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LOCAL DERIVATIONS AND LOCAL AUTOMORPHISMS ON SOME ALGEBRAS

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ABSTRACT. In this paper, we study some algebras that can be generated, as algebras, by their idempotents and discuss local derivations and local automorphisms on these algebras. We prove that if $\mathcal{L}$ is a commutative subspace lattice and $\mathcal{M}$ is a unital Banach alg$\mathcal{L}$-bimodule, then every bounded local derivation from alg$\mathcal{L}$ into $\mathcal{M}$ is a derivation and that if $\mathcal{A}$ is a nest subalgebra in a factor von Neumann algebra $\mathcal{M}$, then every local derivation from $\mathcal{A}$ into $\mathcal{M}$ is a derivation.

KEYWORDS: *Commutative subspace lattice, derivation, idempotent, local automorphism, local derivation.*

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1. INTRODUCTION

Let $\mathcal{A}$ be a complex Banach algebra and $\mathcal{M}$ be a bimodule $\mathcal{A}$-module, that is a Banach space which is an $\mathcal{A}$-bimodule with $\|ax\| \leqslant \|a\| \, \|x\|$ and $\|xa\| \leqslant \|x\| \, \|a\|$ $(x \in \mathcal{M}, a \in \mathcal{A})$. A *derivation* δ of $\mathcal{A}$ into $\mathcal{M}$ is a linear mapping satisfying $\delta(ab) = a\delta(b) + \delta(a)b$, for $a, b \in \mathcal{A}$. A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$ is called a *local derivation* if for every $a \in \mathcal{A}$ there exists a derivation $\delta_a : \mathcal{A} \to \mathcal{M}$ (depending on a) such that $\delta(a) = \delta_a(a)$.

Let X and Y be complex Hausdorff topological linear spaces and let $B(X, Y)$ be the set of continuous linear mappings from X into Y. When $X = Y$, we write $B(X)$ rather than $B(X, X)$. If $\mathcal{S}$ is a subset of $B(X, Y)$, we said that $\mathcal{S}$ is *reflexive* if T belongs to $\mathcal{S}$ whenever $T \in B(X, Y)$ and $Tx \in [\mathcal{S}x]$ for any x in X, where $[\cdot]$ is the topological closure.

Several authors have considered the relationship between local derivations and derivations on selfadjoint algebras or nonselfadjoint algebras. In [16], Kadison shows that norm-continuous local derivations from a von Neumann into any dual bimodule are derivations. In [15], Johnson extends Kadison's result and proves every local derivation from a C^*-algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is a derivation. He also shows that every local derivation from a C^*-algebra $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is bounded.

In [19], Larson and Sourour study local derivations and automorphisms on $B(X)$ for a Banach space X. For a commutative subspace lattice $\mathcal{L}$ on a finite dimensional Hilbert space H, in [3], [4] Crist considers local derivations and local automorphisms on alg$\mathcal{L}$. In [10], for any separable complex Hilbert space H, we investigate local derivations on some reflexive algebras which contain completely distributive commutative subspace lattice algebras, $\mathcal{J}$-subspace lattice algebras and weakly closed unital algebras of $B(H)$ of infinite multiplicity, and we also consider local automorphisms on completely distributive commutative subspace lattice algebras.

In the following we assume that all topological spaces are Hausdorff spaces. An algebra $\mathcal{A}$ is called a *topological algebra* if $\mathcal{A}$ satisfies:

(i) $\mathcal{A}$ is a topological vector space and
(ii) with the product topology of $\mathcal{A}\times\mathcal{A}$, the map $f:(a,b)\to ab$ is continuous.

Let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. If $\mathcal{M}$ is a topological vector space and $\mathcal{A}$ is a topological algebra such that the module multiplications are separately continuous, i.e.,

(i) if $a_v \to a$, then for any $m \in \mathcal{M}$, $a_v m \to am$, $ma_v \to ma$, and
(ii) if $m_t \to m$, then for any $a \in \mathcal{A}$, $am_t \to am$ and $m_t a \to ma$,

then we say that $\mathcal{M}$ is a *topological $\mathcal{A}$-bimodule*.

If $\mathcal{M}$ is a topological $\mathcal{A}$-bimodule, we denote der$(\mathcal{A},\mathcal{M})$ the set of all continuous derivations from $\mathcal{A}$ into $\mathcal{M}$ and der$(\mathcal{A})$ the set of all continuous derivations from $\mathcal{A}$ into itself. If $\mathcal{A}$ is a topological algebra, we say that $\mathcal{A}$ is *topologically generated* by its idempotents if the subalgebra of $\mathcal{A}$ generated by its idempotents is dense in $\mathcal{A}$.

By a *subspace lattice* on X, we mean a collection $\mathcal{L}$ of closed subspaces of X with (0) and X in $\mathcal{L}$ and such that for every family $\{M_r\}$ of elements of $\mathcal{L}$, both $\bigcap M_r$ and $\bigvee M_r$ belong to $\mathcal{L}$, where $\bigvee$ denotes the closed linear span of $\{M_r\}$. For a subspace lattice, we denote $0_+ = \bigcap\{L \in \mathcal{L} : 0 \subsetneq L\}$ and $X_- = \bigvee\{L \in \mathcal{L} : L \subsetneq X\}$. If $\mathcal{L}$ is a subspace lattice, we let alg$\mathcal{L}$ denote the algebra of all operators on X that leave invariant each element of $\mathcal{L}$.

Throughout the paper, H denotes a separable complex Hilbert space, $K(H)$ denotes the ideal of compact operators, and $F(H)$ denotes the set of finite rank operators in $B(H)$. For convenience we disregard the distinction between a closed subspace and the orthogonal projection onto it.

In this paper, we study algebras that can be generated, as an algebra, by their idempotents. We prove that the class of algebras generated by their idempotents contains many interesting algebras. We obtain that if an algebra $\mathcal{A}$ is generated by its idempotents, then every local derivation from $\mathcal{A}$ into any $\mathcal{A}$-bimodule is a derivation and every surjective 2-local automorphism on $\mathcal{A}$ is an automorphism. In particular, for a unital algebra $\mathcal{A}$, and any $n \geqslant 2$, we have that every local derivation from $M_n(\mathcal{A})$ into any $M_n(\mathcal{A})$-bimodule is a derivation. If $\mathcal{M}$ is a topological $\mathcal{A}$-bimodule and $\mathcal{A}$ is topologically generated by its idempotents, we prove that $\mathrm{der}(\mathcal{A}, \mathcal{M})$ is reflexive. We study the local derivations on a nest algebra in a factor von Neumann algebra. We show that if $\mathcal{N}$ is a nest in a factor von Neumann algebra $\mathcal{M}$, then every local derivation δ from $\mathcal{M} \cap \mathrm{alg}\mathcal{N}$ into $\mathcal{M}$ is a derivation. In particular, δ is bounded. If $\mathcal{L}$ is a commutative subspace lattice on H, we prove that every bounded local derivation from $\mathrm{alg}\mathcal{L}$ into a unital Banach $\mathrm{alg}\mathcal{L}$-bimodule is a derivation. For a completely distributive commutative subspace lattice $\mathcal{L}$, we also show that every surjective 2-local automorphism is an automorphism.

