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SIMILARITY PRESERVING LINEAR MAPS

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ABSTRACT. Let H be an infinite-dimensional separable Hilbert space, $B(H)$ the algebra of all bounded linear operators on H, and $\phi : B(H) \to B(H)$ a bijective linear map such that $\phi(A)$ and $\phi(B)$ are similar for every pair of similar operators $A, B \in B(H)$. Then there exist a nonzero complex number c and an invertible operator $T \in B(H)$ such that either $\phi(A) = cTAT^{-1}$, $A \in B(H)$, or $\phi(A) = cTA^{\mathrm{t}}T^{-1}$, $A \in B(H)$. Here, A^{t} denotes the transpose of A with respect to some fixed orthonormal basis in H.

KEYWORDS: *Linear preserver, similarity.*

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

The problem of characterizing linear maps on matrix algebras preserving certain properties, subsets or relations has attracted the attention of many mathematicians in the last few decades [8], [9], [13]. Some of these results have been recently extended to the infinite-dimensional case.

In this paper we will deal with similarity preserving maps. By the results of Hiai [4] and Lim [10], a linear map ϕ defined on the algebra of all $n \times n$ matrices preserving similarity, that is, $\phi(A)$ and $\phi(B)$ are similar whenever A and B are similar, must be either of the form $A \mapsto cTAT^{-1} + d(\operatorname{tr} A)I$, or of the form $A \mapsto cTA^{\mathrm{t}}T^{-1} + d(\operatorname{tr} A)I$ for some complex numbers c, d and some invertible matrix T. Here, $\operatorname{tr} A$ denotes the trace of A. A more difficult infinite-dimensional case was treated by several authors [5], [6], [7], [12]. The best result so far is due to Petek [12] who proved that if H is an infinite-dimensional Hilbert space, $B(H)$ the algebra of all bounded linear operators on H, and $\phi : B(H) \to B(H)$ a surjective linear map such that for every pair $A, B \in B(H)$ the operators $\phi(A)$ and $\phi(B)$ are similar if and only if A and B are similar, then there exist a nonzero complex number c and an invertible operator $T \in B(H)$ such that either $\phi(A) = cTAT^{-1}$, $A \in B(H)$, or $\phi(A) = cTA^{\mathrm{t}}T^{-1}$, $A \in B(H)$. Let us mention that for general Banach spaces the problem of characterizing similarity preserving linear maps seems to be more difficult. Namely, there exist Banach spaces X such that there are nonzero multiplicative linear functionals f on $B(X)$ [11]. Here, $B(X)$ denotes the algebra of all bounded linear operators on X. If X is such a Banach space, then the bijective linear map $\phi : B(X) \to B(X)$ defined by $\phi(A) = A + f(A)I$ has many preserving properties. Among others, every such map preserves similarity in both directions.

It is natural to ask whether we can get the same conclusion as in the result of Petek under the weaker assumption of preserving similarity in one direction only. The aim of this paper is to show that the answer is in the affirmative for infinite-dimensional separable Hilbert spaces.

THEOREM 1.1. *Let H be an infinite-dimensional separable Hilbert space. Assume that $\phi : B(H) \to B(H)$ is a bijective linear map such that $\phi(A)$ and $\phi(B)$ are similar for every pair of similar operators $A, B \in B(H)$. Then there exist a nonzero complex number c and an invertible operator $T \in B(H)$ such that either*

$$\phi(A) = cTAT^{-1}, \quad A \in B(H), \quad \text{or} \quad \phi(A) = cTA^{\mathrm{t}}T^{-1}, \quad A \in B(H).$$

Here, A^{t} denotes the transpose of A with respect to an arbitrary, but fixed orthonormal basis in H.

It turns out that the problem of characterizing linear maps preserving similarity in one direction only is much more difficult than the problem of describing linear maps preserving this relation in both directions. Fortunately, Davidson and Marcoux [3] have recently obtained a result on linear spans of similarity orbits of bounded linear operators acting on a separable Hilbert space which is of great help in solving our problem. They investigated the problem whether every $A \in B(H)$ can be expressed as a linear combination of operators that are similar to a single given operator $C \in B(H)$. Of course, this is impossible if C is of the form scalar plus compact, as the space of such operators is invariant under similarity. Davidson and Marcoux proved the surprising result that for all other operators C the answer to their question is positive. More precisely, they proved that if $C \in B(H)$ is not of the form $\lambda I + K$ for some $\lambda \in \mathbb{C}$ and some compact operator $K \in B(H)$, then every $A \in B(H)$ can be written as a linear combination of at most 6 operators similar to C. Let us remark that we need the separability condition in our main result because the proof depends on the result of Davidson and Marcoux. We conjecture that the result holds true without this assumption.

