J. OPERATOR THEORY
60:1(2008), 3–28

NAGY–FOIAŞ TYPE FUNCTIONAL MODELS OF NONDISSIPATIVE OPERATORS IN PARABOLIC DOMAINS

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Communicated by Nikolai K. Nikolski

ABSTRACT. A functional model for nondissipative unbounded perturbations of an unbounded self-adjoint operator on a Hilbert space X is constructed. This model is analogous to the Nagy–Foiaş model of dissipative operators, but it is linearly similar and not unitarily equivalent to the operator. It is attached to a domain of parabolic type, instead of a half-plane. The transformation map from X to the model space and the analogue of the characteristic function are given explicitly.

All usual consequences of the Nagy–Foiaş construction (the H^∞ calculus, the commutant lifting, etc.) hold true in our context.

KEYWORDS: *Nagy-Foiaş, functional model, Carleman parabola, nonselfadjoint perturbations.*

MSC (2000): 47A45, 47A48, 47A60, 47B44, 35P05.

INTRODUCTION

The paper is devoted to the construction of a functional model of non-dissipative linear operators of the form

$$A = A_0 + \mathrm{i}\psi(A_0)F\psi(A_0), \tag{0.1}$$

where A_0 is a self-adjoint unbounded linear operator on a Hilbert space X, F is bounded on X (not necessarily self-adjoint) and $1 \leqslant \psi(x) \leqslant K(\sqrt{|x|}+1)$ for $x \in \mathcal{D}(\psi)$. We assume that either ψ is defined on $\mathbb{R}$ and is *even* or it is defined on $[0, +\infty)$. The spectrum of A is contained in an unbounded parabolic domain, which is symmetric with respect to the real axis. A precise definition of the unbounded operator A and precise conditions on ψ will be given later. We remark that this definition is a particular case of that of [20].

All Hilbert spaces in this paper are assumed to be complex and separable. If X_1, X_2 are Banach spaces, $\mathcal{B}(X_1, X_2)$ will denote the set of all bounded operators from X_1 to X_2.

It is widely recognized that for understanding the spectral structure of an operator, the method of functional models is one of the most useful tools. The original Nagy–Foiaş functional model exists for contractions and dissipative operators, and is attached, correspondingly, to the unit disc or to the upper halfplane.

This work can be considered as a continuation of author's paper [33], where a linearly similar variant of the Sz.-Nagy–Foiaş model was suggested and studied. A general scheme for constructing such kind of models was presented and several concrete examples were given. This model has many points in common with the original Sz.-Nagy–Foiaş model, but also has important differences. In particular, depending on the operator, it is constructed in a rather general domain in the complex plane (bounded or unbounded) and not only in a disc or a halfplane. We will comment on other differences later on. A related functional model was constructed by A. Tikhonov in [29]; see also his subsequent works [30], [31]. Tikhonov's model, in fact, is closer to the model by Naboko [21], which, in particular, made it possible to develop a stationary scattering theory in the context of non-dissipative operators. We remark that, in general, the function theory that appears in the Nagy–Foiaş model is studied much better than analytic questions that arise from the model by Naboko.

Operators of the form (0.1) frequently appear in applications. Namely, suppose that A_0 is a selfadjoint elliptic operator with regular boundary conditions in $L^2(G)$, where G is a bounded domain in $\mathbb{R}^n$ with smooth boundary. Take $\psi(x) = 1 + |x|^\alpha$, where $0 < \alpha \leqslant 1/2$. Then $L \overset{\text{def}}{=} A - A_0$ has the desired form $L = \mathrm{i}\psi(A_0)F\psi(A_0)$, with a bounded F, if and only if $(I + |A_0|)^{-\alpha}L$ is bounded from $\mathcal{D}((I + |A_0|)^\alpha)$ to $L^2(G)$. Note that $\mathcal{D}((I + |A_0|)^\alpha)$ is a kind of Sobolev class in G. Typically, L can be a differential operator of order at most $4\alpha m$, where $2m$ is the order of A_0. The same is true for elliptic operators on closed manifolds. We refer to [1], [32] and others for details.

The completeness of eigenvectors of operators of a related class was established by Keldysh, see [13].

As is known, the spectrum of A lies in a suitable parabolic domain, see [19] and others; the boundary of this domain is called sometimes the "Carleman parabola". In this work, we apply the general scheme of [33] and construct a Sz.-Nagy–Foiaş type functional model of operator A in a parabolic domain of this type.

In fact, two closely related models were considered in [33], and we will write down both these models of A explicitly. They were called in [33] *the quotient model* and *the resolvent model*. It turns out that these constructions have some points in common with the control theory, in particular, with the theory of L^2 well-posed systems, which was developed in works by Salamon, Curtain, G. Weiss, Staffans and others. In particular, our models are not unique, and their choice depends on the inclusion of our operator in a triple (A, B, C), which is an abstract analogue of linear control system. Here we adopt the systems theory terminology and slightly change the terminology of [33]. We will call here the quotient model *the control model* and the resolvent model *the observation model*. In Section 1 and Section 5, these terms will be explained.

Let us describe briefly the control model (whose connection with the original setting by Nagy and Foiaş is more transparent). Recall first the definition of *the Smirnov class* $E^2(\Omega_{\text{int}})$. It consists of all functions f analytic in Ω_{int} such that $\sup\limits_n \int\limits_{\partial\Omega_n} |f|^2\,|\mathrm{d}z| < \infty$ for a sequence of domains $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ with rectifiable boundary and with $\bigcup\limits_n \Omega_n = \Omega_{\text{int}}$. We refer to [11] for the properties of the Smirnov classes $E^p(\Omega_{\text{int}})$ and their relationship with Hardy classes $H^p(\Omega_{\text{int}})$. The functions in $E^2(\Omega_{\text{int}})$ have nontangential boundary values a.e. on Γ. Equipped with the norm

$$\|f\|^2_{E^2(\Omega_{\text{int}})} \stackrel{\text{def}}{=} \frac{1}{2\pi}\int\limits_{\Gamma} |f(z)|^2\,|\mathrm{d}z|,$$

the class $E^2(\Omega_{\text{int}})$ is a Hilbert space.

For an auxiliary Hilbert space U, the elements of the Hilbert functional space $E^2(\Omega_{\text{int}}, U) \stackrel{\text{def}}{=} E^2(\Omega_{\text{int}}) \otimes U$ are U-valued functions analytic in Ω_{int}. These functions also have nontangential boundary values almost everywhere [28]. The norm in $E^2(\Omega_{\text{int}}, U)$ is given by

$$\|f\|^2_{E^2(\Omega_{\text{int}},U)} \stackrel{\text{def}}{=} \frac{1}{2\pi}\int\limits_{\Gamma} \|f(z)\|^2_U\,|\mathrm{d}z|.$$

The space $E^2(\Omega_{\text{int}}, U)$ can be interpreted either as a closed subspace of $L^2(|\mathrm{d}z|, U)$ or as a space of U-valued analytic functions in Ω_{int}. We will use both interpretations.

We need some more definitions. Let U, Y be two auxiliary Hilbert spaces and Ω_{int} a domain in $\mathbb{C}$ with piecewise smooth boundary Γ. Let δ be a function in $H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$. We call δ *admissible in* Ω_{int} if there is a constant $\varepsilon > 0$ such that $\|\delta(z)y\| \geqslant \varepsilon\|y\|$, $y \in Y$ for a.e. $z \in \Gamma$. This function is *two-sided admissible in* Ω_{int} if $\delta(z)^{-1}$ exists for a.e. $z \in \Gamma$ and $\|\delta^{-1}\| \leqslant C$ a.e. on Γ. Note that the functions in $H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$ have nontangential limits in the strong operator topology a.e. on Γ (see Section V.2 of [28]). If δ is admissible, then the space $\delta E^2(\Omega_{\text{int}}, Y)$ is a closed subspace of $E^2(\Omega_{\text{int}}, U)$.

For a holomorphic function f in some domain in $\mathbb{C}$, we set

$$M_z f(z) \stackrel{\text{def}}{=} z f(z),$$

so that M_z is the operator of multiplication by the independent variable. In general, for any function η, we denote by M_η the multiplication operator by η.

Put

$$\varphi(x) = \psi^2(x).$$

For $0 < \mu < \infty$, consider a parabolic-type domain

$$\Omega_\mu^{\mathrm{int}} = \{z = x + \mathrm{i}y \in \mathbb{C} : x \in \operatorname{int} \mathcal{D}(\varphi), |y| < \mu\varphi(x)\}. \tag{0.2}$$

Let $0 < R < \infty$. We set

$$\Omega_{\mu,R}^{\mathrm{int}} = \Omega_\mu^{\mathrm{int}} \cup B_R(0), \quad \Omega_{\mu,R}^{\mathrm{ext}} = \mathbb{C} \setminus \operatorname{clos} \Omega_{\mu,R}^{\mathrm{int}}, \tag{0.3}$$

where $B_R(\lambda)$ stands for the open disc in the complex plane of radius R centered at λ. It is known that for certain μ and R, $\Omega_{\mu,R}^{\mathrm{int}}$ contains the spectrum of A. The control model of A is given by Theorem 5.6. It asserts that *for suitable μ and R, the operator A is similar to the (unbounded) operator of multiplication by the independent variable on the quotient space*

$$E^2(\Omega_{\mathrm{int}}, X)/\delta \cdot E^2(\Omega_{\mathrm{int}}, X),$$

where $\Omega_{\mathrm{int}} = \Omega_{\mu,R}^{\mathrm{int}}$ and δ is a two-sided admissible H^∞ function on Ω_{int} with values in $\mathcal{B}(X)$. The function δ plays the role of the Nagy–Foiaş characteristic function. It will be given below by an explicit formula.

The model we get is, in fact, an analogue of a C_{00} type model in the domain Ω_{int}, which has no absolutely continuous part corresponding to the boundary curve.

We will derive a few corollaries from our results. In particular, one can assert that there exists an unbounded normal dilation of A (up to similarity), whose spectrum lies on $\partial\Omega_{\mathrm{int}}$. See Corollary 5.8 of Theorem 5.6.

Before formulating the control model, we prove Theorem 1.6, which gives the observation model of A. These two models of A are equivalent. All necessary definitions will be given below.