When we finished the paper (except Lemma 2.21 and Theorem 2.22), we found that Samei [25] independently proved Theorem 2.12 by a different method.

This paper is a continuation of [8]. Some definitions and notation can be found in [8].

2. LOCAL DERIVATIONS ON UNITAL BIMODULES

In this section, we assume that all algebras are unital algebras and all bimodules are unital. Now we study the class of algebras that can be generated by their idempotents. As several applications of the results, we consider local derivations on these algebras.

Let $\mathcal{M}$ be an $\mathcal{A}$-bimodule and let $\mathcal{J}$ be an ideal of $\mathcal{A}$. We say that $\mathcal{J}$ is a *separating set* of $\mathcal{M}$ if for any n, m in $\mathcal{M}$, $m\mathcal{J} = \{0\}$ implies $m = 0$ and $\mathcal{J}n = \{0\}$ implies $n = 0$.

We can easily prove the following result.

PROPOSITION 2.1. *Let $\mathcal{V}$ be the class of unital algebras generated by the idempotents. Then:*

(i) *If $\mathcal{A}_1$ and $\mathcal{A}_2$ belong to $\mathcal{V}$, then $\mathcal{A}_1 \oplus \mathcal{A}_2$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$ belong to $\mathcal{V}$.*
(ii) *$\mathcal{A} \in \mathcal{V}$ and $\mathcal{J}$ is an ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ belongs to $\mathcal{V}$.*
(iii) *$\mathcal{V}$ is closed under algebraic direct limits.*

The technique for the proof of part (i) of the following proposition is taken from Theorem 1 of [17].

PROPOSITION 2.2. *Let $\mathcal{W}$ be the class of all unital algebras $\mathcal{B}$ such that for every unital algebra $\mathcal{A}$, $\mathcal{A} \otimes \mathcal{B}$ is generated by its idempotents. The following are true:*

(i) *For any $2 \leqslant n$, $M_n \in \mathcal{W}$, where M_n is the set of all $n \times n$ complex matrices.*
(ii) *If $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{W}$, then $\mathcal{B}_1 \oplus B_2 \in \mathcal{W}$.*
(iii) *If $\mathcal{B} \in \mathcal{W}$ and $\mathcal{J}$ is an ideal of $\mathcal{B}$, then $\mathcal{B}/\mathcal{J} \in \mathcal{W}$.*
(iv) *$\mathcal{W}$ is closed under algebraic direct limits.*

Proof. (i) Let $\mathcal{M}$ be the algebra generated by the idempotents of $M_n(\mathcal{A})$. Suppose that $A = (a_{ij})_{n\times n} \in M_n(\mathcal{A})$. If $\text{trace}(A) = \sum_{i=1}^{n} a_{ii} = 0$, by Lemma 2 of [11], A is a linear combination of some idempotents of $M_n(\mathcal{A})$; so $A \in \mathcal{M}$. For $a_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$, we abbreviate the matrix

$$\begin{pmatrix} 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & 0 \\ a_1 & a_2 & \ldots & a_n \end{pmatrix}$$

by $\{a_1, \ldots, a_n\}$. Let $I_n = \text{diag}(I, \ldots, I)$ and

$$P = \begin{pmatrix} I & I & & & & \\ & 0 & 0 & & & \\ & & I & I & & \\ & & & 0 & 0 & \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n}, \quad Q = \begin{pmatrix} 0 & 0 & & & & \\ & I & I & & & \\ & & 0 & 0 & & \\ & & & I & I & \\ & & & & \ddots & \ddots \end{pmatrix}_{n\times n}$$

in $M_n(\mathcal{A})$. Since P and Q are idempotents, P, $Q \in \mathcal{M}$. Let $T = P + Q - I_n$. We have that for any $a \in \mathcal{A}$, $\{a, 0, \ldots, 0\}T^{n-1} = \{0, \ldots, 0, a\} \in \mathcal{M}$. Since T and $\{a, 0, \ldots, 0\}$ belong to $\mathcal{M}$, it follows that for any $a \in \mathcal{A}$, $\{0, \ldots, 0, a\} \in \mathcal{M}$. Since the shift matrix $\{I, 0, \ldots, 0\} + T$ and all matrices $\{0, \ldots, 0, a\}$ generate $M_n(\mathcal{A})$, it follows $\mathcal{M} = M_n(\mathcal{A})$.

(ii) This is obvious.

(iii) Since $\mathcal{A} \otimes \mathcal{B}/(\mathcal{A} \otimes \mathcal{J})$ is isomorphic to $\mathcal{A} \otimes (\mathcal{B}/\mathcal{J})$, we have that (iii) is true.

(iv) This is the same as saying if $\{B_i : i \in I\}$ is an increasingly directed family in $\mathcal{W}$ with the same identity then $\bigcup_{i\in I} B_i \in \mathcal{W}$. Since $\mathcal{A} \otimes \big(\bigcup_{i\in I} \mathcal{B}_i\big) = \bigcup_{i\in I} (\mathcal{A} \otimes \mathcal{B})$, it follows that (iv) is true. ∎

PROPOSITION 2.3. *Suppose that $\mathcal{N}$ is a nest in a von Neumann algebra $\mathcal{M}$ and $\mathcal{A} = \mathcal{M} \cap \text{alg}\mathcal{N}$. Then* $\text{span}\{A : A^2 = A, A \in \mathcal{A}\}$ *is weak* dense in* $\mathcal{A}$.

Proof. Let $\phi_u = \text{span}\{Nm(I - N) : m \in \mathcal{M}, N \in \mathcal{N}\}$ and $\mathcal{D}_{\mathcal{N}} = (\text{alg}\mathcal{N}) \cap (\text{alg}\mathcal{N})^*$. By Proposition 2.1 of [8], we have that $\mathcal{M} \cap \mathcal{D}_{\mathcal{N}} + \phi_u$ is weak* dense in $\mathcal{A}$. For $m \in \mathcal{M}$, $N \in \mathcal{N}$, let $T = Nm(I - N)$. Since $\mathcal{N}$ is a nest, we have that T belongs to $\mathcal{A}$ and $T = N - (N - Nm(I - N))$, a difference of two idempotents.

