MULTIPLICATION OPERATORS ON THE BERGMAN SPACE AND WEIGHTEDhiftS

SHUNHUA SUN, DECHAO ZHENG and CHANGYONG ZHONG

Communicated by Kenneth R. Davidson

ABSTRACT. In this paper we show that the multiplication operator on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity if and only if its symbol is a constant multiple of the $N$-th power of a Möbius transform.

KEYWORDS: Multiplication operators, Bergman space, weighted shifts.

MSC (2000): Primary 47B37, 47B35; Secondary 47A15.

INTRODUCTION

Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Let $dA$ denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The Bergman space $L^2_\alpha$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are also in the space $L^2(\mathbb{D}, dA)$ of square integrable functions on $\mathbb{D}$. Because the nonnegative powers $\{z^n\}$ span the Bergman space $L^2_\alpha$, $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ form an orthonormal basis of $L^2_\alpha$.

For a bounded analytic function $\phi$ on the unit disk, the multiplication operator $M_\phi$ is defined on the Bergman space $L^2_\alpha$ by

$$M_\phi h = \phi h$$

for $h \in L^2_\alpha$.

Let $e_n = \sqrt{n+1}z^n$. Then $\{e_n\}_{0}^\infty$ form an orthonormal basis of the Bergman space $L^2_\alpha$. On the basis $\{e_n\}$, the multiplication operator $M_z$ by $z$ is a weighted shift operator:

$$M_z e_n = \sqrt{\frac{n+1}{n+2}} e_{n+1}.$$

So it is usually called the Bergman shift.
A reducing subspace \( M \) for an operator \( T \) on a Hilbert space \( H \) is a subspace \( M \) of \( H \) such that \( TM \subset M \) and \( T^*M \subset M \). In [7] and [8] we have studied reducing subspaces of multiplication operators on the Bergman space via the Hardy space of the bidisk. The multiplication operator \( M_z \) is a weighted shift. The general multiplication operator \( M_\phi \) is a holomorphic calculus of the weighted shift. Shift operators have been studied very extensively [2], [3]. In [4], Stessin and Zhu obtained a complete description of the reducing subspaces of weighted unilateral shift operators of finite multiplicity to shed a light on that \( M_{z_N} \) on the Bergman space has \( N \) nontrivial minimal reducing subspaces, but the multiplication operator by \( z^N \) on the Hardy space has infinitely many reducing subspaces.

A natural question is to characterize the multiplication operators on the Bergman space unitarily equivalent to a weighted unilateral shift operators of finite multiplicity. This paper continues our study on the multiplication operators \( M_\phi \) on the Bergman space in [7], [8] by using the Hardy space of the bidisk to completely answer the question. Our main result of this paper almost says that only \( M_{z_N} \) up to unitary equivalence is a weighted unilateral shift operator of finite multiplicity.

**Theorem 0.1.** If the multiplication operator \( M_\phi \) on the Bergman space is unitarily equivalent to a weighted unilateral shift operator of finite multiplicity, then \( \phi = c\phi_\lambda^N \), for a constant \( c \) and some Möbius transform \( \phi_\lambda(z) = \frac{z-\lambda}{1-\lambda z} \).

Let \( \mathbb{T} \) denote the unit circle. The torus \( \mathbb{T}^2 \) is the Cartesian product \( \mathbb{T} \times \mathbb{T} \). Let \( d\sigma \) be the rotation invariant Lebesgue measure on \( \mathbb{T}^2 \). The Hardy space \( H^2(\mathbb{T}^2) \) is the subspace of \( L^2(\mathbb{T}^2, d\sigma) \), each function in \( H^2(\mathbb{T}^2) \) can be identified with the boundary value of the function holomorphic in the bidisk \( \mathbb{D}^2 \) with the square summable Fourier coefficients. The Toeplitz operator on \( H^2(\mathbb{T}^2) \) with symbol \( f \) in \( L^\infty(\mathbb{T}^2, d\sigma) \) is defined by

\[
T_f(h) = P(fh),
\]
for \( h \in H^2(\mathbb{T}^2) \) where \( P \) is the orthogonal projection from \( L^2(\mathbb{T}^2, d\sigma) \) onto \( H^2(\mathbb{T}^2) \).

For each integer \( n \geq 0 \), let

\[
p_n(z,w) = \sum_{i=0}^{n} z^i w^{n-i}.
\]

Let \( \mathcal{H} \) be the subspace of \( H^2(\mathbb{T}^2) \) spanned by functions \( \{p_n\}_{n=0}^\infty \). Thus

\[
H^2(\mathbb{T}^2) = \mathcal{H} \oplus \text{cl}\{(z-w)H^2(\mathbb{T}^2)\}.
\]

Let

\[
\mathcal{B} = P_{\mathcal{H}} T_z|_{\mathcal{H}} = P_{\mathcal{H}} T_w|_{\mathcal{H}}
\]
where \( P_{\mathcal{H}} \) be the orthogonal projection from \( L^2(\mathbb{T}^2, d\sigma) \) onto \( \mathcal{H} \). So \( \mathcal{B} \) is unitarily equivalent to the Bergman shift \( M_z \) on the Bergman space \( L^2_\alpha \) via the following
unitary operator $U : L^2_a(\mathbb{D}) \to \mathcal{H}$,

$$Uz^n = \frac{p_n(z, w)}{n + 1}.$$  

This implies that the Bergman shift is lifted up as the compression of an isometry on a nice subspace of $H^2(T^2)$. Indeed, for each Blaschke product $\phi(z)$ with finite order, the multiplication operator $M_\phi$ on the Bergman space is unitarily equivalent to $\phi(\mathcal{B})$ on $\mathcal{H}$.

By Lemma 13 in [7], it is easy to see that for each Blaschke product $\phi$ with order $N$, $\mathcal{H}$ can be decomposed as a direct sum of at most $N$ reducing subspaces of $M_\phi$. We will show that if $\phi$ has more than two distinct roots and at least one root is repeated, then $\mathcal{H}$ can not be decomposed as a direct sum of $N$ reducing subspaces of $M_\phi$ (Theorem 3.1).

1. PREMIMINARIES

We need some basic constructions from [7]. Let

$$\mathcal{K}_\phi = \text{span}\{\phi^l(z)\phi^k(w)\mathcal{H}; l, k \geq 0\}.$$  

Then $\mathcal{K}_\phi$ is a reducing subspace for both $T_{\phi(z)}$ and $T_{\phi(w)}$, and so $T_{\phi(z)}$ and $T_{\phi(w)}$ are also a pair of doubly commuting isometries on $\mathcal{K}_\phi$. Introduce the wandering space

$$\mathcal{L}_\phi = \ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{K}_\phi.$$  

Let $L_0 = \ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$. In [7], for each $e \in L_0$, we construct functions $\{d^k_e\}$ and $d^0_e$ in $\mathcal{L}_\phi$ such that for each $l \geq 1$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^l-k_e \in \mathcal{H}$$

and

$$p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))d^0_e \in \mathcal{H}.$$  

We have a precise formula of $d^0_e$ but $d^k_e$ is orthogonal to $\ker T^*_{\phi(z)} \cap \ker T^*_{\phi(w)} \cap \mathcal{H}$, and for a reducing subspace $\mathcal{M}$, and $e \in \mathcal{M}$,

$$p_l(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^l-k_e \in \mathcal{M}.$$  

The relation between $d^1_e$ and $d^0_e$ is given in [7] and stated as follows:

**THEOREM 1.1.** If $\mathcal{M}$ is a reducing subspace of $\phi(\mathcal{B})$ orthogonal to the distinguished reducing subspace $\mathcal{M}_0$, for each $e \in \mathcal{M} \cap L_0$, then there is an element $\tilde{e} \in \mathcal{M} \cap L_0$ and a number $\lambda$ such that

$$d^1_e = d^0_e + \tilde{e} + \lambda e_0.$$  

We will often use the above theorem and the following theorem from [7].
THEOREM 1.2. If \( \phi \) is a finite Blaschke product, then there is a unique reducing subspace \( \mathcal{M}_0 \) for \( \phi(B) \) such that \( \phi(B)|_{\mathcal{M}_0} \) is unitarily equivalent to the Bergman shift. In fact,

\[
\mathcal{M}_0 = \text{span}\{ p_l(\phi(z), \phi(w))e_0 \}_{l \geq 0},
\]

and \( \{ \frac{p_l(\phi(z), \phi(w))e_0}{\sqrt{l+1||e_0||}} \}_{0}^{\infty} \) form an orthonormal basis of \( \mathcal{M}_0 \).