It is very interesting to compare our result with results by Putinar and Sandberg [25] and Badea, Crouzeix, Delyon [4] (see also [8], [7]). The results of [25] imply that for *any bounded operator A* on a Hilbert space and a convex domain Ω_{int} such that the numerical range of A is contained in its closure, one can find a dilation of A to an operator similar to a normal one, whose spectrum is contained in the boundary of Ω_{int}. The same holds true for the case of an unbounded A, if Ω_{int} is a sector, see [17]. If an analogous result were true for a general unbounded operator and a general convex domain Ω_{int}, it would give a better domain Ω_{int} than our results for the case when A_0 is bounded from below and $\|F\| = \|F\|_{\mathrm{ess}}$. On the other side, if the spectrum of A_0 is unbounded from above and from below, the numerical range of A can be the whole complex plane, and the approach of these papers does not apply. We also remark that in these works, no expression for a characteristic function was given, and that our methods are completely different.

We can also mention the work [3] and others by Arlinsky, where characteristic functions of sectorial operators have been investigated.

Our approach is based on *the duality* between (Nagy–Foiaş type) observation models of A and A^* with respect to a two-sided admissible function δ. This notion was introduced and studied in [33]. Once dual observation systems (A, C) and (A^*, B^*) are found, they give rise immediately to dual observation models of A and A^*. In order to prove this duality with respect to δ, one has to find auxiliary operators B and C such that δ and the transfer function of system (A, B, C) are related by a certain algebraic identity. In our case, we are able to give such B and C explicitly.

A serious disadvantage of our results resides in the fact that the values of the generalized characteristic function δ are infinite-dimensional operators and not matrices. In our setting it is inevitable (because the dimensions of eigen-spaces of A need not be uniformly bounded). It can be shown that in some important cases, the characteristic function δ has a scalar multiple. This will be discussed elsewhere. It seems that if more conditions on A_0 are imposed, then one can obtain a finite-dimensional model of the same type.

The plan of the exposition is as follows. In Section 1, the observation models of A and A^* and a duality result will be formulated. In Section 2, we prove the boundedness of the similarity transformation $\mathcal{O}_{A,C}$, which goes from X to the observation model space. In Section 3, we give more background on the duality and formulate the abstract result from [33] that will be used. In Section 4, we finish the proofs of our results on observation models. In Section 5, the control model of A will be introduced, and it will be explained how to pass to it from the observation model. In the end of this section, one can encounter some corollaries and a discussion. Finally, in Section 6 we prove some auxiliary geometric lemmas that have been used earlier.

1. AN OBSERVATION MODEL AND A DUALITY RESULT

1.1. ABSTRACT OBSERVATION SYSTEMS AND ALMOST DIAGONALIZING TRANSFORM. We will have to reproduce some notions and results from [33], which will be used here.

In what follows, we will consider linear systems $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ of possibly unbounded operators, where $\mathbf{A}$ acts on a space X (called a *state space*), $\mathbf{B}$ acts from an *input space* U to X (or to a larger Hilbert space) and $\mathbf{C}$ goes from X (or its dense linear subset) to an *output space* Y. Spaces X, U, Y are Hilbert. A pair $(\mathbf{A}, \mathbf{C})$ of operators as above will be called an *observation system* and a pair $(\mathbf{A}, \mathbf{B})$ will be called a *control system*. Despite the parallelism with the infinite dimensional linear systems theory, in this abstract setting we do not need the assumption that the set $\operatorname{Re}\sigma(A)$ is bounded from above.

We will use bold letters when discussing the abstract constructions of functional models and usual ones when referring to our concrete operator A, given by (0.1), and corresponding auxiliary operators B and C, which will be defined below.

DEFINITION 1.1. A pair of operators $(\mathbf{A}, \mathbf{C})$ (possibly, unbounded) will be called an *abstract observation system* if

(i) $\mathbf{A}$ is a closed densely defined operator on a Hilbert space X with nonempty field of regularity $\rho(\mathbf{A}) = \mathbb{C} \setminus \sigma(\mathbf{A})$;

(ii) $\mathbf{C} : \mathcal{D}(\mathbf{C}) \to Y$, where $\mathcal{D}(\mathbf{C}) = \mathcal{D}(\mathbf{A}) \subset X$ and $\mathbf{C}$ is bounded in the graph norm $\|x\|_G \overset{\text{def}}{=} (\|x\|^2 + \|\mathbf{A}x\|^2)^{1/2}$ in $\mathcal{D}(\mathbf{A})$.

With every abstract observation system $(\mathbf{A}, \mathbf{C})$ we associate the transform $\mathcal{O}_{\mathbf{A},\mathbf{C}}$, defined by

$$\mathcal{O}_{\mathbf{A},\mathbf{C}}x(z) = \mathbf{C}(zI - \mathbf{A})^{-1}x, \quad x \in X, z \in \rho(\mathbf{A}).$$

This map acts from X to the space of Y-valued functions analytic on $\rho(\mathbf{A})$.

Now let Ω_{int} and Ω_{ext} be a pair of open subsets in $\mathbb{C}$ that have a common boundary Γ. In this abstract setting, our requirements are:

(i) $\Omega_{\text{int}} \cap \Omega_{\text{ext}} = \varnothing$; $\mathbb{C} = \Omega_{\text{int}} \cup \Omega_{\text{ext}} \cup \Gamma$;

(ii) Γ is a finite union of piecewise smooth contours, each of them homeomorphic to the unit circle or a real line. In the latter case, both ends of the contour have to go to infinity;

(iii) $1/(|z| + 1) \in L^2(\Gamma, |\mathrm{d}z|)$. If these conditions hold, we will call the open set Ω_{int} *admissible*.

DEFINITION 1.2. We call an abstract observation system $(\mathbf{A}, \mathbf{C})$ *admissible with respect to* Ω_{int} if $\sigma(A) \subset \operatorname{clos} \Omega_{\text{int}}$ and operator $\mathcal{O}_{\mathbf{A},\mathbf{C}}$ is bounded from X to $E^2(\Omega_{\text{ext}}, Y)$.

We call an abstract observation system $(\mathbf{A}, \mathbf{C})$ *exact with respect to* Ω_{int} if it is admissible with respect to Ω_{int} and, moreover, there is a two-sided estimate

$$\|\mathcal{O}_{\mathbf{A},\mathbf{C}}x\|_{E^2(\Omega_{\text{ext}},Y)} \asymp \|x\|, \quad x \in X.$$

The relation with the theory of well-posed control systems is as follows. Suppose now that $\mathbf{A}$ is a generator of a bounded C_0 semigroup. Let

$$\Pi_- = \{z : \operatorname{Re} z < 0\}, \quad \Pi_+ = \{z : \operatorname{Re} z > 0\}$$

be, respectively, the left and the right half-planes. Consider the linear continuous time observation system

$$\dot{x}(t) = \mathbf{A}x(t), \quad x(0) = x_0, \quad y(t) = \mathbf{C}x(t).$$

For any initial value $x_0 \in \mathcal{D}(\mathbf{A})$, the output $y = y(t)$ is a well-defined continuous function on $[0, +\infty)$. Denote $y = \widehat{\mathcal{O}}_{\mathbf{A},\mathbf{C}}x_0$, so that $\widehat{\mathcal{O}}_{\mathbf{A},\mathbf{C}}$ is the space-output map. Then

$$\mathcal{O}_{\mathbf{A},\mathbf{C}}x_0(z) = (\mathcal{L}\widehat{\mathcal{O}}_{\mathbf{A},\mathbf{C}}x_0)(z), \quad x_0 \in \mathcal{D}(\mathbf{A})$$

for all $z \in \Pi_+$, where $\mathcal{L}y(z) = \int_0^\infty \mathrm{e}^{-zt}y(t)\,\mathrm{d}t$ is the Laplace transform. It follows that in this case, abstract observation system $(\mathbf{A}, \mathbf{C})$ is admissible with respect to

$\Omega_{\rm int} := \Pi_-$ if and only if the inequality

$$\int\limits_0^\infty \|y(t)\|^2\,{\rm d}t \leqslant K\|x_0\|^2$$

holds for some constant $K > 0$ and for all initial data $x_0 \in \mathcal{D}(A)$. The system (A, C) is exact with respect to Π_- if and only if the two sides of this inequality are comparable. There is a close connection between this setting and the definition of a well-posed output map, see Theorem 4.4.2 of [27].

Now let us return to the general situation of an abstract observation system and arbitrary admissible domain $\Omega_{\rm int}$.

DEFINITION 1.3 ([33]). Let $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{L}(Y, U))$ be a two-sided admissible function. We introduce *the observation model space* $\mathcal{H}(\delta)$ as the following closed subspace of $E^2(\Omega_{\rm ext}, Y)$:

$$\mathcal{H}(\delta) = \{f \in E^2(\Omega_{\rm ext}, Y) : \widetilde{f} \stackrel{\rm def}{=} \delta \cdot f|\Gamma \in E^2(\Omega_{\rm int}, U)\}. \tag{1.1}$$

We introduce (possibly, unbounded) operators $M_z^{\mathbf{T}}$, j on $\mathcal{H}(\delta)$ as follows. Put

$$\mathcal{D}(j) = \mathcal{D}(M_z^{\mathbf{T}}) = \{f \in \mathcal{H}(\delta) : \exists c \in Y : zf - c \in \mathcal{H}(\delta)\}.$$

For $f \in \mathcal{D}(M_z^{\mathbf{T}})$, the constant c is unique. Therefore, the operators

$$j : \mathcal{D}(M_z^{\mathbf{T}}) \to Y, \quad M_z^{\mathbf{T}} : \mathcal{D}(M_z^{\mathbf{T}}) \to \mathcal{H}(\delta),$$
$$jf \stackrel{\rm def}{=} c, \quad (M_z^{\mathbf{T}} f)(z) \stackrel{\rm def}{=} zf - c,\ f \in \mathcal{D}(M_z^{\mathbf{T}})$$

are well defined. We shall call $M_z^{\mathbf{T}}$ the *operator of truncated multiplication* on $\mathcal{H}(\delta)$.

Let $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{B}(Y, U))$ be a two-sided admissible function. By definition (see [22]), *the spectrum* of δ is the set of points $\lambda \in \operatorname{clos}\Omega_{\rm int}$ such that $\delta^{-1} \notin H^\infty(\Omega_{\rm int} \cap \mathcal{W}, \mathcal{B}(Y, U))$ for any neighbourhood $\mathcal{W}$ of λ. It will be denoted by $\operatorname{spec}\delta$. It is a closed subset of $\operatorname{clos}\Omega_{\rm int}$. Its intersection with $\Omega_{\rm int}$ consists of points λ in $\Omega_{\rm int}$ such that $\delta(\lambda)$ is not invertible.

Any function $f \in \mathcal{H}(\delta)$ can be viewed as an analytic function on $\Omega_{\rm ext} \cup \Omega_{\rm int} \setminus \operatorname{spec}\delta$. On $\Omega_{\rm int} \setminus \operatorname{spec}\delta$, we define it by means of the formula

$$f(z) \stackrel{\rm def}{=} \delta(z)^{-1}\widetilde{f}(z), \tag{1.2}$$

where $\widetilde{f}(z)$ is determined from (1.1).

