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A CHARACTERIZATION OF THE OPERATOR-VALUED TRIANGLE EQUALITY

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ABSTRACT. We will show that for any two bounded linear operators X, Y on a Hilbert space $\mathfrak{H}$, if they satisfy the triangle equality $|X+Y| = |X| + |Y|$, there exists a partial isometry U on $\mathfrak{H}$ such that $X = U|X|$ and $Y = U|Y|$. This is a generalization of Thompson's theorem to the matrix case proved by using a trace.

KEYWORDS: *Operator theory, Hilbert space, triangle inequality.*

MSC (2000): 47A05, 47A10, 47A12.

1. INTRODUCTION

In his interesting paper [3], [4], Thompson showed the triangle inequality for two matrices: for any two matrices X and Y, there exist unitaries V and W such that $|X+Y| \leqslant V|X|V^* + W|Y|W^*$, where $|X| = (X^*X)^{1/2}$. (See [1] for a generalization of the theorem to bounded linear operators on infinite dimensional Hilbert spaces.) Moreover, in [5] he characterized the condition when the triangle equality holds: he showed that for two matrices X and Y, there exist two unitary matrices V, W such that the equality $|X+Y| = V|X|V^* + W|Y|W^*$ holds if and only if there is a unitary matrix U such that $X = U|X|$ and $Y = U|Y|$.

In the proof of Thompson's theorem, the (finite) trace plays a crucial role. Thus his argument does not work in the infinite dimensional case. The aim of this paper is to overcome this difficulty. We shall generalize this theorem to operators on infinite-dimensional Hilbert spaces.

It is easy to see that Thompson's theorem fails in the infinite dimensional setting. Indeed we can easily find two operators X, Y and unitaries V, W satisfying $|X+Y| = V|X|V^* + W|Y|W^*$, but there is no partial isometry U such that $X = U|X|$ and $Y = U|Y|$. For example, consider an orthogonal projection P and its subprojection Q. Suppose that all projections P, Q and $P-Q$ are infinite rank. Then there are (at least countably many) unitaries V, W such that the equality

$|P+(-Q)| = V|P|V^* + W|(-Q)|W^*$ holds. In this case there is no partial isometry U satisfying $P = U|P|$ and $(-Q) = U|(-Q)|$.

To avoid such cases, we consider the following problem: for two bounded linear operators X and Y on an infinite dimensional Hilbert space $\mathfrak{H}$, if the equality $|X+Y| = |X| + |Y|$ holds, does there exists a partial isometry U on $\mathfrak{H}$ such that $X = U|X|$ and $Y = U|Y|$? (Here we remark even in this case we cannot choose U as a unitary. For example, consider the unilateral shift S ([2], p. 40). Then obviously the equality $|S+S| = |S| + |S|$ holds but there exists no unitary U satisfying $S = U|S|$.) The main result of this paper is to answer this problem affirmatively.

2. MAIN RESULT

Throughout this paper we assume that the readers are familiar with basic notations and results on the Hilbert space operator theory. We refer to Halmos's book [2].

We denote by $\mathfrak{H}$ an infinite dimensional complex Hilbert space. The inner product on $\mathfrak{H}$ is denoted by $\langle x, y \rangle$ for two vectors $x, y \in \mathfrak{H}$. Thus the Hilbert space norm $\|\cdot\|$ is defined by $\|x\| = \langle x, x \rangle^{1/2}$. We denote by $B(\mathfrak{H})$ the set of all bounded linear operators on $\mathfrak{H}$. The operator norm on $B(\mathfrak{H})$ is denoted by $\|\cdot\|$. The identity operator on $\mathfrak{H}$ is denoted by I. An operator $W \in B(\mathfrak{H})$ is called a contraction if $\|W\| \leqslant 1$. An operator $A \in B(\mathfrak{H})$ is said to be positive if $\langle Ax, x \rangle \geqslant 0$ for any $x \in \mathfrak{H}$. We denote $A \geqslant 0$ if A is positive. For two selfadjoint operators $A, B \in B(\mathfrak{H})$, we denote $A \leqslant B$ if $B - A \geqslant 0$. A positive operator $A \in B(\mathfrak{H})$ has the unique square root $A^{1/2} \in B(\mathfrak{H})$ which satisfies $(A^{1/2})^2 = A$ and $A^{1/2} \geqslant 0$. For $X \in B(\mathfrak{H})$, we define its "absolute value" by $|X| = (X^*X)^{1/2}$. An operator $U \in B(\mathfrak{H})$ is called a partial isometry if U^*U is an orthogonal projection. For $X \in B(\mathfrak{H})$, we can always express X as a product $X = U|X|$, where $U \in B(\mathfrak{H})$ is a partial isometry. Then U satisfies that $U^*U|X| = |X|$. This expression is called a polar decomposition of X. We define the kernel $\ker(X)$ and the range $\mathrm{ran}(X)$ of X by $\ker(X) = \{a \in \mathfrak{H} : Xa = 0\}$ and $\mathrm{ran}(X) = \{Xa : a \in \mathfrak{H}\}$ respectively. We denote the set of all real numbers by $\mathbb{R}$ and the set of all complex numbers by $\mathbb{C}$. For $z \in \mathbb{C}$ its real part is denoted by $\mathrm{Re}(z)$. The spectrum of $X \in B(\mathfrak{H})$ is defined by $\sigma(X) = \{z \in \mathbb{C} : X - zI$ does not have an inverse in $B(\mathfrak{H})\}$. The numerical range of $X \in B(\mathfrak{H})$ is also defined by $\mathbb{W}(X) = \{\langle Xa, a \rangle : a \in \mathfrak{H},\ \|a\| = 1\}$.

