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COMPOSITION OPERATORS ON THE WIENER-DIRICHLET ALGEBRA

FRÉDÉRIC BAYART, CATHERINE FINET, DANIEL LI and HERVÉ QUEFFÉLEC

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ABSTRACT. We study the composition operators on an algebra of Dirichlet series, the analogue of the Wiener algebra of absolutely convergent Taylor series, which we call the Wiener-Dirichlet algebra. The central issue is to understand the connection between the properties of the operator and of its symbol, with special emphasis on the compact, automorphic, or isometric character of this operator. We are led to the intermediate study of algebras of functions of several, or countably many, complex variables.

KEYWORDS: *Composition operator, Dirichlet series.*

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1. INTRODUCTION

Let $A^+ = A^+(\mathbb{T})$ be the Wiener algebra of absolutely convergent Taylor series in one variable : $f \in A^+$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{with } \|f\|_{A^+} = \sum_{n=0}^{\infty} |a_n| < +\infty.$$

It is well-known that A^+ is a commutative, unital Banach algebra with spectrum $\overline{\mathbb{D}}$, the closed unit disk. If $\phi : \mathbb{D} \to \overline{\mathbb{D}}$ is analytic, the composition operator C_ϕ with symbol ϕ is formally defined by $C_\phi(f) = f \circ \phi$.

Newman [20] studied those symbols ϕ generating bounded composition operators $C_\phi \colon A^+ \to A^+$, and proved in particular the following:

(a) C_ϕ maps A^+ into itself if and only if $\phi \in A^+$ and $\|\phi^n\|_{A^+} = O(1)$ as $n \to \infty$ (e.g. $\phi(z) = 5^{-1/2}(1 + z - z^2)$): this happens if and only if all maximum points θ_0 of $|\phi(\mathrm{e}^{\mathrm{i}\theta})|$ are *ordinary points*, i.e. if and only if we have, as $t \to 0$

$$\log \phi(\mathrm{e}^{\mathrm{i}(\theta_0 + t)}) = \alpha_0 + \alpha_1 t + \alpha_k t^k + \cdots,$$

where $k > 1$ and $\alpha_k \neq 0$ is not purely imaginary;

(b) if moreover $|\phi(\mathrm{e}^{\mathrm{i}t})| = 1$, one must have $\phi(z) = az^d$, with $|a| = 1$ and $d \in \mathbb{N}$;

(c) $C_\phi\colon A^+ \to A^+$ is an automorphism if and only if $\phi(z) = az$, with $|a| = 1$.

Harzallah (see [14]) also proved that:

(d) $C_\phi\colon A^+ \to A^+$ is an isometry if and only if $\phi(z) = az^d$, with $|a| = 1$ and $d \in \mathbb{N}$.

The aim of this paper is to perform a similar study for the *Wiener-Dirichlet* algebra $\mathcal{A}^+$ of absolutely convergent Dirichlet series: $f \in \mathcal{A}^+$ if and only if

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \quad \text{with } \|f\|_{\mathcal{A}^+} = \sum_{n=1}^{\infty} |a_n| < +\infty.$$

$\mathcal{A}^+$ is a commutative, unital Banach algebra, with the following multiplication (quite different from the one for Taylor series):

$$\Big(\sum_{n=1}^{\infty} a_n n^{-s}\Big)\Big(\sum_{n=1}^{\infty} b_n n^{-s}\Big) = \sum_{n=1}^{\infty} c_n n^{-s}, \quad \text{with } c_n = \sum_{ij=n} a_i b_j.$$

$\mathcal{A}^+$ can also be interpreted as a space of analytic functions on $\mathbb{C}_0$ (where in general we denote by $\mathbb{C}_\theta$ the vertical half-plane $\operatorname{Re} s > \theta$). The study of function spaces formed by Dirichlet series has gained some recent interest (see the papers of Hedenmalm-Lindqvist-Seip [10], Gordon-Hedenmalm [8], Bayart [1], [2], Finet-Queffélec-Volberg [7], Finet-Queffélec [6], Finet-Li-Queffélec [5], Mc Carthy [18]). Now, a method due to Bohr (see for example [21]) identifies the algebra $\mathcal{A}^+$ with the algebra $A^+(\mathbb{T}^\infty)$ formed by the absolutely convergent Taylor series in countably many variables (this point of view, which allows to identify the spectrum of $\mathcal{A}^+$ as $\overline{\mathbb{D}}^\infty$, the spectrum of $A^+(\mathbb{T}^\infty)$, has been used by Hewitt and Williamson [11], among others, to prove the following Wiener type tauberian Theorem : *If $f \in \mathcal{A}^+$ and $|f(s)| \geqslant \delta > 0$ for $s \in \mathbb{C}_0$, then $\frac{1}{f} \in \mathcal{A}^+$*).

Let us recall the way this identification is carried out. Let $(p_j)_{j\geqslant 1}$ be the increasing sequence of prime numbers ($p_1 = 2, p_2 = 3, p_3 = 5, \dots$). If

$$f(z) = \sum_{\alpha \in \mathbb{N}_0^{(\infty)}} a_\alpha z^\alpha \quad \text{with } \|f\|_{A^+(\mathbb{T}^\infty)} = \sum_\alpha |a_\alpha| < +\infty,$$

where, as usual, we set $\alpha = (\alpha_1, \dots, \alpha_r, 0, 0, \dots)$ and $z^\alpha = z_1^{\alpha_1} \cdots z_r^{\alpha_r}$ for $z = (z_j)_{j\geqslant 1}$, then $\Delta\colon \mathcal{A}^+ \to A^+(\mathbb{T}^\infty)$ is defined by

$$\Delta\Big(\sum_{n=1}^{\infty} a_n n^{-s}\Big) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r},$$

if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the decomposition of n in prime factors. Δ is an isometric isomorphism. Moreover, we shall need two more facts about Δ. For $s \in \mathbb{C}_0$, we set $z^{[s]} = (p_j^{-s})_j$. We then have

(1.1) $$\Delta f(z^{[s]}) = f(s) \quad \text{for any } f \in \mathcal{A}^+ \text{ and any } s \in \mathbb{C}_0,$$

(1.2) $$\|\Delta f\|_\infty = \|f\|_\infty \quad \text{for each } f \in \mathcal{A}^+,$$

where we set $\|f\|_\infty = \sup_{s\in\mathbb{C}_0} |f(s)|$ and $\|\Delta f\|_\infty = \sup_{z\in\mathbf{B}} |\Delta f(z)|$, with $\mathbf{B} = \big\{z = (z_j)_{j\geqslant 1} \in \mathbb{D}^\infty : z_j \underset{j\to+\infty}{\longrightarrow} 0\big\}$. Indeed, if $f(s) = \sum_1^\infty a_n n^{-s}$, we have

$$\Delta f(z^{[s]}) = \sum_{n=1}^{\infty} a_n (p_1^{-s})^{\alpha_1} \cdots (p_r^{-s})^{\alpha_r} = \sum_{n=1}^{\infty} a_n (p_1^{\alpha_1} \cdots p_r^{\alpha_r})^{-s} = f(s).$$

On the other hand, let $z = (z_j)_{j\geqslant 1} \in \mathbf{B}$. Fix an integer N, let $k = \pi(N)$ be the number of primes not exceeding N, and $S_N(z) = \sum_{n=1}^{N} a_n z_1^{\alpha_1} \cdots z_k^{\alpha_k}$, with $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$. Pick $\sigma > 0$ such that $|z_j| \leqslant p_j^{-\sigma}$, $1 \leqslant j \leqslant k$. Due to the rational independence of $\log p_1, \dots, \log p_k$ and to the Kronecker Approximation Theorem ([12], Corollary 4, page 23), the points $(p_j^{-it})_{1\leqslant j\leqslant k}$, $t \in \mathbb{R}$, are dense in the torus $\mathbb{T}^k$, so that the maximum modulus principle for the polydisk $\mathbb{D}^k$ gives

$$\begin{aligned} |S_N(z)| &\leqslant \sup_{|w_j| = p_j^{-\sigma}} \Big| \sum_{n=1}^{N} a_n w_1^{\alpha_1} \cdots w_k^{\alpha_k} \Big| = \sup_{\operatorname{Re} s = \sigma} \Big| \sum_{n=1}^{N} a_n (p_1^{-s})^{\alpha_1} \cdots (p_k^{-s})^{\alpha_k} \Big| \\ &= \sup_{\operatorname{Re} s=\sigma} \Big| \sum_{n=1}^{N} a_n n^{-s} \Big| \leqslant \Big\| \sum_{n=1}^{N} a_n n^{-s} \Big\|_\infty. \end{aligned}$$

Hence $\|S_N\|_\infty \leqslant \Big\| \sum_1^N a_n n^{-s} \Big\|_\infty$. Letting N tend to infinity gives $\|\Delta f\|_\infty \leqslant \|f\|_\infty$, which proves (1.2), since we trivially have $\|\Delta f\|_\infty \geqslant \|f\|_\infty$.

In this paper, we use the identification proposed above to obtain results similar to (a), (b), (c) and (d) for $\mathcal{A}^+$. This leads to an intermediate study of composition operators on the algebras $A^+(\mathbb{T}^\infty)$ and $A^+(\mathbb{T}^k)$ (the k-dimensional analog of $A^+(\mathbb{T}^\infty)$). Accordingly, the paper is organized as follows:

In Section 2, we give necessary as well as sufficient conditions for boundedness and compactness of $C_\phi : \mathcal{A}^+ \to \mathcal{A}^+$, and study in detail some specific examples. In Section 3, we study the automorphisms of the algebras $A^+(\mathbb{T}^k)$, $A^+(\mathbb{T}^\infty)$, $\mathcal{A}^+$. In Section 4, we study the isometries of those algebras, and we point out some specific differences between the finite and infinite-dimensional cases. Section 5 is devoted to some concluding remarks and questions.

A word on the definitions and notations: we will say that integers $2 \leqslant q_1 < q_2 < \cdots$ are multiplicatively independent if their logarithms are rationally independent in the real numbers; equivalently, if any integer $n \geqslant 2$ can be expressed as $n = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$, $\alpha_j \in \mathbb{N}_0$, in at most one way (e.g. $q_1 = 2$, $q_2 = 6$, $q_3 = 30$). We shall denote by $\mathcal{D}$ the space of functions $\varphi \colon \mathbb{C}_0 \to \mathbb{C}$ which are analytic, and moreover representable as a convergent Dirichlet series $\sum_1^\infty c_n n^{-s}$ for $\operatorname{Re} s$ large enough. $\mathcal{D}$ is also called the space of convergent Dirichlet series. For example, if $\psi(s) = (1 - 2^{1-s})\zeta(s)$, and $\varphi(s) = \psi(s - a)$, φ is entire, and representable as $\sum\limits_{1}^{\infty}(-1)^{n-1}n^a n^{-s}$ for $\mathrm{Re}\, s > a$. $\mathbb{T}$ denotes the unit circle, and plays no role in the definition of $A^+(\mathbb{T}^k)$ and $A^+(\mathbb{T}^\infty)$, although $\mathbb{T}^k$ (respectively $\mathbb{T}^\infty$) might be viewed as the Šhilov boundary of $A^+(\mathbb{T}^k)$ (respectively $A^+(\mathbb{T}^\infty)$). As usual, we set $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. Recall that $\mathbb{C}_\theta$ is the vertical half-plane $\mathrm{Re}\, s > \theta$.

2. BOUNDEDNESS AND COMPACTNESS OF COMPOSITION OPERATORS $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$

2.1. GENERAL RESULTS. We begin by sharpening Newman's result ((a) of the Introduction), under the form of the following (where it is assumed that ϕ is non-constant):

PROPOSITION 2.1. *The composition operator $C_\phi \colon A^+ \to A^+$ is compact if and only if $\|\phi\|_\infty = \sup\limits_{z \in \mathbb{D}} |\phi(z)| < 1$.*

Proof. As will be apparent from the proof of the next proposition, $C_\phi \colon A^+ \to A^+$ is compact if and only if $\|\phi^n\|_{A^+} \to 0$ as $n \to \infty$. On the other hand, by the spectral radius formula, we have $\|\phi\|_\infty = \lim\limits_{n\to\infty} \|\phi^n\|_{A^+}^{1/n} = \inf\limits_{n \geqslant 1} \|\phi^n\|_{A^+}^{1/n}$. That finishes the proof.

Alternatively, we could have applied to $f_n(z) = z^n$ a general criterion of Shapiro [24]: *C_ϕ is compact if and only if $\|C_\phi(f_n)\|_{A^+} \to 0$ for each sequence $(f_n)_n$ in A^+ which is bounded in norm and converges uniformly to zero on compact subsets of $\mathbb{D}$.* ∎

We now turn to the study of composition operators $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ associated with an analytic function $\phi \colon \mathbb{C}_0 \to \mathbb{C}_0$.