For completeness, we give here the formula for the resolvent of $M_z^{\mathbf{T}}$.

PROPOSITION 1.4 ([33], Propositions 1.1 and 2.3). (i) *The operator* $(M_z^{\mathbf{T}} - \lambda I)^{-1}$ *exists and is bounded if and only if* $\lambda \in \mathbb{C} \setminus \operatorname{spec}\delta$.

(ii) *Each function* $f \in \mathcal{H}(\delta)$ *extends analytically to* $\mathbb{C} \setminus \operatorname{spec}\delta$ *according to the rule* $f(\lambda) \stackrel{\rm def}{=} j(\lambda I - M_z^{\mathbf{T}})^{-1} f$. *For* $\lambda \in \Omega_{\rm int} \setminus \operatorname{spec}\delta$, *this extension coincides with that defined in* (1.2).

(iii) *For* λ *in* $\mathbb{C} \setminus \operatorname{spec} \delta$,

$$(M_z^{\mathbf{T}} - \lambda I)^{-1} f(z) = \frac{f(z) - f(\lambda)}{z - \lambda}, \quad f \in \mathcal{H}(\delta).$$

Now this scheme will be concretized in order to give a precise observation model of operator (0.1).

1.2. CONDITIONS ON THE PERTURBATION. Remind that the essential norm of F is defined as

$$\|F\|_{\text{ess}} \stackrel{\text{def}}{=} \inf\{\|F + R\| : R \in \mathcal{B}(X) \text{ is compact}\}.$$

Assume that $\psi : \mathcal{D}(\psi) \to \mathbb{R}$ and A satisfy the following:

(1) Either ψ is defined on $\mathbb{R}$ and is *even* or it is defined on $[0, +\infty)$.
(2) ψ is a continuous function; moreover, ψ is of class C^1 on $\mathcal{D}(\psi) \setminus \{0\}$.
(3) $\psi \geqslant 1$ on $\mathcal{D}(\psi)$ and $\psi(x) \to +\infty$ as $x \to +\infty$.
(4) ψ^2 is concave on $[0, \infty)$.
(5) One has

$$\|F\|_{\text{ess}} \cdot k_0(\psi) < 1, \quad \text{where} \quad k_0(\psi) \stackrel{\text{def}}{=} \lim_{t \to +\infty} \frac{\psi^2(t)}{t} \tag{1.3}$$

(it follows from (4) that this limit exists).
(6) If $\mathcal{D}(\psi) = [0, \infty)$, then $\sigma(A_0) \subset [\varepsilon_0, \infty)$ for some $\varepsilon_0 > 0$.

Notice that condition (5) is automatically fulfilled whenever either F is compact or $k_0(\psi) = 0$.

We put

$$\varphi(t) \stackrel{\text{def}}{=} \psi^2(t).$$

1.3. PRECISE DEFINITION OF A. Put $A_{00} = I + |A_0|$; then $\mathcal{D}(A_0) = \mathcal{D}(A_{00})$. We rewrite A in the form

$$A = A_{00}[A_{00}^{-1} A_0 + \mathrm{i}(A_{00}^{-1} \psi(A_0)) F \psi(A_0)] \tag{1.4}$$

and take it for the precise definition of A. We set

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \{x \in \mathcal{D}(\psi(A_0)) : [A_{00}^{-1} A_0 + \mathrm{i}(A_{00}^{-1} \psi(A_0)) F \psi(A_0)] x \in \mathcal{D}(A_0)\}. \tag{1.5}$$

Notice that operators $A_{00}^{-1} A_0$ and $A_{00}^{-1} \psi(A_0)$ are bounded.

Consider the control system (A, B, C), where

$$C = \mathrm{i}\psi(A_0), \quad B = \psi(A_0).$$

We put $Y = U = X$. Notice that formally, $A = A_0 + L$, where the perturbation L factorizes as

$$L = BFC. \tag{1.6}$$

According to (1.3), the pair (A, C) is an abstract observation system.

DEFINITION 1.5. Let $\mathbf{A}$ be a closed densely defined operator on X with $\sigma(\mathbf{A}) \neq \mathbb{C}$. Take any point $\lambda \in \rho(\mathbf{A})$. We define *the Hilbert space* $X_\lambda(\mathbf{A})$ as the vector space of formal expressions $(\mathbf{A} - \lambda I)x$, where x ranges over the whole space X. Introduce a Hilbert norm on $X_\lambda(\mathbf{A})$ by setting $\|(\mathbf{A} - \lambda I)x\|_{X_\lambda(\mathbf{A})} \stackrel{\text{def}}{=} \|x\|_X$ for all $x \in X$. For $x \in \mathcal{D}(\mathbf{A}) \subset X$, we identify the element $(\mathbf{A} - \lambda I)x$ of $X(\mathbf{A})$ with the element of X, given by the same expression.

It is clear that by this construction, X becomes a dense subset of $X_\lambda(\mathbf{A})$. This construction does not depend on the choice of λ in the sense that for different λ's in $\rho(\mathbf{A})$, the corresponding spaces $X_\lambda(\mathbf{A})$ are naturally isomorphic (and have equivalent norms). If the exact form of the norm in $X_\lambda(\mathbf{A})$ is not important, then we write $X(\mathbf{A})$ instead of $X_\lambda(\mathbf{A})$. Observe that $\mathbf{A}$ is a bounded operator from X to $X(\mathbf{A})$.

1.4. OBSERVATION MODEL OF A. Put

$$\mu_0 = \frac{\|F\|_{\text{ess}}}{\sqrt{1 - \|F\|_{\text{ess}}^2 k_0(\psi)^2}}. \tag{1.7}$$

For $\varkappa \in \mathbb{R}$, we consider the normal (possibly unbounded) operator

$$A_\varkappa = A_0 + \mathrm{i}\varkappa\varphi(A_0).$$

Now we can formulate the "observation form" of our functional model.

THEOREM 1.6 (An observation model of A). *Take any $\mu > \mu_0$. For $\varkappa \in \mathbb{R}$, define*

$$\delta_\varkappa(z) = [A_\varkappa - zI + \mathrm{i}\varphi(A_0)F]^{-1}[A_0 - zI + \mathrm{i}\varphi(A_0)F].$$

Then there exist $R > 0$ and $\varkappa \in \mathbb{R}$ such that for the corresponding function $\delta \stackrel{\text{def}}{=} \delta_\varkappa$ and for the domains

$$\Omega_{\text{int}} \stackrel{\text{def}}{=} \Omega_{\mu,R}^{\text{int}}, \quad \Omega_{\text{ext}} \stackrel{\text{def}}{=} \Omega_{\mu,R}^{\text{ext}} \tag{1.8}$$

(see (0.3)*) the following statements hold:*

(i) *A is a closed densely defined operator, and $\sigma(A) \subset \Omega_{\text{int}}$. The pair (A, C) is an exact observation system.*

(ii) *The function δ is in $H^\infty(\Omega_{\text{int}}, \mathcal{B}(X))$ and is two-sided admissible;*

(iii) *The operator*

$$\mathcal{O}_{A,C} : X \to \mathcal{H}(\delta)$$

is an isomorphism that transforms the operator A into the truncated multiplication operator $M_z^{\mathbf{T}}$ on the observation model space $\mathcal{H}(\delta)$. This means that for any $x \in \mathcal{D}(A)$, $\mathcal{O}_{A,C}\, x$ is in $\mathcal{D}(M_z^{\mathbf{T}})$,

$$\mathcal{O}_{A,C}\, Ax = M_z^{\mathbf{T}} \mathcal{O}_{A,C}\, x \tag{1.9}$$

and that, moreover,

$$\mathcal{O}_{A,C}\mathcal{D}(A) = \mathcal{D}(M_z^{\mathbf{T}}). \tag{1.10}$$

In fact, we will show that there is $\varkappa_0 > 0$ such that one can take any $\varkappa$, $|\varkappa| > \varkappa_0$ in the above theorem. The value of $\varkappa_0$ is given below in (4.1).

The above definition of δ can be understood as follows. It is clear that $A_0 - zI + \mathrm{i}\varphi(A_0)F$ is a bounded operator from X to $X(A_0)$. It will be proved in Lemma 4.5 that for $z \in \Omega_{\text{int}}$ (and for $\varkappa > \varkappa_0$), the operator $A_\varkappa - zI + \mathrm{i}\varphi(A_0)F$ from X to $X(A_0)$ is invertible. Hence $\delta_\varkappa$ is a well-defined bounded operator on X for $z \in \Omega_{\text{int}}$.

The splitting property (1.9) is in fact a matter of algebra and holds true in much more general context, see Proposition 1.2 of [33].

This theorem models the operator A by the operator $M_z^{\mathbf{T}}$ on the model functional space $\mathcal{H}(\delta)$. As it will be seen in Section 5, this model is closely related to the Nagy and Foiaş model.

1.5. THE DUALITY RESULT. Let $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$ be a two-sided admissible operator function in an admissible domain Ω_{int}. As before, we put $\Omega_{\text{ext}} = \mathbb{C} \setminus (\Omega_{\text{int}} \cup \Gamma)$. We orient the curves that constitute $\Gamma = \partial\Omega_{\text{int}}$ in such a way that, under the movement along them, the domain Ω_{int} remain on the left. Put

$$\overline{\Omega}_{\text{int}} \stackrel{\text{def}}{=} \{\bar{z} : z \in \Omega_{\text{int}}\}; \quad \delta^{\mathbf{T}}(z) = \delta(\bar{z})^*, \quad z \in \overline{\Omega}_{\text{int}}.$$

Then $\delta^{\mathbf{T}}$ is a two-sided admissible function in $\overline{\Omega}_{\text{int}}$. We will need the model space

$$\mathcal{H}(\delta^T) \stackrel{\text{def}}{=} \{f \in E^2(\overline{\Omega}_{\text{ext}}, Y) : \delta^{\mathbf{T}} \cdot f|\partial\overline{\Omega}_{\text{int}} \in E^2(\overline{\Omega}_{\text{int}}, U)\},$$

which is associated to the function $\delta^{\mathbf{T}}$ and the domain $\overline{\Omega}_{\text{int}}$.

We start with the following fact.