Since $\mathcal{M} \cap \mathcal{D}_{\mathcal{N}}$ is a von Neumann algebra, it is topologically generated by its idempotents in the weak* topology. So $\mathcal{A}$ is topologically generated by its idempotents in the weak* topology. ∎

Suppose that $\{H_i : i \in \mathbb{N}\}$ is a family of Hilbert spaces and u_i is a unit vector in H_i for each i in $\mathbb{N}$. Let $H = \bigotimes_{i=1}^{\infty} (H_i, u_i)$ be an infinite tensor product

Hilbert space associated with H_i and u_i; define $\mathcal{A} = \mathcal{A}_1 \otimes^{\sigma}_{(u_1)} \mathcal{A}_2 \otimes^{\sigma}_{(u_2)} \cdots$ to be the weak* closed operator algebra on H generated by the elementary operator $\bigotimes_{i=1}^{\infty} A_i$, where $A_i \in \mathcal{A}_i$ and $A_i = I$ for all i in a cofinite set. We call $\mathcal{A}$ the *tensor product algebra* associated with $\mathcal{A}_i$ and the unit vectors u_i.

PROPOSITION 2.4. *Suppose that $\mathcal{A}_i = \mathrm{alg}\mathcal{L}_i$ with $\mathcal{L}_i$ a completely distributive commutative subspace lattice on H_i and u_i is a unit vector in H_i for $i = 1, 2, \ldots$. Then the tensor product algebra $\mathcal{A}$ associated with $\mathcal{A}_i$ and u_i is generated by idempotents in the weak* topology.*

Proof. By Lemma 2.3 of [10], it follows that every rank one operator of $\mathcal{A}_i$ is a linear combination of some idempotents of $\mathcal{A}_i$. For $j \neq i$, let $A_j = I$ and A_i be a rank one operator in $B(H_i)$; define $T_i = \bigotimes_{j=1}^{\infty} A_j$. By Theorem 1 of [15], it follows that span$\{A : A$ is a rank one operator in $\mathcal{A}_i\}$ is weak* dense in $\mathcal{A}_i$. Since T_i is a linear combination of some idempotents of $\mathcal{A}$, by the definition of $\mathcal{A}$, it follows that $\mathcal{A}$ is generated by its idempotents in the weak* topology. ∎

To show our main results, we need several lemmas.

LEMMA 2.5 ([10]). *Let δ be a linear mapping from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. Then the following are equivalent:*

(i) *The mapping $(I - P)\delta(PaQ)(I - Q) = 0$, for every $a \in \mathcal{A}$ and any idempotents $P, Q \in \mathcal{A}$.*

(ii) *δ satisfies $\delta(PaQ) = \delta(Pa)Q + P\delta(aQ) - P\delta(a)Q$, for every $a \in \mathcal{A}$ and any idempotents $P, Q \in \mathcal{A}$.*

Let δ be a linear mapping from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$. We say that δ satisfies the Condition $(*)$ if

$$\delta(PaQ) = \delta(Pa)Q + P\delta(aQ) - P\delta(a)Q \quad \text{and} \quad \delta(I) = 0$$

hold for each a and any idempotents P, Q in $\mathcal{A}$.

LEMMA 2.6. *Suppose that δ is a linear mapping from an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ satisfying the Condition $(*)$. Then for any idempotents $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ in $\mathcal{A}$ and every $a \in \mathcal{A}$*

$$\begin{aligned} \delta(P_1 \cdots P_n a Q_1 \cdots Q_m) = \delta(P_1 \cdots P_n a)Q_1 \cdots Q_m + P_1 \cdots P_n \delta(aQ_1 \cdots Q_m) \\ - P_1 \cdots P_n \delta(a) Q_1 \cdots Q_m. \end{aligned} \tag{2.1}$$

Proof. We first show that for any positive integer

$$\delta(P_1 \cdots P_n a Q) = \delta(P_1 \cdots P_n a)Q + P_1 \cdots P_n \delta(aQ) - P_1 \cdots P_n \delta(a)Q. \tag{2.2}$$

If $n = 1$, by the condition $(*)$, (2.2) is obvious.

Suppose if $n = k$, (2.2) is true.

For $n = k+1$, by the Condition $(*)$, it follows

$$\begin{aligned}\delta(P_1\cdots P_{k+1}aQ) &= \delta(P_1P_2\cdots P_{k+1}a)Q + P_1\delta(P_2\cdots P_{k+1}aQ) - P_1\delta(P_2\cdots P_{k+1}a)Q\\ &= \delta(P_1\cdots P_{k+1}a)Q + P_1[\delta(P_2\cdots P_{k+1}a)Q + P_2\cdots P_{k+1}\delta(aQ)\\ &\quad - P_2\cdots P_{k+1}\delta(a)Q] - P_1\delta(P_2\cdots P_{k+1}a)Q\\ &= \delta(P_1\cdots P_{k+1}a)Q + P_1\cdots P_{k+1}\delta(aQ) - P_1\cdots P_{k+1}\delta(a)Q.\end{aligned}$$

Now we show that (2.1) is true. For $m = 1$, by (2.2) we have that (2.1) is true. Suppose that if $m = k$, (2.1) is true.

For $m = k+1$, by the induction assumption, the Condition $(*)$ and (2.2), we have

$$\begin{aligned}&\delta(P_1\cdots P_naQ_1\cdots Q_kQ_{k+1})\\ &=\delta(P_1\cdots P_naQ_1\cdots Q_k)Q_{k+1}+P_1\cdots P_n\delta(aQ_1\cdots Q_kQ_{k+1})-P_1\cdots P_n\delta(aQ_1\cdots Q_k)Q_{k+1}\\ &=[\delta(P_1\cdots P_na)Q_1\cdots Q_k+P_1\cdots P_n\delta(aQ_1\cdots Q_k)-P_1\cdots P_n\delta(a)Q_1\cdots Q_k)]Q_{k+1}\\ &\quad + P_1\cdots P_n\delta(aQ_1\cdots Q_kQ_{k+1}) - P_1\cdots P_n\delta(aQ_1\cdots Q_k)Q_{k+1}\\ &=\delta(P_1\cdots P_na)Q_1\cdots Q_{k+1}+P_1\cdots P_n\delta(aQ_1\cdots Q_{k+1})-P_1\cdots P_n\delta(a)Q_1\cdots Q_{k+1}. \quad ∎\end{aligned}$$

The following theorem generalizes Theorem 2.2 of [10]. The proof uses Lemma 2.6 and arguments similar to those in the proof of Theorem 2.2 of [10], and is left to the reader.