LEMMA 2.1. *Let $A, B \in B(\mathfrak{H})$ be two positive operators and let $V, W \in B(\mathfrak{H})$ be partial isometries such that $V^*VA = A$ and $W^*WB = B$. Then there exists a partial isometry $U \in B(\mathfrak{H})$ such that $UA = VA$ and $UB = WB$ if and only if the equality $A(V^*W - I)B = 0$ holds.*

Proof. The "only if" part is obvious.

Assume that the equality $A(V^*W-I)B=0$ holds. Since for any $x,y\in\mathfrak{H}$

$$\begin{aligned}\|VAx+WBy\|^2 &= \|Ax\|^2+\|By\|^2+2\mathrm{Re}\langle AV^*WBy,x\rangle\\ &= \|Ax\|^2+\|Bx\|^2+2\mathrm{Re}\langle ABy,x\rangle = \|Ax+By\|^2,\end{aligned}$$

we can define the desired partial isometry U by $U(Ax+By)=VAx+WBy$. ∎

LEMMA 2.2. *Let $W\in B(\mathfrak{H})$ be a contraction. Then for any operator $D\in B(\mathfrak{H})$ we have*

$$\sigma(D^*(W-I)D)\subset\{z\in\mathbb{C}:|z+\|D\|^2|\leqslant\|D\|^2\},$$

and hence

$$\sigma(D^*(W-I)D)\cap i\mathbb{R}\subset\{0\}.$$

Proof. For any unit vector $x\in\mathfrak{H}$, we see that

$$\langle D^*(W-I)Dx,x\rangle=\|Dx\|^2\Big\langle(W-I)\frac{Dx}{\|Dx\|},\frac{Dx}{\|Dx\|}\Big\rangle.$$

Since W is a contraction, the numerical range $\mathbb{W}(W-I)$ is contained in the set $\{z\in\mathbb{C}:|z+1|\leqslant 1\}$. Combining this with $\|Dx\|^2\leqslant\|D\|^2$, we conclude that $\langle D^*(W-I)Dx,x\rangle\in\{z\in\mathbb{C}:|z+\|D\|^2|\leqslant\|D\|^2\}$. In other words, the numerical range $\mathbb{W}(D^*(W-I)D)$ is contained in $\{z\in\mathbb{C}:|z+\|D\|^2|\leqslant\|D\|^2\}$.

Recall that in general, for any operator $X\in B(\mathfrak{H})$ the spectrum $\sigma(X)$ is a subset of the closure of its numerical range $\mathbb{W}(X)$. (See p. 111, Problem 169 of [2].) Hence we get

$$\sigma(D^*(W-I)D)\subset\{z\in\mathbb{C}:|z+\|D\|^2|\leqslant\|D\|^2\}. \quad ∎$$

THEOREM 2.3. *For two bounded linear operators $X,Y\in B(\mathfrak{H})$, the triangle equality $|X+Y|=|X|+|Y|$ holds if and only if there exists a partial isometry U such that $X=U|X|$ and $Y=U|Y|$.*

Proof. For notational simplicity we write $A\equiv|X|$ and $B\equiv|Y|$.

First assume that there exists a partial isometry U such that $X=UA$ and $Y=UB$. Then since

$$U^*UA=A \quad\text{and}\quad U^*UB=B,$$

we have

$$|X+Y|^2=(AU^*+BU^*)(UA+UB)=A^2+AB+BA+B^2=(A+B)^2=(|X|+|Y|)^2$$

and so $|X+Y|=|X|+|Y|$.