PROPOSITION 1.7 ([33], Proposition 4.2). *For any two-sided admissible function $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$, the model spaces $\mathcal{H}(\delta)$ and $\mathcal{H}(\delta^T)$ are dual to each other with respect to the Hermitian pairing*

$$\langle f, g\rangle_\delta \stackrel{\text{def}}{=} \frac{1}{2\pi\mathrm{i}} \int_\Gamma \langle \delta(z) f(z), g(\bar{z})\rangle \, \mathrm{d}z.$$

In fact, in our case, Ω_{int} is symmetric with respect to the real line, that is $\Omega_{\text{int}} = \overline{\Omega}_{\text{int}}$.

DEFINITION 1.8. Suppose we are given a triple of (possibly unbounded) operators $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ and Hilbert spaces X, U, Y, which have the meaning of the state space, the input space and the output space, respectively. We say that the triple $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is *a full abstract system* if the following condition hold:

(i) $(\mathbf{A}, \mathbf{C})$ is an abstract observation system, whose state space is X and output space is Y;

(ii) $\mathbf{B} : U \to X(\mathbf{A})$ is a bounded operator.

Each linear continuous functional in $X(\mathbf{A})^*$ can be considered in the same time as a linear continuous functional on X, that is, an element of X^*. In this sense, $X(\mathbf{A})^*$ coincides with $\mathcal{D}(\mathbf{A}^*)$, see, for example, [27]. It follows that whenever

$(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a full abstract system, $(\mathbf{A}^*, \mathbf{C}^*, \mathbf{B}^*)$ is also a full abstract system, with input and output space interchanged.

DEFINITION 1.9. Suppose Ω_{int} is fixed and $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is a full abstract system. Let $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$ be two-sided admissible. We say that the *observation systems* $(\mathbf{A}, \mathbf{C})$ *and* $(\mathbf{A}^*, \mathbf{B}^*)$ *are in duality* with respect to δ if

(i) the following operators are isomorphisms

$$\mathcal{O}_{\mathbf{A},\mathbf{C}} : X \to \mathcal{H}(\delta), \quad \mathcal{O}_{\mathbf{A}^*,\mathbf{B}^*} : X \to \mathcal{H}(\delta^T);$$

(ii) for all x_1, x_2 in X,

$$\langle x_1, x_2 \rangle_X = \langle \mathcal{O}_{\mathbf{A},\mathbf{C}} x_1, \mathcal{O}_{\mathbf{A}^*,\mathbf{B}^*} x_2 \rangle_\delta. \tag{1.11}$$

In addition to Theorem 1.6, we will prove the following result.

THEOREM 1.10. *Take any $\mu > \mu_0$. Then there exist $R > 0$ and $\varkappa \in \mathbb{R}$ such that all statements of Theorem* 1.6 *hold and, moreover, systems $(A, -\varkappa C)$ and (A^*, B^*) are in duality with respect to the function $\delta = \delta_\varkappa$. It is assumed here that Ω_{int} is defined by* (1.8).

This result implies that the transform $\mathcal{O}_{A^*,B^*}$ is an isomorphism that converts the action of A^* into the action of $M_z^{\mathbf{T}}$ on the model space $\mathcal{H}(\delta^T)$.

Observation models and control models are closely related. One of these relations is given below in Lemma 5.3. Another one is Proposition 4.1 of [33].

REMARKS. (i) If δ is a two-sided admissible function and $\mathcal{O}_{\mathbf{A},\mathbf{C}}$ is an isomorphism of X onto $\mathcal{H}(\delta)$, then we called δ a *generalized characteristic function of an observation system* $(\mathbf{A}, \mathbf{C})$ in [33]. Its determination is far from unique, to the opposite to the classical notion of the Nagy–Foiaş characteristic function, which is essentially unique. In fact, it is easy to see that $\mathcal{H}(\delta) = \mathcal{H}(\beta \cdot \delta)$ for any function β, which is invertible in the algebra $H^\infty(\Omega_{\text{int}}, \mathcal{B}(X))$. Therefore for any β of this kind, $\beta \cdot \delta$ is a generalized characteristic function of system $(\mathbf{A}, \mathbf{C})$ together with δ.

We would obtain formally a closer analogue of the classical Nagy–Foiaş construction if we required δ to be two-sided inner. However, it is this freedom of the choice of δ that permits us to give an explicit formula for the generalized characteristic function of the operator A in study.

In [33], in fact, we discussed functional models for a more general class of $*$-admissible functions.

(ii) Full systems $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ such that the observation systems $(\mathbf{A}, \mathbf{C})$ and $(\mathbf{A}^*, \mathbf{B}^*)$ are in duality are very special ones. Their consideration is motivated by our scheme of constructing functional models rather than by the control theory. If a generalized characteristic function δ of system $(\mathbf{A}, \mathbf{C})$ is fixed, then, by Proposition 9.3 of [33], there is a unique operator $\mathbf{B}$ such that $(\mathbf{A}, \mathbf{C})$ and $(\mathbf{A}^*, \mathbf{B}^*)$ are in duality with respect to δ.

2. ADMISSIBILITY OF THE OBSERVATION SYSTEM (A, C)

We fix some $\mu > \mu_0$. From now on, let us also fix a number $r' > \|F\|_{\mathrm{ess}}$, close to $\|F\|_{\mathrm{ess}}$, and a number $k > k_0(\psi)$, close to $k_0(\psi)$, so that

$$r'k < 1, \qquad \frac{r'}{\sqrt{1 - r'^2 k^2}} < \mu \tag{2.1}$$

(see (1.3) and (1.7)). Take any decomposition $F = F' + F''$ such that F'' is a finite rank operator and

$$\|F'\| < r'. \tag{2.2}$$

It is possible, because any compact operator in X can be approximated in norm by finite rank operators.

First let us formulate two technical lemmas, whose proofs will be given in the last section.

LEMMA 2.1. *There exists $R_0 > 0$ such that for all $t \in \mathcal{D}(\varphi)$,*

$$B(t, r'\varphi(t)) \subset \Omega^{\mathrm{int}}_{\mu,R_0}. \tag{2.3}$$

LEMMA 2.2. *Let $\Omega_{\mathrm{int}} = \Omega^{\mathrm{int}}_{\mu,R}$ for some positive R, and let $\Gamma = \partial\Omega_{\mathrm{int}}$. Then there is a positive constant K such that the following inequality holds for all $x \in \sigma(A_0)$:*

$$\varphi(x) \int\limits_{\Gamma} \frac{|\mathrm{d}\lambda|}{|x - \lambda|^2} \leqslant K.$$

LEMMA 2.3. *The system (A_0, C) is admissible with respect to the domain Ω_{int}.*

Proof. By the Spectral Theorem, A_0 is unitarily equivalent to the operator $\widetilde{A}_0 f(t) = t f(t)$, acting on a direct integral

$$\widetilde{\mathcal{X}} \overset{\mathrm{def}}{=} \int^{\oplus} \mathcal{X}(t)\, \mathrm{d}\nu(t),$$

where ν is a positive Borel measure on $\sigma(A_0)$ (a scalar spectral measure of A_0) and $\{\widetilde{\mathcal{X}}(t)\}$ is a ν-measurable family of Hilbert spaces [6]. The same unitary isomorphism converts $C = \mathrm{i}\psi(A_0)$ into $\widetilde{C} = M_{\mathrm{i}\psi}$. We prove our statement by passing to this model of the pair (A_0, C). For $f = f(t) \in \widetilde{\mathcal{X}}$,

$$\begin{aligned}\|(\mathcal{O}_{\widetilde{A}_0,\widetilde{C}} f)\|^2_{E^2(\Omega_{\mathrm{ext}},Y)} &= \int\limits_{\Gamma}\int\limits_{\mathbb{R}} \left\|\frac{\psi(t) f(t)}{t - z}\right\|^2 \mathrm{d}\nu(t)|\mathrm{d}z| = \int\limits_{\mathbb{R}} \mathrm{d}\nu(t) \|f(t)\|^2 |\psi(t)|^2 \int\limits_{\Gamma} \frac{|\mathrm{d}z|}{|t - z|^2} \\ &\leqslant K \int\limits_{\mathbb{R}} \|f(t)\|^2 |\, \mathrm{d}\nu(t) = K\|f\|^2_{\widetilde{\mathcal{X}}}.\end{aligned}$$

The inequality is due to Lemma 2.2. ∎

LEMMA 2.4. (i) $\limsup_{z\in\Omega_\mu^{\mathrm{ext}},\, z\to\infty} \|FC(A_0-zI)^{-1}B\| \leqslant \dfrac{\|F'\|}{r'} < 1$;

(ii) $\limsup_{z\in\Omega_\mu^{\mathrm{ext}},\, z\to\infty} \|C(A_0-zI)^{-1}BF\| \leqslant \dfrac{\|F'\|}{r'} < 1$.

Proof. By Lemma 2.1, if $|z-t| < r'\varphi(t)$ for some $t \in \mathcal{D}(\varphi)$, then $z \in \Omega^{\mathrm{int}}_{\mu,R_0}$. It follows that for $z \in \operatorname{clos}\Omega^{\mathrm{ext}}_{\mu,R_0}$,

$$\frac{\varphi(t)}{|z-t|} \leqslant \frac{1}{r'}, \quad \forall t \in \mathcal{D}(\varphi). \tag{2.4}$$

Hence

$$\|\varphi(A_0)(A_0-zI)^{-1}\| \leqslant \frac{1}{r'} \tag{2.5}$$

for $z \in \operatorname{clos}\Omega^{\mathrm{ext}}_{\mu,R_0}$. Moreover, if we put $\widetilde{\mathcal{A}}(z) = \varphi(A_0)(A_0-zI)^{-1}$, then it is easy to check that

$$\widetilde{\mathcal{A}}(z)^* \to 0 \quad \text{as} \quad z\to\infty,\ z \in \operatorname{clos}\Omega^{\mathrm{ext}}_{\mu,R_0} \tag{2.6}$$

in the strong operator topology. (One can apply here the Spectral Theorem in the same way as in the proof of Lemma 2.3, the Lebesgue dominated convergence theorem and (2.4).) We have

$$\|FC(A_0-zI)^{-1}B\| \leqslant \|F'\varphi(A_0)(A_0-zI)^{-1}\| + \|F''\varphi(A_0)(A_0-zI)^{-1}\|.$$

The relation (2.6) and the fact that F'' has a finite rank imply that $\|F''(A_0 - zI)^{-1}\varphi(A_0)\| \to 0$ as $z \to \infty$, $z \in \operatorname{clos}\Omega^{\mathrm{out}}_{\mu,R_0}$. By applying the estimate $\|F'\| < r'$ and (2.5), we obtain (i). Assertion (ii) is obtained similarly. ∎

From now on, we fix some $\varepsilon > 0$ and a radius $R > R_0$ such that

$$\|FC(A_0-zI)^{-1}B\| \leqslant 1-\varepsilon, \quad \|C(A_0-zI)^{-1}BF\| \leqslant 1-\varepsilon \tag{2.7}$$

for all $z \in \Omega^{\mathrm{ext}}_{\mu,R}$. It is possible due to Lemma 2.4. According to (0.3), we put $\Omega_{\mathrm{int}} = \Omega^{\mathrm{int}}_{\mu,R}$, $\Omega_{\mathrm{ext}} = \mathbb{C}\setminus\operatorname{clos}\Omega_{\mathrm{int}}$.