We call \( \mathcal{M}_0 \) the distinguished reducing subspace for \( \phi(B) \). \( \mathcal{M}_0 \) is unitarily equivalent to a reducing subspace of \( \mathcal{M}_\phi \) contained in the Bergman space, denoted by \( \mathcal{M}_0(\phi) \). The space plays an important role in classifying the minimal reducing subspaces of \( \mathcal{M}_\phi \) [7], [8]. If \( 0 \) is a zero of \( \phi \), it was shown [5] that

\[
\mathcal{M}_0(\phi) = \text{span}\{ \phi^n : n = 0, 1, \ldots, m, \ldots \}.
\]

The following lemmas give some properties for functions in \( \mathcal{H} \) or \( \mathcal{H}^\perp \).

**Lemma 1.3.** If \( f \) is in \( H^2(\mathbb{T}^2) \) and continuous on the closed bidisk and \( e \) is in \( \mathcal{H} \), then

\[
\langle f(z,w), e(z,w) \rangle = \langle f(z,z), e(z,0) \rangle = \langle f(w,w), e(0,0) \rangle.
\]

**Proof.** Since \( f(z,w) \) is continuous on the closed bidisk, there are a sequence \( \{ P_n \} \) of polynomials of \( z \) and \( w \) converging uniformly to \( f(z,w) \) on the closed bidisk. Thus it suffices to show

\[
\langle P_n(z,w), e(z,w) \rangle = \langle P_n(z,z), e(z,0) \rangle = \langle P_n(w,w), e(0,0) \rangle.
\]

Noting that \( T_{z^l}^*|\mathcal{H} = T_{w^l}^*|\mathcal{H} \), we see that

\[
T_{P_n(z,w)}^* e = T_{P_n(z,z)}^* e = T_{P_n(w,w)}^* e.
\]

This gives

\[
\langle P_n(z,w), e(z,w) \rangle = \langle 1, P_n(z,w) \rangle e(z,w) = \langle 1, T_{P_n(z,w)}^* e \rangle = \langle 1, T_{P_n(z,z)}^* e \rangle = \langle 1, P_n(z,z) e(z,w) \rangle = \langle P_n(z,z), e(z,w) \rangle = \langle P_n(z,z), e(z,0) \rangle.
\]

Similarly we also obtain the following which completes the proof:

\[
\langle P_n(z,w), e(z,w) \rangle = \langle P_n(w,w), e(0,0) \rangle.
\]

The proofs of the following lemmas are easy and left for readers.

**Lemma 1.4.** For \( h(z,w) \in H^2(\mathbb{T}^2) \), \( h \) is in \( \mathcal{H}^\perp \) if and only if \( h(z,z) = 0 \), for \( z \in \mathbb{D} \).

**Lemma 1.5.** Suppose that \( e(z,w) \) is in \( \mathcal{H} \). If \( e(z,z) = 0 \) for each \( z \) in the unit disk, then \( e(z,w) = 0 \) for \( (z,w) \) on the torus.

The above lemma tells us that a function in \( \mathcal{H} \) is completely determined by its value on the diagonal. The following result says that \( e(z,w) \) is symmetric with respect to \( z \) and \( w \).
LEMMA 1.6. If \( e(z, w) \) is in \( \mathcal{H} \), then
\[
e(z, w) = e(w, z).
\]

LEMMA 1.7. Suppose \( f(z, w) \) is in \( \mathcal{H} \). Let \( F(z) = f(z, 0) \). Then, for each \( \lambda \in \mathbb{D} \),
\[
f(\lambda, \lambda) = \lambda F'(\lambda) + F(\lambda).
\]

For \( \alpha \in \mathbb{D} \), let \( k_\alpha \) be the reproducing kernel of the Hardy space \( H^2(\mathbb{T}) \) at \( \alpha \). That is, for each function \( f \) in \( H^2(\mathbb{T}) \),
\[
f(\alpha) = \langle f, k_\alpha \rangle.
\]

For an integer \( s \geq 0 \), define
\[
k^s_\alpha(z) = \frac{s!z^s}{(1 - \overline{\alpha}z)^{s+1}}.
\]

Let \( \phi \) be a Blaschke product with zeros \( \{\alpha_k\}_{k=0}^K \) and \( \alpha_k \) repeats \( n_k+1 \) times. That is,
\[
\phi(z) = \prod_{k=0}^{K} \left( \frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \right)^{n_k+1}.
\]
The order of \( \phi \) is given by
\[
N = \sum_{i=0}^{K} (n_i + 1).
\]

We assume that \( \alpha_0 = 0 \), and so \( \phi(z) = z\phi_0(z) \) where \( \phi_0 \) is the following Blaschke product:
\[
\phi_0(z) = z^{n_0} \prod_{k=1}^{K} \left( \frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \right)^{n_k+1}.
\]

For each \( \alpha \in \mathbb{D} \) and integer \( t \geq 0 \), let
\[
(1.1) \quad e^t_{\alpha}(z, w) = \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} k^s_\alpha(z) k^{t-s}_\alpha(w).
\]

The Mittag-Leffler expansion of the finite Blaschke product \( \phi_0 \) is
\[
\phi_0(z) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} c_i^t k^t_{\alpha_i}(z),
\]
for some constants \( \{c_i^t\} \). Define
\[
e_0(z, w) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} c_i^t e^t_{\alpha_i}(z, w).
\]

Clearly,
\[
e_0(z, 0) = \phi_0(z).
\]

Simple calculations give the following lemmas.
Lemma 1.8. For each \( \alpha \in \mathbb{D} \) and \( t \geq 0 \), then
\[
e_{\alpha}^t(z, z) = \frac{(t + 1)!z^t}{(1 - \alpha z)^{t+2}}.
\]

Lemma 1.9. For each \( F(z, w) \in H^2(\mathbb{T}^2) \),
\[
\langle F, e_{\alpha}^t \rangle = \left[ (\partial_z + \partial_w)f(z, w) \right]_{z = w = \alpha}.
\]

Noting that the dimension of \( L_0 \) is \( N \) and \( \{ e_{t_i}^i(z, w) : 0 \leq i \leq K, 0 \leq t_i \leq n_i \} \) are linearly independent, we immediately have the following lemma.

Lemma 1.10. We have
\[
L_0 = \text{span}\{ e_{t_i}^i(z, w) : 0 \leq i \leq K, 0 \leq t_i \leq n_i \}.
\]

Consequently, the above lemma gives the following lemma.

Lemma 1.11. For each function \( F(z, w) \in \ker T^*_\phi(z) \cap \ker T^*_\phi(w) \), there is a function \( E(z, w) \in L_0 \) such that
\[
F(z, 0) = E(z, 0).
\]

Theorem 18 in [7] only gives the existence of the family of functions \( \{ d_\nu^{(k)} \} \subset \mathcal{L}_\phi \ominus L_0 \). It will be useful to know how those functions are constructed from \( e \).

Theorem 1.14 will give a recursive formula of \( \{ d_\nu^{(k)} \} \). First we need the following simple but useful lemma.

