_{ii}) = f_i$ for all $i \in \mathbb{Z}_{>0}$ and φ_1 induces $\alpha|_{V_*(I(E))}$. By Lemma 3.2 in [17], there exists a unital injective $*$-homomorphism $\widetilde{\varphi}_1 : \mathcal{M}(I(E)) \to \mathcal{M}(I(E'))$ extending φ_1. Set $\varphi_2 = \widetilde{\varphi}_1|_E$. Then $C^*(W, I(E'))$ and $C^*(\varphi_2(S_1), I(E'))$ are equivalent unital essential extensions of $C(S^1)$ by $I(E')$. Then, by Proposition 1.11, there exists a unitary $U \in \mathcal{M}(I(E'))$ such that $\pi'(U)(\pi' \circ \varphi_2(S_1))\pi'(U)^* = \pi'(W)$. Then $\varphi' = \mathrm{Ad}(U) \circ \varphi_2$ is the desired $*$-homomorphism. This proves the claim.

We will now show that there exists a unital $\varphi : E \to E'$ which induces α and $\pi' \circ \varphi = \psi \circ \pi$. Suppose $\ker \delta_1^E = \{0\}$. Then $K_1(E) \cong K_1(I(E))$, where the isomorphism is induced by the inclusion map from $I(E)$ to E. Hence, the $*$-homomorphism constructed above is the desired $*$-homomorphism. Suppose $\ker \delta_1^E \neq \{0\}$. Let e be a nonzero projection in $I(E')$ such that $[e]$ is the zero element in $K_0(I(E'))$. Note that the inclusion $j : (1_{E'} - e)E'(1_{E'} - e) \to E'$ induces an isomorphism from $V_*((1_{E'} - e)E'(1_{E'} - e))$ onto $V_*(E')$. Let π' denote the quotient map from E' onto $Q(E')$ and from $(1_{E'} - e)E'(1_{E'} - e)$ onto $Q((1_{E'} - e)E'(1_{E'} - e))$. Since $Q((1_{E'} - e)E'(1_{E'} - e)) = Q(E')$, we have $Q(j) = \mathrm{id}_{Q(E')}$. Therefore, we may choose a non-unitary isometry $W' \in (1_{E'} - e)E'(1_{E'} - e)$ such that $\pi'(W') = \psi(z)$. By the above claim, there exists a unital injective $*$-homomorphism $\varphi' : E \to (1_{E'} - e)E'(1_{E'} - e)$ such that $I(\varphi')$ induces $\alpha|_{V_*(I(E))}$ and $\pi' \circ \varphi' = \psi \circ \pi$.

By Lemma 3.9, $K_1(E)$ can be identified as a subsemigroup of $k(E)_+$. Since $\ker \delta_1^E \neq \{0\}$, we have $K_1(E) \cong K_1(I(E)) \oplus \mathbb{Z}$. By Lemma 1.16, there exists $w \in U(E)$ such that $[w] = (0, 1)$. By Theorem 3.13, $k(eI(E')e)_+ \cong K_1(eI(E')e) \sqcup \{\mathcal{S}\}$. It is easy to see there exists a homomorphism $\beta : V_*(E) \to V_*(eI(E')e)$ such that $\beta(x) = [e]$ for all $x \neq 0$ in $V(E)$, $\beta(y) = \mathcal{S}$ for all $y \in k(E)_+ \setminus (K_1(E) \cup K_1(I(E)))$, and $\beta([w]) = \alpha_k([w]) - (\varphi')_{*,1}([w])$. Therefore, by Case 1, there exists $\varphi'' \in \mathrm{Hom}(E, eE'e)$ such that φ'' induces β and $\varphi''(1_E) = e$. Then $\varphi = \varphi' + \varphi''$ is the desired $*$-homomorphism.

(iii) Suppose E is a trivial extension and $\alpha(V(I(E))) \neq \{0\}$. Since E is a trivial extension, $k(E)_+ \cong k(I(E))_+ \oplus \mathbb{Z}$, where the copy of $\mathbb{Z}$ is generated by a unitary in E. By Lemma 3.18, $\alpha(0, 1) \in K_1(E')$. So, by Lemma 1.16, there exists $W \in U(E')$ such that $\alpha(0, 1) = [W]$ in $K_1(I(E')) \oplus \mathbb{Z} \cong K_1(E') \subset k(E')_+$.

We first assume that $\mathrm{sp}(\pi'(W)) = S^1$. As in the proof of Case 2(ii), there exist an approximate identity $\{q_i\}_{i=1}^\infty$ of $I(E')$ consisting of projections such that $[q_i] - [q_{i-1}] = \alpha([e_{ii}])$ for all $i \in \mathbb{Z}_{>0}$ and a unital injective $*$-homomorphism $\varphi' : \mathcal{M}(I(E)) \to \mathcal{M}(I(E'))$ such that $I(\varphi')$ induces $\alpha|_{V_*(I(E))}$ and $\varphi'(e_{ii}) = d_i = q_i - q_{i-1}$ for all $i \in \mathbb{Z}_{>0}$. By Proposition 1.15, we may assume E is generated by w and $I(E)$, where $w = \sum_{k=1}^{\infty} \lambda_k e_{kk}$ (convergence is in the strict topology). Let $V = \varphi'(w)$. Note that $\mathrm{sp}(\pi'(V)) = S^1$. Set $E_1 = C^*(V, I(E'))$ and set $E_2 = C^*(W, I(E'))$. Then E_1 and E_2 are trivial essential extensions of $C(S^1)$ by $I(E')$. Hence, by Theorem 8.3.1, pp. 125 in [25] ([13]), there exists a unitary $U \in \mathcal{M}(I(E'))$ such that $\|U^*VU - W\| < 1$ and $U^*VU - W \in I(E')$. Let $\varphi = (\mathrm{Ad}\, U \circ \varphi')|_E$. Then φ is the desired $*$-homomorphism.