THEOREM 2.7. *Let $\mathcal{J}$ be a separating set of $\mathcal{M}$. Suppose that $\mathcal{J}$ is contained in the algebra generated by all idempotents in $\mathcal{A}$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the Condition $(*)$ (in particular, if δ is a local derivation), then δ is a derivation.*

COROLLARY 2.8. *If $\mathcal{A}$ is any algebra and $\mathcal{B} \in \mathcal{W}$, then every local derivation from $\mathcal{A}\otimes\mathcal{B}$ into itself is a derivation. In particular, for $2 \leqslant n$, every local derivation from $M_n(\mathcal{A})$ into an $M_n(\mathcal{A})$-bimodule $\mathcal{M}$ is a derivation.*

An algebra $\mathcal{B}$ is called a *local matrix algebra* if any finite subset of $\mathcal{B}$ can be embedded in a subalgebra which is a matrix algebra $M_n(\mathcal{A})$ for some $2 \leqslant n$.

COROLLARY 2.9. *If for any $a, b \in \mathcal{A}$, there exists a unital subalgebra $\mathcal{B}$ of $\mathcal{A}$ containing a and b such that $\mathcal{B}$ is isomorphic to a matrix algebra, then every local derivation from $\mathcal{A}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$ is a derivation.*

COROLLARY 2.10. *Suppose that $\mathcal{L}$ is a subspace lattice and $2 \leqslant n$. If δ is a local derivation from $\mathcal{A} = \mathrm{alg}\mathcal{L}^{(n)}$ into an $\mathcal{A}$-bimodule $\mathcal{M}$, then δ is a derivation.*

THEOREM 2.11. *Suppose that $\{\mathcal{J}_\lambda : \lambda \in \Lambda\}$ is a collection of two-sided ideals in $\mathcal{A}$ such that:*

(i) *$\mathcal{A}/\mathcal{J}_\lambda$ is generated by its idempotents and*

(ii) $\bigcap_{\lambda\in\Lambda} \mathcal{J}_\lambda = 0.$

If δ is a linear mapping from $\mathcal{A}$ into itself satisfying Condition $(*)$ and $\delta(\mathcal{J}_\lambda) \subseteq \mathcal{J}_\lambda$, then δ is a derivation.

Proof. For each λ in Λ, δ induces a linear mapping δ_λ on $\mathcal{A}/\mathcal{J}_\lambda$ satisfying the Condition $(*)$. By Theorem 2.7 and assumptions, it follows that δ_λ is a derivation. Hence for any a, b in $\mathcal{A}$, we have that $\delta(ab) - a\delta(b) - \delta(a)b \in \mathcal{J}_\lambda$, for any $\lambda \in \Lambda$. It follows that from (ii), $\delta(ab) = \delta(a)b + a\delta(b)$. ∎

THEOREM 2.12. *Suppose that $\mathcal{M}$ is a topological $\mathcal{A}$-bimodule and $\mathcal{A}$ is topologically generated by its idempotents. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the Condition $(*)$ (in particular, if δ is a local derivation), then δ is a derivation.*

Proof. If $a = \sum_{i=1}^{m} \alpha_i \prod_{j=1}^{t_i} P_j^{(i)}$, $b = \sum_{s=1}^{n} \beta_s \prod_{l=1}^{v_s} Q_l^{(s)}$, where $P_j^{(i)}, Q_l^{(s)}$ are idempotents of $\mathcal{A}$, α_i, $\beta_s \in \mathbb{C}$, by Lemma 2.6, it follows $\delta(ab) = \delta(a)b + a\delta(b)$. Since δ is continuous and $\mathcal{A}$ is topologically generated by its idempotents, we have that for any a, b in $\mathcal{A}$, $\delta(ab) = \delta(a)b + a\delta(b)$. ∎

By Proposition 2.3 and Theorem 2.12, we have

COROLLARY 2.13. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Proposition* 2.3. *If δ is a weak* continuous local derivation from $\mathcal{A}$ into $\mathcal{M}$, then δ is a derivation.*

COROLLARY 2.14. *Let $\mathcal{A}$ and $\mathcal{M}$ be as in Theorem* 2.7. *Then* $\mathrm{der}(\mathcal{A}, \mathcal{M})$ *is reflexive.*

Proof. Suppose that δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying for any $x \in \mathcal{A}$,

$$\delta(x) \in [\mathrm{der}(\mathcal{A}, \mathcal{M})x]. \tag{2.3}$$

By (2.3), there exists a sequence δ_n(depending on x) in $\mathrm{der}(\mathcal{A}, \mathcal{M})$ such that

$$\lim_{n\to\infty} \delta_n(x) = \delta(x). \tag{2.4}$$

Let P, Q be idempotents of $\mathcal{A}$. By (2.4), for any $a \in \mathcal{A}$, take $x = PaQ$, it follows that $(I - P)\delta(PaQ)(I - Q) = 0$. By Lemma 2.4 and Theorem 2.7, it follows that δ is a derivation. Hence $\mathrm{der}(\mathcal{A}, \mathcal{M})$ is reflexive. ∎

REMARK 2.15. Let $\mathcal{M}_2$ be the algebra of 2×2 matrices over $L^\infty[0, 1]$. By [7], we know that $\mathcal{M}_2$ is not the linear span of its projections. By Theorem 3.6 of [21], we have that every idempotent in $\mathcal{M}_2$ is a linear combination of at most 5 projections in $\mathcal{M}_2$. Hence $\mathcal{M}_2$ is not the linear span of its idempotents. By Theorem 2.7, $\mathcal{M}_2$ is generated by its idempotents. Thus Theorem 2.7 improves Theorem 2.2 of [10].

Now we consider the local derivations on a reflexive subalgebra in a factor von Neumann algebra.

LEMMA 2.16 ([7]). *A von Neumann algebra $\mathcal{M}$ is generated by its projections if and only if $\mathcal{M}$ has no infinite dimensional abelian summand.*

THEOREM 2.17. *Let $\mathcal{L}$ be a subspace lattice in a factor von Neumann algebra $\mathcal{M}$ with $0_+ \neq 0$ and $H_- \neq H$. If δ is a linear mapping (not necessarily bounded) from $\mathcal{M} \cap \mathrm{alg}\mathcal{L}$ into $\mathcal{M}$ such that $(*)$ holds (in particular, if δ is a local derivation), then δ is a derivation.*

Proof. Let E be the projection on 0_+ and let F be the projection on H_-.