We first recall the following:

THEOREM 2.2 ([8], Theorem 4). *Let $\phi \colon \mathbb{C}_0 \to \mathbb{C}$ be an analytic function such that $k^{-\phi} \in \mathcal{D}$ for $k = 1, 2, \ldots$. Then we have necessarily:*

$$\phi(s) = c_0 s + \varphi(s), \quad \text{with } c_0 \in \mathbb{N}_0 \text{ and } \varphi \in \mathcal{D}. \tag{2.1}$$

We will therefore restrict ourselves, in the sequel, to symbols ϕ of the form given by (2.1). To avoid trivialities, we will also assume once and for all that ϕ is non-constant.

THEOREM 2.3. *Let $\phi \colon \mathbb{C}_0 \to \mathbb{C}$ be an analytic function of the form (2.1). Then:*

(a) (i) *If C_ϕ maps $\mathcal{A}^+$ into itself then $n^{-\phi} \in \mathcal{A}^+$ and $\|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, $n = 1, 2, \ldots$, for a positive constant C independent of n.*

(ii) *Conversely, if $(n^{-\phi})_{n=1}^\infty$ is a bounded sequence in $\mathcal{A}^+$, then ϕ maps $\mathbb{C}_0$ into $\mathbb{C}_0$ and C_ϕ is a bounded composition operator on $\mathcal{A}^+$.*

(b) (i) $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *is compact if and only if* $\|n^{-\phi}\|_{\mathcal{A}^+} \xrightarrow[n\to\infty]{} 0$. *Then* $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ *for some* $\delta > 0$.

(ii) *Assume that* $\phi(s) = c_0 s + \sum\limits_1^\infty c_n n^{-s}$, *with* $\sum\limits_1^\infty |c_n| < +\infty$. *Then* C_ϕ *is compact if and only if* $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ *for some* $\delta > 0$.

Proof. (a) (i) Suppose that C_ϕ maps $\mathcal{A}^+$ into itself. C_ϕ is an algebra homomorphism and $\mathcal{A}^+$ is semi-simple, therefore (see p. 263 of [22]) C_ϕ is continuous. Thus

$$\|n^{-\phi}\|_{\mathcal{A}^+} = \|C_\phi(n^{-s})\|_{\mathcal{A}^+} \leqslant \|C_\phi\|\,\|n^{-s}\|_{\mathcal{A}^+} = \|C_\phi\| =: C.$$

(ii) Conversely, suppose that $n^{-\phi} \in \mathcal{A}^+$ and $\|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, $n = 1, 2, \ldots$. We first see that, for $s \in \mathbb{C}_0$, we have: $n^{-\operatorname{Re}\phi(s)} = |n^{-\phi(s)}| \leqslant \|n^{-\phi}\|_\infty \leqslant \|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C$, whence $\operatorname{Re}\phi(s) \geqslant -\frac{\log C}{\log n}\cdot$ Letting n tend to infinity gives $\operatorname{Re}\phi(s) \geqslant 0$, and the open mapping theorem gives $\operatorname{Re}\phi(s) > 0$, since ϕ is not constant. If now $f(s) = \sum\limits_1^\infty a_n n^{-s} \in \mathcal{A}^+$, the series $\sum\limits_1^\infty a_n n^{-\phi(s)}$ is absolutely convergent in $\mathcal{A}^+$, so that $f \circ \phi \in \mathcal{A}^+$, with $\|f\circ\phi\|_{\mathcal{A}^+} \leqslant \sum\limits_1^\infty |a_n|\|n^{-\phi}\|_{\mathcal{A}^+} \leqslant C\sum\limits_1^\infty |a_n| = C\|f\|_{\mathcal{A}^+}$.

(b) (i) Suppose that $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is compact. Let $f \in \mathcal{A}^+$ be a cluster point of $n^{-\phi(s)} = C_\phi(n^{-s})$, and let $(n_k)_k$ be a sequence of integers such that $\|n_k^{-\phi} - f\|_{\mathcal{A}^+} \to 0$. For fixed $s \in \mathbb{C}_0$, we have $|n_k^{-\phi(s)} - f(s)| \leqslant \|n_k^{-\phi} - f\|_{\mathcal{A}^+}$. But $n_k^{-\phi(s)} \to 0$ (since $\operatorname{Re}\phi(s) > 0$, by part (a)), so that $f(s) = 0$. Hence $f = 0$. This implies $\|n^{-\phi}\|_{\mathcal{A}^+} \to 0$.

Now, since $\|n^{-\phi}\|_\infty \leqslant \|n^{-\phi}\|_{\mathcal{A}^+}$, we get $n^{-\inf\operatorname{Re}\phi(s)} = \|n^{-\phi}\|_\infty \to 0$, and so $\inf\limits_{s\in\mathbb{C}_0} \operatorname{Re}\phi(s) > 0$.

Conversely, suppose that $\varepsilon_n = \|n^{-\phi}\|_{\mathcal{A}^+} \to 0$ and set $\delta_n = \sup\limits_{k>n} \varepsilon_k$. Let $T_n\colon \mathcal{A}^+ \to \mathcal{A}^+$ be the finite-rank operator defined by $(T_n f)(s) = \sum\limits_{k=1}^n a_k k^{-\phi(s)}$ if $f(s) = \sum\limits_{k=1}^\infty a_k k^{-s}$. We have

$$\|C_\phi f - T_n f\|_{\mathcal{A}^+} \leqslant \sum_{k>n} |a_k|\|k^{-\phi}\|_{\mathcal{A}^+} \leqslant \delta_n \sum_{k>n} |a_k| \leqslant \delta_n \|f\|_{\mathcal{A}^+},$$

showing that $\|C_\phi - T_n\| \leqslant \delta_n$, and therefore that C_ϕ is compact.

(ii) For any $v \in \mathcal{A}^+$, and for any real number $r \geqslant 1$, we have

$$\|r^{-v}\|_{\mathcal{A}^+} \leqslant r^{\|v\|_{\mathcal{A}^+}}\,. \tag{2.2}$$

Indeed

$$r^{-v} = \exp(-v\log r) = \sum_{k=0}^\infty \frac{(-\log r)^k}{k!}\, v^k \in \mathcal{A}^+$$

since v belongs to the algebra $\mathcal{A}^+$. Moreover

$$\|r^{-v}\|_{\mathcal{A}^+} \leqslant \sum_{k=0}^{\infty} \frac{(\log r)^k}{k!} \|v\|_{\mathcal{A}^+}^k = r^{\|v\|_{\mathcal{A}^+}}$$

(we may remark that when $v(s) = c_j j^{-s}$ is a monomial, we have equality; in particular: $\|n^{-c_j j^{-s}}\|_{\mathcal{A}^+} = n^{|c_j|}$ for every positive integer n).

We shall use the following:

PROPOSITION 2.4 (see [8]). *Let θ and τ be real numbers and suppose that ϕ maps $\mathbb{C}_\theta$ into $\mathbb{C}_\tau$. Then, if $\phi(s) = c_0 s + \varphi(s)$, and φ is not constant, c_0 must be a non-negative integer and φ maps $\mathbb{C}_\theta$ into $\mathbb{C}_{\tau - c_0\theta}$.*

Now, assume that φ is non-constant (since otherwise the result is trivial), and that $\varepsilon = \inf_{s\in\mathbb{C}_0} \operatorname{Re}\phi(s) > 0$. By Proposition 2.4, φ maps $\mathbb{C}_0$ into $\mathbb{C}_\varepsilon$. The spectral radius formula and Bohr's theory (as seen in the Introduction) give, with $\psi = 2^{-\varphi}$:

$$\lim_{j\to+\infty} \|\psi^j\|_{\mathcal{A}^+}^{1/j} = \sup_{h\in\operatorname{sp}\mathcal{A}^+} |h(\psi)| = \sup_{s\in\mathbb{C}_0} |\psi(s)| = 2^{-\varepsilon};$$

and, in particular, $\|2^{-j\varphi}\|_{\mathcal{A}^+} \underset{j\to+\infty}{\longrightarrow} 0$. Now, if n is any positive integer, let $j = j(n)$ be the integer such that $2^j \leqslant n < 2^{j+1}$, and set $r = n\,2^{-j}$, so that $1 \leqslant r < 2$. By using (2.2), we get

$$\begin{aligned}\|n^{-\phi}\|_{\mathcal{A}^+} = \|n^{-\varphi}\|_{\mathcal{A}^+} &= \|2^{-j\varphi} r^{-\varphi}\|_{\mathcal{A}^+} \leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} \|r^{-\varphi}\|_{\mathcal{A}^+} \\ &\leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} r^{\|\varphi\|_{\mathcal{A}^+}} \leqslant \|2^{-j\varphi}\|_{\mathcal{A}^+} 2^{\|\varphi\|_{\mathcal{A}^+}}.\end{aligned}$$

This shows that $\|n^{-\phi}\|_{\mathcal{A}^+} \underset{n\to\infty}{\longrightarrow} 0$ (more precisely, we have $\|n^{-\phi}\|_{\mathcal{A}^+} = \mathrm{O}\,(n^{-\delta})$ for some $\delta > 0$), and so C_ϕ is compact, by part (b) (i) of the theorem. ∎

REMARK 2.5. Using the notation of Theorem 2.2, we have

$$\|n^{-\phi}\|_{\mathcal{A}^+} = \|n^{-\varphi}\|_{\mathcal{A}^+},$$

and, in particular, the integer c_0 plays no role for the continuity or the compactness of the composition operator C_ϕ on $\mathcal{A}^+$. This is quite amazing, since c_0 intervenes decisively in the study of composition operators on the Hilbert space $\mathcal{H}^2$ of the square-summable Dirichlet series (so much so that Gordon and Hedenmalm [8] called it *characteristic*).

COROLLARY 2.6. *Let $\phi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s}$. Then C_ϕ is bounded if $\operatorname{Re} c_1 \geqslant \sum_{n=2}^{\infty} |c_n|$, and is compact if $\operatorname{Re} c_1 > \sum_{n=2}^{\infty} |c_n|$.*

Proof. Let $\varphi_0 \in \mathcal{A}^+$ be defined by $\varphi_0(s) = \sum_{n=2}^{\infty} c_n n^{-s}$. For each positive integer N, we have: $N^{-\phi(s)} = (N^{c_0})^{-s} N^{-c_1} N^{-\varphi_0(s)}$, and so the inequality (2.2) with $r = N$ gives

$$\|N^{-\phi}\|_{\mathcal{A}^+} = N^{-\operatorname{Re} c_1}\|N^{-\varphi_0}\|_{\mathcal{A}^+} \leqslant N^{-\operatorname{Re} c_1} N^{\|\varphi_0\|_{\mathcal{A}^+}} = N^{-\operatorname{Re} c_1 + \sum_{n=2}^{\infty}|c_n|};$$

thus $\|N^{-\phi}\|_{\mathcal{A}^+}$ is less than 1 in the first case, and tends to 0 in the second case. Theorem 2.3 ends the proof. ∎

Note that under the assumption of Corollary 2.6, $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is actually a contraction: $\|C_\phi\| \leqslant 1$.

2.2. SOME SPECIFIC EXAMPLES. One of the main differences between the study of composition operators on $\mathcal{A}^+$ and those on $A^+(\mathbb{T})$ is the fact that the function $z \mapsto z$ does not belong to $\mathcal{A}^+$. Therefore, it is not clear that if C_ϕ is a composition operator on $\mathcal{A}^+$, we must have $\sum_n |c_n| < +\infty$. In some cases, it is however true. The next proposition contains a partial result of this type.

Recall (see [14]) that $(\lambda_j)_{j\geqslant 1}$ is a Sidon set if

$$\sum_{j=1}^{N}|a_j| \leqslant C_0 \sup_{t\in\mathbb{R}}\Big|\sum_{j=1}^{N} a_j \mathrm{e}^{\mathrm{i}\lambda_j t}\Big|$$

for some finite positive constant C_0.

PROPOSITION 2.7. (i) *If* $2 \leqslant q_1 < q_2 < \cdots$ *are multiplicatively independent integers and* $\phi(s) = c_0 s + c_1 + \sum_{j=1}^{\infty} d_j q_j^{-s}$, *then the boundedness of* $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *implies that* $\operatorname{Re} c_1 \geqslant \sum_{j=2}^{\infty}|d_j|$, *and its compactness implies* $\operatorname{Re} c_1 > \sum_{j=2}^{\infty}|d_j|$.

(ii) *Let* $(\lambda_j)_{j\geqslant 1}$ *be a Sidon set of positive integers, r an integer* $\geqslant 2$, *and* $\phi(s) = c_0 s + \varphi(s)$, *where* $\varphi \in \mathcal{D}$ *and* $\varphi(s) = c_1 + \sum_{j=1}^{\infty} d_j r^{-\lambda_j s}$ *for* $\operatorname{Re} s$ *large. Then the boundedness of* $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ *requires that* $\sum_{j=1}^{\infty}|d_j| < +\infty$.