Suppose $\mathrm{sp}(\pi'(W)) = X \neq S^1$. Let $J = \{a \in E : \pi(a)|_X = 0\}$. Since E is a trivial extension, $k(E)_+ \cong \{(x, y) : x \in k(I(E))_+, y \in \mathbb{Z}\}$ and $V(E) \cong K_0(I(E)) \oplus \mathbb{Z}_{\geqslant 0}$. Let f be a nonzero projection in $I(E')$ such that $\alpha([e_{11}]) = [f]$. Let $\eta_v : V(E) \to V(I(E'))$ be $\eta_v(a, b) = \alpha(a, 0)$ and let $\eta_k : k(E)_+ \to k(I(E'))_+$ be $\eta_k(x, y) = \alpha(x, 0)$. Note that $\eta_v|_{V(I(E))} = \alpha|_{V(I(E))}$ and $\eta_k|_{k(I(E))_+} = \alpha|_{k(I(E))_+}$. It is easy to check that $\eta = (\eta_v, \eta_k)$ is a homomorphism from $V_*(E)$ to $V_*(fI(E')f)$.

Therefore, by Case 1, there exists $\gamma \in \mathrm{Hom}(E, fI(E')f)$ such that γ induces η and $\gamma(1_E) = f$.

Let B be the hereditary C^*-subalgebra of $fI(E')f$ generated by $\gamma(J)$. By Corollary 2.6 in [2] and Theorem 1.2 in [30], B is a stable purely infinite simple C^*-algebra in $\mathcal{N}$. Hence, there exist approximate identities $\{e_n\}_{n=1}^{\infty}$ and $\{e'_n\}_{n=1}^{\infty}$ for B and for $I(E')$ respectively such that e_n and e'_n are projections with the following property: $e_0 = e'_0 = 0$, $\{e_n - e_{n-1}\}_{n=1}^{\infty}$ and $\{e'_n - e'_{n-1}\}_{n=1}^{\infty}$ are collections of mutually orthogonal projections, and for all $n \in \mathbb{Z}_{\geqslant 0}$, $[e_n - e_{n-1}] = [e'_n - e'_{n-1}]$. So, there exists $v_n \in I(E')$ such that $v_n^* v_n = e'_n - e'_{n-1}$ and $v_n v_n^* = e_n - e_{n-1}$ for all $n \in \mathbb{Z}_{>0}$. Set $V = \sum_{n=1}^{\infty} v_n$, where the sum converges in the strict topology. Then V is an element in the double commutant of $I(E')$. Note that $bV \in I(E')$ for all $b \in B$ and $V^*a \in B$ for all $a \in I(E')$. Therefore, the map $\sigma : B \to I(E')$ defined by $\sigma(b) = V^*bV$ is a $*$-homomorphism and $[\sigma(p)] = [p]$ in $K_0(I(E'))$ for all projections $p \in B$. Also, $[\sigma(u)] = [u]$ in $K_1(I(E'))$ for all unitaries $u \in \widetilde{B}$. Let $\gamma_1 = (\sigma \circ \gamma)|_J$. Then γ_1 maps an approximate identity of J to an approximate identity of $I(E')$. Hence, by Lemma 3.2 in [17], there exists a unital extension $\widetilde{\gamma}_1 : \mathcal{M}(J) \to \mathcal{M}(I(E'))$ of γ_1.

Note that J is an essential ideal of E since $I(E)$ is an essential ideal of E. Let $w \in U(E)$ be as in the case $\mathrm{sp}(\pi'(W)) = S^1$. It is now easy to check that $\mathrm{sp}(\pi' \circ \widetilde{\gamma}_1(w)) = X = \mathrm{sp}(\pi'(W))$. Therefore, $C^*(\widetilde{\gamma}_1(w), I(E'))$ and $C^*(W, I(E'))$ are unital essential trivial extensions of $C(X)$ by $I(E')$. So, by Theorem 8.3.1, pp. 125 in [25] ([13]), there exists a unitary $U \in \mathcal{M}(I(E'))$ such that $\|U^*\widetilde{\gamma}_1(w)U - W\| < 1$ and $U^*\widetilde{\gamma}_1(w)U - W \in I(E')$. Let $\varphi = (\mathrm{Ad}\, U \circ \widetilde{\gamma}_1)|_E$. Then φ is the desired $*$-homomorphism. ∎

Let A and B be separable nuclear C^*-algebras satisfying the UCT. The subgroup of $\mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B))$ consisting of all pure extensions will be denoted by $\mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B))$. Set $\mathrm{KL}(A, B) = \mathrm{KK}(A, B)/\, \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B))$ and set $\mathrm{ext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B)) = \mathrm{Ext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B))/\, \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B))$. Then

$$0 \longrightarrow \mathrm{ext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B)) \longrightarrow \mathrm{KL}(A, B) \xrightarrow{\Gamma} \mathrm{Hom}(K_*(A), K_*(B)) \longrightarrow 0$$

is an exact sequence. See Rørdam, Section 5 in [24].