For two functions \( x, y \) in \( H^2(\mathbb{T}^2) \), the symbol \( x \otimes y \) is the operator on \( H^2(\mathbb{T}^2) \) defined, for \( g \in H^2(\mathbb{T}^2) \), by
\[
(x \otimes y)g = [\langle g, y \rangle_{H^2(\mathbb{T}^2)}] x.
\]

Lemma 1.12. On the Hardy space \( H^2(\mathbb{T}^2) \), the identity operator equals
\[
I = T_zT_z^* + \sum_{l \geq 0} w^l \otimes w^l = T_wT_w^* + \sum_{l \geq 0} z^l \otimes z^l.
\]

Lemma 1.13. Suppose that \( \phi(z) = z\phi_0(z) \) for some Blaschke product \( \phi_0(z) \) with finite order. If \( f \) is a function in \( H^2(\mathbb{T}^2) \), then for each \( l \geq 1 \),
\[
T_{z-w}^*(p_l(\phi(z), \phi(w))f) = p_l(\phi(z), \phi(w))T_{z-w}^*f + \phi_0(z)p_{l-1}(\phi(z), \phi(w))f(0, w) - \phi_0(w)p_{l-1}(\phi(z), \phi(w))f(z, 0).
\]

Proof. Let \( f \in H^2(\mathbb{T}^2) \). By Lemma 1.12, we have
\[
T^*_z(p_l(\phi(z), \phi(w))f) = T^*_z(p_l(\phi(z), \phi(w))\left(T_zT_z^* + \sum_{i \geq 0} w^i \otimes w^i\right)f)
\]
\[
= T^*_z[p_l(\phi(z), \phi(w))(T_zT_z^*f)] + T^*_z[p_l(\phi(z), \phi(w))\left(\sum_{i \geq 0} w^i \otimes w^i\right)f]
\]
\[
= p_1(\phi(z), \phi(w))(T_z^* f) + T_z^* \left[ p_1(\phi(z), \phi(w)) \left( \sum_{i \geq 0} w^i \otimes w^i \right) f \right].
\]

Noting
\[
p_1(\phi(z), \phi(w)) = \sum_{k=0}^l \phi(z)^k \phi(w)^{l-k} = \phi(w)^l + \phi(z) \sum_{k=1}^l \phi(z)^{k-1} \phi(w)^{l-k}
\]
\[
= \phi(w)^l + z\phi_0(z) \sum_{k=1}^l \phi(z)^{k-1} \phi(w)^{l-k},
\]
and
\[
\left( \sum_{i \geq 0} w^i \otimes w^i \right) f = f(0, w),
\]
we obtain
\[
T_z^* \left[ p_1(\phi(z), \phi(w)) \left( \sum_{i \geq 0} w^i \otimes w^i \right) f \right] = T_z^* [p_1(\phi(z), \phi(w)) f(0, w)]
\]
\[
= T_z^* [\phi(w)^l f(0, w)] + T_z^* \left[ z\phi_0(z) \sum_{k=1}^l \phi(z)^{k-1} \phi(w)^{l-k} f(0, w) \right]
\]
\[
= \phi_0(z) \left[ \sum_{k=1}^l \phi(z)^{k-1} \phi(w)^{l-k} \right] f(0, w) = \phi_0(z) p_{l-1}(\phi(z), \phi(w)) f(0, w).
\]

This gives
\[
(1.2) \quad T_z^* p_1(\phi(z), \phi(w)) f = p_1(\phi(z), \phi(w))(T_z^* f) + \phi_0(z) p_{l-1}(\phi(z), \phi(w)) f(0, w).
\]

Similarly, we also have
\[
(1.3) \quad T_w^* p_1(\phi(z), \phi(w)) f = p_1(\phi(z), \phi(w))(T_w^* f) + \phi_0(w) p_{l-1}(\phi(z), \phi(w)) f(z, 0).
\]
Combining (1.2) and (1.3) yields as desired
\[
T_{z-w}^* p_1(\phi(z), \phi(w)) f = p_1(\phi(z), \phi(w)) T_{z-w}^* f + \phi_0(z) p_{l-1}(\phi(z), \phi(w)) f(0, w) - \phi_0(w) p_{l-1}(\phi(z), \phi(w)) f(z, 0).
\]

The following theorem gives a recursive formula for those functions \(\{d^k_e\}\), which will be used in the construction of \(d_e\).

**THEOREM 1.14.** Suppose that \(e\) is in \(L_0\) and \(\{d^k_e\}\) are a family of functions in \(H^2(\mathbb{T}^2)\). Then for a given integer \(n \geq 1\),
\[
p_1(\phi(z), \phi(w)) e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w)) d_{l-k}^e \in H,
\]
for each \(1 \leq l \leq n\), if and only if the following recursive formula holds
\[
\phi_0(z) e(0, w) - \phi_0(w) e(z, 0) + T_{z-w}^* d_1^e(z, w) = 0;
\]
and, for $1 \leq k \leq n - 1$,
\[\phi_0(z)d^k_e(0, w) - \phi_0(w)d^k_e(z, 0) + T^*_{z-w}(d^{k+1}_e)(z, w) = 0.\]

**Proof.** For a given $e \in L_0$ and a family of functions $\{d^k_e\} \subset H^2(\mathbb{T}^2)$, for each integer $l \geq 1$, let
\[E_l = p_1(\phi(z), \phi(w))e + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-k}_e.\]

$E_l$ is in $\mathcal{H}$ for each $1 \leq l \leq n$, if and only if $T^*_{z-w}E_l = 0$ for each $1 \leq l \leq n$. We need only show that for each $1 \leq l \leq n$, $T^*_{z-w}E_l = 0$ is equivalent to the recursive formula in the theorem.

By Lemma 1.13, we have
\[T^*_{z-w}E_l = T^*_{z-w}[p_1(\phi(z), \phi(w))e] + \sum_{k=0}^{l-1} T^*_{z-w}[p_k(\phi(z), \phi(w))d^{l-k}_e] = p_1(\phi(z), \phi(w))T^*_{z-w}e + \phi_0(z)p_{l-1}(\phi(z), \phi(w))e(0, w) - \phi_0(w)p_{l-1}(\phi(z), \phi(w))e(0, w) + \sum_{k=1}^{l-1} [p_k(\phi(z), \phi(w))T^*_{z-w}d^{l-k}_e + \phi_0(z)p_{k-1}(\phi(z), \phi(w))d^{l-k}_e(z, 0)] + \sum_{k=0}^{l-2} [p_k(\phi(z), \phi(w))(T^*_{z-w}d^{l-k}_e + \phi_0(z)d^{l-k-1}_e(0, w) - \phi_0(w)d^{l-1-k}_e(z, 0))]
\]
since $e$ is in $L_0$. Thus $T^*_{z-w}E_l = 0$ for each $1 \leq l \leq n$ if and only if
\[\phi_0(z)e(0, w) - \phi_0(w)e(z, 0) + T^*_{z-w}d^1_e = 0,
\]
and
\[T^*_{z-w}d^{l-k}_e + \phi_0(z)d^{l-k-1}_e(0, w) - \phi_0(w)d^{l-1-k}_e(z, 0)) = 0,
\]
for $1 \leq k < l \leq n$. This completes the proof. 

**Lemma 1.15.** If for a function $f \in \mathcal{H}$, $p_1(\phi(z), \phi(w))f \in \mathcal{H}$, for each $l \geq 0$, then $f(z, 0) = \lambda \phi_0(z)$, for constant $\lambda$.