2. NOTATION AND PRELIMINARY RESULTS

Let H be a separable complex Hilbert space. By $B(H)$, $K(H)$, $F(H)$, and $F_0(H)$ we denote the algebra of all bounded linear operators on H, the ideal of all compact operators, the ideal of all finite rank operators, and the subspace of all trace zero finite rank operators, respectively. We write $A \sim B$ when $A, B \in B(H)$

are similar. For $x, y \in H$ we denote by y^*x the inner product of x and y. By $x^\perp$ we denote the orthogonal complement of the vector x. If $x, y \in H$ are nonzero vectors, then xy^* stands for the rank one operator defined by $(xy^*)z = (y^*z)x$, $z \in H$. Note that every rank one operator can be expressed in this way and that xy^* is an idempotent if and only if $y^*x = 1$. Further, xy^* is a square-zero operator (nilpotent of rank one) if and only if $y^*x = 0$.

LEMMA 2.1. *Let $A \in B(H)$ be an operator that is not of the form scalar plus finite rank. Then for every positive integer n there exist vectors $x_1, \ldots, x_n \in H$ such that the set of vectors $x_1, \ldots, x_n, Ax_1, \ldots, Ax_n$ is linearly independent.*

Proof. We will prove the statement by induction on n. In the case $n = 1$ the statement follows from the fact that A is not a scalar operator. Assume that we have already found vectors $x_1, \ldots, x_n \in H$ such that $x_1, \ldots, x_n, Ax_1, \ldots, Ax_n$ are linearly independent. Assume also that for every $y \in H$ the set of vectors $x_1, \ldots, x_n, y, Ax_1, \ldots, Ax_n, Ay$ is linearly dependent. In other words, for every $y \in H$ the vectors y and Ay are linearly dependent modulo the subspace spanned by $x_1, \ldots, x_n, Ax_1, \ldots, Ax_n$. It follows from [1], [2] that there exist complex numbers λ, μ, not both of them zero, such that $\lambda A + \mu I = F$ for some finite rank operator F. Clearly, λ cannot be zero. This contradiction completes the proof of the induction step. ∎

LEMMA 2.2. *Let $A \in B(H)$, $A \notin \mathbb{C}I$, be of the form scalar plus compact. Then there exist trace zero operators $B_1, B_2 \in F(H)$ such that:*
(i) *B_1 and B_2 are linearly independent, and*
(ii) *$A + B_k \sim A$, $k = 1, 2$.*

Proof. Every compact operator has a nontrivial invariant subspace. Thus, there exists a direct sum decomposition $H = H_1 \oplus H_2$ such that with respect to this decomposition A has a matrix representation

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}.$$

At least one of the subspaces H_1 and H_2, say H_1, has dimension greater than one.

We may further assume that $A_2 \neq 0$. Indeed, if $A_2 = 0$, then

$$\begin{bmatrix} I & M \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} I & -M \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & MA_3 - A_1M \\ 0 & A_3 \end{bmatrix}.$$

If $MA_3 - A_1M = 0$ for every bounded linear operator $M : H_2 \to H_1$, then $A_1 = \lambda I$ and $A_3 = \lambda I$ for some λ, and consequently, $A = \lambda I$, a contradiction.

Thus we may and we will assume that $A_2 \neq 0$. Let $N : H_1 \to H_1$ be any nilpotent of rank one. Then

$$\begin{bmatrix} I+N & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} I-N & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} + \begin{bmatrix} NA_1 - A_1N - NA_1N & NA_2 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $NA_1 - A_1N - NA_1N$ is a trace zero operator of rank at most two. Since $\dim H_1 > 1$ and $A_2 \neq 0$ we can find two nilpotents $N_1, N_2 : H_1 \to H_1$ of rank one such that N_1A_2 and N_2A_2 are linearly independent. ∎

LEMMA 2.3. *Let $A \in B(H)$ be any non-scalar operator and α any complex number. Then there exists an idempotent $P \in B(H)$ of rank one such that α is an eigenvalue of $A + 2P$.*

Proof. As $A \notin \mathbb{C}I$ we can find $x \in H$ such that x and Ax are linearly independent. Define $P \in B(H)$ by

$$Px = \frac{\alpha}{2}x - \frac{1}{2}Ax, \quad PAx = \alpha\Big(\frac{\alpha}{2} - 1\Big)x + \Big(1 - \frac{\alpha}{2}\Big)Ax,$$

and

$$Pz = 0$$

for every $z \in \{x, Ax\}^\perp$. Clearly, P is an idempotent of rank one and $(A + 2P)x = \alpha x$. ∎

REMARK 2.4. The analogue statement with $A - 2P$ instead of $A + 2P$ can be proved in the same way.

LEMMA 2.5. *Let $\phi : B(H) \to B(H)$ be a bijective linear map preserving similarity. Assume that there exists $A \in B(H)$ such that $A \notin \mathbb{C}I + F(H)$ and $\phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{C}$ and some finite rank operator F. Denote $r =$ rank F. Then for every finite rank square-zero operator $B \in B(H)$ we have* rank $\phi(B) \leqslant 3r$.