THEOREM 5.2 (Theorem 1.4 in [6]). *Let A be a C^*-algebra in $\mathcal{N}$ and let B be a σ-unital C^*-algebra. Then there is a short exact sequence*

$$0 \longrightarrow \mathrm{Pext}^1_{\mathbb{Z}}(K_*(A), K_{1-*}(B)) \xrightarrow{d} \mathrm{KK}(A, B) \xrightarrow{\Gamma} \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B)) \longrightarrow 0$$

which is natural in each variable. Therefore, $\mathrm{KL}(A, B) \cong \mathrm{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B))$, *where the isomorphism is natural.*

We are now ready to prove the main result of this section.

THEOREM 5.3. *Let* $E = \bigoplus_{i=1}^{n} E_i$ *and* $E' = \bigoplus_{j=1}^{k} E'_j$ *be two finite direct sums of* $\mathcal{E}_0$-*algebras. Suppose* $\alpha : (V_*(E), \widetilde{K}(E)) \to (V_*(E'), \widetilde{K}(E'))$ *is a homomorphism satisfying* $\alpha_v([1_E]) = [1_{E'}]$ *and* $\alpha|_{V(I(E_i))} \neq 0$ *for all* $i = 1, 2, \ldots, n$. *Suppose* $\psi : Q(E) \to Q(E')$ *is a* $*$-*homomorphism as in Theorem* 5.1. *Then there exists a unital* $*$-*homomorphism* $\varphi : E \to E'$ *such that* φ *induces* α *and* $\pi' \circ \varphi = \psi \circ \pi$.

Proof. It is clear that we may assume E' has only one summand. Note that if $K_*(A)$ and $K_*(B)$ are finitely generated, then $\mathrm{KK}(A, B)$ is naturality isomorphic to $\mathrm{KL}(A, B)$.

Suppose E' is a unital purely infinite simple C^*-algebra. Then the existence of φ follows from Theorem 5.2 and Theorem 6.7 in [19]. Suppose that E' is not a purely infinite simple C^*-algebra. Let $\{\alpha_i\}_{i=1}^{6}$ be as in Theorem 5.1. By Theorem 5.2, there exist $\beta \in \mathrm{KK}(E, E'), \beta_I \in \mathrm{KK}(I(E), I(E'))$, and $\beta_Q \in \mathrm{KK}(Q(E), Q(E'))$ such that:

(1) $\Gamma(\beta_I) = (\alpha_1, \alpha_4)$, $\Gamma(\beta) = (\alpha_2, \alpha_5)$, and $\Gamma(\beta_Q) = (\alpha_3, \alpha_6) = (\psi_{*,1}, \psi_{*,0})$, and

(2) $\beta_I = \alpha|_{\underline{K}(I(E))}, \beta = \alpha|_{\underline{K}(E)}, \beta_Q = \alpha|_{\underline{K}(Q(E))}$, using the identification in Theorem 5.2.

Let $[i]$ be the element in $\mathrm{KK}(I(E), E)$ induced by $i : I(E) \to E$. We define $[i'], [\pi'], [\pi]$ in a similar fashion. Since α is a homomorphism from $(V_*(E), \widetilde{K}(E))$ to $(V_*(E'), \widetilde{K}(E'))$ and since the isomorphism in Theorem 5.2 is natural, we have $[i] \times \beta = \beta_I \times [i']$ and $\beta \times [\pi'] = [\pi] \times \beta_Q$, where $\times$ represent the Kasparov product.

Let p be a nonzero projection $I(E')$ such that $[p] = 0$ in $K_0(I(E'))$. It is easy to see that the embedding $j : (1-p)E'(1-p) \to E'$ induces an isomorphism from $V_*((1-p)E'(1-p))$ onto $V_*(E')$. By Theorem 1.17 in [26], there exists $[j]^{-1} \in \mathrm{KK}(E', (1-p)E'(1-p))$ such that $[j] \times [j]^{-1} = [\mathrm{id}_{(1-p)E'(1-p)}]$ and $[j]^{-1} \times [j] = [\mathrm{id}_{E'}]$. By Theorem 5.1, there exists a unital $*$-homomorphism $\varphi' : E \to (1-p)E'(1-p)$ such that φ' induces $j_*^{-1} \circ \alpha$ and $\pi' \circ j \circ \varphi' = \psi \circ \pi$. Note that $\beta - [\varphi'] \times [j]$ is an element of $\mathrm{Ext}^1_{\mathbb{Z}}(K_0(E), K_1(E'))$ and $\beta_I - [I(\varphi')] \times [I(j)]$ is an element of $\mathrm{Ext}^1_{\mathbb{Z}}(K_0(I(E)), K_1(I(E')))$. Also, we have $\beta_Q = [\psi] = [Q(\varphi')]$ in $\mathrm{KK}(Q(E), Q(E'))$.