Proof. (i) Write $\varphi_0(s) = \sum_{j=1}^{\infty} d_j q_j^{-s}$, as in the proof of Corollary 2.6. For every integer $n \geqslant 2$, we have, for $\operatorname{Re} s$ large enough:

$$n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1} \exp(-\varphi_0(s)\log n) = (n^{c_0})^{-s} n^{-c_1} \sum_{k=0}^{\infty} \frac{(-\log n)^k}{k!}\varphi_0(s)^k.$$

Since C_ϕ is assumed to be bounded on $\mathcal{A}^+$, we know that $n^{-\phi} \in \mathcal{A}^+$, and so

$$n^{-\phi(s)} = \sum_{j=1}^{\infty} a_{n,j} j^{-s}, \quad \text{with } \sum_{j=1}^{\infty}|a_{n,j}| < +\infty.$$

But the supports (spectra) of the φ_0^k's do not intersect: in fact, the spectrum of φ_0^k only involves finite products $\prod_j q_j^{\alpha_j}$, where $\sum \alpha_j = k$, and these products are all distinct. In particular, for $k=1$, $(-\log n)\varphi_0(s)$ is part of the expansion of $n^{-\phi(s)}$, which means that $(-d_j \log n)_j$ is a subsequence of $(a_{n,j})_j$. Therefore, $\sum_j |d_j| < +\infty$ (and so $\varphi_0 \in \mathcal{A}^+$), and the series expansion of $\varphi_0(s)$ holds for every $s \in \mathbb{C}_0$. Finally, since the $\log q_j$'s are rationally independent, Kronecker's Approximation Theorem implies that, for each $\sigma > 0$, we have

$$\inf_{t\in\mathbb{R}} \operatorname{Re}\phi(\sigma+\mathrm{i}t) = c_0\sigma + \operatorname{Re} c_1 - \sum_{j=1}^{\infty} |d_j| q_j^{-\sigma}.$$

Since the left-hand side is $\geqslant 0$ by the first part of Theorem 2.3, we get $\operatorname{Re} c_1 \geqslant \sum_{j=1}^{\infty} |d_j|$, by letting σ go to zero. The compact case is similar.

(ii) We have (see [17])

$$\inf_{\tau\in\mathbb{R}} \sum_{j=1}^{N} \rho_j \cos(\lambda_j \tau + \xi_j) \leqslant -\delta \sum_{j=1}^{N} \rho_j \tag{2.3}$$

for some other constant $\delta > 0$, where the ρ_j's (non-negative) and the (real) ξ_j's are arbitrary. Without loss of generality, we can assume that $r = 2$. Fix an integer $J \geqslant 1$, and let $B : \mathbb{R} \to \mathbb{R}^+$ (see [15], p. 165) be a non-negative Dirichlet polynomial (of the form $\sum \alpha_k \mathrm{e}^{\mathrm{i}\beta_k t}$, $\beta_k \in \mathbb{R}$, $\alpha_k \in \mathbb{C}$) such that

$$\widehat{B}(0) = \widehat{B}(\lambda_j \log 2) = 1, \quad 1 \leqslant j \leqslant J \tag{2.4}$$

(recall that $\widehat{B}(\lambda) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} B(t)\mathrm{e}^{-\mathrm{i}\lambda t}\,\mathrm{d}t$).

For large $\sigma > 0$, we have an absolutely convergent expansion

$$\varphi(\sigma + \mathrm{i}(t+\tau)) = c_1 + \sum_{j=1}^{\infty} d_j 2^{-\lambda_j \sigma} 2^{-\lambda_j \mathrm{i}t} \mathrm{e}^{-\mathrm{i}\lambda_j \tau \log 2},$$

so that, for $\operatorname{Re} s$ large enough (say $\operatorname{Re} s \geqslant \sigma_0 > 0$)

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \varphi(s+\mathrm{i}\tau) B(\tau)\,\mathrm{d}\tau = c_1 + \sum_{j=1}^{\infty} d_j 2^{-\lambda_j s} \widehat{B}(\lambda_j \log 2). \tag{2.5}$$

Actually, (2.5) holds for every s with positive real part σ. To see this, set

$$f_T(s) = \frac{1}{2T} \int_{-T}^{T} \varphi(s+\mathrm{i}\tau) B(\tau)\,\mathrm{d}\tau.$$

Proposition 2.4 shows that $\operatorname{Re}\varphi(s+\mathrm{i}\tau) > 0$ for $s \in \mathbb{C}_0$, and thus that $\operatorname{Re} f_T(s) > 0$ for $s \in \mathbb{C}_0$. Moreover, f_T as well as the right-hand side of (2.5), since B is a Dirichlet polynomial, are holomorphic in $\mathbb{C}_0$; hence a normal family argument gives the above statement.

Therefore, if we take the real part of both sides of (2.5), we get, for every $\sigma > 0$ and $t \in \mathbb{R}$:

$$\operatorname{Re} c_1 + \sum_{j\geqslant 1} 2^{-\lambda_j \sigma} \operatorname{Re}\big(d_j 2^{-\lambda_j \mathrm{i}t} \widehat{B}(\lambda_j \log 2)\big) = \lim_{T\to+\infty} \operatorname{Re} f_T(\sigma + \mathrm{i}t) \geqslant 0.$$

Letting σ tend to zero gives $\operatorname{Re} c_1 + \sum_{j\geqslant 1} \operatorname{Re}\big(d_j 2^{-\lambda_j \mathrm{i}t} \widehat{B}(\lambda_j \log 2)\big) \geqslant 0$, for any $t \in \mathbb{R}$.

Taking the infimum with respect to t and using (2.3), we get $\operatorname{Re} c_1 - \delta \sum_{j=1}^{\infty} |d_j|$ $|\widehat{B}(\lambda_j \log 2)| \geqslant 0$ and therefore, using (2.4)

$$\operatorname{Re} c_1 - \delta \sum_{j=1}^{J} |d_j| \geqslant 0.$$

It follows that $\sum_{j=1}^{\infty} |d_j| \leqslant \frac{1}{\delta} \operatorname{Re} c_1$, and this ends the proof of Proposition 2.7. ∎

REMARK 2.8. The above proof gives the following information about Dirichlet series, which is actually not connected to composition operators: *let φ be a Dirichlet series which can be written as $\varphi(s) = c_1 + \sum_{j\geqslant 1} d_j r^{-\lambda_j s}$, where $(\lambda_j)_j$ is a Sidon sequence; if there is a $\beta \in \mathbb{R}$ such that $\varphi(\mathbb{C}_0) \subseteq \mathbb{C}_\beta$, then $\sum_{j\geqslant 1} |d_j| < +\infty$.*

However, in general, conditions like $\sum_{n\geqslant 2} |c_n| \leqslant \operatorname{Re} c_1$ (respectively $< \operatorname{Re} c_1$) are not necessary to have boundedness or compactness of the composition operator $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ (with $\phi(s) = c_0 s + c_1 + \sum_{n\geqslant 2} c_n n^{-s}$), as shown by the following examples.

PROPOSITION 2.9. *Let $\phi(s) = c_0 s + c_1 + c_r r^{-s} + c_{r^2} r^{-2s}$, where $r \geqslant 2$ and c_r, c_{r^2} are > 0. Then:*

(i) *If we have the following, then $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is bounded and even compact:*

$$\operatorname{Re} c_1 > \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}. \tag{2.6}$$

(ii) *Conversely, if $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is bounded, and moreover $c_r \leqslant 4c_{r^2}$, we must have*

$$\operatorname{Re} c_1 \geqslant \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}. \tag{2.7}$$

In fact, we must have (2.6) *whenever C_ϕ is compact.*

(iii) *In the following case, $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$ is bounded if and only if $c_r \neq 4c_{r^2}$:*

$$\operatorname{Re} c_1 = \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}. \tag{2.8}$$

Proof. (i) and (ii) follow immediately from Theorem 2.3, since (2.6) implies $\operatorname{Re}\phi(s) > c_0 \operatorname{Re} s + \delta \geqslant \delta$ for every $s \in \mathbb{C}_0$, with $\delta = \operatorname{Re} c_1 - \big[\frac{(c_r)^2}{8c_{r^2}} + c_{r^2}\big]$, and, under the assumption that $c_r \leqslant 4c_{r^2}$, the converse is true. ∎

However, we shall give another proof, because we think that it sheds additional light.

Second proof. (i) Without loss of generality, we may and shall assume that $r = 2$. We will make use (see p. 60 of [16]) of the Hermite polynomials $H_0, H_1, \dots$ defined by

$$H_k(\lambda) = (-1)^k \mathrm{e}^{\lambda^2} \frac{\mathrm{d}^k}{\mathrm{d}\lambda^k}(\mathrm{e}^{-\lambda^2}) = (2\lambda)^k + \text{terms of lower degree.} \tag{2.9}$$

The exponential generating function of the H_k's is

$$\sum_{k=0}^{\infty} \frac{H_k(\lambda)}{k!} x^k = \exp(2\lambda x - x^2). \tag{2.10}$$

Following Indritz [13], we have the sharp estimate

$$|H_k(\lambda)| \leqslant (2^k k!)^{1/2} \mathrm{e}^{\lambda^2/2}, \tag{2.11}$$

for each $k \in \mathbb{N}_0$ and each $\lambda \in \mathbb{R}$. The estimate (2.11) implies the following:

LEMMA 2.10. *Let λ be a real number, and x be a non-negative real number. Then we have the following, where C is a positive constant:*

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C(1+x)^{1/2} \exp\Big(x^2 + \frac{\lambda^2}{2}\Big). \tag{2.12}$$

Proof. (2.11) implies that

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant \sum_{k=0}^{\infty} \frac{(x\sqrt{2})^k}{(k!)^{1/2}} \mathrm{e}^{\lambda^2/2}.$$

We now make use of the classical estimate (see e.g. Dieudonné p. 195 of [3]):

$$\sum_{k=0}^{\infty} \frac{y^k}{(k!)^p} \sim \frac{1}{\sqrt{p}} (2\pi)^{(1-p)/2} y^{(1-p)/2p} \exp(p y^{1/p}) \text{ as } y \to \infty \ (p > 0 \text{ fixed}). \tag{2.13}$$

Using the above, with $p = \frac{1}{2}$ and $y = x\sqrt{2}$, we obtain for some constant C the following, proving the lemma:

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C\mathrm{e}^{\lambda^2/2}(1+x)^{1/2}\mathrm{e}^{x^2}. \quad ∎$$

Note that, if we wish to avoid the use of (2.13), we can easily obtain the slightly weaker estimate:

$$\sum_{k=0}^{\infty} \frac{|H_k(\lambda)|}{k!} x^k \leqslant C_a \exp\Big(ax^2 + \frac{\lambda^2}{2}\Big), \quad \text{for each } a > 1. \tag{2.14}$$

Indeed, we have by the Cauchy-Schwarz inequality and by (2.11) :

$$\sum_{k=0}^{\infty}\frac{|H_k(\lambda)|}{k!}x^k=\sum_{k=0}^{\infty}\frac{|H_k(\lambda)|}{(k!)^{1/2}(2a)^{k/2}}\frac{(2a)^{k/2}x^k}{(k!)^{1/2}}\leqslant\Big(\sum_{k=0}^{\infty}\frac{|H_k(\lambda)|^2}{k!(2a)^k}\Big)^{1/2}\Big(\sum_{k=0}^{\infty}\frac{(2a)^k x^{2k}}{k!}\Big)^{1/2}$$
$$\leqslant e^{\lambda^2/2}\Big(\sum_{k=0}^{\infty}a^{-k}\Big)^{1/2}\exp(ax^2)=(1-a^{-1})^{-1/2}\exp\Big(ax^2+\frac{\lambda^2}{2}\Big).$$

We now finish the proof of Proposition 2.9. First, we notice that $n^{-\phi(s)} = (n^{c_0})^{-s}n^{-c_1}\exp(-c_2 2^{-s}\log n - c_4 4^{-s}\log n)$. We then set

$$x_n = \sqrt{c_4\log n},\quad \lambda_n = \frac{-c_2}{2\sqrt{c_4}}\sqrt{\log n},\quad x = 2^{-s}x_n, \tag{2.15}$$

which allows us to write $n^{-\phi(s)}$ under the form

$$n^{-\phi(s)} = (n^{c_0})^{-s}n^{-c_1}\exp(2\lambda_n x - x^2) = (n^{c_0})^{-s}n^{-c_1}\sum_{k=0}^{\infty}\frac{H_k(\lambda_n)}{k!}x_n^k(2^k)^{-s}.$$

This implies that we have the equality

$$\|n^{-\phi}\|_{\mathcal{A}^+} = n^{-\operatorname{Re}c_1}\sum_{k=0}^{\infty}\frac{|H_k(\lambda_n)|}{k!}x_n^k. \tag{2.16}$$

If we now use Lemma 2.10 and change C (if necessary), we get for $n\geqslant 2$,

$$\|n^{-\phi}\|_{\mathcal{A}^+}\leqslant Cn^{-\operatorname{Re}c_1}(\log n)^{1/4}\exp\Big(x_n^2+\frac{\lambda_n^2}{2}\Big)=C(\log n)^{1/4}n^{-\operatorname{Re}c_1}n^{c_2^2/8c_4+c_4}=:\varepsilon_n.$$

By (2.6), we have $\varepsilon_n\to 0$, implying that $C_\phi\colon\mathcal{A}^+\to\mathcal{A}^+$ is compact as a consequence of Theorem 2.3.