DEFINITION 2.5. Let η be a real Borel function on $\mathcal{D}(\psi)$ such that $\eta(t) \neq 0$ for all $t \in \mathcal{D}(\psi)$. We define the Hilbert space X_η as the set of formal expressions $\eta(A_0)x$, $x \in X$. Recall that the self-adjoint operator $\eta(A_0)$ is bounded if and only if η is essentially bounded with respect to the spectral measure of A_0. We introduce a Hilbert norm on X_η by setting $\|\eta(A_0)x\|_{X_\eta} \overset{\mathrm{def}}{=} \|x\|_X$ for all $x \in X$. For $x \in \mathcal{D}(\eta(A_0)) \subset X$, we identify the element $\eta(A_0)x$ of X_η with the element of X, given by the same expression. Notice that if η is essentially bounded, then $X_\eta = \mathcal{D}(\eta^{-1}(A_0)) \subset X$.

This definition is very close to the definition of spaces $X_\lambda(\mathbf{A})$, given earlier. In fact, if $\eta > \varepsilon > 0$, then $X_\eta = X_0(\eta(A_0))$.

Consider, in particular, the Hilbert space $X_\psi \supset X$. Since B is an isometric isomorphism of X onto X_ψ, for any $T \in \mathcal{B}(X_\psi)$,

$$\|T\|_{\mathcal{B}(X_\psi)} = \|B^{-1}TB\|_{\mathcal{B}(X)}. \tag{2.8}$$

Recalling the notation from (1.6), by the first inequality in (2.7) we obtain

$$\|L(A_0 - zI)^{-1}\|_{\mathcal{B}(X_\psi)} \leqslant 1 - \varepsilon, \quad z \in \operatorname{clos} \Omega_{\text{ext}}. \tag{2.9}$$

LEMMA 2.6. (i) $\sigma(A) \subset \Omega_{\text{int}}$.

(ii) *For $z \in \operatorname{clos} \Omega_{\text{ext}}$, $(A - zI)^{-1}$ is an isomorphism of X_ψ onto $X_{\psi(t)/(|t|+1)}$.*

(iii) *The identity*

$$\mathcal{O}_{A_0,C}x(z) = H(z)\mathcal{O}_{A,C}x(z), \quad x \in X, z \in \Omega_{\text{ext}} \tag{2.10}$$

holds, where

$$H(z) = I + C(A_0 - zI)^{-1}BF. \tag{2.11}$$

(iv) *One has $H, H^{-1} \in H^\infty(\Omega_{\text{ext}}, \mathcal{B}(X))$.*

(v) *The observation system (A, C) is admissible with respect to the domain Ω_{int}.*

Notice that $\psi^2(t) \leqslant K(|t| + 1)$ implies that $X_{\psi(t)/(|t|+1)} \hookrightarrow X_{1/\psi}$.

Proof of Lemma 2.6. The definition (1.3) of $\mathcal{D}(A)$ can be rewritten as

$$\mathcal{D}(A) \stackrel{\text{def}}{=} \{x \in X_{1/\psi} : [A_0 + \mathrm{i}\psi(A_0)F\psi(A_0)]x \in X\},$$

where $[A_0 + \mathrm{i}\psi(A_0)F\psi(A_0)]x$ is understood a priori as an element of $X_{|t|+1}$. Recall that $X_{1/\psi} = \mathcal{D}(\psi(A_0))$. Hence for all $y \in X_{\psi(t)/(|t|+1)}$, the equality

$$(A - zI)y = (I + L(A_0 - zI)^{-1})(A_0 - zI)y$$

between elements of X_ψ holds for all $z \notin \sigma(A_0)$. By (2.9), $I + L(A_0 - zI)^{-1}$ is invertible in X_ψ for $z \in \operatorname{clos} \Omega_{\text{ext}}$. Hence for these z, $A - zI$ has a bounded inverse in X, given by

$$(A - zI)^{-1} = (A_0 - zI)^{-1}(I + L(A_0 - zI)^{-1})^{-1} \tag{2.12}$$

(notice that the immersion $X \hookrightarrow X_\psi$ is bounded). This proves (i). Formula (2.12) also gives (ii).

Similarly, we have

$$(A - zI)y = (A_0 - zI)(I + (A_0 - zI)^{-1}BFC)y, \quad y \in \mathcal{D}(\psi(A_0)),$$

which implies that

$$(A_0 - zI)^{-1} = (I + (A_0 - zI)^{-1}BFC)(A - zI)^{-1},$$

where $I + (A_0 - zI)^{-1}BFC$ is a bounded operator from $\mathcal{D}(\psi(A_0))$ to X. By multiplying this equality by C from the left, we obtain (iii).

Assertion (iv) follows from the second inequality in (2.7). At last, (2.10), (iv) and Lemma 2.3 imply (v). ∎

Assertion (ii) of lemma implies that A is closed and densely defined.

Notice that inequality (2.9) implies that

$$\|(I + L(A_0 - zI)^{-1})^{-1}\|_{\mathcal{B}(X_\psi)} \leqslant \varepsilon^{-1} < \infty, \quad z \in \Omega_{\text{ext}}. \tag{2.13}$$

3. OUTLINE OF THE PROOF OF THEOREM 1.10

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ be a full abstract system with the state space X, input space U and output space Y. Let Φ be a holomorphic operator-valued function on $\rho(A)$ with values in $\mathcal{B}(U, Y)$.

DEFINITION 3.1. We call Φ *a transfer function* of system $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ if the following identity holds for all $z, w \in \rho(A)$:

$$\Phi(z) - \Phi(w) = \mathbf{C}[(zI - \mathbf{A})^{-1} - (wI - \mathbf{A})^{-1}]\mathbf{B}.$$

This definition is standard in the theory of well-posed systems, see [27]. The point is that the difference of the resolvents of $\mathbf{A}$ in points z and w is a bounded map from $X(\mathbf{A})$ to $\mathcal{D}(\mathbf{A})$, which implies that the right hand part is always in $\mathcal{B}(X)$. The transfer function of a system is determined uniquely up to adding an arbitrary operator constant.

We need the following definition from [33].

DEFINITION 3.2. We say that a function $\Phi \in H^\infty(\Omega_{\text{ext}}, \mathcal{B}(U, Y))$ *corresponds to* a two-sided admissible function $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$ if there is a function $\tau \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(U, Y))$ such that the following two conditions hold:

1) $\Phi|\Omega_{\text{ext}} \in H^\infty(\Omega_{\text{ext}}, \mathcal{B}(U, Y))$;

2) $\Phi_e = (\delta^{-1} + \tau)_i$ a.e. on Γ.

Our main tool in proving Theorem 1.10 will be the following result from [33].

THEOREM A. (see Theorem 9.5 of [33]). *Let* $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ *be a full abstract system and* Φ *its transfer function. Suppose that* Ω_{int} *is an admissible domain,* $\sigma(\mathbf{A}) \subset \operatorname{clos}\Omega_{\text{int}}$*, and let* δ *be a two-sided admissible function in* $H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$*. Suppose that the following conditions hold:*

(i) $\mathcal{O}_{\mathbf{A},\mathbf{C}} : X \to E^2(\Omega_{\text{ext}}, Y)$ *and* $\mathcal{O}_{\mathbf{A}^*,\mathbf{B}^*} : X \to E^2(\Omega_{\text{ext}}, U)$ *are bounded injective operators;*

(ii) Φ *corresponds to* δ*.*

Then the observation systems $(\mathbf{A}, \mathbf{C})$ *and* $(\mathbf{A}^*, \mathbf{B}^*)$ *are dual with respect to* δ*.*

REMARK. In [33], we gave a wider definition of the correspondence between Φ and δ. It was required there that τ and Φ belong to wider functional classes than classes H^∞. Theorem 9.5 from [33] gives *a necessary and sufficient condition* for the duality of observation systems $(\mathbf{A}, \mathbf{C})$ and $(\mathbf{A}^*, \mathbf{B}^*)$, and in this sense the definition in [33] is the adequate one. For our purpose, the above formulation in Theorem A will suffice.

The proof of Theorem 1.10 will consist in checking conditions (1) and (2) of Theorem A for the triple $(A, B, -\varkappa C)$. The transfer function of this system can be simply expressed as

$$\Phi(z) = \varkappa C(A - zI)^{-1}B.$$

Indeed, for $z \in \rho(A)$, $(A - zI)^{-1}B$ is bounded from X to $X_{\psi(t)/(|t|+1)}$ (see Lemma 2.6, (ii)), and C is bounded from $X_{\psi(t)/(|t|+1)}$ to X.

4. THEOREM 1.10: DETAILS OF PROOF

LEMMA 4.1. *The transfer function Φ of the system $(A, B, -\varkappa C)$ belongs to $H^\infty(\Omega_{\text{ext}}, \mathcal{B}(X))$.*

Proof. Since $A - zI$ is bounded from $X_{\psi^{-1}}$ to X_ψ, $(A - zI)^{-1}$ is bounded from X_ψ to $X_{\psi^{-1}}$ for $z \in \rho(A)$. It follows from (2.12) that

$$\Phi(z) = \varkappa CB(A_0 - zI)^{-1} \cdot B^{-1}[I + L(A_0 - zI)^{-1}]^{-1}B.$$

By (2.13), $B^{-1}[I + L(A_0 - zI)^{-1}]^{-1}B$ is an $H^\infty(\Omega_{\text{ext}}, \mathcal{B}(X))$ function and by (2.5), $CB(A_0 - zI)^{-1}$ is an $H^\infty(\Omega_{\text{ext}}, \mathcal{B}(X))$ function. ∎

We will need a domain

$$\Omega'_{\text{int}} \stackrel{\text{def}}{=} \Omega^{\text{int}}_{\mu-\sigma,R-\sigma} \subset \Omega^{\text{int}}_{\mu,R}\,,$$

where a small parameter $\sigma > 0$ is chosen in such a way that (2.7) and (2.9) still hold true for $z \in \mathbb{C} \setminus \operatorname{clos} \Omega'_{\text{int}}$ for some $\varepsilon > 0$. Then Φ belongs to $H^\infty(\mathbb{C} \setminus \operatorname{clos} \Omega'_{\text{int}}, \mathcal{B}(X))$.