Note that the following diagram is commutative where the rows are split exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(E), K_1(I(E'))) & \xrightarrow{i'_*} & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(E), K_1(E')) & \xrightarrow{\pi'_*} & \mathrm{Ext}^1_{\mathbb{Z}}((K_0(E), \mathrm{ran}(\pi')_{*,1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*} & & \\
0 & \longrightarrow & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(I(E)), K_1(I(E'))) & \xrightarrow{i'_*} & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(I(E)), K_1(E')) & \xrightarrow{\pi'_*} & \mathrm{Ext}^1_{\mathbb{Z}}(K_0(I(E)), \mathrm{ran}(\pi')_{*,1}) & \longrightarrow & 0
\end{array}$$

Let $b = \pi'_*(\beta - [\varphi'] \times [j])$. Since $(\beta_I - [I(\varphi')] \times [I(j)]) \times [i'] = [i] \times (\beta - [\varphi'] \times [j])$, by a diagram chase in the above diagram, $i^*(b) = 0$. Note that $K_0(E) \cong K_0(I(E)) / \operatorname{ran} \delta_1^E \oplus \mathbb{Z}$. By considering the long exact sequence between Hom and

Ext induced by

$$0 \to \operatorname{ran} \delta_1^E \to K_0(I(E)) \to K_0(I(E))/ \operatorname{ran} \delta_1^E \to 0,$$

we see that $i^* : \operatorname{Ext}^1_{\mathbb{Z}}(K_0(E), \operatorname{ran}(\pi')_{*,1}) \to \operatorname{Ext}^1_{\mathbb{Z}}(K_0(I(E)), \operatorname{ran}(\pi')_{*,1})$ is injective. So, $b = 0$. Therefore, there exists $y \in \operatorname{Ext}^1_{\mathbb{Z}}(K_0(E), K_1(I(E')))$ such that $\beta_E - [\varphi'] \times [j] = y \times [i']$. By Theorem 6.7 in [19], there exists a unital $*$-homomorphism $\varphi'' : E \to pI(E')p$ such that $[\varphi''] = y$. Then $\varphi = \varphi' + \varphi''$ is a unital $*$-homomorphism such that φ induces α and $\pi' \circ \varphi = \psi \circ \pi$. ∎

6. A CLASSIFICATION RESULT

THEOREM 6.1 (Classification Theorem). *Let E and E' be unital $\mathcal{AE}_0$-algebras with real rank zero. Suppose $\alpha : (V_*(E), \widetilde{K}(E)) \to (V_*(E'), \widetilde{K}(E'))$ is an isomorphism such that $\alpha_v([1_E]) = [1_{E'}]$. Then there exists a unital isomorphism $\varphi : E \to E'$ such that φ induces α. By Proposition* 1.7, *the converse is also true.*

Proof. Let $E = \varinjlim(E_i, \varphi_{i,i+1})$ and let $E' = \varinjlim(E'_i, \varphi'_{i,i+1})$. Since E and E' are unital C^*-algebras, we may assume all maps are unital. Since $\operatorname{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$ is naturally isomorphic to $KL(A, B)$ and $KL(A, \varinjlim B_n) = \varinjlim KL(A, B_n)$ whenever $K_*(A)$ is finitely generated (see Proposition 7.13 in [26] and Theorem 5.1 in [27]), by Proposition 3.20 and by passing to subsystems and reindexing, we get the following commutative diagram:

$$\begin{array}{ccccccc}
(V_*(E_1), \widetilde{K}(E_1)) & \longrightarrow & (V_*(E_2), \widetilde{K}(E_2)) & \longrightarrow & \cdots & \longrightarrow & (V_*(E), \widetilde{K}(E)) \\
\downarrow{\scriptstyle \alpha^{(1)}} & \nearrow{\scriptstyle \beta^{(1)}} & \downarrow{\scriptstyle \alpha^{(2)}} & \nearrow{\scriptstyle \beta^{(2)}} & & & {\scriptstyle \beta}\uparrow \downarrow{\scriptstyle \alpha} \\
(V_*(E'_1), \widetilde{K}(E'_1)) & \longrightarrow & (V_*(E'_2), \widetilde{K}(E'_2)) & \longrightarrow & \cdots & \longrightarrow & (V_*(E'), \widetilde{K}(E'))
\end{array}$$

where $\alpha_i([1_{E_i}]) = [1_{E'_i}]$ and $\beta_i([1_{E'_i}]) = [1_{E_{i+1}}]$. Recall from Section 3 that $\mathbf{K}(E_i)$ represent the six-term exact sequence in K-theory induced by the extension E_i. Therefore, the above diagram induces the following commutative diagram:

$$\begin{array}{ccccccc}
\mathbf{K}(E_1) & \longrightarrow & \mathbf{K}(E_2) & \longrightarrow & \cdots & \longrightarrow & \mathbf{K}(E) \\
\downarrow{\scriptstyle \alpha^{(1)}} & \nearrow{\scriptstyle \beta^{(1)}} & \downarrow{\scriptstyle \alpha^{(2)}} & \nearrow{\scriptstyle \beta^{(2)}} & & & {\scriptstyle \beta}\uparrow \downarrow{\scriptstyle \alpha} \\
\mathbf{K}(E'_1) & \longrightarrow & \mathbf{K}(E'_2) & \longrightarrow & \cdots & \longrightarrow & \mathbf{K}(E')
\end{array}$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccccccc}
K_*(Q(E_1)) & \longrightarrow & K_*(Q(E_2)) & \longrightarrow & \cdots & \longrightarrow & K_*(Q(E)) \\
\downarrow & \nearrow & \downarrow & \nearrow & & & \uparrow \downarrow \\
K_*(Q(E'_1)) & \longrightarrow & K_*(Q(E'_2)) & \longrightarrow & \cdots & \longrightarrow & K_*(Q(E'))
\end{array}$$