Assume conversely that the triangle equality $|X+Y|=|X|+|Y|$ holds. Let $X=V|X|=VA$ and $Y=W|Y|=WB$ be the polar decompositions of X and Y respectively. Then the triangle equality implies

$$(AV^*+BW^*)(VA+WB)=(A+B)^2.$$

Combining this with the relations $V^*VA = A$ and $W^*WB = B$ we have $A(V^*W - I)B + B(W^*V - I)A = 0$. This means that the operator $\mathrm{i}A(V^*W - I)B$ is selfadjoint. We claim

$$\mathrm{i}A(V^*W - I)B = 0. \tag{2.1}$$

Since $A, B \leqslant A + B$, there are uniquely contractions K, L such that

$$A^{1/2} = K(A+B)^{1/2} \quad \text{and} \quad B^{1/2} = L(A+B)^{1/2} \tag{2.2}$$

and $\ker(A+B)^{1/2} \subset \ker(A) \cap \ker(B)$. By taking adjoints of the both sides of (2.2) we also have

$$A^{1/2} = (A+B)^{1/2}K^* \quad \text{and} \quad B^{1/2} = (A+B)^{1/2}L^*. \tag{2.3}$$

Let P be the orthogonal projection to the closure of $\mathrm{ran}(A+B)^{1/2}$. (Remark that $I - P$ is the orthogonal projection to $\ker(A+B)^{1/2}$ by selfadjointness of $A + B$.) Since by (2.2) and (2.3)

$$PK^*K = K^*KP = K^*K \quad \text{and} \quad PL^*L = L^*LP = L^*L$$

and

$$(A+B)^{1/2}(K^*K + L^*L - I)(A+B)^{1/2} = 0$$

we conclude

$$K^*K + L^*L = P,$$

because $\mathrm{ran}(A+B)^{1/2}$ is dense in $\mathrm{ran}(P)$. This implies that K^*K and L^*L commute, and hence $(L^*L)(K^*K)$ is a positive operator.

Now by (2.2) and (2.3) we can write

$$\mathrm{i}A(V^*W - I)B = \mathrm{i}(A+B)^{1/2}K^*K(A+B)^{1/2}(V^*W - I)(A+B)^{1/2}L^*L(A+B)^{1/2}.$$

Since $\mathrm{i}A(V^*W - I)B$ is selfadjoint, as in the proof of $K^*K + L^*L = P$ we see that $\mathrm{i}K^*K(A+B)^{1/2}(V^*W - I)(A+B)^{1/2}L^*L$ is also selfadjoint, and for (2.1) it suffices to show that

$$\mathrm{i}(K^*K)(A+B)^{1/2}(V^*W - I)(A+B)^{1/2}(L^*L) = 0. \tag{2.4}$$

Then since in general a selfadjoint operator whose spectrum is a one-point set $\{0\}$ must be 0 (see [2], p. 61), for (2.4) it suffices to show

$$\sigma(\mathrm{i}(K^*K)(A+B)^{1/2}(V^*W - I)(A+B)^{1/2}(L^*L)) = \{0\}.$$

Since every spectrum of a selfadjoint operator is real, it is further reduced to showing that

$$\sigma((K^*K)(A+B)^{1/2}(V^*W - I)(A+B)^{1/2}(L^*L)) \cap (\mathrm{i}\mathbb{R}) = \{0\}. \tag{2.5}$$

To prove (2.5) we use a general fact that for two bounded linear operators S, T

$$\sigma(ST) \setminus \{0\} = \sigma(TS) \setminus \{0\}. \tag{2.6}$$

(See p. 39, Problem 61 of [2].) Then we have by (2.6)

$$\sigma((K^*K)(A+B)^{1/2}(V^*W-I)(A+B)^{1/2}(L^*L))\setminus\{0\}$$
$$=\sigma((V^*W-I)(A+B)^{1/2}(L^*L)(K^*K)(A+B)^{1/2})\setminus\{0\}.$$

Since $(L^*L)(K^*K)$ is a positive operator, so is $(A+B)^{1/2}(L^*L)(K^*K)(A+B)^{1/2}$. Let

$$D \equiv [(A+B)^{1/2}(L^*L)(K^*K)(A+B)^{1/2}]^{1/2}$$

and apply (2.6) once again to get $\sigma((V^*W-I)(A+B)^{1/2}(L^*L)(K^*K)(A+B)^{1/2}) \setminus \{0\} = \sigma(D(V^*W-I)D) \setminus \{0\}$. Therefore we have

$$\sigma((K^*K)(A+B)^{1/2}(V^*W-I)(A+B)^{1/2}(L^*L))\cap(\mathrm{i}\mathbb{R})\setminus\{0\}$$
$$=\sigma(D(V^*W-I)D)\cap(\mathrm{i}\mathbb{R})\setminus\{0\}.$$

On the other hand since V^*W is a contraction we have by Lemma 2.2

$$\sigma(D(V^*W-I)D)\cap(\mathrm{i}\mathbb{R})\subset\{0\}$$

which establishes (2.5) and hence (2.1).

Now by Lemma 2.1 it follows from (2.1) that there is a partial isometry U such that

$$U|X| = UA = VA = V|X| \quad \text{and} \quad U|Y| = UB = WB = W|Y|.$$

This completes the proof. ∎

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