Define $\mathcal{J} = \mathrm{span}\{Em_1, m_2(I-F) : m_1, m_2 \in \mathcal{M}\}$. Since $Em_1 = Em_1E + Em_1(I-E)$, $m_2(I-F) = Fm_2(I-F) + (I-F)m_2(I-F)$ and $\mathcal{M}$ is a factor, we have that $\mathcal{J} \subseteq \mathcal{A}$ and by Lemma 2.15, $\mathcal{J}$ is in the algebra generated by its idempotents of $\mathcal{M} \cap \mathrm{alg}\mathcal{L}$. It is easy to show that $\mathcal{J}$ is an ideal of $\mathcal{M} \cap \mathrm{alg}\mathcal{L}$. We claim that $\mathcal{J}$ is a separating set of $\mathcal{M}$.

For any $m \in \mathcal{M}$, suppose that $m\mathcal{J} = 0$. We have $m\mathcal{M}(I-F) = 0$. Since $\mathcal{M}$ is a factor, it follows that $m = 0$. If $m \in \mathcal{M}$ and $\mathcal{J}m = 0$, similarly we can show that $m = 0$. By Theorem 2.7, we have that δ is a derivation. ∎

If $\mathcal{M}$ is a factor von Neumann algebra, we can improve Corollary 2.5.

THEOREM 2.18. *Suppose that $\mathcal{N}$ is a nest in a factor von Neumann algebra $\mathcal{M}$ on H and $\mathcal{A} = \mathcal{M} \cap \mathrm{alg}\mathcal{N}$. Let δ be a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying the Condition $(*)$. Then δ is a derivation. In particular, δ is bounded.*

Proof. Let $E = 0_+$ and $F = H_-$.

We divide the proof into four cases.

Case 1. Suppose that $E \neq 0$ and $F \neq H$. By Theorem 2.16, we have the theorem is true.

Case 2. Suppose that $E = 0$ and $F = H$. Define $\mathcal{J} = \mathrm{span}\{Nm(I-N) : m \in \mathcal{M}, N \in \mathcal{N}\}$. $\mathcal{J}$ is an ideal of $\mathcal{M} \cap \mathrm{alg}\mathcal{N}$ and by $Nm(I-N) = N - (N - Nm(I-N))$, $\mathcal{J}$ is contained in the linear span of the idempotents of $\mathcal{M} \cap \mathrm{alg}\mathcal{N}$. In the following we show that $\mathcal{J}$ is a separating set of $\mathcal{M}$.

For any $m \in \mathcal{M}$, suppose that $m\mathcal{J} = 0$. Hence

$$mN\mathcal{M}(I-N) = 0, \quad \text{for any } N \in \mathcal{N}. \tag{2.5}$$

Since $F = H$, we can choose a sequence $\{N_i\}$ with $N_i \subseteq N_{i+1} \subsetneq I$ satisfying $N_i \to I$ in the strong operator topology. By (2.5) and $\mathcal{M}$ is a factor, it follows that $mN_i = 0$ for any i. Hence $m = 0$. If $m \in \mathcal{M}$ and $\mathcal{J}m = 0$, by $E = 0$, similarly we can show that $m = 0$. Hence $\mathcal{J}$ is a separating set of $\mathcal{M}$.

Case 3. Suppose that $F = H$ and $E \neq 0$. Define

$$\mathcal{J} = \mathrm{span}\{Em_1, Nm_2(I-N) : m_1, m_2 \in \mathcal{M}, N \in \mathcal{N}\}.$$

Then $\mathcal{J}$ is an ideal of $\mathcal{M} \cap \mathrm{alg}\mathcal{N}$ and $\mathcal{J}$ is contained in the linear span of the idempotents of $\mathcal{M} \cap \mathrm{alg}\mathcal{N}$. We claim that $\mathcal{J}$ is a separating set of $\mathcal{M}$.

For $m \in \mathcal{M}$, suppose that $\mathcal{J}m = 0$. Since $E\mathcal{M} \subseteq \mathcal{J}$, from the proof of Case 1, we know that $m = 0$.

For any $m \in \mathcal{M}$, suppose that $m\mathcal{J} = 0$. Since $F = H$, from the proof of Case 2, we know that $m = 0$.

Case 4. Suppose that $F \neq H$ and $E = 0$. Define

$$\mathcal{J} = \text{span}\{m_1(I - F), Nm_2(I - N) : m_1, m_2 \in \mathcal{M}, N \in \mathcal{N}\}.$$

The proof is similar to that in Case 3.

By the above proof, δ is a derivation and by Proposition II1 of [22], we have δ is bounded. ∎

REMARK 2.19. (i) From the proofs of Theorems 2.16 and 2.17, we know that if $\mathcal{B}$ is a subalgebra of $\mathcal{M}$, $\mathcal{B}$ is an $\mathcal{A}$-bimodule, $\mathcal{J}$ is a separating set of $\mathcal{B}$, and δ is a linear mapping from $\mathcal{A} \cap \mathcal{M}$ into $\mathcal{B}$ satisfying Condition $(*)$, then the inclusions of Theorems 2.16 and 2.17 are true.

(ii) In [13], Jing Wu shows that if $\mathcal{L}$ is a subspace lattice with $0_+ \neq 0$ and $X_- \neq X$ for a Banach space X, then every local derivation from alg$\mathcal{L}$ into itself is a derivation.

THEOREM 2.20. *Let $\mathcal{L}$ be a commutative subspace lattice and let $\mathcal{A} = \text{alg}\mathcal{L}$. Suppose that $\mathcal{M}$ is a unital Banach right (respectively left) alg$\mathcal{L}$-module. If T is a bounded linear mapping from $\mathcal{A}$ into $\mathcal{M}$ satisfying*

$$T(aP) = n_{aP}P \quad (\textit{respectively } T(Pa) = Pn_{Pa}) \tag{2.6}$$

for any a in $\mathcal{A}$ and any idempotent P in $\mathcal{A}$, where n_{aP} (respectively n_{Pa}) depends on aP (respectively Pa) and belongs to $\mathcal{M}$, then $Tx = T(I)x$ (respectively $Tx = xT(I)$), for any x in $\mathcal{A}$.

Proof. Suppose that $\mathcal{M}$ is a unital Banach right alg$\mathcal{L}$-module. Let $\mathcal{J} = \text{span}\{PaP^\perp : a \in \mathcal{A}, P \in \mathcal{L}\}$. We have that $\mathcal{J}$ is an ideal of $\mathcal{A}$. Since $PmP^\perp = P - (P - PmP^\perp)$ is a difference of two idempotents, we have that every element in $\mathcal{J}$ is a linear combination of some idempotents in $\mathcal{J}$.