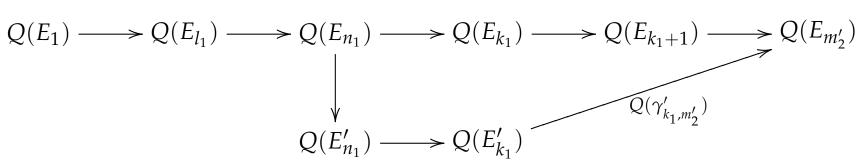
where each homomorphism lifts to a $*$-homomorphism at the level of C^*-algebras. By Theorem 5.3, there are unital $*$-homomorphisms $\eta'_k : E_k \to E'_k$ and $\gamma'_k : E'_k \to E_{k+1}$ such that η'_k induces $\alpha^{(k)}$ and γ'_k induces $\beta^{(k)}$.

Let $\{\varepsilon_n\}_{n=1}^{\infty} \subset \mathbb{R}_{>0}$ be a decreasing sequence such that $\sum\limits_{n=1}^{\infty} \varepsilon_n < \infty$. Let $\mathcal{F}_n$ be a finite subset of the unit ball of E_n and let $\mathcal{F}'_n$ be a finite subset of the unit ball of E'_n such that (1) $\varphi_{n,n+1}(\mathcal{F}_n) \subset \mathcal{F}_{n+1}$ and $\varphi'_{n,n+1}(\mathcal{F}'_n) \subset \mathcal{F}'_{n+1}$; (2) $\bigcup\limits_{n=1}^{\infty} \varphi_{n,n+1}(\mathcal{F}_n)$ is dense in the unit ball of E; and (3) $\bigcup\limits_{n=1}^{\infty} \varphi'_{n,n+1}(\mathcal{F}'_n)$ is dense in the unit ball of E'. Let π_k and π'_k denote the quotient maps from E_k onto $Q(E_k)$ and from E'_k onto $Q(E'_k)$ respectively. We may assume that for all $n \in \mathbb{Z}_{>0}$, $\pi_n(\mathcal{F}_n)$ and $\pi'_n(\mathcal{F}'_n)$ contain the standard generators for $Q(E_n)$ and $Q(E'_n)$ respectively.

Let $\{E_{1,i}\}_{i=1}^{l(1)}$ be the summands of E_1. Let $\delta_{1,i} > 0$ be the positive number given in Theorem 4.14 corresponding to $\varepsilon_1/2$, $E_{1,i}$, and the image of $\mathcal{F}_1$ in $E_{1,i}$. Let $0 < \delta_1 < \min\{\delta_{1,i}\}$, where the minimum is taken over all summand of E_1. By Lemma 1.4.14 in [12], $Q(\varphi_{1,l_1})(\pi_1(\mathcal{F}_1))$ is weakly approximately constant to within $\delta_1/140$ for some $l_1 \in \mathbb{Z}_{>0}$. By Lemma 1.4.14 in [12], there exists $n_1 > l_1$ such that $Q(\varphi_{l_1,n_1})(\pi_{l_1}(\varphi_{1,l_1}(\mathcal{F}_1) \cup \mathcal{F}_{l_1}))$ is weakly approximately constant to within $\delta_1/140$.

Let $\{E'_{n_1,i}\}_{i=1}^{l(n_1)}$ be the summands of E_{n_1}. Let $\lambda_{1,i}$ be the positive number given in Theorem 4.14 corresponding to $\varepsilon_2/2$, $E'_{n_1,i}$, and the image of $\mathcal{F}'_{n_1} \cup (\eta'_{n_1} \circ \varphi_{1,n_1})(\mathcal{F}_1)$ in $E'_{n_1,i}$. Let $0 < \lambda_1 < \min\{\lambda_{1,i}\}$, where the minimum is taken over all summand of E'_{n_1}. Then $Q(\varphi'_{n_1,l'_1})(\pi'(\mathcal{F}'_{n_1} \cup \eta'_{n_1} \circ Q(\varphi_{1,n_1})(\mathcal{F}_1)))$ is weakly approximately constant to within $\lambda_1/140$ for some $l'_1 \in \mathbb{Z}_{>0}$ by Lemma 1.4.14 in [12]. Using Lemma 1.4.14 in [12] again, $Q(\varphi'_{l'_1,k_1})(\pi'_{l_1}(\mathcal{F}'_{l'_1}) \cup Q(\varphi'_{n_1,l'_1})(\mathcal{F}'_{n_1} \cup Q(\eta'_{n_1}) \circ Q(\varphi_{1,n_1})(\mathcal{F}_1)))$ is weakly approximate constant to within $\lambda_1/140$ for some $k_1 > l'_1$.