**Proof.** Suppose that $p_1(\phi(z), \phi(w))f \in \mathcal{H}$, for each $l \geq 0$. Let $d^k_f = 0$. Then
\[p_1(\phi(z), \phi(w))f + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-k}_f \in \mathcal{H},
\]
for each $l \geq 1$. By Theorem 1.14, we have
\[\phi_0(z)f(0, w) - \phi_0(w)f(z, 0) = 0.
\]
This gives
\[
\frac{f(z,0)}{\phi_0(z)} = \frac{f(0,w)}{\phi_0(w)}
\]
holds for all \((z,w) \in \mathbb{D} \times \mathbb{D}\) except for a finite vertical or horizontal lines. Thus the equality holds for an open subset of \(D^2\), and so there is a constant \(\lambda\) such that \(f(z,0) = \lambda \phi_0(z)\) on the unit disk. This completes the proof.  

The following theorem is proved in [7] and is used in the proof of Theorem 1.17.

**Theorem 1.16.** If for a function \(f \in \mathcal{H}\), \(p_1(\phi(z),\phi(w))f \in \mathcal{H}\), for each \(l \geq 0\), then there exists a constant \(\lambda\) such that \(f = \lambda e_0\).

Next for a given \(e \in L_0\), we will show that there is a unique function \(d_e \in \mathcal{L}_{\phi} \oplus e_0\) such that, for each \(l \geq 1\),
\[
p_1(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \in \mathcal{H}.
\]

**Theorem 1.17.** For a given \(e \in L_0\), there is a unique function \(d_e \in \mathcal{L}_{\phi} \oplus e_0\) such that
\[
p_1(\phi(z),\phi(w))e + p_{l-1}(\phi(z),\phi(w))d_e \in \mathcal{H}
\]
for each \(l \geq 1\). If \(e\) is linearly independent of \(e_0\), then \(d_e \neq 0\). Moreover, the mapping
\[
e \to d_e
\]
is a linear operator from \(L_0\) into \(\mathcal{L}_{\phi} \oplus e_0\).

**Proof.** First we show the existence of \(d_e\). For the given \(e\), by Theorem 18 in [7], there is a function \(d_1^e \in \mathcal{L}_{\phi}\) such that
\[
p_1(\phi(z),\phi(w))e + d_1^e \in \mathcal{H}.
\]
By Theorem 1.14 we have
\[
(1.4) \quad \phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}d_1^e(z,w) = 0.
\]
Since \(e(z,w)\) is in \(\mathcal{H}\), by Lemma 1.6, \(d_1^e(z,w)\) is symmetric with respect to \(z\) and \(w\). In addition, \(p_1(\phi(z),\phi(w))\) is also symmetric with respect to \(z\) and \(w\). This gives
\[
d_1^e(z,w) = d_1^e(w,z).
\]
Hence \(d_1^e(z,0) = d_1^e(0,z)\). By Lemma 1.11, choose a function \(\tilde{e}(z,w) \in L_0\) such that \(d_1^e(z,0) = \tilde{e}(0,z)\). Hence \(d_1^e(0,z) = \tilde{e}(0,z)\), because \(\tilde{e}(z,w)\) is also symmetric with respect to \(z\) and \(w\). Let \(d_e = d_1^e - \tilde{e}\). Clearly,
\[
p_1(\phi(z),\phi(w))e + d_e \in \mathcal{H}, \quad \text{and} \quad d_e(z,0) = d_e(0,z) = d_1^e(z,0) - \tilde{e}(z,0) = 0.
\]
Letting \(d_1^e = d_e\) and \(d_k^e = 0\), for \(k > 1\), by (1.4), we have the following equations:
\[
\phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}d_1^e(z,w)
\]
\[
= \phi_0(z)e(0,w) - \phi_0(w)e(z,0) + T^*_{z-w}[d_1^e(z,w) - \tilde{e}(z,w)] = 0,
\]
\[
\phi_0(z)d_k^e(0,w) - \phi_0(w)d_k^e(z,0) + T^*_{z-w}(d_k^e(z,w)) = 0 - 0 - 0 = 0,
\]

for $1 \leq k \leq l-1$. The last equality in the first equation follows from $T_{z-w^*}(z, w) = 0$. By Theorem 1.14, we conclude that, as desired,

$$
p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))d_e \in \mathcal{H}.
$$

Next we show that if there is another function $b_e \in \mathcal{L} \phi$ such that

$$
p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))b_e \in \mathcal{H},
$$

for each $l \geq 1$, then $d_e - b_e = \mu e_0$ for some constant $\mu$.

Since

$$
p_{l-1}(\phi(z), \phi(w))[d_e - b_e] = p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))d_e - (p_l(\phi(z), \phi(w))e + p_{l-1}(\phi(z), \phi(w))b_e) \in \mathcal{H},
$$

letting $f = d_e - b_e$, we have that $f \in \mathcal{H}$ and $p_l(\phi(z), \phi(w))f \in \mathcal{H}$. By Theorem 1.16, we obtain that $f = \lambda e_0$ to conclude

$$
d_e = b_e + \lambda e_0.
$$

If $d_e = 0$, i.e.,

$$
p_l(\phi(z), \phi(w))e \in \mathcal{H},
$$

then Theorem 1.16 again implies that $e = \lambda e_0$. This gives that if $e$ is linearly independent of $e_0$, then $d_e \neq 0$.

As showed above, we know that the mapping $e \to d_e$ is well-defined from $L_0$ into $\mathcal{L} \phi \oplus e_0$. To finish the proof we need to show that the mapping is linear. To do so, let $e_1$ and $e_2$ be in $L_0$. For given constants $c_1$ and $c_2$, we have

$$
p_l(\phi(z), \phi(w))e_1 + p_{l-1}(\phi(z), \phi(w))d_{e_1} \in \mathcal{H},
$$

$$
p_l(\phi(z), \phi(w))e_2 + p_{l-1}(\phi(z), \phi(w))d_{e_2} \in \mathcal{H},
$$

$$
p_l(\phi(z), \phi(w))[c_1 e_1 + c_2 e_2] + p_{l-1}(\phi(z), \phi(w))d_{c_1 e_1 + c_2 e_2} \in \mathcal{H}.
$$

Thus $p_{l-1}(\phi(z), \phi(w))[c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2}] \in \mathcal{H}$, for each $l \geq 1$. By Theorem 1.16,

$$
c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2} = c_3 e_0,
$$

for some constant $c_3$. But $d_{e_1}$, $d_{e_2}$, and $d_{c_1 e_1 + c_2 e_2}$ are orthogonal to $e_0$. We conclude

$$
c_1 d_{e_1} + c_2 d_{e_2} - d_{c_1 e_1 + c_2 e_2} = 0. \blacksquare
$$

2. Weighted Shifts

In this section we will characterize multiplication operators on the Bergman space which are unitarily equivalent to a weighted shift of finite multiplicity to prove our main result.

A weighted shift $T$ of finite multiplicity $n$ on Hilbert space $H$ is an operator that maps each vector in some orthonormal basis $\{e_k\}_{k=0}^\infty$ into a scalar multiple of the next $n$th vector

$$
Te_k = w_k e_{k+n},
$$

where $w_k$ are weights defined on $\mathbb{N}$.
for all $k$. The sequence $\{w_k\}$ is called the weight of the weighted shift $T$. In fact, $T$ is unitarily equivalent to the multiplication operator by $z^n$ on some Hilbert space of analytic functions on the unit disk. [2] and [3] contain many results on the shift operators, which will be used in this paper.

Indeed, a weighted shift of finite multiplicity is unitarily equivalent to a direct sum of finite weighted shifts. The following theorem tells us that if a multiplication operator on the Bergman space is unitarily equivalent to a weighted shift of finite multiplicity, then the first construction in [7] will become much simpler.