(ii) The identities (2.16) and (2.10) imply that we have, for each real θ,

$$n^{\operatorname{Re}c_1}\|n^{-\phi}\|_{\mathcal{A}^+}\geqslant\Big|\sum_{k=0}^{\infty}\frac{H_k(\lambda_n)}{k!}x_n^k e^{ik\theta}\Big| = |\exp(2\lambda_n x_n e^{i\theta}-x_n^2e^{2i\theta})|$$
$$=\exp(2\lambda_n x_n\cos\theta - x_n^2\cos 2\theta).$$

Setting $t=\cos\theta$, we see that $2\lambda_n x_n\cos\theta - x_n^2\cos 2\theta = 2\lambda_n x_n t - x_n^2(2t^2-1)$ is maximum for $t=\frac{\lambda_n}{2x_n}=\frac{-c_2}{4c_4}$, and this t will be admissible if $\left|\frac{-c_2}{4c_4}\right|\leqslant 1$, i.e. if $c_2\leqslant 4c_4$ (recall that c_2, c_4 are positive). For this value of t, we get

$$\|n^{-\phi}\|_{\mathcal{A}^+}\geqslant n^{-\operatorname{Re}c_1+c_2^2/8c_4+c_4},\quad n=1,2,\ldots,\ c_2\leqslant 4c_4. \tag{2.17}$$

Now, if C_ϕ is bounded, $\|n^{-\phi}\|_{\mathcal{A}^+}$ is bounded from above, and (2.17) implies that $\operatorname{Re}c_1\geqslant\frac{(c_2)^2}{8c_4}+c_4$. If C_ϕ is compact, $\|n^{-\phi}\|_{\mathcal{A}^+}\to 0$ and (2.17) implies that $\operatorname{Re}c_1>\frac{(c_2)^2}{8c_4}+c_4$. ∎

REMARK 2.11. Condition (2.6) is a more general sufficient condition for the boundedness of C_ϕ than the *trivial* sufficient condition $\operatorname{Re} c_1 \geqslant |c_2| + |c_4|$ of Corollary 2.6 if and only if $c_r < 8c_{r^2}$. This might be due to the highly oscillatory character of the Hermite polynomials $H_k(\lambda)$, involving a term $\cos\big(\sqrt{2k+1}\lambda - k\frac{\pi}{2}\big)$ (see p. 67 of [16]), which we ignore when we majorize $|H_k(\lambda)|$ as in (2.11).

End of proof of Proposition 2.9. (iii) We still assume that $r = 2$. First, if $c_2 > 4c_4$, then (2.8) implies that $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_\delta$ for some $\delta > 0$, and we are done. So, we assume that $c_2 \leqslant 4c_4$. We have

$$\begin{aligned}\|(2^j)^{-\phi}\|_{\mathcal{A}^+} &= \|2^{-j\phi}\|_{\mathcal{A}^+} = \|(2^{-c_1-c_2 2^{-s}-c_4 4^{-s}})^j\|_{\mathcal{A}^+} \\ &= \|(\exp[(-c_1 - c_2 2^{-s} - c_4 4^{-s})\log 2])^j\|_{\mathcal{A}^+} = \|\psi^j\|_{A^+(\mathbb{T})},\end{aligned}$$

with $\psi(z) = \exp(-(c_1+c_2z+c_4z^2)\log 2)$. We then apply Newman's result (quoted as (a) in the Introduction: see [20]) to check whether the sequence $(\|\psi^j\|_{A^+(\mathbb{T})})_j$ is bounded. Let $\theta_0 \in [0, 2\pi[$ be such that $|\psi(\mathrm{e}^{\mathrm{i}\theta_0})| = 1$. We look for the coefficient of t^2 in the Taylor expansion of

$$\log\psi(\mathrm{e}^{\mathrm{i}\theta_0+\mathrm{i}t}) = -(c_1 + c_2\mathrm{e}^{\mathrm{i}\theta_0}\mathrm{e}^{\mathrm{i}t} + c_4\mathrm{e}^{2\mathrm{i}\theta_0}\mathrm{e}^{2\mathrm{i}t})\log 2.$$

This term is $\big(\frac{c_2}{2}\mathrm{e}^{\mathrm{i}\theta_0} + 2c_4\mathrm{e}^{2\mathrm{i}\theta_0}\big)\log 2$, and its real part is

$$\Big(\frac{c_2}{2}\cos\theta_0 + 2c_4(2\cos^2\theta_0 - 1)\Big)\log 2. \tag{2.18}$$

Now, remark that the condition $|\psi(\mathrm{e}^{\mathrm{i}\theta_0})| = 1$ means that

$$\operatorname{Re} c_1 = -c_2\cos\theta_0 - c_4(2\cos^2\theta_0 - 1),$$

which gives, using (2.8), $(4c_4\cos\theta_0 + c_2)^2 = 0$, that is $\cos\theta_0 = -\frac{c_2}{4c_4}\cdot$ Hence (2.18) is equal to 0 if and only if $c_2 = 4c_4$.

But in this case, $\theta_0 = \pi$, and Taylor's expansion becomes

$$\log\psi(\mathrm{e}^{\mathrm{i}(\theta_0+t)}) = d_1 + d_2t + 0\cdot t^2 + \mathrm{i}\log 2\frac{2c_4}{3}t^3 + \cdots.$$

Hence, in Newman's terminology (see [20], and see (a) in the Introduction), the point $\mathrm{e}^{\mathrm{i}\theta_0}$ is not an ordinary point, and hence the sequence $(\|\psi^j\|_{A^+(\mathbb{T})})_j$ is not bounded. It follows that the sequence $(\|2^{-j\phi}\|_{\mathcal{A}^+})_j$ is not bounded either.

In the case $c_2 < 4c_4$, the point $\mathrm{e}^{\mathrm{i}\theta_0}$ is ordinary, and therefore $(\|2^{-j\phi}\|_{\mathcal{A}^+})_j$ is bounded. Since $\sum_n |c_n| = |c_1| + |c_2| + |c_4| < +\infty$, the argument used in the proof in Theorem 2.3, (b) (ii) gives the boundedness of C_ϕ. ∎

REMARK 2.12. Part (iii) of Proposition 2.9 shows that, if $\operatorname{Re} c_1 = \frac{(c_r)^2}{8c_{r^2}} + c_{r^2}$ and $c_r = 4c_{r^2}$ (so $\operatorname{Re} c_1 = 3c_4$, and $\psi(s) = \mathrm{i}a + c(3 + 4\cdot 2^{-s} + 4^{-s})$, with $a \in \mathbb{R}$ and $c > 0$), then C_ϕ is not bounded on $\mathcal{A}^+$, though $\phi(\mathbb{C}_0) \subseteq \mathbb{C}_0$ (and $\sum_n |c_n| < +\infty$).

3. AUTOMORPHISMS OF $A^+(\mathbb{T}^k), A^+(\mathbb{T}^\infty), \mathcal{A}^+$

In this section, we will make repeated use of the following lemma (see (b) of the Introduction):

LEMMA 3.1. *Let* $\phi(z) = \prod_{j=1}^{J} \varepsilon_j \frac{z-a_j}{1-\overline{a}_j z}$, *where* $|\varepsilon_j| = 1$ *and* $a_j \in \mathbb{D}$. *Suppose that* $\|\phi^n\|_{A^+}$ *remains bounded* $(n = 1, 2, \ldots)$. *Then,* $a_j = 0$ *for each* j.

Proof. This lemma is well-known (see page 38, assertion (3) of [20] or page 77 of [14]). For example, if $a_j \neq 0$ for some j, we have $\phi(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}g(t)}$, where g is a $\mathcal{C}^2$, real, non affine function; and the Van der Corput inequalities show that we even have: $\|\phi^n\|_{A^+} \geqslant \delta\sqrt{n}$. ∎

Since $|\phi(\mathrm{e}^{\mathrm{i}t})| = 1$, Lemma 3.1 can be viewed as a special case of the following lemma (which will be needed only in Section 4, but which we state here because it is the natural extension of Lemma 3.1), due to Beurling and Helson, and this lemma is itself a special case of Cohen's Theorem (see page 93, corollary of Theorem 4.7.3 in [22]). We shall use the following definition:

Let G be a discrete abelian group, and Γ be its (compact) dual group; the *Wiener algebra* $A(\Gamma)$ is the set of functions $f : \Gamma \to \mathbb{C}$ which can be written as an absolutely convergent series $f(\gamma) = \sum_1^\infty a_n (x_n, \gamma)$, with the norm $\|f\|_{A(\Gamma)} = \sum_1^\infty |a_n|$, and where (x_n, γ) denotes the action of $\gamma \in \Gamma$ on the element x_n of G. We are now ready to state:

LEMMA 3.2 (Beurling-Helson). *Let G be a discrete abelian group, with connected dual group Γ. Let* $\phi \in A(\Gamma)$, *which does not vanish on* Γ, *and such that* $\|\phi^n\|_{A(\Gamma)} \leqslant C$ *for some constant* C $(n = 0, \pm 1, \pm 2, \ldots)$. *Then* ϕ *is affine, i.e. there exist a complex number* a *with* $|a| = 1$ *and an element* x *of* G *such that* $\phi(\gamma) = a(x, \gamma)$ *for any* $\gamma \in \Gamma$.

Let us now consider the Wiener algebra $A^+(\mathbb{T}^k)$ in k variables, i.e. the algebra of functions $f : \overline{\mathbb{D}}^k \to \mathbb{C}$ which can be written as

$$f(z) = \sum_{n_1, \ldots, n_k \geqslant 0} a(n_1, \ldots, n_k) z_1^{n_1} \cdots z_k^{n_k}, \quad z = (z_1, \ldots, z_k),$$

with the norm $\|f\|_{A^+(\mathbb{T}^k)} = \sum_{n_1, \ldots, n_k \geqslant 0} |a(n_1, \ldots, n_k)| < +\infty$.

If $\phi = (\phi_1, \ldots, \phi_k) : \mathbb{D}^k \to \mathbb{C}^k$ is an analytic function, the composition operator C_ϕ will be bounded on $A^+(\mathbb{T}^k)$ if and only if

$$\|\phi_j^n\|_{A^+(\mathbb{T}^k)} \leqslant C, \quad j = 1, \ldots, k, \text{ and } n = 0, 1, 2, \ldots \tag{3.1}$$

(the proof is the same as in Newman's case $k = 1$).

Then, since $\|\phi_j\|_\infty = \lim_{n \to \infty} \|\phi_j^n\|_{A^+(\mathbb{T}^k)}^{1/n}$, we see that ϕ necessarily maps $\mathbb{D}^k$ into $\overline{\mathbb{D}}^k$. We can now state:

THEOREM 3.3. *Assume that the map* $\phi\colon \mathbb{D}^k \to \overline{\mathbb{D}}^k$ *induces a bounded composition operator* $C_\phi\colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$. *Then* C_ϕ *is an automorphism of* $A^+(\mathbb{T}^k)$ *if and only if* $\phi(z) = (\varepsilon_1 z_{\sigma(1)}, \ldots, \varepsilon_k z_{\sigma(k)})$ *for some permutation* σ *of* $\{1, \ldots, k\}$ *and some complex signs* $\varepsilon_1, \ldots, \varepsilon_k$.

Proof. The sufficient condition is trivial. For the necessary one, we first observe that, for $j = 1, \ldots, k$, $\phi_j \in A^+(\mathbb{T}^k)$, since $\phi_j = C_\phi(z_j)$; hence ϕ can be continuously extended to a continuous map, still denoted by ϕ, from $\overline{\mathbb{D}}^k$ to $\overline{\mathbb{D}}^k$. We are going to show that this map is bijective.

Assume first that $a, b \in \overline{\mathbb{D}}^k$ and that $\phi(a) = \phi(b)$. Let $f \in A^+(\mathbb{T}^k)$; since C_ϕ is bijective we can find $g \in A^+(\mathbb{T}^k)$ such that $f = g \circ \phi$, so that $f(a) = f(b)$. Since $A^+(\mathbb{T}^k)$ obviously separates the points of $\overline{\mathbb{D}}^k$, we have $a = b$. In particular, ϕ is injective on $\mathbb{D}^k$ and by Osgood's Theorem (see Theorem 5, page 86 of [19]) $\det(\phi'(z)) \neq 0$ for each $z \in \mathbb{D}^k$, implying that ϕ is an open mapping on $\mathbb{D}^k$. Therefore, $\phi(\mathbb{D}^k) \subseteq \mathbb{D}^k$.