By Theorem 2.29 and Theorem 3.25 in [12], there exists $m'_2 > k_1 + 1$ such that $Q(\varphi_{l_1,m})$ and $Q(\varphi_{k_1+1,m} \circ \gamma'_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{l_1,n_1})$ are approximately unitarily equivalent to within $\delta_1/2$ on $\pi_{l_1}(\varphi_{1,l_1}(\mathcal{F}_1) \cup \mathcal{F}_{l_1})$ for any $m \geqslant m'_2$. In particular, there exists $v_1 \in U(Q(E_{m'_2}))$ such that if $Q(\gamma'_{k_1,m'_2}) = \mathrm{Ad}(v_1) \circ Q(\varphi_{k_1+1,m'_2}) \circ Q(\gamma'_{k_1})$, then we have the following diagrams:



$$\begin{array}{ccccccccc}
Q(E_{l_1}) & \longrightarrow & Q(E_{n_1}) & \longrightarrow & Q(E_{k_1}) & \longrightarrow & Q(E_{k_1+1}) & \longrightarrow & Q(E_{m_2'}) \\
 & & \downarrow & & & & & \nearrow{\scriptstyle Q(\gamma'_{k_1,m_2'})} & \\
 & & Q(E'_{n_1}) & \longrightarrow & Q(E'_{k_1}) & & & &
\end{array}$$

where the first diagram is approximately commutative on $\pi_1(\mathcal{F}_1)$ to within δ_1 and the second diagram is approximately commutative on $\pi_{l_1}(\varphi_{1,l_1}(\mathcal{F}_1) \cup \mathcal{F}_{l_1})$ to within δ_1. By Theorem 5.3, there exists a unital $*$-homomorphism γ''_{k_1} from E'_{k_1} to $E_{m_2'}$ such that γ''_{k_1} and $\varphi_{k_1+1,m_2'} \circ \gamma'_{k_1}$ induce the same map on $(V_*(E'_{k_1}), \widetilde{K}(E'_{k_1}))$ and $Q(\gamma''_{k_1}) = Q(\gamma'_{k_1,m_2'})$.

Let $m_1 = 1$. We will show that there exists $m_2 \geqslant m_2'$ such that $\varphi_{m_2',m_2} \circ \varphi_{1,m_2'}$ and $\varphi_{m_2',m_2} \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately unitarily equivalent on $\mathcal{F}_1$ within ε_1. Let $E^j_{m_2'}$ be the jth summand of $E_{m_2'}$ and let $P_j : E_{m_2'} \to E^j_{m_2'}$ be the projection onto the jth summand. Suppose that $P : E_1 \to H$ is the projection of E_1 onto one of its summands.

(a) Suppose $P_j \circ \varphi_{1,m_2'}$ is injective on H and $Q(H) \neq 0$. By the choice of $\delta_1 > 0$ and by Theorem 4.14(i), there exists a unitary $W_j \in (P_j \circ \varphi_{1,m_2'})(1_{E_1})E^j_{m_2'}(P_j \circ \varphi_{1,m_2'})(1_{E_1})$ such that $P_j \circ \varphi_{1,m_2'}$ and $\mathrm{Ad}(W_j) \circ P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately the same on $P(\mathcal{F}_1)$ to within ε_1. Set $m(j) = m_2'$.

(b) Suppose $Q(H) = 0$. In this case, $P \circ \varphi_{1,l_1}(\mathcal{F}_1)$ is a finite subset of a corner of $eI(E_{n_1})e$ for some nonzero projection $e \in I(E_{n_1})$. Note that $eI(E_{n_1})e$ will be mapped to $e'I(E^j_{m_2'})e'$ for some nonzero projection $e' \in I(E^j_{m_2'})$. So, from the commutative diagrams involving $(V_*(\cdot), \widetilde{K}(\cdot))$ and by Theorem 4.10 in [18], there is a unitary $W_j \in (P_j \circ \varphi_{1,m_2'})(1_{E_1})E^j_{m_2'}(P_j \circ \varphi_{1,m_2'})(1_{E_1})$ such that $P_j \circ \varphi_{1,m_2'}$ and $\mathrm{Ad}(W_j) \circ P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately the same on $P(\mathcal{F}_1)$ to within ε_1. Set $m(j) = m_2'$.

(c) Suppose $P_j \circ \varphi_{1,m_2'}$ is not injective on H but $Q(P_j \circ \varphi_{1,m_2'})$ is injective on $Q(H)$. By Theorem 4.14 (ii), there exists a unitary $W_j \in (P_j \circ \varphi_{1,m_2'})(1_{E_1})E^j_{m_2'}(P_j \circ \varphi_{1,m_2'})(1_{E_1})$ such that $P_j \circ \varphi_{1,m_2'}$ and $\mathrm{Ad}(W_j) \circ P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately the same on $P(\mathcal{F}_1)$ to within ε_1. Set $m(j) = m_2'$.

(d) Suppose $P_j \circ \varphi_{1,m_2'}$ is not injective on H and $Q(P_j \circ \varphi_{1,m_2'})|_{Q(H)} = 0$. Then by Theorem 4.15(ii), there exists a unitary $W_j \in (P_j \circ \varphi_{1,m_2'})(1_{E_1})E^j_{m_2'}(P_j \circ \varphi_{1,m_2'})(1_{E_1})$ such that $P_j \circ \varphi_{1,m_2'}$ and $\mathrm{Ad}(W_j) \circ P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately the same on $P(\mathcal{F}_1)$ to within ε_1. Set $m(j) = m_2'$.