**Theorem 2.1.** Suppose that $\phi$ is a Blaschke product with order $N$. If there are $N$ mutually orthogonal reducing subspaces $\{M_i\}$ of $\phi(B)$ such that $\phi(B)|_{M_i}$ is unitarily equivalent to a weighted shift, then for each $e_i \in M_i \cap L_0$ and each $l > 1$,

$$d^{l-1}_{e_i} = 0.$$  

**Proof.** By Theorem 1.2 we may assume that $\phi(B)|_{M_1}$ is unitarily equivalent to the Bergman shift. Let $e_i$ be a nonzero vector in $M_i \cap L_0$. By Theorem 19 in [7], there are functions $d^{l-1}_{e_i} \in L_\phi \otimes L_0$ such that

$$p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-1-k}_{e_i} \in M_i.$$  

**Theorem 1.2** implies that $d^{l-1}_{e_i} = 0$ for $l \geq 1$ and $d^{l-1}_{e_i} \neq 0$, for $i > 1$. Let

$$E_{il} = p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-1-k}_{e_i}.$$  

Then $E_{il}$ is in $M_i$ and

$$\phi(B)^*E_{il} = T_{\phi(z)}^*E_{il} = P\left[\bar{\phi}(z)\left(p_l(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))d^{l-1-k}_{e_i}\right)\right]$$

$$= p_{l-1}(\phi(z), \phi(w))e_i + \sum_{k=0}^{l-2} p_k(\phi(z), \phi(w))d^{l-1-k}_{e_i} = E_{i(l-1)}.$$  

The last equality follows from $P(\bar{\phi}(z)e_i) = 0$ and $P(\bar{\phi}(z)d^{l-1}_{e_i}) = 0$. Thus $\{E_{il}\}_l$ are orthogonal to $\{E_{jl}\}_l$ for $i \neq j$ and so $\{d^{l-1}_{e_i}\}_l$ are orthogonal to $\{d^{l-1}_{e_i}\}_l$. Since $\dim[L_\phi \otimes L_0]$ equals $N - 1$ and $d^{l-1}_{e_i}$ does not equal zero for $i > 1$, $\{d^{l-1}_{e_i}\}$ form an orthogonal basis of $L_\phi \otimes L_0$. This gives that there are constants $\beta_{il}$ such that

$$d^{l-1}_{e_i} = \beta_{il}d^{l-1}_{e_i}.$$  

Because $\phi(B)|_{M_i}$ is a weighted shift, there is an orthonormal basis $\{F_l\}$ of $M_i$ such that

$$\phi(B)F_l = a_l F_{l+1}$$

where $\{a_l\}$ are weights of $\phi(B)$ on $M_i$. Thus $F_0$ is in the kernel of $[\phi(B)|_{M_i}]^*$, and so $F_0 = \lambda_0 e_i$ for some constant $\lambda_0$. Since $\phi(B)^*F_1 = a_0 F_0$, we have $\phi(B)^*[F_1 -
By induction, we obtain that there are constants \( \lambda \) such that
\[ F_1 = \lambda_1 E_{i1}. \]

By Lemma 1.9, there is a vector \( \alpha_0 \) such that
\[ \langle \alpha, \phi(z), \phi(w) \rangle w = 0 \]
for \( w = 0 \). Thus \( F_1 = \lambda_1 E_{i1} + \mu_1 e_i \). But both \( F_1 \) and \( E_{i1} \) are orthogonal to \( e_i \). So \( \mu_1 = 0 \). Hence there is a constant \( \lambda_1 \) such that
\[ F_1 = \lambda_1 E_{i1}. \]

By induction, we obtain that there are constants \( \lambda_l \) such that
\[ F_l = \lambda_l E_{i_l}. \]

This implies that \( \{E_{i_l}\} \) form an orthogonal set. Note that
\[ E_{i_l} = p_1(\phi(z), \phi(w))e_i + \left[ \sum_{k=0}^{l-1} p_k(\phi(z), \phi(w))\beta_i(l-k) \right] d_{e_i}^1. \]

We conclude that \( \beta_{i_l} = 0 \) for \( l > 1 \). This gives
\[ E_{i_l} = p_1(\phi(z), \phi(w))e_i + p_{l-1}(\phi(z), \phi(w))d_{e_i}^1 \in M_i \]
and \( d_{e_i}^1 = 0 \) for \( l > 1 \). This completes the proof. \( \blacksquare \)

**Theorem 2.2.** Suppose that \( \phi \) is a finite Blaschke product and \( \phi(0) = 0 \). If \( \phi \) has a nonzero root \( \alpha \), then there is a function \( e \in L_0 \) such that \( d_e^0 \) is not orthogonal to \( L_0 \).

**Proof.** Recall that \( L_0 \) equals \( \ker T_{\phi(0)}^* \cap \ker T_{\phi(w)}^* \) \( \cap \mathcal{H} \). Assuming that for each \( e \in L_0 \), \( d_e^0 \) is orthogonal to \( L_0 \), we will derive a contradiction.

Observe that \( \{\{e_{\alpha_k}^k\}_{k=0}^{n_k}\}_{k=0}^{K} \) form a basis for \( L_0 \). So for each \( e \in L_0 \) there is a vector
\[ (u_0^0, \ldots, u_0^{n_0}, \ldots, u_k^0, \ldots, u_{nk}^0, \ldots, u_{nk}^n) \in C^N \]
such that
\[ e(z, w) = \sum_{i=0}^{K} \sum_{t=0}^{n_i} u_i^t e_{\alpha_i}^t(z, w). \]

Noting that \( \dim L_0 = N \), we see that \( e \rightarrow (u_0^0, \ldots, u_0^{n_0}, \ldots, u_k^0, \ldots, u_{nk}^n) \) is a linear invertible mapping from \( L_0 \) onto \( C^N \).

Let \( \alpha_j \) be a nonzero root of \( \phi \) with multiplicity \( n_j + 1 \). Then
\[ \phi^{(t)}(\alpha_j) = \langle \phi, k_{\alpha_j}^t \rangle = 0, \quad \text{for } 0 \leq t \leq n_j \quad \text{and} \quad \phi^{(n_j+1)}(\alpha_j) = \langle \phi, k_{\alpha_j}^{n_j+1} \rangle \neq 0. \]

Because \( d_e^0 \) is orthogonal to \( L_0 \) and \( \{e_{\alpha_j}^t\}_{t=0}^{l} \) is in \( L_0 \), we have
\[ 0 = \langle d_e^0, e_{\alpha_j}^t \rangle = \langle [w\phi_0(w)e(z, w) - we(0, w)e_0(z, w)], e_{\alpha_j}^t \rangle \]
\[ = \langle w\phi_0(w)e(z, w), e_{\alpha_j}^t \rangle - \langle we(0, w)e_0(z, w), e_{\alpha_j}^t \rangle. \]

By Lemma 1.9,
\[ \langle w\phi_0(w)e(z, w), e_{\alpha_j}^t \rangle = \left\{ [\partial_z + \partial_w]^t \phi(w)e(z, w) \right\}_{z=w=\alpha_j} \]
\[ = \sum_{s=0}^{t} \frac{t!}{s!(t-s)!} \phi^{(s)}(\alpha_j) \left\{ [\partial_z + \partial_w]^{t-s} e(z, w) \right\}_{z=w=\alpha_j} = 0. \]
Thus
\[
\langle we(0, w)e_0(z, w), e_{\alpha_j}^t \rangle = 0
\]
for \(0 \leq t \leq n_j\). By Lemma 1.9 again, we have
\[
0 = \langle we(0, w)e_0(z, w), e_{\alpha_j}^t \rangle = \{[\partial_z + \partial_w]^t we(0, w)e_0(z, w)\}|_{z=w=\alpha_j}
\]
(2.1)
\[
= \sum_{s=0}^{t} \frac{t!}{s!(t-s)!}(we(0, w))^{(s)}(\alpha_i) \{[\partial_z + \partial_w]^{t-s}e_0(z, w)\}|_{z=w=\alpha_j}
\]
for \(0 \leq t \leq n_j\). When \(t = 0\), the above equation gives \(\alpha_j e(0, \alpha_j)e_0(\alpha_j, \alpha_j) = 0\).