Now, let $u \in \overline{\mathbb{D}}^k$. Define an element L of the spectrum of $A^+(\mathbb{T}^k)$ by $L(f) = g(u)$ if $f = g \circ \phi$. Since the spectrum of $A^+(\mathbb{T}^k)$ is clearly $\overline{\mathbb{D}}^k$, we can find $a \in \overline{\mathbb{D}}^k$ such that $L(f) = f(a)$, so that $g(\phi(a)) = g(u)$ for any $g \in A^+(\mathbb{T}^k)$, implying $u = \phi(a)$. ϕ is therefore a homeomorphism : $\overline{\mathbb{D}}^k \to \overline{\mathbb{D}}^k$.

Since $\phi(\overline{\mathbb{D}}^k) = \overline{\mathbb{D}}^k$ and $\phi(\mathbb{D}^k) \subseteq \mathbb{D}^k$, we get $\phi(\mathbb{D}^k) = \mathbb{D}^k$. In particular, $\phi \in \operatorname{Aut} \mathbb{D}^k$, the group of analytic automorphisms of $\mathbb{D}^k$.

Recall that ([19], Proposition 3, page 68):

LEMMA 3.4. *The analytic map* $\phi\colon \mathbb{D}^k \to \mathbb{D}^k$ *belongs to* Aut $\mathbb{D}^k$ *if and only if*

$$\phi(z) = \Big(\varepsilon_1 \frac{z_{\sigma(1)} - a_1}{1 - \overline{a}_1 z_{\sigma(1)}}, \ldots, \varepsilon_k \frac{z_{\sigma(k)} - a_k}{1 - \overline{a}_k z_{\sigma(k)}}\Big),$$

for some permutation σ *of* $\{1, \ldots, k\}$, *for some* $(a_1, \ldots, a_k) \in \mathbb{D}^k$ *and some complex signs* $\varepsilon_1, \ldots, \varepsilon_k$.

We therefore see that $\phi_j(z) = \varepsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a}_j z_{\sigma(j)}}$, so that for each $n \in \mathbb{N}$, we have in view of (3.1):

$$\Big\| \Big(\varepsilon_j \frac{z - a_j}{1 - \overline{a}_j z}\Big)^n \Big\|_{A^+} = \Big\| \Big(\varepsilon_j \frac{z_{\sigma(j)} - a_j}{1 - \overline{a}_j z_{\sigma(j)}}\Big)^n \Big\|_{A^+(\mathbb{T}^k)} \leqslant C.$$

Lemma 3.1 now implies that $a_j = 0$, $j = 1, \ldots, k$, so that $\phi_j(z) = \varepsilon_j z_{\sigma(j)}$, and this ends the proof of Theorem 3.3. ∎

We now consider the Wiener algebra $A^+(\mathbb{T}^\infty)$ in countably many variables. It will be convenient to consider holomorphic functions on the open unit ball $\mathbf{B} = \mathbb{D}^\infty \cap \mathbf{c}_0$ of the Banach space $\mathbf{c}_0$ of sequences $z = (z_n)_{n \geqslant 1}$ tending to zero at infinity, with its natural norm $\|z\| = \sup_{n\geqslant 1} |z_n|$. We then have the following extension of Cartan's Lemma 3.4 to the case of **B**, which is due to Harris [9]:

LEMMA 3.5 (Analytic Banach-Stone Theorem). *The analytic automorphisms $\phi\colon \boldsymbol{B} \to \boldsymbol{B}$ are exactly the maps of the form $\phi = (\phi_j)_{j\geqslant 1}$, with $\phi_j(z) = \varepsilon_j \frac{z_{\sigma(j)} - a_j}{1-\overline{a}_j z_{\sigma(j)}}$, for some permutation σ of $\mathbb{N}$, some point $a = (a_j)_{j\geqslant 1} \in \boldsymbol{B}$, and some sequence $(\varepsilon_j)_{j\geqslant 1}$ of complex signs.*

Recall that the linear Banach-Stone Theorem states: if $L\colon \mathbf{c}_0 \to \mathbf{c}_0$ is a surjective isometry fixing the origin, then L has the form

$$L(z_1, \ldots, z_n, \ldots) = (\varepsilon_1 z_{\sigma(1)}, \ldots, \varepsilon_n z_{\sigma(n)}, \ldots).$$

If we want to exploit Lemma 3.5 for describing the composition automorphisms of $A^+(\mathbb{T}^\infty)$, we have to make an extra assumption, the reason for which is the following: if C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$, then ϕ is an automorphism of $\overline{\mathbb{D}}^\infty$, but there is no reason, *a priori*, why ϕ should be an automorphism of **B**.

THEOREM 3.6. *Let $\phi = (\phi_j)_j\colon \boldsymbol{B} \to \boldsymbol{B}$ be an analytic map such that C_ϕ maps $A^+(\mathbb{T}^\infty)$ into itself. Then:*

(i) *If $\phi(z) = (\varepsilon_j z_{\sigma(j)})_{j\geqslant 1}$ for some permutation σ of $\mathbb{N}$ and some sequence $(\varepsilon_j)_{j\geqslant 1}$ of complex signs, then C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$, and it is isometric.*

(ii) *If C_ϕ is an automorphism of $A^+(\mathbb{T}^\infty)$ and if we moreover assume that $\phi_k(z) = z_k^{d_k} u_k(z)$, with $d_k \geqslant 1$ and $u_k(0) \neq 0$, for each $k \in \mathbb{N}$ and each $z \in \boldsymbol{B}$, then $\phi(z) = (\varepsilon_j z_j)_{j\geqslant 1}$ for some sequence $(\varepsilon_j)_{j\geqslant 1}$ of complex signs.*

Proof. (i) is trivial. For (ii), consider the compact set $K = \overline{\mathbb{D}}^\infty$, endowed with the product topology (K is nothing but the spectrum of $A^+(\mathbb{T}^\infty)$); clearly, **B** is dense in K, and since $\phi_j = C_\phi(z_j) \in A^+(\mathbb{T}^\infty)$, $\phi_j\colon \mathbf{B} \to \mathbb{D}$ extends continuously to K, and $\phi = (\phi_j)_j$ extends continuously to a map, still denoted by ϕ, from K to K, and we still can write, for every $k \in \mathbb{N}$, $\phi_k(z) = z_k^{d_k} u_k(z)$ for each $z \in K$. Exactly as in the proof of Theorem 3.3, we can show that ϕ is bijective, since K is the spectrum of $A^+(\mathbb{T}^\infty)$. Let now $\psi\colon K \to K$ be the inverse map of ϕ. Since K is compact, ψ is continuous on K, and so on **B**; it is then easy to see, as usual, that ψ is holomorphic in **B** (alternatively, $\psi_k = (C_\phi)^{-1}(z_k) \in A^+(\mathbb{T}^\infty)$, and so is analytic in $\mathbb{D}^\infty$, and it is clear that $\psi = (\psi_k)_k$).

Now, it suffices to show that ψ maps **B** into **B**; indeed, it will follow that ϕ maps **B** onto **B**, and so the map ϕ will appear as an analytic automorphism of **B** (since we already know that $\psi = \phi^{-1}$ is analytic in **B**), and Lemma 3.5 shows that $\phi_j(z)$ has the form $\varepsilon_j \frac{z_{\sigma(j)} - a_j}{1-\overline{a}_j z_{\sigma(j)}}\cdot$ Now, $\|\phi_j^n\|_{A^+(\mathbb{T}^\infty)} = \|C_\phi(z_j^n)\|_{A^+(\mathbb{T}^\infty)} \leqslant \|C_\phi\|$, and as in the proof of Theorem 3.3, we shall conclude that $a_j = 0$ for each j. Finally, the assumption $\phi_k(z) = z_k^{d_k} u_k(z)$ for each k will imply that σ is the identity map.

So we have to show that $\psi(\mathbf{B}) \subseteq \mathbf{B}$. If it were not the case, there would exist an element $w = (w_j)_j \in \mathbf{B}$ such that $\psi(w) \notin \mathbf{B}$. Hence there would exist $\delta > 0$ and an infinite subset $J \subseteq \mathbb{N}$ such that

$$|\psi_j(w)| > \delta \quad \text{for every } j \in J. \tag{3.2}$$

Let $\delta' = \frac{\delta}{\|C_\psi\|}$.

Since $w \in \mathbf{B}$, we should find an integer $N \geqslant 1$ such that

$$n \geqslant N \quad \Rightarrow \quad |w_n| \leqslant \delta'.$$

Let $\kappa = \max_{1 \leqslant n \leqslant N} |w_n|$. Since $\kappa < 1$, there would exist $p \geqslant 1$ such that $\kappa^p < \delta'$. Consider the finite set

$$F = \{\alpha = (m_1, \ldots, m_N, 0, \ldots) : m_1 + \cdots + m_N \leqslant p\}.$$

We assert that

$$F \text{ intersects the spectrum of } \psi_j \text{ for every } j \in J. \tag{3.3}$$

Indeed, writing $\psi_j(z) = \sum a_j(n_1, \ldots, n_k, 0, \ldots) z_1^{n_1} \cdots z_k^{n_k}$, we have:

- if $\alpha = (n_1, \ldots, n_l, \ldots)$ with $l > N$ and $n_l \neq 0$, then $|w_l| \leqslant \delta'$, and so
$$|w^\alpha| \leqslant |w_1^{n_1} \cdots w_l^{n_l}| \leqslant |w_l^{n_l}| \leqslant |w_l| \leqslant \delta';$$
- if $n_1 + \cdots + n_N \geqslant p$, then
$$|w_1^{n_1} \cdots w_N^{n_N}| \leqslant \kappa^{n_1 + \cdots + n_N} \leqslant \kappa^p < \delta'.$$

Hence, in both cases, $\alpha \notin F$ implies $|w^\alpha| < \delta'$. Therefore, if F does not intersect the spectrum of ψ_j, we get

$$|\psi_j(w)| \leqslant \sum_{\alpha \notin F} |a_j(\alpha)|\, |w^\alpha| \leqslant \delta' \|\psi_j\|_{A^+(\mathbb{T}^\infty)} \leqslant \delta' \|C_\psi\| = \delta$$

(since $\|\psi_j\|_{A^+(\mathbb{T}^\infty)} = \|C_\psi(z_j)\|_{A^+(\mathbb{T}^\infty)} \leqslant \|C_\psi\|\, \|z_j\|_{A^+(\mathbb{T}^\infty)} = \|C_\psi\|$), which contradicts (3.2). To end the proof, remark now that the assumption $\phi_k(z) = z_k^{d_k} u_k(z)$ for every $k \in \mathbb{N}$ implies that

$$z_k = \phi_k[\psi(z)] = [\psi_k(z)]^{d_k} u_k[\psi(z)].$$

But this is impossible, since J is infinite and, for $k \in J$, $\psi_k(z)$ depends on $(z_1, \ldots, z_N)$, and hence $\phi_k[\psi(z)] = [\psi_k(z)]^{d_k} u_k[\psi(z)]$ also (since $d_k \geqslant 1$ and $u_k(0) \neq 0$).

That ends the proof of Theorem 3.6. ∎

REMARK 3.7. We shall see later, in Section 4, Theorem 4.5, that the converse of (i) in Theorem 3.6 is true.

Although Theorem 3.6 is not completely satisfactory, it will be sufficient for characterizing the composition automorphisms of the Wiener-Dirichlet algebra $\mathcal{A}^+$. In fact, we have:

THEOREM 3.8. *Let $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ be a composition operator. Then C_ϕ is an automorphism of $\mathcal{A}^+$ if and only if ϕ is a vertical translation: $\phi(s) = s + \mathrm{i}\tau$, where τ is a real number.*

Note that a similar result was obtained by F. Bayart [1] for the Hilbert space $\mathcal{H}^2$ of square-summable Dirichlet series $f(s) = \sum_1^\infty a_n n^{-s}$ such that $\sum_1^\infty |a_n|^2 < +\infty$, but his proof does not seem to extend to our setting, and our strategy for proving Theorem 3.8 will be to deduce it from Theorem 3.6, with the help of the transfer operator Δ mentioned in the Introduction. The following lemma (with the notation used in the Introduction) allows the transfer from composition operators on $\mathcal{A}^+$ to composition operators on $A^+(\mathbb{T}^\infty)$.