(e) Suppose $P_j \circ \varphi_{1,m_2'}$ is not injective on H and $Q(P_j \circ \varphi_{1,m_2'})|_{Q(H)} \neq 0$ (but $Q(H) \neq 0$). Therefore, H is not a purely infinite simple C^*-algebra. Let u be the canonical unitary generator of $Q(H)$. Since $P_j \circ \varphi_{1,m_2'}$ is not injective on H, $P_j \circ \varphi_{1,m_2'}$ factors through $Q(H)$. Let κ be the $*$-homomorphism from $Q(H)$ to

$E^j_{m'_2}$ induced by $P_j \circ \varphi_{1,m'_2}$. The commutativity at the level of $(V_*(\cdot), \widetilde{K}(\cdot))$ shows that $P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}|_H$ also factors through $Q(H)$. Denote the induced map by κ'. Let $u_1 = \kappa(u)$ and $u_2 = \kappa'(u)$. Note that $[u_1] = [u_2]$ in $K_1((P_j \circ \varphi_{1,m'_2})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m'_2})(1_{E_1}))$ and u_1, u_2 are map to the same element in $Q((P_j \circ \varphi_{1,m'_2})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m'_2})(1_{E_1}))$. If $\mathrm{sp}(u_1) = \mathrm{sp}(u_2) = S^1$, by Lemma 2.9, there exists a unitary $W_j \in I((P_j \circ \varphi_{1,m'_2})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m'_2})(1_{E_1}))$ such that $\|W_j u_2 W_j^* - u_1\| < \varepsilon_1$. Set $m(j) = m'_2$.

Suppose $\mathrm{sp}(u_1) \neq S^1$ or $\mathrm{sp}(u_2) \neq S^1$. Then u_1 and u_2 are connected to the identity in the unitary group of $(P_j \circ \varphi_{1,m'_2})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m'_2})(1_{E_1})$. Since $RR(E) = 0$, by Theorem 5 in [14], there exists $m(j) > m'_2$ such that for $i = 1, 2$ we have that $\|\varphi_{m'_2,m(j)}(u_i) - v_i\| < \varepsilon_1/2$, where v_1, v_2 are unitaries in $(P_j \circ \varphi_{1,m(j)})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m(j)})(1_{E_1})$ with finite spectrum. By replacing u_i by v_i for $i = 1, 2$, we obtain two $*$-homomorphisms $h_1, h_2 : H \to (P_j \circ \varphi_{1,m(j)})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m(j)})(1_{E_1})$ such that h_1 and $\varphi_{l_1,m(j)}|_H$ are approximately the same to within $\varepsilon_1/2$ on $P(\mathcal{F}_1)$ and h_2 and $P_j \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{l_1,n_1}|_H$ are approximately the same to within $\varepsilon_1/2$ on $P(\mathcal{F}_1)$. It follows from Theorem 4.15 (i) that there exists a unitary $W_j \in (P_j \circ \varphi_{1,m(j)})(1_{E_1})E^j_{m'_2}(P_j \circ \varphi_{1,m(j)})(1_{E_1})$ such that $P_j \circ \varphi_{1,m(j)}$ and $\mathrm{Ad}(W_j) \circ P_j \circ \circ\varphi_{m'_2,m(j)} \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately the same on $P(\mathcal{F}_1)$ to within ε_1. Set $m(j) = m'_2$.

If $m''_{2,H} = \max_j\{m(j)\}$, then by the above observations, there exists a unitary $W \in (\varphi_{1,m''_2} \circ P)(1_{E_1})E_{m''_2}(\varphi_{1,m''_2} \circ P)(1_{E_1})$ such that

$$\|\varphi_{1,m''_2}(f) - \mathrm{Ad}(W) \circ \varphi_{m'_2,m''_2} \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}(f)\| < \varepsilon_1$$

for all $f \in P(\mathcal{F}_1)$. Then let $m_2 = \max\{m''_{2,H}\}$, where the maximum is taken over all direct summands of E_1. So φ_{1,m_2} and $\varphi_{m'_2,m_2} \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1} \circ \eta'_{n_1} \circ \varphi_{1,n_1}$ are approximately unitarily equivalent on $\mathcal{F}_1$ to within ε_1. Hence, there exists $U \in U(E_{m_2})$ such that if $\eta_1 = \eta'_{n_1} \circ \varphi_{1,n_1}$ and $\gamma_1 = \mathrm{Ad}(U) \circ \varphi_{m'_2,m_2} \circ \gamma''_{k_1} \circ \varphi'_{n_1,k_1}$, then φ_{1,m_2} and $\gamma_1 \circ \eta_1$ are approximately the same on $\mathcal{F}_1$ to within ε_1.