Put $\varphi_* : \mathbb{R} \to \mathbb{R}$ to be the even continuation of φ if φ is defined on $[0, \infty)$, and $\varphi_* = \varphi$, if φ is already defined on $\mathbb{R}$.

LEMMA 4.2. (i) *For all $s > 1$ and $t \in \mathbb{R}_+$, $\varphi(st) \leqslant s\varphi(t)$.*
(ii) *$\varphi_*(s) \leqslant (|s| + 1)\varphi_*(1)$ for $s \in \mathbb{R}$.*
(iii) *$\varphi(s + t) \leqslant \varphi(s) + \varphi(t)$ for $s, t \geqslant 0$.*

Proof. (i) and (iii) follow from the concavity of φ on $[0, \infty)$ and the fact that $\varphi(0) \geqslant 0$. Conditions on ψ imply that ψ grows on $[0, \infty)$. Therefore (ii) is obtained by putting $t = 1$ in (i). ∎

It is clear that there is $\mu_1 > \mu$ such that

$$\Omega^{\text{int}}_{*,\mu_1} \supset \operatorname{clos} \Omega^{\text{int}}_{\mu,R},$$

where

$$\Omega^{\text{int}}_{*,\mu_1} \stackrel{\text{def}}{=} \{z = x + \mathrm{i}y \in \mathbb{C} : x \in \mathbb{R},\ |y| < \mu_1\varphi_*(x)\}.$$

For the rest of this section, we fix a real number $\ell > \|F\|$ and put

$$\varkappa_0 = \ell + \mu_1(\alpha + \varphi(\alpha)), \quad \text{where } \alpha = 1 + \ell\varphi(1). \tag{4.1}$$

We also fix any $\varkappa > \varkappa_0$. Our aim is to prove that for any such $\varkappa$, the conclusions of Theorems 1.6 and 1.10 hold.

LEMMA 4.3. *For any $x, t \in \mathbb{R}$,*

$$|t + \mathrm{i}\varkappa\varphi_*(t) - (x + \mathrm{i}\mu_1\varphi_*(x))| \geqslant \ell\varphi_*(t). \tag{4.2}$$

Proof. Note that $\varkappa > \mu_1$. Take any $x, t \in \mathbb{R}$. We distinguish two cases.

(i) Suppose that $|x| < \alpha(|t| + 1)$. Then, by the previous lemma,

$$\varphi_*(x) \leqslant \alpha\varphi_*(t) + \varphi_*(\alpha),$$

which gives that

$$|\varkappa\varphi_*(t) - \mu_1\varphi_*(x)| \geqslant \varkappa\varphi_*(t) - \mu_1\varphi_*(x) \geqslant (\varkappa - \mu_1\alpha)\varphi_*(t) - \mu_1\varphi_*(\alpha) \geqslant \ell\varphi_*(t).$$

The last inequality is due to the facts that $\varkappa > \varkappa_0$ and $\varphi_*(t) \geqslant 1$. Now (4.2) follows.

(ii) Suppose that $|x| \geqslant \alpha(|t| + 1)$. Since $\alpha - 1 = \ell\varphi(1)$, we get

$$|t - x| \geqslant |x| - |t| \geqslant (\alpha - 1)|t| + \alpha > \ell\varphi(1)|t| + \ell\varphi(1) \geqslant \ell\varphi_*(t),$$

and this again gives (4.2). ∎

LEMMA 4.4. *For any $z \in \Omega^{\mathrm{int}}_{*,\mu_1}$ and any $t \in \mathcal{D}(\varphi)$,*

$$|t + \mathrm{i}\varkappa\varphi(t) - z| \geqslant \ell\varphi(t). \tag{4.3}$$

Proof. Since $\varkappa > \mu_1$, $t + \mathrm{i}\varkappa\varphi(t)$ is outside $\operatorname{clos}\Omega^{\mathrm{int}}_{*,\mu_1}$. Hence the straight line interval with endpoints in $t + \mathrm{i}\varkappa\varphi(t)$ and z contains a boundary point of $\Omega^{\mathrm{int}}_{*,\mu_1}$, which has a form $x + \mathrm{i}\mu_1\varphi_*(x)$ for some x. Therefore (4.3) follows from (4.2). ∎

LEMMA 4.5. (i) *$A_\varkappa - zI + \mathrm{i}\varphi(A_0)F$ is an isomorphism from X onto $X_{|t|+1}$ for all $z \in \Omega^{\mathrm{int}}_{\mu_1}$.*

(ii) *$\delta \in H^\infty(\Omega^{\mathrm{int}}_{\mu_1}, \mathcal{B}(X))$.*

(iii) *For any $z \in \Omega^{\mathrm{int}}_{*,\mu_1} \setminus \Omega'_{\mathrm{int}}$, $\delta(z)$ is invertible, and*

$$\delta^{-1}(z) = I + \Phi(z).$$

For these values of z, the norms of $\delta^{-1}(z)$ are uniformly bounded. In particular, the norms $\|\delta^{-1}(\cdot)\|$ are uniformly bounded on $\partial\Omega_{\mathrm{int}}$.

Proof. (i) Since $A_\varkappa - zI$ is an isomorphism from X to $X_{|t|+1}$ for $z \in \Omega^{\mathrm{int}}_{*,\mu_1}$, one has to check that $I + \mathrm{i}\varphi(A_0)(A_\varkappa - zI)^{-1}F$ is an invertible operator on X. By (4.3),

$$\|\varphi(A_0)(A_\varkappa - zI)^{-1}F\| \leqslant \|F\| \sup_{t \in \mathcal{D}(\varphi)} \frac{\varphi(t)}{t + \mathrm{i}\varkappa\varphi(t) - z} \leqslant \frac{\|F\|}{\ell} < 1$$

for $z \in \Omega^{\text{int}}_{*,\mu_1}$. We obtain that for $z \in \Omega^{\text{int}}_{*,\mu_1}$,

$$\|[I + \mathrm{i}\varphi(A_0)(A_\varkappa - zI)^{-1}F]^{-1}\| \leqslant \frac{1}{1 - \|F\|/\ell} < \infty, \tag{4.4}$$

and our assertion follows.

(ii) It is easy to check that

$$\delta(z) = I - \mathrm{i}\varkappa[I + \mathrm{i}\varphi(A_0)(A_\varkappa - zI)^{-1}F]^{-1}(A_\varkappa - zI)^{-1}\varphi(A_0). \tag{4.5}$$

Since $\sup_{z \in \Omega^{\text{int}}_{*,\mu_1}} \|(A_\varkappa - zI)^{-1}\varphi(A_0)\| < \infty$, the assertion follows from (4.4).

(iii) By (2.11),

$$\delta(z) = [A_\varkappa - zI + \mathrm{i}\varphi(A_0)F]^{-1}(A_0 - zI)H(z).$$

For $z \in \Omega^{\text{int}}_{*,\mu_1} \setminus \Omega'_{\text{int}}$, $(A_0 - zI)H(z)$ is an isomorphism from X to $X(A_0)$ (see Lemma 2.6, (iv)), and $[A_\varkappa - zI + \mathrm{i}\varphi(A_0)F]^{-1}$ is an isomorphism from $X(A_0)$ to X. Hence for these z, $\delta(z)$ is invertible, and

$$\begin{aligned} \delta(z)^{-1} &= [A_0 - zI + \mathrm{i}\varphi(A_0)F]^{-1}[A_\varkappa - zI + \mathrm{i}\varphi(A_0)F] \\ &= I + \mathrm{i}\varkappa[A_0 - zI + \mathrm{i}\varphi(A_0)F]^{-1}\varphi(A_0) \\ &= I + \mathrm{i}\varkappa\psi(A_0)[A - zI]^{-1}\psi(A_0) = I + \Phi(z). \end{aligned} \tag{4.6}$$

The definition of Ω'_{int} implies that Φ is an operator-valued H^∞ function in $\mathbb{C} \setminus \operatorname{clos}\Omega'_{\text{int}}$. Hence $\|\delta^{-1}(z)\| \leqslant K < \infty$ in $\Omega^{\text{int}}_{\mu_1} \setminus \Omega'_{\text{int}}$. ∎

Proofs of Theorems 1.10 *and* 1.6. Let R, $\varkappa_0$ be chosen as above, and take any $\varkappa > \varkappa_0$. Put $\delta = \delta_\varkappa$. We check the hypotheses of Theorem A for the system $(A, B, -\varkappa C)$. By Lemma 2.6, (v), $\mathcal{O}_{A,C}$ is bounded. Since $C = \mathrm{i}\psi(A_0)$ and $\psi \neq 0$ on $\sigma(A_0)$, $\mathcal{O}_{A_0,C}$ is injective. Statements (iii) and (iv) of Lemma 2.6 imply that $\mathcal{O}_{A,C}$ is injective. By symmetry, the same can be said about $\mathcal{O}_{A^*,B^*}$.

By Lemma 4.5, (ii) and (iii), δ is two-sided admissible. By (4.6), Φ corresponds to δ, with $\tau(z) \equiv I$. Hence all the hypotheses of Theorem A are fulfilled. Therefore the observation systems $(A, -\varkappa C)$ and (A^*, B^*) are dual to each other with respect to δ. In particular, $\mathcal{O}_{A,C}$ is an isomorphism.

Statements (i) and (ii) of Theorem 1.6 have been already verified. Now statement (iii) of this theorem follows immediately from Theorem 3.1 and Proposition 5.1 in [33]. ∎

It is easy to see that the same results hold for $\varkappa < -\varkappa_0$.

5. THE CONTROL MODEL

In this section, we give a dual formulation of our result in terms of what we call the control model. Let us give first a control theory motivation.