Noting that \(\alpha_j e(0, \alpha_j) = 0\) is equivalent to \(\sum_{i=0}^{K} \sum_{t=0}^{n_i} u_i^t e_{\alpha_j}^t (0, \alpha_j) = 0\), we see that there is a function \(e\) in \(L_0\) such that \(\alpha_j e(0, \alpha_j) \neq 0\). Hence \(e_0(\alpha_j, \alpha_j) = 0\). Letting \(t = 1\), (2.1) gives
\[
\alpha_j e(0, \alpha_j) \{[\partial_z + \partial_w]e_0(z, w)\}|_{z=w=\alpha_j} = (we(0, w))^{(1)}|_{w=\alpha_j}e_0(\alpha_j, \alpha_j) = 0,
\]
Thus \(\{[\partial_z + \partial_w]e_0(z, w)\}|_{z=w=\alpha_j} = 0\). By induction we obtain
\[
\{[\partial_z + \partial_w]^t e_0(z, w)\}|_{z=w=\alpha_j} = 0,
\]
for \(0 \leq t \leq n_j\). In particular, \(0 = \{[\partial_z + \partial_w]^n e_0(z, w)\}|_{z=w=\alpha_j}\). A simple calculation gives
\[
\{[\partial_z + \partial_w]^n e_0(z, w)\}|_{z=w=\alpha_j} = \langle e_0, e_{\alpha_j}^n \rangle = \langle e_{\alpha_j}^n, e_0(z, w) \rangle, 1 = \langle P_\mathcal{H}[e_{\alpha_j}^n(z, w)e_0(z, w)], 1 \rangle.
\]
Because \(e_{\alpha_j}^n\) is in \(H^\infty(\mathbb{T}^2)\) and \(e_0(z, w)\) is in \(\mathcal{H}\), we have
\[
P_\mathcal{H}[e_{\alpha_j}^n(z, w)e_0(z, w)] = P_\mathcal{H}[e_{\alpha_j}^n(z, z)e_0(z, w)]
\]
Thus
\[
\{[\partial_z + \partial_w]^n e_0(z, w)\}|_{z=w=\alpha_j} = \langle P_\mathcal{H}[e_{\alpha_j}^n(z, z)e_0(z, w)], 1 \rangle = \langle e_{\alpha_j}^n(z, z)e_0(z, w), 1 \rangle = \langle e_0(z, w), e_{\alpha_j}^n(z, z) \rangle
\]
\[
= \langle e_0(z, 0), e_{\alpha_j}^n(z, z) \rangle = \langle \phi_0(z), \frac{n_j!z^n_j}{(1 - \alpha_j z)^{n_j+2}} \rangle.
\]
On the other hand, we also have
\[
0 = \phi_0^{(n_j)}(\alpha_j) = \langle \phi_0, k_{\alpha_j}^{n_j} \rangle = \langle \phi_0, \frac{n_j!z^n_j}{(1 - \alpha_j z)^{n_j+1}} \rangle.
\]
Combining the above equalities gives
\[
0 = \langle \phi_0(z), \left[ \frac{z^n_j}{(1 - \alpha_j z)^{n_j+2}} - \frac{z^n_j}{(1 - \alpha_j z)^{n_j+1}} \right] \rangle = \langle \phi_0(z), \frac{\alpha_j z^{n_j+1}}{(1 - \alpha_j z)^{n_j+2}} \rangle.
\]
Hence
\[ \phi_0^{(n_j+1)}(\alpha_j) = \langle \phi_0(z), k_{\alpha_j}^{n_j+1}(z) \rangle = \frac{(n_j + 1)!}{\overline{\alpha_j}} \left( \phi_0(z), \frac{\overline{\alpha_j} z^{n_j+1}}{(1 - \alpha_j z)^{n_j+2}} \right) = 0. \]

This contradicts the fact that \( \alpha_j \) is a nonzero root of \( \phi_0 \) with multiplicity \( n_j + 1 \).

We are ready to prove our main result.

**Proof of Theorem 0.1.** We may assume that \( \| M_\phi \| = 1 \). Suppose that \( M_\phi \) is unitarily equivalent to the direct sum \( \bigoplus_{i=1}^{N} W_i \) where \( W_i \) is a weighted shift. Then
\[ \dim \ker M_\phi^* = \sum_i \dim \ker W_i^* \]
and the essential spectrum of \( M_\phi \) is
\[ \sigma_e(M_\phi) = \bigcup_{i=1}^{N} \sigma_e(W_i). \]

Noting that \( W_i \) is subnormal, we see that the essential spectrum of \( W_i \) is a circle with center at origin. So \( \bigcup_{i=1}^{N} \sigma_e(W_i) \) is a union of circles with the same center at origin. On the other hand, by Corollary 20 of [6], the essential spectrum of \( M_\phi \) is connected. Thus \( \bigcup_{i=1}^{N} \sigma_e(W_i) \) is the unit circle and \( |\phi(z)| = 1 \) on \( \mathbb{T} \). So \( \phi \) is an inner function.

We claim that \( \phi \) is a Blaschke product with \( N \) zeros in the unit disk. If \( \phi \) is not so, there is a singularity \( z_0 \in \mathbb{T} \) of \( \phi(z) \) (that is a point that \( \phi(z) \) does not extend analytically), by Theorem 6.6 in [1], the cluster set of \( \phi(z) \) is the closed unit disk. Note that a point \( \eta \) in the cluster set of \( \phi(z) \) at \( z_0 \) if and only if there are points \( z_n \in \mathbb{D} \) tending to \( z_0 \) such that \( \phi(z_n) \) converges to \( \eta \). This implies that the cluster set of \( \phi(z) \) at every point \( z_0 \) on the unit circle is contained in the essential spectrum of \( M_\phi \), which is a contradiction.

By Theorem 1.17, there are \( N \) linearly independent functions \( \{e_i\} \) of \( L_0 \) such that \( \{d_{e_i}\} \) are orthogonal to \( e_0 \) and
\[ p_l(\phi(z), \phi(w))e_i + p_{l-1}(\phi(z), \phi(w))d_{e_i} \in \mathcal{H}. \]

Also we have \( p_l(\phi(z), \phi(w)e_i + p_{l-1}(\phi(z), \phi(w))d_{e_i}^0 \in \mathcal{H}, \) for \( l \geq 0 \). Thus \( p_l(\phi(z), \phi(w))(d_{e_i} - d_{e_i}^0) \in \mathcal{H} \) for \( l \geq 0 \). Since \( d_{e_i} - d_{e_i}^0 \) is in \( L_0 \) and hence Theorem 1.16 gives that there are constants \( \lambda_i \) such that \( d_{e_i} = d_{e_i}^0 + \lambda_i e_0 \). Since \( e_0^n \) is in \( L_0 \) and \( d_{e_i} \) is orthogonal to \( L_0 \), we have
\[ 0 = \langle d_{e_i}, e_0^n \rangle = \langle d_{e_i}^0, e_0^n \rangle + \lambda_i \langle e_0, e_0^n \rangle. \]
On the other hand, Lemma 1.9 gives
\[ \langle e_0, e_0^{n_0} \rangle = \langle e_0(z, w), e_0^{n_0}(z, z) \rangle = \langle e_0(z, 0), e_0^{n_0}(z, z) \rangle = (n_0 + 1)! \langle \phi_0(z), z^{n_0} \rangle = (n_0 + 1)! \phi_0^{(n_0)}(0) \neq 0, \]
\[ \langle d_{e_i}^0, e_0^{n_0} \rangle = \langle w\phi_0(w)e_i(z, w) - we_i(0, w)e_0(z, w), e_0^{n_0}(z, w) \rangle = \langle \phi(w)e_i(z, w), e_0^{n_0}(z, w) \rangle - \langle w\phi_0(w)e_i(z, w), e_0^{n_0}(z, w) \rangle. \]