Let Q be the projection onto the subspace $\overline{\mathcal{J}H}$. It is easy to show that Q belongs to $\mathcal{L}$.

For any $a \in \mathcal{A}$ and for any $P \in \mathcal{L}$, since $Q^\perp PaP^\perp = 0 = PQ^\perp aP^\perp$, it follows that $P^\perp a^* Q^\perp P \in \text{alg}\mathcal{L}$. Hence $Q^\perp a \in \text{alg}\mathcal{L}^\perp$ and $Q^\perp a \in (\text{alg}\mathcal{L}) \cap (\text{alg}\mathcal{L})^* = \mathcal{L}'$. By (2.6), we have that for any $a \in \text{alg}\mathcal{L}$ and any idempotent $P \in \text{alg}\mathcal{L}$

$$T(a)P = [T(aP) + T(a - aP)]P = [T(aP) + T(aP^\perp)]P = T(aP). \tag{2.7}$$

Since every element in $\mathcal{J}$ is a linear combination of some idempotents in $\mathcal{J}$, by (2.7), it follows that for any $t \in \mathcal{J}$ and any $a \in \mathcal{A}$,

$$T(at) = T(a)t. \tag{2.8}$$

By (2.8), it follows that for a, b in $\mathcal{A}$,

$$T(abPmP^\perp) = T(aPbPmP^\perp) = T(a)bPmP^\perp \quad \text{and} \quad T(abPmP^\perp) = T(ab)PmP^\perp.$$

Hence for any a, b in $\mathcal{A}$,

$$[T(ab) - T(a)b]PmP^\perp = 0, \quad [T(ab) - T(a)b]Q = 0. \tag{2.9}$$

Since $Q^\perp bQ^\perp$ belongs to $\mathcal{L}'$ and $\mathcal{L}'$ is a von Neumann algebra, it follows that there is a sequence $\{x_n\}$ such that $\lim_{n\to\infty} x_n = Q^\perp bQ^\perp$ in the operator norm, where x_n is a linear combination of some projections in $\mathcal{L}'$. Since T is bounded, by (2.8), we have $T(aQ^\perp bQ^\perp) = T(a)Q^\perp bQ^\perp$, for any $a, b \in \mathrm{alg}\mathcal{L}$.

Hence

$$\begin{aligned} T(ab)Q^\perp &= T(abQ^\perp) = T(aQ^\perp bQ^\perp) + T(aQbQ^\perp) \\ &= T(a)Q^\perp bQ^\perp + T(a)QbQ^\perp = T(a)bQ^\perp. \end{aligned} \tag{2.10}$$

By (2.9) and (2.10), we have that $T(a) = T(I)a$.

Suppose that $\mathcal{M}$ is a unital Banach left $\mathrm{alg}\mathcal{L}$-module. Similarly, we can prove that $T(a) = aT(I)$. ∎

COROLLARY 2.21. *Let $\mathcal{L}$ be a commutative subspace lattice and let $\mathcal{A} = \mathrm{alg}\mathcal{L}$. If T is a bounded linear mapping from $\mathcal{A}$ into itself satisfying*

$$T(a) = n_a a \quad (\textit{respectively } T(a) = an_a)$$

for every a in $\mathcal{A}$, where n_a depends on a and belongs to $\mathcal{A}$, then $T(x) = T(I)x$ (respectively $T(x) = xT(I)$), for every x in $\mathcal{A}$.

DEFINITION 2.22. Let $\mathcal{B}$ be a Banach algebra with identity and let $\mathcal{M}$ be a unital Banach right $\mathcal{B}$-module. Suppose that T is a bounded linear transformation from $\mathcal{B}$ into $\mathcal{M}$. T is said to be a *left* (respectively *right*) *multiplier*, if $T(a) = T(I)a$ (respectively $T(a) = aT(I)$). T is called an *approximately local left* (respectively *right*) *multiplier* if for each $a \in \mathcal{B}$, there exists a sequence of left (respectively right) multipliers $\{T_{n,a}\}$ from $\mathcal{B}$ into $\mathcal{M}$ such that $\lim_{n\to\infty} T_{n,a}(a) = T(a)$.

LEMMA 2.23. *Let $\mathcal{L}$ be a commutative subspace lattice and $\mathcal{M}$ be a unital Banach left $\mathrm{alg}\mathcal{L}$-module. If T is an approximately local right multiplier from $\mathrm{alg}\mathcal{L}$ into $\mathcal{B}$, then $T(a) = aT(I)$ for any $a \in \mathrm{alg}\mathcal{L}$.*

Proof. By the assumption, for any $a \in \mathrm{alg}\mathcal{L}$ and any idempotent P in $\mathrm{alg}\mathcal{L}$, there exists a sequence of right multipliers $\{T_{n,Pa}\}$ such that

$$T(Pa) = \lim_{n\to\infty} T_{n,Pa}(Pa) = \lim_{n\to\infty} T_{n,Pa}(PPa) = P\lim_{n\to\infty} T_{n,Pa}(Pa) = P(\lim_{n\to\infty} T_{n,Pa}(Pa)).$$

By Theorem 2.18, it follows that $T(a) = aT(I)$, for any $a \in \mathcal{L}$. ∎

The proof technique of the following theorem is almost the same as the proof of Theorem 3.2 in [25]. For completeness, we give its proof.

THEOREM 2.24. *Let $\mathcal{L}$ be a commutative subspace lattice and let $\mathcal{M}$ be a unital Banach $\mathrm{alg}\mathcal{L}$-bimodule. If δ is a bounded local derivation from $\mathrm{alg}\mathcal{L}$ into $\mathcal{M}$, then δ is a derivation.*

Proof. For a in $\mathrm{alg}\mathcal{L}$, let $\mathcal{Y}$ be the norm closure of $a\mathcal{M}$. Then $\mathcal{Y}$ is a unital Banach right $\mathrm{alg}\mathcal{L}$-module and $\mathcal{M}/\mathcal{Y}$ is a unital Banach right $\mathrm{alg}\mathcal{L}$-module.

Define

$$\widetilde{\delta} : \mathrm{alg}\mathcal{L} \to \mathcal{M}/\mathcal{Y}$$

by $\widetilde{\delta}(b) = \delta(ab) + \mathcal{Y}$, for any b in $\mathrm{alg}\mathcal{L}$. It is easy to show that $\widetilde{\delta}$ is bounded. Since δ is a local derivation, there is derivation δ_{ab} such that $\delta(ab) = \delta_{ab}(ab) = \delta_{ab}(a)b + a\delta_{ab}(b)$.