Let $\mathcal{G}'_1 = \mathcal{F}'_{n_1} \cup \eta_1(\mathcal{F}_1)$ and let $\mathcal{G}_2 = \mathcal{F}_{m_2} \cup \varphi_{1,m_2}(\mathcal{F}_1) \cup \gamma_1(\mathcal{G}'_1)$. Denote the summands of E_{m_2} by $\{E_{m_2,i}\}_{i=1}^{l(m_2)}$. Let $\delta_{2,i} > 0$ be the positive number that is given in Theorem 4.14 which corresponds to $\varepsilon_3/2$, $E^i_{m_2}$, and the image of $\mathcal{G}_2$ in $E^i_{m_2}$. Let $0 < \delta_2 < \min\{\delta_{2,i}\}$, where the minimum is taken over all direct summands of E_{m_2}. By Lemma 1.4.14 in [12], there exists $l_2 > m_2$ such that $Q(\varphi_{m_2,l_2})(\mathcal{G}_2)$ is weakly approximately constant to within $\delta_2/140$. By Lemma 1.4.14 in [12], we get $n'_2 > l_2$ such that $Q(\varphi_{l_2,n'_2})(\pi_{l_2}(\mathcal{F}_{l_2} \cup \varphi_{m_2,l_2}(\mathcal{G}_2)))$ is weakly approximately constant to within $\delta_2/140$.

Note $Q(\varphi'_{l'_1,k_1})(\pi'(\varphi_{n_1,l'_1}(\mathcal{G}'_1)))$ is weakly approximately constant to within $\lambda_1/140$. Hence, by Theorem 2.29 and Theorem 3.25 in [12], there exists $n''_2 > n'_2$ such that $Q(\varphi_{l'_1,m})$ and $Q(\varphi_{n'_2,m} \circ \eta'_{n_2} \circ \mathrm{Ad}(U) \circ \varphi_{m_2,n'_2} \circ \gamma''_{k_1} \circ \varphi_{l'_1,k_1})$ are approximately unitarily equivalent to within $\lambda_1/2$ on $\pi'_{l'_1}(\varphi_{n_1,l'_1}(\mathcal{G}'_1))$ for any $m \geqslant n''_2$. Set $\kappa_1 = \mathrm{Ad}(U) \circ \varphi_{m_2,n'_2} \circ \gamma''_{k_1} \circ \varphi_{l'_1,k_1}$.

By the choice of l'_1 and k_1 and using the same argument as above we get a positive integer $n_2 \geqslant n''_2$ and a unitary $V \in E'_{n_2}$ such that φ'_{n_1,n_2} and $\mathrm{Ad}(V) \circ \varphi_{n'_2,n_2} \circ \eta'_{n'_2} \circ \varphi_{m_2,n'_2} \circ \gamma_1$ are approximately the same on $f \in \mathcal{G}'_1$ to within ε_2.

Set $\eta_2 = \mathrm{Ad}(V) \circ \varphi_{n'_2,n_2} \circ \eta'_{n'_2} \circ \varphi_{m_2,n'_2}$ and $\kappa_2 = \mathrm{Ad}(V) \circ \varphi'_{n'_2,n_2} \circ \eta'_{n'_2}$. Then $\gamma_1 \circ \eta_1$ and φ_{1,m_2} are approximately the same on $\mathcal{F}_1$ to within ε_1 and $\eta_2 \circ \gamma_1$ and φ'_{n_1,n_2} are approximately the same on $\mathcal{F}'_{n_1} \cup \eta_1(\mathcal{F}_1)$ to within ε_2. Furthermore, η_2 can be rewritten as $\varphi'_{n'_2,n_2} \circ \eta_{n'_2} \circ \varphi_{m_2,n'_2}$ or $\kappa_2 \circ \varphi_{m_2,n'_2}$. The choice of l_2 and n'_2 ensures that the above construction can continue. Therefore, we get the following approximately intertwining diagram:

$$\begin{array}{ccccccc}
E_1 & \longrightarrow & E_{m_2} & \longrightarrow & \cdots & \longrightarrow & E \\
{\scriptstyle \eta_1}\downarrow & \overset{\gamma_1}{\nearrow} & {\scriptstyle \eta_2}\downarrow & \overset{\gamma_2}{\nearrow} & & & \\
E'_{n_1} & \longrightarrow & E'_{n_2} & \longrightarrow & \cdots & \longrightarrow & E'
\end{array}$$

Hence, there exists a unital isomorphism $\varphi : E \to E'$ such that φ induces α. ∎

COROLLARY 6.2. *Suppose $K_1(I(E_n)) = 0 = K_1(I(E'_n))$ for all $n \in \mathbb{Z}_{\geqslant 0}$ in Theorem 6.1. If $\alpha : V_*(E) \to V_*(E')$ is an isomorphism such that $\alpha_v([1_E]) = [1_{E'}]$, then there exists a unital isomorphism $\varphi : E \to E'$ such that φ induces α.*

Proof. Note that there exists an isomorphism (β_1, β_2) from $(V_*(E), \widetilde{K}(E))$ to $(V_*(E), \widetilde{K}(E))$ such that $\beta_1 = \alpha$ since $K_1(I(E_n)) = 0 = K_1(I(E'_n))$. ∎

REMARK 6.3. Every unital AT-algebra with real rank zero is an $\mathcal{AE}_0$-algebra.

The result follows from Theorem 6.1 and Theorem 4.3 in [17].

REMARK 6.4. A unital separable nuclear purely infinite simple C^*-algebra satisfying the UCT with torsion free K_1 is an $\mathcal{AE}_0$-algebra. See [13] and [23].

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