The Leibniz rule and Lemma 1.9 give
\[ \langle \phi(w)e_i(z, w), e_0^{n_0}(z, w) \rangle = \langle (\partial_z + \partial_w)^{n_0}(\phi(w)e_i(z, w)) \rangle \bigg|_{z=w=0} = \sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0-s)!} \phi^{(s)}(0)[(\partial_z + \partial_w)^{n_0-s}e_i](0, 0) = 0. \]

The last equality follows from the fact that 0 is a root of \( \phi \) with multiplicity \( n_0 + 1 \). Similarly, we have
\[ \langle we_i(0, w)e_0(z, w), e_0^{n_0}(z, w) \rangle = \langle (\partial_z + \partial_w)^{n_0}(we_i(0, w)e_0(z, w)) \rangle \bigg|_{z=w=0} = \sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0-s)!} (we_i(0, w))^{(s)}(0)[(\partial_z + \partial_w)^{n_0-s}e_0](0, 0). \]

Lemmas 1.3 and 1.9 give
\[ [(\partial_z + \partial_w)^{n_0-s}e_0](0, 0) = \langle e_0(z, w), e_0^{n_0-s}(z, z) \rangle = \langle e_0(z, 0), e_0^{n_0-s}(z, z) \rangle = \langle \phi_0(z), (n_0 - s + 1)!z^{n_0-s} \rangle = 0 \]
for \( 0 < s \leq n_0 \). The second equality follows from \( P_{\mathcal{H}}[e_0^{n_0-s}(z, w)e_0(z, w)] = P_{\mathcal{H}}[e_0^{n_0-s}(z, z)e_0(z, w)] \). Thus
\[ \sum_{s=0}^{n_0} \frac{n_0!}{s!(n_0-s)!} (we_i(0, w))^{(s)}(0)[(\partial_z + \partial_w)^{n_0-s}e_0](0, 0) = 0, \]
and so
\[ \langle we_i(0, w)e_0(z, w), e_0^{n_0}(z, w) \rangle = 0. \]

Hence we have that the constant \( \lambda_i = 0 \). Therefore \( d_{e_i}^0 \) is orthogonal to \( L_0 \) for each \( i \). Noting that \( \{e_i\} \) form a basis for \( L_0 \) we see that \( d_{e_i}^0 \) is orthogonal to \( L_0 \) for each \( e \in L_0 \). By Theorem 2.2, we conclude that \( \phi = \phi_0^N \), to complete the proof.

3. DECOMPOSITION OF \( \mathcal{H} \)

The proof of Theorem 0.1 in the previous section suggests a more general result stating that if \( \phi \) has more than two distinct roots and at least one root is repeated, then \( \mathcal{H} \) can not be decomposed as a direct sum of \( N \) reducing subspaces of \( M_\phi \). In this section we will prove the result.
THEOREM 3.1. Suppose that \( \phi \) is a Blaschke product of order \( N \). If 0 is a zero and a critical point of \( \phi \) and the zero set of \( \phi \) contains at least one nonzero point in the unit disk, then \( \mathcal{H} \) cannot be decomposed as a direct sum \( \bigoplus_{i=0}^{N-1} M_i \) of \( N \) mutually orthogonal nontrivial reducing subspaces \( \{M_i\}_{i=0}^{N-1} \) of \( \mathcal{B} \).

Proof. By the assumption, we may write

\[
\phi = z \phi_0 = z^{n_0+1} \phi_1,
\]

where

\[
\phi_0 = z_{n_0}^{\alpha_1} \cdots \alpha_K^{n_K+1} \quad \text{and} \quad \phi_1 = \alpha_1^{n_1+1} \cdots \alpha_K^{n_K+1},
\]

for some nonzero points \( \alpha_1, \ldots, \alpha_K \) in the unit disk and nonnegative integers \( n_0, \ldots, n_K \).

Recall that \( L_0 \) is equal to \( \ker T_{\phi_0(z)}^* \cap \ker T_{\phi_0(w)}^* \cap \mathcal{H} \). Then

\[
L_0 = \text{span}\{1, p_1, \ldots, p_{n_0}, e_{\alpha_1}, \ldots, e_{\alpha_K}, \ldots, e_{\alpha_K}^0, e_{\alpha_K}^1, \ldots, e_{\alpha_K}^n\}.
\]

Assume that \( \mathcal{B} \) has \( N \) mutually orthogonal nontrivial reducing subspaces \( \{M_i\}_{i=0}^{N-1} \) such that

\[
\mathcal{H} = \bigoplus_{i=0}^{N-1} M_i
\]

where \( M_0 \) is the distinguished reducing subspace \( M_0 \) in Theorem 1.2.

By Lemma 1.10, for each \( i \), there is an \( e_i \neq 0 \) such that \( e_i \in M_i \cap L_0 \), and

\[
L_0 = \text{span}\{e_0, e_1, \ldots, e_{N-1}\}.
\]

By Theorems 19 in [7], there are functions \( \{d_{e_i}^1\} \subset \mathcal{L}_\phi \cap L_0 \) such that

\[
p_1(\phi(z), \phi(w))e_i + d_{e_i}^1 \in M_i.
\]

Since \( M_i \) is orthogonal to \( M_j \) for distinct \( i \) and \( j \), we have

\[
\langle p_1(\phi(z), \phi(w))e_i + d_{e_i}^1, p_1(\phi(z), \phi(w))e_j + d_{e_j}^1 \rangle = 0.
\]

On the other hand, a simple calculation gives

\[
\langle p_1(\phi(z), \phi(w))e_i + d_{e_i}^1, p_1(\phi(z), \phi(w))e_j + d_{e_j}^1 \rangle = \langle p_1(\phi(z), \phi(w))e_i + d_{e_i}^1, p_1(\phi(z), \phi(w))e_j + d_{e_j}^1 \rangle + \langle p_1(\phi(z), \phi(w))e_i + d_{e_i}^1, d_{e_j}^1 \rangle + \langle d_{e_i}^1, p_1(\phi(z), \phi(w))e_j + d_{e_j}^1 \rangle + \langle d_{e_i}^1, d_{e_j}^1 \rangle = \langle d_{e_i}^1, d_{e_j}^1 \rangle.
\]

The second equality follows from the fact that \( d_{e_i} \) and \( d_{e_j} \) are in \( \mathcal{L}_\phi \cap L_0 \). The equality follows since \( e_i \) and \( e_j \) are in \( L_0 \). Thus,

\[
\langle d_{e_i}^1, d_{e_j}^1 \rangle = 0.
\]

By Theorems 19 in [7], each \( d_{e_i}^1 \neq 0 \) for \( i > 0 \) and

\[
\{d_{e_i}^1\}_{i=1}^{N-1} \subset \mathcal{L}_\phi \cap L_0.
\]
are linearly independent.

By Theorem 1.1, there are numbers $\beta_i, \lambda_i$ such that

$$d_{e_i}^1 = d_{e_i}^0 + \beta_i e_i + \lambda_i e_0 \quad i = 1, \ldots, N - 1. \quad (3.1)$$

We will show that $d_{e_i}^0$ and $e_0$ are in

$$\{1, p_1, \ldots, p_{n_0-1}, e_{a_1}^0, \ldots, e_{a_1}^{n_1-1}, \ldots, e_{a_K}^0, \ldots, e_{a_K}^{n_K-1}\} \perp.$$

To do this, observe that for $0 \leq k \leq n_0$,

$$-\langle d_{e_i}^0, p_k \rangle = \langle \phi(w) e_i - w e_i(0, w) e_0, p_k \rangle = \langle \phi(w) e_i(w, w), p_k(0, w) \rangle - \langle w e_i(0, w) e_0(w, w), p_k(0, w) \rangle = \langle \phi(w) e_i(w, w), w^k \rangle - \langle w e_i(0, w)(w \phi_0(w) + \phi_0(w)), w^k \rangle = \langle w^{n_0+1-k} \phi_1(w) e_i(w, w), 1 \rangle - \langle w^{n_0+1-k} [w \phi_1(w) + (n_0+1) \phi_1(w)] e_i(0, w), 1 \rangle = 0.$$

The second equality follows from Lemma 1.3 and the third equality follows from Lemma 1.7.