So

$$\widetilde{\delta}(b) = \delta_{ab}(a)b + \mathcal{Y} = (\delta_{ab}(a) + \mathcal{Y})b.$$

Thus $\widetilde{\delta}$ is a local left multiplier. By Theorem 2.18, $\widetilde{\delta}(b) = \widetilde{\delta}(I)b$, for any $b \in \mathrm{alg}\mathcal{L}$. Therefore $\delta(ab) - \delta(a)b \in \mathcal{Y}$.

Hence for any $b \in \mathrm{alg}\mathcal{L}$, there exists a sequence $\{x_n\}$ in $\mathcal{Y}$ such that

$$\delta(ab) - \delta(a)b = \lim_{n\to\infty} ax_n. \tag{2.11}$$

Fix b, define

$$\overline{\delta} : \mathcal{A} \to \mathcal{M}$$

by $\overline{\delta}(a) = \delta(ab) - \delta(a)b$ for any $a \in \mathrm{alg}\mathcal{L}$. It is easy to prove that $\overline{\delta}$ is bounded. By (2.11), it follows that $\overline{\delta}$ is a bounded approximately local right multiplier. By Lemma 2.21, it follows that $\overline{\delta}(a) = a\overline{\delta}(I)$. Thus $\delta(ab) - \delta(a)b = a(\delta(b) - \delta(I)b)$. Since δ is a local derivation, we have $\delta(I) = 0$. So $\delta(ab) = \delta(a)b + a\delta(b)$. ∎

3. LOCAL DERIVATIONS ON ANY BIMODULES

In this section, we assume that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is any $\mathcal{A}$-bimodule (not necessary a unital bimodule).

THEOREM 3.1. *Let $\mathcal{M}$ be any $\mathcal{A}$-bimodule and $\mathcal{J}$ be a left ideal of $\mathcal{A}$ that is contained in the algebra generated by its idempotents such that for every $x \in \mathcal{M}$, $x\mathcal{J} = 0$ implies $x = 0$. Then for every linear mapping $T : \mathcal{A} \to \mathcal{M}$ such that*

$$T(bP) = n_{bP}P \tag{3.1}$$

for any idempotents P in $\mathcal{A}$, where n_{bP} depends on bP, we have $T(x) = T(1)x$ for every $x \in \mathcal{A}$.

Proof. Suppose $a \in \mathcal{A}$ and P is an idempotent. By (3.1), it follows that

$$T(a)P = [T(aP) + T(a - aP)]P = T(aP).$$

A simple induction shows that if $P_1, \ldots, P_n$ are idempotents in $\mathcal{A}$, then

$$T(aP_1 \cdots P_n) = T(a)P_1 \cdots P_n. \tag{3.2}$$

By (3.2), we see that $T(ax) = T(a)x$ for every x in the algebra generated by the idempotents of $\mathcal{A}$, and thus for every element of $\mathcal{J}$.

Hence if $a \in \mathcal{A}$ and $x \in \mathcal{J}$, we have (since $ax \in \mathcal{J}$),

$$T(a)x = T(ax) = T(I(ax)) = T(I)ax. \tag{3.3}$$

(3.3) implies $[T(a) - T(I)a]\mathcal{J} = 0$. By the assumption, $T(a) = T(I)a$ for any $a \in \mathcal{A}$. ∎

With a proof similar to the proof of Theorem 3.1, we can show that

THEOREM 3.2. *Let $\mathcal{M}$ be any $\mathcal{A}$-bimodule and $\mathcal{J}$ be a right ideal of $\mathcal{A}$ that is contained in the algebra generated by its idempotents such that for every $x \in \mathcal{M}$, $\mathcal{J}x = 0$ implies $x = 0$. Then for every linear mapping $T : \mathcal{A} \to \mathcal{M}$ such that*

$$T(Pb) = Pn_{Pb}$$

for any idempotents P in $\mathcal{A}$, where n_{Pb} depends on Pb, we have $T(x) = xT(1)$ for every $x \in \mathcal{A}$.

THEOREM 3.3. *Let $\mathcal{A}$ be a unital algebra and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. If $\mathcal{J}$ is a separating set of $\mathcal{M}$, $\mathcal{J}$ is contained in the algebra generated by all idempotents in $\mathcal{A}$ and δ is a local derivation from $\mathcal{A}$ into $\mathcal{M}$, then δ is a derivation.*

Proof. Define $L : \mathcal{A} \to \mathcal{M}$ by $L(a) = Ia$ and $R : \mathcal{A} \to \mathcal{M}$ by $R(a) = aI$. Let $\mathcal{M}_0 = I\mathcal{M}I$. Then $\mathcal{M}_0$ is a unital $\mathcal{A}$-bimodule.

Since δ is a local derivation, it follows that $\delta(I) = \delta_I(I \cdot I) = \delta_I(I)I + I\delta_I(I)$. Hence $I\delta(I)I = 0$.

Similar to the proof of Lemma 3 of [16], we can show that $RL\delta$ is a local derivation from $\mathcal{A}$ into the unital $\mathcal{A}$-bimodule $\mathcal{M}_0$. By Theorem 2.7, we have that $RL\delta$ is a derivation.

Let $T = (i - L)R\delta$, where i is the identity mapping from $\mathcal{M}$ into itself. For any $a \in \mathcal{A}$, let δ_a be a local derivation such that $\delta(a) = \delta_a(a)$. We have that

$$\begin{aligned} T(a) &= \delta(a)I - I\delta(a)I = \delta_a(Ia)I - I\delta_a(a)I \\ &= \delta_a(I)aI + I\delta_a(a)I - I\delta_a(a)I = \delta_a(a)a. \end{aligned} \tag{3.4}$$

By Theorem 3.1, we have that $T(a) = T(I)a$. By (3.4), it follows that $T(I) = \delta_I(I)I = \delta(I)I$. So we have that $T(a) = \delta(I)a$.

Since $a\delta(I)b = aI\delta(I)Ib = 0$, we have that

$$T(ab) = \delta(I)ab = \delta(I)ab + a\delta(I)b = T(a)b + aT(b).$$

Thus T is a derivation. Let $K = L(i - R)$. Similarly, we can show that $K(a) = a\delta(I)$, by Theorem 3.2, from which, we have K is a derivation.