Suppose we are given an abstract control system $(\mathbf{A}, \mathbf{B})$ such that $\mathbf{A}$ is a generator of a bounded C_0 semigroup. Associate with the pair $(\mathbf{A}, \mathbf{B})$ the linear system

$$\dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t).$$

Consider the mapping $\mathcal{C}_{\mathbf{A},\mathbf{B}}$, which sends an input $u \in L^2(\mathbb{R}_-, U)$ to $x(0)$. We define it first for smooth functions u with compact support, assuming that $x(t) = 0$ for large negative times. For these functions, $\mathcal{C}_{\mathbf{A},\mathbf{B}}$ is always well-defined. Next we make the assumption that $\mathcal{C}_{\mathbf{A},\mathbf{B}}$ extends continuously to $L^2(\mathbb{R}_-, U)$. In the control theory, a system with this property is called *infinite time admissible* [27]. Denote the extended map by the same symbol $\mathcal{C}_{\mathbf{A},\mathbf{B}}$. For an admissible system, define the controllability map

$$W_{\mathbf{A},\mathbf{B}} : E^2(\Omega_{\text{int}}, U) \to X$$

by taking the composition map in the diagram

$$E^2(\Omega_{\text{int}}, U) \underset{\mathcal{L}^{-1}}{\longrightarrow} L^2(\mathbb{R}_-, U) \underset{\mathcal{C}_{\mathbf{A},\mathbf{B}}}{\longrightarrow} X$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform and $\Omega_{\text{int}} = \Pi_-$. By the usual convention, $E^2(\Pi_-) = H^2(\Pi_-)$. We put $\Omega_{\text{ext}} = \Pi_+$. It is easy to see that

$$W_{\mathbf{A},\mathbf{B}}(z-\lambda)^{-1}u = (\mathbf{A} - \lambda I)^{-1}\mathbf{B}u, \quad \lambda \in \Omega_{\text{ext}},\ u \in U.$$

Let us return to the general case when $(\mathbf{A}, \mathbf{B})$ is an arbitrary abstract control system and Ω_{int} an arbitrary admissible domain such that $\sigma(\mathbf{A}) \subset \operatorname{clos} \Omega_{\text{int}}$. We take the last formula as a starting point of the general definition of the transform $W_{\mathbf{A},\mathbf{B}}$.

Consider the linear set $\mathcal{H}$ of rational holomorphic functions from Ω_{int} to U that are representable as finite sums

$$f(z) = \sum_j (z - \lambda_j)^{-1} u_j$$

with $u_j \in U$ and $\lambda_j \in \Omega_{\text{ext}}$. For each such $f \in \mathcal{H}$, we put

$$\overset{o}{W}_{\mathbf{A},\mathbf{B}} f \overset{\text{def}}{=} \sum_j (\mathbf{A} - \lambda_j I)^{-1}\mathbf{B}u_j.$$

It is easy to prove that $\mathcal{H}$ is a dense subset of $E^2(\Omega_{\text{int}}, U)$.

DEFINITIONS 5.1. (i) The abstract control system $(\mathbf{A}, \mathbf{B})$ is called *admissible with respect to* Ω_{int} if $\overset{o}{W}_{\mathbf{A},\mathbf{B}}$ extends to a continuous operator

$$W_{\mathbf{A},\mathbf{B}} : E^2(\Omega_{\text{int}}, U) \to X.$$

(ii) The abstract control system $(\mathbf{A}, \mathbf{B})$ is called *exact with respect to* Ω_{int} if it is admissible with respect to this domain and the image of the extended map $W_{\mathbf{A},\mathbf{B}}$ is the whole space X.

It is easy to see that $W_{\mathbf{A},\mathbf{B}}$ splits the multiplication operator by z on $E^2(\Omega_{\rm int}, U)$ with the operator $\mathbf{A}$; more exactly,

$$W_{\mathbf{A},\mathbf{B}}[q(z)f(z)] = q(\mathbf{A})(W_{\mathbf{A},\mathbf{B}}f), \quad f \in E^2(\Omega_{\rm int}, U) \tag{5.1}$$

for any rational scalar function $q \in H^\infty(\Omega_{\rm int})$. For these functions, $q(\mathbf{A})$ is bounded. It follows from this equation that $\ker W_{\mathbf{A},\mathbf{B}}$ is invariant under the multiplication by rational functions in $H^\infty(\Omega_{\rm int})$.

For any admissible abstract control system $(\mathbf{A}, \mathbf{B})$, we define the quotient operator $\widehat{W}_{\mathbf{A},\mathbf{B}}$ by factoring $W_{\mathbf{A},\mathbf{B}}$ by its kernel:

$$\widehat{W}_{\mathbf{A},\mathbf{B}} : E^2(\Omega_{\rm int}, U) / \ker W_{\mathbf{A},\mathbf{B}} \to X,$$
$$\widehat{W}_{\mathbf{A},\mathbf{B}}(f + \ker W_{\mathbf{A},\mathbf{B}}) \overset{\rm def}{=} W_{\mathbf{A},\mathbf{B}}f.$$

By the Beurling–Lax–Halmos theorem, $\ker W_{\mathbf{A},\mathbf{B}}$ has the form

$$\ker W_{\mathbf{A},\mathbf{B}} = \delta E^2(\Omega_{\rm int}, Y) \tag{5.2}$$

for a Hilbert space Y and an admissible function $\delta \in H^\infty(\Omega_{\rm int}, \mathcal{B}(Y, U))$.

DEFINITION 5.2. Any admissible function δ satisfying (5.2) will be called *a generalized characteristic function* of abstract control system $(\mathbf{A}, \mathbf{B})$.

Put (as before) $\overline{\Omega}_{\rm int} = \{\bar z : z \in \Omega_{\rm int}\}$ and $\overline{\Omega}_{\rm ext} = \{\bar z : z \in \Omega_{\rm ext}\}$. Notice that the pairing

$$\langle f, g\rangle \overset{\rm def}{=} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \langle f(z), g(\bar z)\rangle \, \mathrm{d}z, \quad f \in E^2(\Omega_{\rm int}, U),\ g \in E^2(\overline{\Omega}_{\rm ext}, U) \tag{5.3}$$

defines a duality between Hilbert spaces $E^2(\Omega_{\rm int}, U)$ and $E^2(\overline{\Omega}_{\rm ext}, U)$, see [33].

LEMMA 5.3. (i) *The abstract control system $(\mathbf{A}, \mathbf{B})$ is admissible with respect to $\Omega_{\rm int}$ if and only if abstract observation system $(\mathbf{A}^*, \mathbf{B}^*)$ is admissible with respect to $\overline{\Omega}_{\rm int}$.*

(ii) *If any of the above two assertions holds, then*

$$\mathcal{O}_{\mathbf{A}^*,\mathbf{B}^*} = W^*_{\mathbf{A},\mathbf{B}}$$

with respect to the pairing (5.3).

Proof. The Cauchy pairing (5.3) extends to rational functions $f \in \mathcal{H}$ and *all* holomorphic functions $g : \Omega_{\rm ext} \to U$ by putting

$$\langle (z-\lambda)^{-1}u, g\rangle \overset{\rm def}{=} \langle u, g(\lambda)\rangle$$

and extending this formula by linearity. For functions $g \in E^2(\overline{\Omega}_{\rm ext}, U)$, it is the same pairing. It is plain to check the identity $\langle f, \mathcal{O}_{\mathbf{A}^*,\mathbf{B}^*}x\rangle = \langle \overset{o}{W}_{\mathbf{A},\mathbf{B}}f, x\rangle$ for $f \in \mathcal{H}$ and $x \in X$. Both statements of lemma are easy consequences of this formula. ∎

LEMMA 5.4. *Suppose we are given an abstract control system* $(\mathbf{A}, \mathbf{B})$, *which is admissible with respect to an admissible domain* Ω_{int} *and a two-sided admissible function* $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$.

Then the following are equivalent:

(i) $\mathcal{O}_{\mathbf{A}^*, \mathbf{B}^*} : X \to \mathcal{H}(\delta^T)$ *is an isomorphism;*

(ii) $(\mathbf{A}, \mathbf{B})$ *is an exact abstract control system and* $\ker W_{\mathbf{A}, \mathbf{B}} = \delta E^2(\Omega_{\text{int}}, Y)$;

(iii) $\ker W_{\mathbf{A}, \mathbf{B}} = \delta E^2(\Omega_{\text{int}}, Y)$ *and* $\widehat{W}_{\mathbf{A}, \mathbf{B}}$ *is an isomorphism.*

Proof. If follows from Lemma 5.3 that the closure of the range of $\mathcal{O}_{\mathbf{A}^*, \mathbf{B}^*}$ in $E^2(\Omega_{\text{ext}}, U)$ equals to the annihilator of $\ker W_{\mathbf{A}, \mathbf{B}}$ with respect to the Cauchy pairing (5.3). By the Banach theorem, the range of $\mathcal{O}_{\mathbf{A}^*, \mathbf{B}^*}$ is closed if and only if $W_{\mathbf{A}, \mathbf{B}} = \mathcal{O}^*_{\mathbf{A}^*, \mathbf{B}^*}$ is onto. Finally,

$$\mathcal{H}(\delta^T) = (\delta E^2(\Omega_{\text{int}}, Y))^\perp$$

(see Proposition 2.5 of [33]). These remarks imply the equivalence of statements (i)–(iii). ∎

DEFINITION 5.5. For a domain Ω_{int}, Hilbert spaces U, Y and a fixed two-sided admissible function $\delta \in H^\infty(\Omega_{\text{int}}, \mathcal{B}(Y, U))$, we consider *the control model space*, which is the quotient space

$$E^2(\Omega_{\text{int}}, U)/\delta E^2(\Omega_{\text{int}}, Y). \tag{5.4}$$

For a function f in $E^2(\Omega_{\text{int}}, Y)$, we put $[f] = f + \delta E^2(\Omega_{\text{int}}, Y)$ to be its coset in this quotient space.

The model operator $\widehat{M}_z$ on this space is simply the quotient operator of multiplication by the independent variable z. It is given by

$$\mathcal{D}(\widehat{M}_z) = \{[f] : f, M_z f \in E^2(\Omega_{\text{int}}, Y)\},$$
$$\widehat{M}_z[f] \stackrel{\text{def}}{=} [zf], \quad \text{if } [f] \in \mathcal{D}(\widehat{M}_z) \text{ and } f, zf \in E^2(\Omega_{\text{int}}, Y).$$

THEOREM 5.6. *Let* A *be an operator given by* (0.1), (1.4), *where* ψ *is a function satisfying* (i)–(vi). *Put* $B = \psi(A)$, *and define* $\delta_\varkappa$ *as in Theorem* 1.6. *Then there exist* $R > 0$ *and* $\varkappa \in \mathbb{R}$ *such that for the corresponding function* $\delta = \delta_\varkappa$ *and for the domains* Ω_{int}, Ω_{ext}, *given by* (1.8), *the following are true:*

(i) (A, B) *is an exact control system in* Ω_{int} *and* δ *is its generalized characteristic function (that is,* $\ker W_{A,B} = \delta E^2(\Omega_{\text{int}}, X)$*).*

(ii) *The operator*

$$\widehat{W}_{A,B} : E^2(\Omega_{\text{int}}, X)/\delta E^2(\Omega_{\text{int}}, X) \to X$$

is an isomorphism that transforms the operator A *into the quotient multiplication operator* $\widehat{M}_z$ *on the control model space. In particular,* $\widehat{W}_{A,B}\, \mathcal{D}(\widehat{M}_z) = \mathcal{D}(A)$, *and*

$$A\, \widehat{W}_{A,B}[f] = \widehat{W}_{A,B}\, \widehat{M}_z[f], \quad \forall f \in \mathcal{D}(\widehat{M}_z). \tag{5.5}$$

Proof. Take μ, R and $\varkappa_0$ as in Sections 1–4, and put $\Omega_{\rm int} = \Omega^{\rm int}_{\mu,R}$. Take any $\varkappa > \varkappa_0$, and put $\delta = \delta_\varkappa$. By Theorem 1.10, the observation system (A^*, B^*) is exact and $\mathcal{O}_{A^*,B^*}$ is an isomorphism from X onto $\mathcal{H}(\delta^T)$. Hence, by Lemma 5.4, (i) holds, and $\widehat{W}_{A,B}$ is an isomorphism. The splitting properties of this isomorphism, stated in (ii), follow easily from (5.1). ∎

The calculus $q \mapsto q(A) \in \mathcal{B}(X)$ is defined for any rational function q in $H^\infty(\Omega_{\rm int})$.