LEMMA 3.9. *Suppose that $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$ is a composition operator, with $\phi(s) = c_0 s + \varphi(s)$, $c_0 \in \mathbb{N}_0$, $\varphi \in \mathcal{D}$. Let $T = \Delta C_\phi \Delta^{-1}\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$. Then:*

(i) *$T = C_{\widetilde{\phi}}$, where $\widetilde{\phi}\colon \boldsymbol{B} \to \mathbb{D}^\infty$ is an analytic map such that $\widetilde{\phi}(z^{[s]}) = z^{[\phi(s)]}$, for any $s \in \mathbb{C}_0$.*

(ii) *If moreover $c_0 \geqslant 1$ (which is the case if C_ϕ is surjective), $\widetilde{\phi}$ maps $\boldsymbol{B}$ into $\boldsymbol{B}$.*

Proof. (i) Define $f_k(s) = p_k^{-\phi(s)} \in \mathcal{A}^+$, $\phi_k = \Delta f_k$ and

$$\widetilde{\phi} = (\phi_1, \phi_2, \ldots). \tag{3.4}$$

We have

$$\widetilde{\phi}(z^{[s]}) = (\Delta f_k(z^{[s]}))_{k\geqslant 1} = (f_k(s))_{k\geqslant 1} = z^{[\phi(s)]}$$

by (1.1), and $\|\phi_k\|_\infty = \|f_k\|_\infty \leqslant 1$ by (1.2). Moreover, no ϕ_k is constant, so the open mapping theorem implies that $|\phi_k(z)| < 1$ for $z \in \mathbf{B}$, i.e. $\widetilde{\phi}(z) \in \mathbb{D}^\infty$. Finally, if $f(z) = \sum_{n=1}^\infty a_n z_1^{\alpha_1} \cdots z_r^{\alpha_r} \in A^+(\mathbb{T}^\infty)$ (where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ is the decomposition in prime factors), we have the following "diagram":

$$f \xmapsto{\Delta^{-1}} \sum_{n=1}^\infty a_n n^{-s} \xmapsto{C_\phi} \sum_{n=1}^\infty a_n f_1^{\alpha_1} \cdots f_r^{\alpha_r} \xmapsto{\Delta} \sum_{n=1}^\infty a_n \phi_1^{\alpha_1} \cdots \phi_r^{\alpha_r} = f \circ \widetilde{\phi},$$

i.e. $T(f) = C_{\widetilde{\phi}}(f)$.

(ii) First observe that C_φ also maps $\mathcal{A}^+$ into $\mathcal{A}^+$ (see the remark before Corollary 2.6). Secondly, we have $f_k(s) = p_k^{-c_0 s} p_k^{-\varphi(s)} = p_k^{-c_0 s} g_k(s)$, with $g_k \in \mathcal{A}^+$ and $\|g_k\|_{\mathcal{A}^+} = \|C_\varphi(p_k^{-s})\|_{\mathcal{A}^+} \leqslant C$. It follows that, for $z \in \mathbf{B}$: $\Delta f_k(z) = z_k^{c_0} \Delta g_k(z)$, and via (1.2) that

$$|\Delta f_k(z)| \leqslant |z_k|^{c_0} \|\Delta g_k\|_\infty = |z_k|^{c_0} \|g_k\|_\infty \leqslant |z_k|^{c_0} \|g_k\|_{\mathcal{A}^+} \leqslant C|z_k|^{c_0}.$$

Since $c_0 \geqslant 1$, we see that $\Delta f_k(z) \to 0$ as $k \to \infty$, i.e. $\widetilde{\phi}(z) \in \mathbf{B}$. Finally, whenever C_ϕ is surjective, $\phi : \mathbb{C}_0 \to \mathbb{C}_0$ is injective: indeed, $\mathcal{A}^+$ separates the points of $\mathbb{C}_0$ ($2^{-a} = 2^{-b}$ and $3^{-a} = 3^{-b}$ imply $a = b$, since $\frac{\log 2}{\log 3}$ is irrational), and we can argue as in Theorem 3.3.

To end the proof of Lemma 3.9, it remains to remark that if $c_0 = 0$, ϕ is never injective on $\mathbb{C}_0$, according to well-known results on the theory of analytic, almost-periodic functions (see e.g. p. 13 of [4]). Therefore, we have $c_0 \geqslant 1$ if C_ϕ is surjective. ∎

Proof of Theorem 3.8. The sufficiency of the condition is trivial. Conversely, if C_ϕ is an automorphism of $\mathcal{A}^+$, let $C_{\widetilde{\phi}} = \Delta C_\phi \Delta^{-1}$, as in Lemma 3.9. Since C_ϕ is surjective, we know from Lemma 3.9 that $\widetilde{\phi}$ maps **B** into **B**; we can apply Theorem 3.6, because $C_{\widetilde{\phi}}$ is an automorphism of $A^+(\mathbb{T}^\infty)$ onto itself and moreover $\widetilde{\phi}_k(z) = \Delta f_k(z) = z_k^{c_0} \Delta g_k(z)$, with $c_0 \geqslant 1$ (by Lemma 3.9 again) and

$$\Delta g_k(0) = \lim_{\operatorname{Re} s \to +\infty} g_k(s) = \lim_{\operatorname{Re} s \to +\infty} p_k^{-\varphi(s)} = p_k^{-c_1} \neq 0.$$

We conclude that

$$\widetilde{\phi}(z) = (\varepsilon_1 z_1, \dots, \varepsilon_n z_n, \dots), \tag{3.5}$$

for some sequence of signs $(\varepsilon_n)_n$, where $z = (z_1, \dots, z_n, \dots)$.

If we now test this equality at the points $z^{[s]} = (p_j^{-s})_j$, $s \in \mathbb{C}_0$, and use (1.1), we see that

$$p_j^{-\phi(s)} = \varepsilon_j p_j^{-s}, \quad s \in \mathbb{C}_0,\ j \in \mathbb{N}. \tag{3.6}$$

Taking the moduli in (3.6), we get $\operatorname{Re} \phi(s) = \operatorname{Re} s$. Since $\phi(s) - s$ is analytic on the domain $\mathbb{C}_0$, this implies $\phi(s) - s = \mathrm{i}\tau$, with $\tau \in \mathbb{R}$, thus ending the proof of Theorem 3.8. ∎

4. ISOMETRIES OF $A^+(\mathbb{T}^k), A^+(\mathbb{T}^\infty), \mathcal{A}^+$

In this section, we shall characterize the composition operators which are isometric on $A^+(\mathbb{T}^k)$ and then those which are isometric on $A^+(\mathbb{T}^\infty)$ (under an additional assumption) and on $\mathcal{A}^+$. If $f(z) = \sum a_\alpha z^\alpha \in A^+(\mathbb{T}^k)$, it will be convenient to note $a_\alpha = \widehat{f}(\alpha)$. The spectrum of f (denoted by $\operatorname{Sp} f$) is the set of α's such that $\widehat{f}(\alpha) \neq 0$. e will denote the point $(1, \dots, 1)$ of $\overline{\mathbb{D}}^k$. An elaboration of the method of Harzallah [14] allows us to show:

THEOREM 4.1. *Assume that* $\phi = (\phi_j)_j \colon \mathbb{D}^k \to \overline{\mathbb{D}}^k$, *induces a composition operator* $C_\phi \colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$. *Then* $C_\phi \colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$ *is an isometry if and only if there exists a square matrix* $A = (a_{ij})_{1 \leqslant i,j \leqslant k}$, *with* $a_{ij} \in \mathbb{N}_0$ *and* $\det A \neq 0$, *and complex signs* $\varepsilon_1, \dots, \varepsilon_k$ *such that*

$$\phi_i(z) = \varepsilon_i z_1^{a_{i1}} \cdots z_k^{a_{ik}}, \quad 1 \leqslant i \leqslant k,\ z = (z_1, \dots, z_k) \in \mathbb{D}^k. \tag{4.1}$$

To prove this theorem, it will be convenient to use the next two lemmas.

LEMMA 4.2. *C_ϕ is an isometry if and only if:*

(i) *$\phi_i = \varepsilon_i F_i$, $1 \leqslant i \leqslant k$, where ε_i is a complex sign, $\widehat{F_i} \geqslant 0$, and $F_i(e) = \|F_i\|_\infty = 1$;*

(ii) *if $\alpha, \alpha' \in \mathbb{N}_0^k$ are distinct, the spectra of ϕ^α and $\phi^{\alpha'}$ are disjoint.*

Proof. Suppose that (i) and (ii) hold, and let $f(z) = \sum \widehat{f}(\alpha) z^\alpha \in A^+(\mathbb{T}^k)$. We have by (ii)

$$\|C_\phi f\|_{A^+(\mathbb{T}^k)} = \sum |\widehat{f}(\alpha)|\, \|\phi^\alpha\|_{A^+(\mathbb{T}^k)} = \sum |\widehat{f}(\alpha)|\, \|F^\alpha\|_{A^+(\mathbb{T}^k)},$$

since, with the obvious notation, $\phi^\alpha = \varepsilon^\alpha F^\alpha$. Since $\widehat{F^\alpha} \geqslant 0$, we have, using (i), $\|F^\alpha\|_{A^+(\mathbb{T}^k)} = F^\alpha(e) = 1$, so that

$$\|C_\phi f\|_{A^+(\mathbb{T}^k)} = \sum |\widehat{f}(\alpha)| = \|f\|_{A^+(\mathbb{T}^k)}.$$

Conversely, suppose that C_ϕ is an isometry. For each $i \in [1, k]$ and each $n \in \mathbb{N}$, we have $\|\phi_i^n\|_{A^+(\mathbb{T}^k)} = \|z_i^n\|_{A^+(\mathbb{T}^k)} = 1$, whence $\|\phi_i\|_\infty = \lim_{n\to\infty} \|\phi_i^n\|_{A^+(\mathbb{T}^k)}^{1/n} = 1$, by the spectral radius formula. Since $\|\phi_i\|_\infty \leqslant \|\phi_i\|_{A^+(\mathbb{T}^k)} = 1$, the only possibility is that $\phi_i = \varepsilon_i F_i$, with $|\varepsilon_i| = 1$, $\widehat{F_i} \geqslant 0$, and $\|\phi_i\|_{A^+(\mathbb{T}^k)} = 1 = \|F_i\|_{A^+(\mathbb{T}^k)} = F_i(e)$. Therefore, (i) holds. Now suppose that we can find $\alpha \neq \alpha'$ such that $\operatorname{Sp} \phi^\alpha \cap \operatorname{Sp} \phi^{\alpha'}$ contains an element $\beta_0 \in \mathbb{N}_0^k$, and set $\rho = \widehat{\phi}^\alpha(\beta_0)$, $\rho' = \widehat{\phi}^{\alpha'}(\beta_0)$. Without loss of generality, we may assume that $|\rho| \geqslant |\rho'|$. Let θ be a complex sign such that $|\rho + \theta\rho'| = |\rho| - |\rho'|$. Then, we have $\|z^\alpha + \theta z^{\alpha'}\|_{A^+(\mathbb{T}^k)} = 2$, whereas

$$\begin{aligned}
\|C_\phi(z^\alpha + \theta z^{\alpha'})\|_{A^+(\mathbb{T}^k)} = \|\phi^\alpha + \theta\phi^{\alpha'}\|_{A^+(\mathbb{T}^k)} &= \sum_{\beta \neq \beta_0} |\widehat{\phi}^\alpha(\beta) + \theta\widehat{\phi}^{\alpha'}(\beta)| + |\rho + \theta\rho'| \\
&\leqslant \sum_{\beta \neq \beta_0} |\widehat{\phi}^\alpha(\beta)| + \sum_{\beta \neq \beta_0} |\widehat{\phi}^{\alpha'}(\beta)| + |\rho| - |\rho'| \\
&= 1 - |\rho| + 1 - |\rho'| + |\rho| - |\rho'| = 2(1 - |\rho'|) < 2,
\end{aligned}$$

contradicting the isometric character of C_ϕ. ∎

LEMMA 4.3. *If $\phi = (\phi_i)_i$ and if one of the ϕ_i's is not a monomial, then we can find a pair of distinct elements α, $\alpha' \in \mathbb{N}_0^k$ such that the spectra of ϕ^α and $\phi^{\alpha'}$ intersect.*

Proof. To avoid awkward notation, we will assume that $k = 3$, but it will be clear that the reasoning works for any value of k. Since only the spectra of the ϕ_i's are involved, we can assume without loss of generality that we have

$$\begin{aligned}
&\phi_1(z) = z_1^{s_1} z_2^{s_2} z_3^{s_3} + z_1^{t_1} z_2^{t_2} z_3^{t^3}, \\
&\text{with } (s_1, s_2, s_3) \neq (t_1, t_2, t_3),\ \phi_2(z) = z_1^{u_1} z_2^{u_2} z_3^{u_3},\ \phi_3(z) = z_1^{v_1} z_2^{v_2} z_3^{v_3}
\end{aligned}$$

(in short, $\phi_1(z) = z^s + z^t$; $\phi_2(z) = z^u$; $\phi_3(z) = z^v$).