Since $e_{a_j}^t$ is in the kernel of $T_{\phi(w)}^*$ and $\phi^{(s)}(a_j) = 0$ for $0 \leq s \leq n_j$, we have that for $0 \leq t \leq n_j - 1$ and $j = 1, \ldots, K$,

$$\langle d_{e_i}^0, e_{a_j}^t \rangle = \langle w e_i(0, w) e_0(w, w) - \phi(w) e_i, e_{a_j}^t \rangle = \langle w e_i(0, w) e_0(w, w), e_{a_j}^t(0, w) \rangle = \langle w e_i(0, w) (w \phi_0(w) + \phi_0(w)), e_{a_j}^t(0, w) \rangle = \langle w e_i(0, w) \phi'(w) + \lambda_j e_{a_j}^t(0, w) \rangle = \langle w e_i(0, w) \phi'(w) \rangle^{(n_j)} |_{a_j} = \alpha_j e_i(0, a_j) \phi^{(n_j+1)}(a_j).$$

These give that

$$d_{e_i}^0 \perp \{1, p_1, \ldots, p_{n_0-1}, e_{a_1}^0, \ldots, e_{a_1}^{n_1-1}, \ldots, e_{a_K}^0, \ldots, e_{a_K}^{n_K-1}\}. \quad (3.2)$$

We also have that for $0 \leq k \leq n_0 - 1$

$$\langle e_0, p_k \rangle = \langle e_0(0, w), p_k(0, w) \rangle = \langle \phi'(w), w^k \rangle = 0,$$

$$\langle e_0, p_{n_0} \rangle = \frac{1}{n_0!} \phi^{(n_0+1)}(0) \neq 0.$$

A simple calculation shows that for $j = 1, \ldots, K, 0 \leq t \leq n_j - 1$

$$\langle e_0, e_{a_j}^t \rangle = [e_0(w, w)]^{(t)} |_{a_j} = \phi^{(t+1)}(a_j) = 0,$$

$$\langle e_0, e_{a_j}^{n_j} \rangle = \phi^{(n_j+1)}(a_j) \neq 0.$$

These give

$$e_0 \perp \{1, p_1, \ldots, p_{n_0-1}, e_{a_1}^0, \ldots, e_{a_1}^{n_1-1}, \ldots, e_{a_K}^0, \ldots, e_{a_K}^{n_K-1}\}. \quad (3.3)$$
We claim that there are at most $K$ nonzero $\beta_i$'s. If $\beta_{i_0}$ does not equal 0 for some $i_0$, (3.1) yields
\[
e_{i_0} = \frac{1}{\beta_{i_0}}[d^1_{i_0} - d^0_{i_0} - \lambda_{i_0}e_0].\]
Noting that $d^1_{i_0}$ is orthogonal to $L_0$, by (3.2) and (3.3) we have
\[
e_{i_0} \perp \{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^1, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}\}.\]
Thus
\[
(3.4) \quad e_{i_0} \perp \{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^0, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}, e_0\}.
\]
So there are at most $K$ nonzero $\beta_i$'s and hence our claim holds.

On the other hand if $\beta_i = 0$, then (3.1) gives
\[
d^1_{i_0} = d^0_{i_0} + \lambda_i e_0.
\]
Since $p_{n_0}$ is in $L_0$ and $d^1_{i_0} \perp L_0$, we have that $d^0_{i_0} \perp p_{n_0}$, and
\[
\langle e_0, p_{n_0} \rangle \neq 0,
\]
to obtain that $\lambda_i = 0$ and $d^0_{i_0} = d^1_{i_0}$ is orthogonal to $L_0$. By Theorem 2.2, there is at least one nonzero $\beta_i$.

Without loss of generality, assume that for some $m$, $\beta_{N-j} \neq 0$ for $1 \leq j \leq m$ and $\beta_j = 0$ for $1 \leq j \leq N - m - 1$, (3.4) gives
\[
e_{N-j} \perp \{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^0, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}, e_0\}
\]
for $1 \leq j \leq m$. Now we extend
\[
\{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^0, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}, e_0, e_{N-1}, \ldots, e_{N-m}\}
\]
to a basis of $L_0$
\[
\{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^0, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}, e_{N-m}, f_1, \ldots, f_{K-m}\}
\]
by adding some elements $f_1, \ldots, f_{K-m}$ in $L_0$. Let $\{g_j\}_{j=1}^{N-m-1}$ denote
\[
\{1, p_1, \ldots, p_{n_0-1}, e_{\alpha_1}^0, \ldots, e_{\alpha_1}^{n_1-1}, \ldots, e_{\alpha_K}^0, \ldots, e_{\alpha_K}^{n_K-1}, f_1, \ldots, f_{K-m}\}.
\]
Since for $1 \leq j \leq N - m - 1$, $e_j$ is in $L_0$ and
\[
e_j \perp \{e_0, e_{N-1}, \ldots, e_{N-m}\}
\]
we have that $e_j$ is in the subspace span $\{1, g_2, \ldots, g_{N-m-1}\}$ of $L_0$. This implies that there are numbers $\{c_{j,l}\}_{j,l=1}^{N-m-1}$ such that for $1 \leq j \leq N - m - 1$
\[
(3.5) \quad e_j = c_{j1} + c_{j2}g_2 + \cdots + c_{j,N-m-1}g_{N-m-1}.
\]
On the other hand, because $\beta_j = 0$ for $1 \leq j \leq N - m - 1$, we have that $d^0_{i_0} = d^1_{i_0}$ is orthogonal to $L_0$, and
\[
\langle d^0_{i_0}, e_{\alpha_1}^{n_1} \rangle = \alpha_1 e_j(0, \alpha_1) \phi^{(n_1+1)}(\alpha_1) = 0.
\]
This implies that $e_j(0, \alpha_1) = 0$. Hence (3.5) gives

$$e_j(0, \alpha_1) = c_1 + c_2g_2(0, \alpha_1) + \cdots + c_{N-m-1}g_{N-m-1}(0, \alpha_1) = 0$$

for $1 \leq j \leq N - m - 1$. Thus the determinant $\det[c_{jk}]$ of the coefficient matrix of the above system must be zero. So there is a nonzero vector $(x_1, \ldots, x_{N-m-1})$ such that

$$c_1x_1 + c_2x_2 + \cdots + c_{N-m-1}x_{N-m-1} = 0$$

for $1 \leq l \leq N - m - 1$. This implies

$$x_1e_1 + x_2e_2 + \cdots + x_{N-m-1}e_{N-m-1} = 0.$$  

We obtain a contradiction that $e_1, \ldots, e_{N-m-1}$ are linearly independent to complete the proof. □

Acknowledgements. We would like to thank the referee for his useful comments. The first author was supported partially by the National Natural Science Foundation 10471041 of China. The second author was partially supported by the National Science Foundation.

REFERENCES


SHUNHUA SUN, INSTITUTE OF MATHEMATICS, JIAxing UNIVERSITY, JIAxing, ZHEJIANG, 314001, P. R. CHINA  
E-mail address: shsun@mail.zjxu.edu.cn
DECHAO ZHENG, DEPARTMENT OF MATHEMATICS, VANDERBILT UNIVERSITY, NASHVILLE, TN 37240, USA  
E-mail address: dechao.zheng@vanderbilt.edu

CHANGYONG ZHONG, DEPARTMENT OF MATHEMATICS AND STATISTICS, GEORGIA STATE UNIVERSITY, ATLANTA, GA 30303, USA  
E-mail address: matcyz@langate.gsu.edu

Received January 10, 2006.