COROLLARY 5.7. *The above functional calculus $q \mapsto q(A)$ extends by continuity to an $H^\infty(\Omega_{\rm int})$ functional calculus for A. In particular,*

$$\|f(A)\| \leqslant K \sup_{z\in\Omega_{\rm int}} |f(z)| \tag{5.6}$$

for $f \in H^\infty(\Omega_{\rm int})$, where $K = \|\widehat{W}_{A,B}\|\, \|\widehat{W}^{-1}_{A,B}\|$.

COROLLARY 5.8. *Operator A admits a skew normal dilation on $\partial\Omega_{\rm int}$ in the following sense. There exists a Hilbert space K and an unbounded operator N, acting on K, that has the following properties:*

(i) *N is similar to an unbounded normal operator;*

(ii) *$\sigma(N)$ is contained in $\partial\Omega_{\rm int}$ and is absolutely continuous with respect to the arc length measure;*

(iii) *$q(A) = P_X q(A)|X$ for any rational function q in $H^\infty(\Omega_{\rm int})$.*

REMARKS. (i) The quotient operator $\widehat{M}_z$ of multiplication by z can also be defined for the case of two-sided admissible function δ on a *bounded* domain $\Omega_{\rm int}$; in this case $\widehat{M}_z$ is bounded. For the case of a simply connected $\Omega_{\rm int}$, one can substitute δ by its inner part δ_i, which comes from a canonical factorization $\delta = \delta_i\delta_e$ in this model. In particular, in the case when $\Omega_{\rm int}$ is the unit disc $\mathbb{D}$, the model operator of Theorem 5.6 becomes exactly the Nagy–Foiaş model operator. In the general case of bounded or unbounded (simply connected) admissible domain $\Omega_{\rm int}$, one can identify $\widehat{M}_z$ with $\gamma(T)$, where $\gamma : \mathbb{D} \to \Omega_{\rm int}$ is a conformal mapping and T is the Nagy–Foiaş model operator, whose Nagy–Foiaş characteristic function is $\delta_i \circ \gamma$ (more precisely, the pure part of this function). We understand $\gamma(T)$ in the sense of the Nagy–Foiaş theory. See Section 5 of [33] for more details.

(ii) Inequality (5.6) implies that clos $\Omega_{\rm int}$ is a so-called K-spectral set of A. As Pisier proved in 1997 (see [24]), the fact that a set T is K-spectral of an operator does not imply the existence of a skew dilation of this operator to a normal operator whose spectrum is contained in ∂T. We refer to [26], [12], [16], [18], [23] and others for more information on K-spectral sets of operators and positive and negative results on similarity.

The existence of invariant subspaces for Banach space operators such that the unit disc is their K-spectral set (with a certain additional condition) has been proved in [2]. In our situation, by applying the results by Nagy and Foiaş ([28], Chapter VII), we can describe all invariant subspaces of operator A under consideration in terms of regular factorizations of δ. By an invariant subspace of A we mean here an invariant subspace of $(A-\lambda I)^{-1}$ for all $\lambda \in \Omega_{\text{ext}}$.

The results by Nagy and Foiaş are also applicable to the lifting of the commutant of A.

It would be interesting to use the results of [14] to give a necessary and sufficient condition for similarity of A to a normal operator. We refer to [5] and [15] for additional information.

(iii) Suppose we have an unbounded operator $A = A_0 + L$, where L has been represented as $L = \mathrm{i}\psi(A_0)F\psi(A_0)$, so that conditions (i)–(vi) on ψ and A_0 are fulfilled. Take any function ψ_1 that satisfies (i)–(iv) and such that $\psi_1 \geqslant \psi$ on $\mathcal{D}(\psi)$. Then $L = \mathrm{i}\psi_1(A_0)F_1\psi_1(A_0)$, where $\|F_1\|_{\text{ess}} \leqslant \|F\|_{\text{ess}}$, so that conditions (i)–(vi) are also fulfilled for ψ_1 and A_0. Hence, whenever our construction yields a model in some parabolic domain, it also gives a model in larger parabolic domains, with other auxiliary operators B and C.

6. PROOFS OF AUXILIARY LEMMAS

Proof of Lemma 2.1. Remind that r' and k were chosen so as to satisfy (2.1). There exists $t_0 > 0$ such that

$$\frac{\varphi(t)}{t} < k \quad \text{for } t \geqslant t_0. \tag{6.1}$$

Let us prove that for these t, the disc $B(t, r'\varphi(t))$ is contained in

$$\Omega_+^{\text{int}} \stackrel{\text{def}}{=} \Omega_\mu^{\text{int}} \cap \{z : \operatorname{Re} z > 0\}.$$

The vertical line $\operatorname{Re} z = t$ divides the disc $B(t, r'\varphi(t))$ into two halves. First notice that the right half-disc is contained in Ω_+^{int}. Indeed, if $z = x + \mathrm{i}y \in B(t, r'\varphi(t))$ and $x \geqslant t$, then $|y| \leqslant r'\varphi(t) \leqslant \mu\varphi(x)$. It remains to prove that the left half-disc is contained in Ω_+^{int}. Consider the triangle T with vertices at points 0 and $\tau_\pm = t \pm \mathrm{i}\mu\varphi(t)$. Since Ω_+^{int} is convex, $\operatorname{int} T \subset \Omega_+^{\text{int}}$. It is easy to check, using (1.7), (6.1) and the second inequality in (2.1) that the distances from the point t to the sides $[0, \tau_\pm]$ of the triangle T are equal to

$$\frac{t\mu\varphi(t)}{\sqrt{t^2 + \mu^2\varphi^2(t)}},$$

which is greater than $r'\varphi(t)$. This proves that the left half of the disc is contained in Ω_+^{int}. We conclude that (2.3) holds for $|t| \geqslant t_0$. The union of discs $B(t, r'\varphi(t))$, $|t| < t_0$ is bounded, and the statement of lemma follows. ∎

Proof of Lemma 2.2. Fix some $t_0 > 0$ such that (6.1) holds. Put $\rho = 1/(2k)$. For a fixed x, divide $\mathcal{D}(\varphi) \setminus (-2R, 2R)$ into a countable union of sets $I_n = I_n(x)$,

$n \geqslant 0$, where

$$I_0 = \{t \in \mathcal{D}(\varphi) : |t| \geqslant 2R,\ |t-x| \leqslant \rho\varphi(x)\},$$
$$I_n = \{t \in \mathcal{D}(\varphi) : |t| \geqslant 2R,\ 2^{n-1}\rho\varphi(x) \leqslant |t-x| \leqslant 2^n\rho\varphi(x)\}, \quad n \geqslant 1.$$

Then $|I_n| \leqslant 2^n\rho\varphi(x)$ for $n \geqslant 1$.

Put

$$\Gamma'_n = \{x + \mathrm{i}y \in \Gamma : x \in I_n\}, \quad n \geqslant 0,$$

and $\Gamma'' = \{x + \mathrm{i}y \in \Gamma : |x| \leqslant 2R\}$. Then $\Gamma = \Big(\bigcup_{n\geqslant 0} \Gamma'_n\Big) \cup \Gamma''$. We parametrize Γ'_n by $z(t) = t \pm \mathrm{i}\mu\varphi(t)$, $t \in I_n$. We have $|\mathrm{d}z(t)|/\mathrm{d}t \leqslant C_1$ on all curves Γ'_n. For $n \geqslant 1$,

$$\int_{\Gamma'_n} \frac{|\mathrm{d}z|}{|x-z|^2} \leqslant 2C_1 \int_{I_n} \frac{\mathrm{d}t}{|x-z(t)|^2} \leqslant 2C_1 \int_{I_n} \frac{\mathrm{d}t}{|x-t|^2} \leqslant \frac{2C_1|I_n|}{2^{2n-2}\rho^2\varphi(x)^2} \leqslant \frac{2^{3-n}C_1}{\rho\varphi(x)}. \tag{6.2}$$

Next, for all $x > 0$ sufficiently large, φ increases on the interval $[x/2, 2x]$, which contains $I_0(x)$ (we use (6.1)). Hence for $t \in I_0(x)$, $\varphi(t) \geqslant \varphi(x/2) \geqslant \varphi(x)/2$. Similar estimates hold for $x < 0$ with large $|x|$, and we obtain that

$$\int_{\Gamma'_0} \frac{|\mathrm{d}z|}{|x-z|^2} \leqslant 2C_1 \int_{I_0} \frac{\mathrm{d}t}{|x-z(t)|^2} \leqslant \frac{8C_1|I_0|}{\mu^2\rho^2\varphi(x)^2} \leqslant \frac{16C_1}{\mu^2\rho\varphi(x)}. \tag{6.3}$$

By (6.2) and (6.3),

$$\int_{\Gamma'} \frac{|\mathrm{d}z|}{|x-z|^2} \leqslant \frac{16C_1}{\mu^2\rho\varphi(x)} + \sum_1^\infty \frac{2^{3-n}C_1}{\rho\varphi(x)} \stackrel{\mathrm{def}}{=} \frac{C_2}{\varphi(x)}.$$

Since $\int_{\Gamma''} |\mathrm{d}z|/|x-z|^2 \sim C_3/|x|^2 \leqslant 1/\varphi(x)$ for large $|x|$, we obtain the statement of lemma. ∎

Acknowledgements. This work was finished under the support of the Ramón and Cajal Programme (2002) by the Ministry of the Science and Technology of Spain, the FEDER and the MEC grants MTM2004-03822, MTM 2005 - 08350-C03-01.

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Received January 23, 2006.