By

$$\begin{aligned} (i - L)(i - R)\delta(a) = \delta(Ia) - \delta(Ia) - I\delta(a) - I\delta(a)I \\ - \delta_a(I)a + I\delta_a(a) - [\delta_a(I)a - I\delta_a(a)I] - I\delta(a) = 0 \end{aligned}$$

and $\delta = LR\delta + K + T + (i - L)(i - R)\delta$, we have that δ is a derivation. ∎

REMARK 3.4. Suppose that $\mathcal{A}$ is an algebra without identity and $\mathcal{M}$ is a topological $\mathcal{A}$-bimodule. If $\widetilde{\mathcal{A}}$ is the algebra obtained by adjointing an identity I to $\mathcal{A}$, then $\mathcal{M}$ becomes a unital $\widetilde{\mathcal{A}}$-bimodule by defining $Ix = xI = x$ for all x in $\mathcal{M}$. If $\mathcal{M}$ is a topological $\mathcal{A}$-bimodule, then we can make $\mathcal{M}$ a unital topological

$\mathcal{A}$-bimodule. By this way, we may generalize several results in Section 2 and Theorems 3.1, 3.2 and 3.3. We omit it.

4. LOCAL AUTOMORPHISMS

DEFINITION 4.1. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital algebras. Let α be a linear mapping from $\mathcal{A}$ into $\mathcal{B}$. We say that α is a *zero-product preserving mapping* if $\alpha(a)\alpha(b) = 0$ in $\mathcal{B}$ whenever $ab = 0$ in $\mathcal{A}$.

PROPOSITION 4.2. *Suppose that $\mathcal{L}$ is a completely distributive commutative subspace lattice and α is a zero product preserving mapping from* alg$\mathcal{L}$ *into itself. If $\alpha(I) = I$ and α is weak* continuous, then α is a homomorphism.*

Proof. By Lemma 2.3 of [10] and Theorem 1 of [20], it follows that alg$\mathcal{L}$ is generated by idempotents in the weak* topology. By Lemma 2.1 of [2] and $\alpha(I) = I$, we have that α is a homomorphism. ∎

Let $\mathcal{A}$ be a unital algebra. A linear mapping θ from $\mathcal{A}$ into itself is called a 2-*local automorphism* if for any $a, b \in \mathcal{A}$, there is an automorphism $\theta_{a,b} : \mathcal{A} \to \mathcal{A}$ such that $\theta(a) = \theta_{a,b}(a), \theta(b) = \theta_{a,b}(b)$.

COROLLARY 4.3. *Suppose that $\mathcal{L}$ is a completely distributive subspace lattice on a finite dimensional Hilbert space H. Then every 2-local automorphism α on* alg$\mathcal{L}$ *is an automorphism.*

Proof. By Corollaries 7.1 and 8.1 of [12], we have that $\mathcal{L}$ is similar to a commutative subspace lattice on a finite dimensional Hilbert space. Without loss of generality, we can assume that $\mathcal{L}$ is a commutative subspace lattice. Since α is a 2-local automorphism, it follows that α is a zero product preserving mapping, $\alpha(I) = I$ and α is injective. By Proposition 4.2, we have that α is an automorphism. ∎

THEOREM 4.4. *If $\mathcal{L} = \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n$, where $\mathcal{N}_i$ is a nest on H_i, and α is a surjective 2-local automorphism of* alg$\mathcal{L}$*, then α is an automorphism.*

Proof. By Theorem 6.5 of [6], we know that every automorphism of alg$\mathcal{L}$ is spatially implemented. Since α is a surjective 2-local automorphism, it follows

$$\alpha((\mathrm{alg}\mathcal{L}) \cap F(H)) = (\mathrm{alg}\mathcal{L}) \cap F(H). \tag{4.1}$$

By the assumption, we have that:

(i) α is a Jordan homomorphism,
(ii) $\alpha(I) = I$,
(iii) α is zero-product preserving.

By Lemma 3.4 of [10], we have that for any idempotent p and any m in $\mathcal{A}$, $\alpha(pm) = \alpha(p)\alpha(m)$. Hence by Lemma 2.3 of [10] for any rank one operator n in alg$\mathcal{L}$

$$\alpha(nm) = \alpha(n)\alpha(m). \tag{4.2}$$

For any u, q and any rank one operator n in alg$\mathcal{L}$, by (4.2) we have

$$\alpha(nuq) = \alpha(n)\alpha(uq) = \alpha(nu)\alpha(q) = \alpha(n)\alpha(u)\alpha(q).$$

Hence

$$\alpha(n)(\alpha(uq) - \alpha(u)\alpha(q)) = 0. \tag{4.3}$$

By Theorem 1 of [20], (4.1) and (4.3), we have that $\alpha(uq) = \alpha(u)\alpha(q)$. ∎

THEOREM 4.5. *Suppose that $\mathcal{L}$ is a completely distributive commutative subspace lattice. If α is a bounded surjective* 2*-local automorphism of* alg$\mathcal{L}$*, then α is an automorphism.*

Proof. With a proof similar to the proof of Theorem 4.4, we have

$$\alpha(n)(\alpha(uq) - \alpha(u)\alpha(q)) = 0 \tag{4.4}$$

for any rank one operator n in alg$\mathcal{L}$ and any u, q in alg$\mathcal{L}$. Since α is continuous, by Theorem 23.16 of [5] and Theorem 6.5 of [6], we have that $\alpha((\mathrm{alg}\mathcal{L}) \cap F(H))$ is dense in $(\mathrm{alg}\mathcal{L}) \cap K(H)$ in the norm. By (4.4), we have $\alpha(uq) = \alpha(u)\alpha(q)$ for any u, q in alg$\mathcal{L}$. ∎

PROPOSITION 4.6. *Suppose that $\mathcal{A}$ is a unital algebra. For* $2 \leqslant n$*, let ϕ be a zero-product preserving mapping from $M_n(\mathcal{A})$ into itself with $\phi(I) = I$. Then ϕ is a homomorphism.*

Proof. Since $M_n(\mathcal{A})$ can be generated by its idempotents, by Theorem 2.6 (iii) of [2] and $\phi(I) = I$, we have that ϕ is a homomorphism. ∎

COROLLARY 4.7. *Let $\mathcal{A}$ and $M_n(\mathcal{A})$ be as in Proposition* 4.6*. If ϕ is a surjective* 2*-local automorphism of $M_n(\mathcal{A})$ for* $2 \leqslant n$*, then ϕ is an automorphism.*

Proof. Since ϕ is a 2-local automorphism, we have that ϕ is a zero-product preserving mapping. By Proposition 4.6, we have that ϕ is a homomorphism. Since ϕ is a 2-local automorphism, it follows that ϕ is injective. Hence ϕ is an automorphism. ∎

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