If $\alpha = (a, b, c)$, the spectrum of $\phi^\alpha = (z^s + z^t)^a z^{bu} z^{cv}$ consists of the triples

$$\rho s_j + (a - \rho) t_j + b u_j + c v_j = \rho(s_j - t_j) + a t_j + b u_j + c v_j,$$

with $j = 1, 2, 3$ and $0 \leqslant \rho \leqslant a$. Therefore, if $\alpha' = (a', b', c')$, the spectra of ϕ^α and $\phi^{\alpha'}$ will intersect if and only if we can find $0 \leqslant \rho \leqslant a$ and $0 \leqslant \rho' \leqslant a'$ such that

$$\rho(s_j - t_j) + at_j + bu_j + cv_j = \rho'(s_j - t_j) + a't_j + b'u_j + c'v_j, \quad j = 1, 2, 3,$$

or equivalently

$$(\rho - \rho')(s_j - t_j) + (a - a')t_j + (b - b')u_j = (c' - c)v_j, \quad j = 1, 2, 3. \tag{4.2}$$

In (4.2), we can drop the conditions $\rho \leqslant a$, $\rho' \leqslant a'$, since we can always replace a and a' by $a + N$ and $a' + N$, where N is a large integer, without affecting the result. Now, let M be the matrix

$$M = \begin{bmatrix} s_1 - t_1 & t_1 & u_1 \\ s_2 - t_2 & t_2 & u_2 \\ s_3 - t_3 & t_3 & u_3 \end{bmatrix}.$$

To solve equation (4.2), we distinguish two cases.

Case 1. $\det M = 0$.

We decide then to take $c' = c$. Since the field $\mathbb{Q}$ of rational numbers is the quotient field of $\mathbb{Z}$, we can find $\lambda, \mu, \nu \in \mathbb{Z}$, not all zero, such that

$$\lambda(s_j - t_j) + \mu t_j + \nu u_j = 0, \quad j = 1, 2, 3.$$

If μ and ν are both zero, then $\lambda = 0$, since $s_j - t_j \neq 0$ for some j. Therefore, we may assume for example that $\mu \neq 0$, and write $\lambda = \rho - \rho'$, $\mu = a - a'$, $\nu = b - b'$, with $\alpha = (a, b, c) \in \mathbb{N}_0^3$, $\alpha' = (a', b', c') \in \mathbb{N}_0^3$, and $\alpha \neq \alpha'$ since $a \neq a'$. By construction, we have (4.2), so that the spectra of ϕ^α and $\phi^{\alpha'}$ are not disjoint.

Case 2. $\det M \neq 0$.

We can then find rational numbers q, r, s such that

$$q(s_j - t_j) + rt_j + su_j = v_j, \quad j = 1, 2, 3,$$

and we can write $q = \frac{\lambda}{N}$, $r = \frac{\mu}{N}$, $s = \frac{\nu}{N}$, where $\lambda, \mu, \nu \in \mathbb{Z}$ and where N is a positive integer. Therefore, we have

$$\lambda(s_j - t_j) + \mu t_j + \nu u_j = N v_j, \quad 1 \leqslant j \leqslant 3,$$

and writing $\lambda = \rho - \rho'$, $\mu = a - a'$, $\nu = b - b'$, $c = 0$, $c' = N$, we get (4.2) with distinct triples $\alpha = (a, b, c)$ and $\alpha' = (a', b', c')$ of non-negative integers. Once again, the spectra of ϕ^α and $\phi^{\alpha'}$ are not disjoint. ∎

Proof of Theorem 4.1. If the condition holds, C_ϕ is an isometry by Lemma 4.2.

Conversely, suppose that C_ϕ is an isometry. Then, by Lemma 4.2, the spectra of ϕ^α and $\phi^{\alpha'}$ are disjoint if $\alpha \neq \alpha'$, and by Lemma 4.3 each ϕ_i is a monomial, necessarily of the form (4.1) by (i) of Lemma 4.2. Finally, if we denote by A the square matrix (a_{ij}), by $A^* = (a_{ji})$ its adjoint matrix, and if we let A, A^* act on $\mathbb{Z}^k$ by the formulas

$$A(\alpha) = \beta, \quad A^*(\alpha) = \gamma, \tag{4.3}$$

with $\beta_i = \sum_{j=1}^{k} a_{ij}\alpha_j$ and $\gamma_j = \sum_{i=1}^{k} a_{ij}\alpha_i$, we see that

$$C_\phi(z^\alpha) = \phi^\alpha = \varepsilon^\alpha z^{A^*(\alpha)}. \tag{4.4}$$

In fact,

$$C_\phi(z^\alpha) = \prod_i \phi_i^{\alpha_i} = \prod_i \varepsilon_i^{\alpha_i}\Big(\prod_j z_j^{a_{ij}}\Big)^{\alpha_i} = \varepsilon^\alpha \prod_j z_j^{\gamma_j}.$$

Now, by Lemma 4.2, the ϕ^α's have disjoint spectra, so that the $A^*(\alpha)$'s are distinct, implying $\det A \neq 0$. ∎

If we now turn to the case of $A^+(\mathbb{T}^\infty)$, Lemma 4.2 clearly still holds, but Lemma 4.3 no longer holds : for example, if $I_1, \dots, I_n, \dots$ are disjoint subsets of $\mathbb{N}$, c_{ij} positive numbers such that $\sum_{j\in I_i} c_{ij}=1$, $i=1,2,\dots$ and if the map $\widetilde{\phi}$ is defined by

$$\widetilde{\phi} = (\phi_i)_i, \quad \text{where } \phi_i(z) = \sum_{j\in I_i} c_{ij} z_j, \tag{4.5}$$

then $C_{\widetilde{\phi}}$ is an isometry by Lemma 4.2 and yet no ϕ_i is a monomial if each I_i has more than one element. We have however a weaker result:

THEOREM 4.4. *Let $\phi\colon \overline{\mathbb{D}}^\infty \to \overline{\mathbb{D}}^\infty$ be a map inducing an operator $C_\phi\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$, and such that moreover $\phi(\mathbb{T}^\infty) \subseteq \mathbb{T}^\infty$. Then*

(i) *There exists a matrix $A = (a_{ij})_{i,j\geqslant 1}$, with $a_{ij} \in \mathbb{N}_0$ and $\sum_j a_{ij} < \infty$ for each i, and complex signs ε_i such that $\phi = (\phi_i)_i$ and*

$$\phi_i(z) = \varepsilon_i \prod_{j=1}^{\infty} z_j^{a_{ij}}, \quad i = 1, 2, \dots. \tag{4.6}$$

(ii) *C_ϕ is an isometry if and only if $A^* = (a_{ji})$, acting on $\mathbb{Z}^{(\infty)}$ as in* (4.3), *is injective.*

Proof. (i) If we apply Lemma 3.2 to the (connected) group $\Gamma = \mathbb{T}^\infty$ and its dual $G = \mathbb{Z}^{(\infty)}$, we see that for each $i \in \mathbb{N}$ there exists $L_i = (a_{i1}, a_{i2}, \dots) \in \mathbb{Z}^{(\infty)}$, necessarily in $\mathbb{N}_0^{(\infty)}$, and a complex sign ε_i such that, for each $z \in \mathbb{T}^\infty$, we have:

$$\phi_i(z) = \varepsilon_i \langle L_i, z\rangle = \varepsilon_i \prod_j z_j^{a_{ij}}$$

(note that, for $n \in \mathbb{N}$, setting $C = \|C_\phi\|$, we have $\|\phi_i^n\|_{A^+(\mathbb{T}^\infty)} = \|C_\phi(z_i^n)\|_{A^+(\mathbb{T}^\infty)} \leqslant C$, and also, since $|\phi_i(\mathrm{e}^{it})| = 1$: $\|\phi_i^{-n}\|_{A^+(\mathbb{T}^\infty)} = \|\overline{\phi}_i^n\|_{A^+(\mathbb{T}^\infty)} = \|\phi_i^n\|_{A^+(\mathbb{T}^\infty)} \leqslant C$).

This proves (4.6).

(ii) We know from (4.4) (which clearly still holds for $k = \infty$) that $\phi^\alpha = \varepsilon^\alpha z^{A^*(\alpha)}$, and we know from Lemma 4.2 that C_ϕ is an isometry if and only if the spectra of the ϕ^α's are disjoint. This gives the result. ∎

We shall prove here the announced converse of part (i) of Theorem 3.6.

THEOREM 4.5. *Let $\phi = (\phi_j)_j \colon \boldsymbol{B} \to \boldsymbol{B}$ be an analytic function which induces a composition operator C_ϕ on $A^+(\mathbb{T}^\infty)$. If C_ϕ is an isometric automorphism of $A^+(\mathbb{T}^\infty)$, then $\phi(z) = (\varepsilon_j z_{\sigma(j)})_j$, for some permutation σ of $\mathbb{N}$ and some some sequence $(\varepsilon_j)_{j\geqslant 1}$ of complex signs.*

Proof. It suffices to look at the proof of Theorem 3.6, (ii): as in that proof, and with the same notation, it suffices to show that $\psi(\mathbf{B}) \subseteq \mathbf{B}$; but if it is not the case, it follows from (3.3), since the set J is infinite, that there exist at least two distinct integers $j_1, j_2 \in J$ such that the spectra of ϕ_{j_1} and ϕ_{j_2} are not disjoint. By Lemma 4.2, this contradicts the isometric nature of C_ϕ. ∎

REMARK 4.6. It is easy to see that the composition operator $C_{\widetilde{\phi}}$ on $A^+(\mathbb{T}^\infty)$ given by (4.5) does not correspond in general to a $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$.

For example, if

$$\phi_i(z) = \frac{z_{2i-1} + z_{2i}}{2} \quad i = 1, 2, \ldots, \tag{4.7}$$

the equation $\widetilde{\phi}(z^{[s]}) = z^{[\phi(s)]}$ would give

$$\frac{p_{2i-1}^{-s} + p_{2i}^{-s}}{2} = p_i^{-\phi(s)} \quad i = 1, 2, \ldots;$$

taking equivalents of both members as $s \to \infty$ would give that

$$\frac{\phi(s)}{s} \underset{s\to+\infty}{\longrightarrow} \frac{\log p_{2i-1}}{\log p_i},$$

and it is impossible to have that, even for one i, since $\frac{\phi(s)}{s} \to c_0 \in \mathbb{N}_0$!

On the other hand, the additional assumption made in Theorem 4.4 does not allow to use the Bohr's transfer operator Δ to characterize the isometric composition operators on $\mathcal{A}^+$. Nevertheless, we have:

THEOREM 4.7. *Let $\phi \colon \mathbb{C}_0 \to \mathbb{C}_0$ inducing a composition operator $C_\phi \colon \mathcal{A}^+ \to \mathcal{A}^+$. Then C_ϕ is an isometry if and only if $\phi(s) = c_0 s + i\tau$, with $c_0 \in \mathbb{N}$ and $\tau \in \mathbb{R}$.*

Proof. One direction is trivial. For the other, let us introduce the following notation: if $f(s) = \sum_{k=1}^{\infty} a_k k^{-s} \in \mathcal{A}^+$, denote by $\operatorname{Sp} f$ (the spectrum of f) the set of indices k such that $a_k \neq 0$. Now, the technique of the proof of Lemma 4.2 clearly works to show that:

(4.8) *If m and n are distinct integers, the spectra of $m^{-\phi}$ and $n^{-\phi}$ are disjoint.*

This automatically implies $c_0 \neq 0$, since, otherwise, the integer 1 would belong to the spectra of all the $n^{-\phi}$'s. Suppose now that ϕ is not of the form $c_0 s + c_1$, and write:

$$\phi(s) = c_0 s + c_1 + \omega(s), \quad \text{with } \omega(s) = c_r r^{-s} + c_{r+1}(r+1)^{-s} + \cdots,\ r \geqslant 2,\ c_r \neq 0.$$

Then

$$n^{-\phi(s)} = (n^{c_0})^{-s} n^{-c_1} \exp(-\omega(s)\log n) = (n^{c_0})^{-s} n^{-c_1}\Big[1 + \sum_{k=1}^{\infty} \frac{(-\log n)^k}{k!}(\omega(s))^k\Big]$$
$$= (n^{c_0})^{-s} n^{-c_1}\Big[1 + \cdots + \sum_{k=1}^{\infty} \frac{(-\log n)^k}{k!}(c_r r^{-s} + \cdots)^k\Big].$$

For $\operatorname{Re} s$ large enough, all the series involved will be absolutely convergent; therefore the Dirichlet series of $n^{-\phi}$ will be obtained by expanding $(c_r r^{-s} + \cdots)^k$ and grouping terms. In particular, the coefficient λ_n of $n^{c_0} r^{c_0}$ in $n^{-\phi}$ can be obtained only by expanding $(c_r r^{-s} + \cdots)^k$ for $k = 1, \ldots, c_0$, so that $\lambda_n = P(\log n)$, where P is a non-zero polynomial. This implies that, for large n, $\lambda_n \neq 0$, and $(nr)^{c_0} \in \operatorname{Sp} n^{-\phi}$. Moreover, it is clear that $l^{c_0} \in \operatorname{Sp} l^{-\phi}$ for every positive integer l. Hence $(nr)^{c_0} \in \operatorname{Sp} n^{-\phi} \cap \operatorname{Sp} (nr)^{-\phi}$ for large n, which contradicts (4.8).

Therefore $\phi(s) = c_0 s + c_1$, and c_1 clearly has to be purely imaginary if C_ϕ is an isometry. ∎

5. CONCLUDING REMARKS AND QUESTIONS

Corollary 2.6 does not answer, in general, the natural question: if C_ϕ maps $\mathcal{A}^+$ into $\mathcal{A}^+$, is it true that $\phi(s) = c_0 s + \sum_1^\infty c_n n^{-s}$, with $\sum_1^\infty |c_n| < \infty$?

Proposition 2.9 does not apply to the case of complex coefficients c_r, c_{r^2}. Here, recent estimates due to Rusev [23] might help.

The estimate $\|\phi^n\|_{A^+} \geqslant \delta\sqrt{n}$ of Lemma 3.1 is best possible. In fact (see p. 76 of [14]) it is fairly easy to see that $\|\phi^n\|_{A^+} \leqslant C\sqrt{n}$ if $\phi = \mathrm{e}^{\mathrm{i}g}$ and g is $\mathcal{C}^\infty$ (say), and a similar computation in dimension k (i.e. if we work with $A^+(\mathbb{T}^k)$) easily gives the estimate $\|\phi^n\|_{A^+(\mathbb{T}^k)} \leqslant C_k n^{k/2}$ if $\phi = \mathrm{e}^{\mathrm{i}g}$ and g is $\mathcal{C}^\infty$. It would be interesting to know whether the converse holds, i.e. if we have the following quantitative version of Lemma 3.1: if $\phi = \mathrm{e}^{\mathrm{i}g}$, where g is a $\mathcal{C}^\infty$, non-affine, real function, then $\|\phi^n\|_{A^+} \geqslant \delta n^{1/2}$?

In the proof of Theorem 3.8, we used the fact that an analytic function defined on a vertical half-plane which is almost-periodic is never injective to show that $c_0 > 0$, and therefore that the assumption (ii) in Theorem 3.6 naturally holds. This raises two questions:

(a) Can an almost-periodic function defined only on a vertical line be injective, i.e. can an almost-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ be injective? (of course, if f is real-valued, this is impossible: if f is injective, it is monotonic and therefore non almost-periodic).

(b) Can one, at the price of using a different Banach-Stone type Theorem, dispense with the condition $\phi_k(z) = z_k^{d_k} u_k(z)$, with $d_k \geqslant 1$ and $u_k(0) \neq 0$ of (ii) in Theorem 3.6, i.e. is the converse of (i) in this theorem always true?

In view of the examples (4.5) in Section 4, a complete description of the isometric composition operators $C_\phi\colon A^+(\mathbb{T}^\infty) \to A^+(\mathbb{T}^\infty)$ seems out of reach.

We gave a proof of Theorem 4.7 which does not use Theorem 4.1. Using this theorem, we can give a variant of Theorem 4.7: fix an integer $k \geqslant 1$, and denote by $\mathcal{A}_k^+$ the subalgebra of $\mathcal{A}^+$ consisting of the functions $f(s) = \sum_{P^+(n)\leqslant k} a_n n^{-s}$, where $P^+(n)$ denotes the largest prime factor of n. Equivalently, $f \in \mathcal{A}_k^+$ if the Dirichlet expansion of f only involves the primes $p_1, \dots, p_k$. Define similarly the subspace $\mathcal{D}_k$ of $\mathcal{D}$. With those definitions, we can state:

THEOREM 5.1. *Let $\phi(s) = c_0 s + \varphi(s)$, $\varphi \in \mathcal{D}_k$, inducing a composition operator $C_\phi\colon \mathcal{A}_k^+ \to \mathcal{A}_k^+$. Then $C_\phi\colon \mathcal{A}_k^+ \to \mathcal{A}_k^+$ is an isometry if and only if $\phi(s) = c_0 s + \mathrm{i}\tau$, with $c_0 \in \mathbb{N}$ and $\tau \in \mathbb{R}$.*

Proof. The sufficiency is trivial. For the necessity, define an isometry $\Delta\colon \mathcal{A}_k^+ \to \mathcal{A}^+(\mathbb{T}^k)$ by

$$\Delta\Big(\sum_{n=1}^{\infty} a_n n^{-s}\Big) = \sum_{n=1}^{\infty} a_n z_1^{\alpha_1} \cdots z_k^{\alpha_k},$$

where $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition of n in prime factors. Setting $z^{[s]} = (p_1^{-s}, \dots, p_k^{-s}) \in \mathbb{D}^k$, we easily check that $\Delta C_\phi \Delta^{-1} = T$ is a composition operator $C_{\widetilde{\phi}}\colon A^+(\mathbb{T}^k) \to A^+(\mathbb{T}^k)$, which is isometric if C_ϕ is isometric, and such that

$$\widetilde{\phi}(z^{[s]}) = z^{[\phi(s)]}. \tag{5.1}$$

We now use Theorem 4.1 to conclude that $\widetilde{\phi} = (\phi_1, \dots, \phi_k)$, with $\phi_1(z) = \varepsilon_1 z_1^{a_{11}} \cdots z_k^{a_{1k}}$, and where $a_{11}, \dots, a_{1k}$ are non-negative integers. Exactly as in the proof of Theorem 3.3, we then conclude that $\phi(s) = c_0 s + \mathrm{i}\tau$. ∎

In the next theorem, we shall see that there are few composition operators whose symbols preserve the boundary $\mathrm{i}\mathbb{R}$.

THEOREM 5.2. *Let $\phi\colon \mathbb{C}_0 \to \mathbb{C}_0$ inducing a composition operator $C_\phi\colon \mathcal{A}^+ \to \mathcal{A}^+$, and such that moreover ϕ has a continuous extension to $\overline{\mathbb{C}}_0$, preserving the boundary of $\mathbb{C}_0$, i.e. $\phi(\mathrm{i}\mathbb{R}) \subseteq \mathrm{i}\mathbb{R}$. Then $\phi(s) = c_0 s + \mathrm{i}\tau$, where $c_0 \in \mathbb{N}_0$ and $\tau \in \mathbb{R}$.*

Proof. Let $\widetilde{\phi}$ be associated with ϕ as in Theorem 3.3. By continuity, the equation $\widetilde{\phi}(z^{[s]}) = z^{[\phi(s)]}$, $s \in \mathbb{C}_0$, still holds for $s = \mathrm{i}t$, $t \in \mathbb{R}$, to give $\widetilde{\phi}((p_j^{-\mathrm{i}t})_j) = (p_j^{-\phi(\mathrm{i}t)})_j$, and so $\widetilde{\phi}(\mathbb{T}^\infty) \subseteq \mathbb{T}^\infty$ since, by the Kronecker Approximation Theorem and the definition of the product topology on $\mathbb{T}^\infty$, the points $(p_j^{-\mathrm{i}t})_j$, $t \in \mathbb{R}$, are dense in $\mathbb{T}^\infty$. Now, by Theorem 4.4, we have in particular $\widetilde{\phi} = (\phi_i)_i$, with

$$\phi_1(z) = \varepsilon_1 z_1^{a_{11}} \cdots z_k^{a_{1k}},$$

for some complex sign ε_1 and some integer k. In particular, the equation $\widetilde{\phi}(z^{[s]}) = z^{[\phi(s)]}$ implies that

$$\varepsilon_1 (p_1^{-s})^{a_{11}} \cdots (p_k^{-s})^{a_{1k}} = p_1^{-\phi(s)}, \quad s \in \mathbb{C}_0.$$

Passing to the moduli gives $\operatorname{Re} \phi(s) = c \operatorname{Re} s$, with $c = \sum_{j=1}^{k} a_{1j} \frac{\log p_j}{\log p_1}$.

Therefore, $\phi(s) - cs = \mathrm{i}\tau$, $\tau \in \mathbb{R}$, and we know that $c = c_0$ is necessarily an integer. ∎

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REFERENCES

[1] F. BAYART, Hardy spaces of Dirichlet series and their composition operators, *Monat. Math.* **136**(2002), 203–236.

[2] F. BAYART, Compact composition operators on a Hilbert space of Dirichlet series, *Illinois J. Math.* **47**(2003), 725–743.

[3] J. DIEUDONNÉ, *Calcul infinitésimal*, Hermann, Paris 1968.

[4] J. FAVARD, *Leçons sur les fonctions presque périodiques*, Gauthier-Villars, Paris 1933.

[5] C. FINET, D. LI, H. QUEFFÉLEC, Opérateurs de composition sur l'algèbre de Wiener-Dirichlet, *C.R. Acad.Sci. Paris Sér. I Math.* **339**(2004), 109–114.

[6] C. FINET, H. QUEFFÉLEC, Numerical Range of Composition Operators on a Hilbert space of Dirichlet series, *Linear Algebra Appl.* **377**(2004), 1–10.

[7] C. FINET, H. QUEFFÉLEC, A. VOLBERG, Compactness of Composition Operators on a Hilbert space of Dirichlet series, *J. Funct. Anal.* **211**(2004), 271–287.

[8] J. GORDON, H. HEDENMALM, The composition operators on the space of Dirichlet series with square summable coefficients, *Michigan Math. J.* **46**(1999), 313–329.

[9] L.A. HARRIS, Schwarz's lemma in normed linear spaces, *Proc. Math. Acad. Sci. USA* **62**(1969), 1014–1017.

[10] H. HEDENMALM, P. LINDQVIST, K. SEIP, A Hilbert space of Dirichlet series and a system of dilated functions in $L^2(0,1)$, *Duke Math. J.* **86**(1997), 1–36.

[11] E. HEWITT, J.H. WILLIAMSON, Note on absolutely convergent Dirichlet series, *Proc. Amer. Math. Soc.* **8**(1957), 863–868.

[12] E. HLAWKA, J. SCHOISSENGEIER, R. TASCHNER, *Geometric and Analytic Number Theory*, Universitext, Springer-Verlag, Berlin 1991.

[13] J. INDRITZ, An inequality for Hermite polynomials, *Proc. Amer. Math. Soc.* **12**(1961), 981–983.

[14] J.-P. KAHANE, *Séries de Fourier absolument convergentes*, Ergeb. Math. Grenzgeb., vol. 50, Springer-Verlag, Berlin-New York 1970.

[15] Y. KATZNELSON, *An Introduction to Harmonic Analysis*, John Wiley & Sons, Inc., New York-London-Sydney 1968.

[16] N.M. LEBEDEV, *Special Functions and their Applications*, Dover Publ., Inc., New York 1972.

[17] J.M. LOPEZ, K.A. ROSS, *Sidon Sets*, Lecture Notes Pure Appl. Math., vol. 13, Marcel Dekker, Inc., New York 1975.

[18] J.E. MC CARTHY, Hilbert spaces of Dirichlet series and their multipliers, *Trans. Amer. Math. Soc.* **356**(2004), 881–893.

[19] R. NARASIMHAN, *Several Complex Variables*, Chicago Lectures Math., The Univ. of Chicago Press, Chicago, Ill.-London 1971.

[20] D.J. NEWMAN, Homomorphisms of ℓ_+, *Amer. J. Math.* **91**(1969), 37–46.

[21] H. QUEFFÉLEC, Harald Bohr's vision of Dirichlet series: Old and new results, *J. Anal.* **3**(1995), 43–60.

[22] W. RUDIN, *Fourier Analysis on Groups*, Interscience, Tracts Pure Appl. Math., vol. 12, Interscience Publ., Inc., New York-London 1962.

[23] P. RUSEV, An inequality for Hermite's polynomials in the complex plane, *C. R. Acad. Bulgare Sci.* **53**(2000), 13–16.

[24] J. SHAPIRO, *Composition Operator and Classical Function Theory*, Universitext Tracts Math., Springer-Verlag, New York 1991.

FRÉDÉRIC BAYART, LABAG, UNIVERSITÉ BORDEAUX 1, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE
E-mail address: Frederic.Bayart@math.u-bordeaux1.fr

CATHERINE FINET, INSTITUT DE MATHÉMATIQUE, UNIVERSITÉ DE MONS-HAINAUT, "LE PENTAGONE", AVENUE DU CHAMP DE MARS, 6, 7000 MONS, BELGIQUE
E-mail address: catherine.finet@umh.ac.be

DANIEL LI, LABORATOIRE DE MATHÉMATIQUES DE LENS EA 2462, FÉDÉRATION CNRS NORD-PAS-DE-CALAIS FR 2956, UNIVERSITÉ D'ARTOIS, RUE JEAN SOUVRAZ, SP18, 62307 LENS CEDEX, FRANCE
E-mail address: daniel.li@euler.univ-artois.fr

HERVÉ QUEFFÉLEC, UFR DE MATHÉMATIQUES, UNIVERSITÉ DE LILLE 1, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE
E-mail address: queff@math.univ-lille1.fr

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