Proof. Let k be any positive integer. By Lemma 2.1 there exist vectors $x_1, \dots, x_k \in H$ such that the set of vectors $x_1, \dots, x_k, Ax_1, \dots, Ax_k$ is linearly independent. Let H_1 be the linear span of $x_1, \dots, x_k$ and H_2 the orthogonal complement of H_1. If

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

is the matrix representation of A with respect to the decomposition $H = H_1 \oplus H_2$, then A_3 is of rank k. Let

$$C = \begin{bmatrix} 0 & A_2 \\ -\frac{1}{2}A_3 & 0 \end{bmatrix} = \begin{bmatrix} 2I & 0 \\ 0 & I \end{bmatrix} A \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & I \end{bmatrix} - A,$$

$$D = \begin{bmatrix} 0 & 2A_2 \\ -\frac{2}{3}A_3 & 0 \end{bmatrix} = \begin{bmatrix} 3I & 0 \\ 0 & I \end{bmatrix} A \begin{bmatrix} \frac{1}{3}I & 0 \\ 0 & I \end{bmatrix} - A.$$

The operator

$$2C - D = \begin{bmatrix} 0 & 0 \\ -\frac{1}{3}A_3 & 0 \end{bmatrix}$$

is a square-zero operator of rank k. Furthermore, $\phi(2C - D) = (2TFT^{-1} + 2\lambda I - 2F - 2\lambda I) - (SFS^{-1} - F) = 2TFT^{-1} - SFS^{-1} - F$ is of rank at most $3r$. Since all square-zero operators of rank k are similar and since k was an arbitrary positive integer, we are done. ∎

3. LINEAR PRESERVERS OF NILPOTENTS OF RANK ONE

This section is devoted to the study of linear maps preserving nilpotents of rank one. The proof is quite standard in the study of linear preservers. We include it for the sake of completeness.

PROPOSITION 3.1. *Let H be a separable Hilbert space. Let $\phi : B(H) \to B(H)$ be an injective linear map such that $\phi(N)$ is a nilpotent of rank one for every rank one nilpotent operator $N \in B(H)$. Then either:*

(i) *there exist a nonzero $x \in H$ and an injective conjugate-linear map $\tau : F_0(H) \to x^\perp$ such that $\phi(A) = x(\tau(A))^*$ for every $A \in F_0(H)$, or*

(ii) *there exist a nonzero $y \in H$ and an injective linear map $\delta : F_0(H) \to y^\perp$ such that $\phi(A) = \delta(A)y^*$ for every $A \in F_0(H)$, or*

(iii) *there exist linear injective maps $T, S : H \to H$ such that $\phi(xy^*) = (Tx)(Sy)^*$ for every rank one nilpotent operator xy^*, or*

(iv) *there exist conjugate-linear injective maps $T, S : H \to H$ such that $\phi(xy^*) = (Ty)(Sx)^*$ for every rank one nilpotent operator xy^*.*

REMARK 3.2. Consider the third possibility above. For every pair of orthogonal vectors x, y we have $Tx \perp Sy$ since ϕ preserves nilpotents of rank one. Thus, $y^*x = 0$ yields $(Sy)^*(Tx) = 0$. The linear maps T and S may be discontinuous. For example, decompose H into an orthogonal sum of subspaces $H = K \oplus L$ with K and L both isomorphic to H and choose $T, S : H \to H$ to be any injective linear unbounded maps with images in K and L, respectively. Define $\phi(xy^*) = (Tx)(Sy)^*$ for every rank one nilpotent $xy^* \in B(H)$. The map ϕ is well-defined on the set of all nilpotents of rank one and can be uniquely extended to a linear map on $F_0(H)$. Using Zorn's lemma this map can be further extended to an injective linear map $\phi : B(H) \to B(H)$. Of course, the obtained extension preserves rank one nilpotents.

However, if we assume in addition that T and S are bijective, then they must be continuous and S is the adjoint of some scalar multiple of the inverse of T which further yields that $\phi(A) = cTAT^{-1}$ for every $A \in F_0(H)$. Indeed, choose $u, v \in H$ such that $v^*u = 1$ and define a complex number c by $c = (Sv)^*(Tu)$. Consider $z, w \in H$ such that $w^*z = 1$ and $w^*u = 0$ and $v^*z = 0$. Then $(u - z)(v + w)^*$ is square-zero, and hence, $(S(v + w))^*(Tu - Tz) = 0$, which, because of $(Sw)^*(Tu) = 0$ and $(Sv)^*(Tz) = 0$, implies that $(Sw)^*(Tz) = (Sv)^*(Tu) = c$.

Let now $x, y \in H$ be arbitrary vectors with $y^*x = 1$. We want to show that $(Sy)^*(Tx) = c$. To do this we choose $s \in H$ such that $v^*s = y^*s = 0$ and $s \notin \operatorname{span}\{x, u\}$. We further choose $t \in H$ with $t^*s = 1$ and $t^*u = t^*x = 0$. As before we prove that $(Sv)^*(Tu) = (St)^*(Ts)$ and $(Sy)^*(Tx) = (St)^*(Ts)$ which yields the desired equality $(Sy)^*(Tx) = c$.

By linearity we have

$$(Sy)^*(Tx) = cy^*x \tag{3.1}$$

for every pair of vectors $x, y \in H$. Surjectivity of linear operators T and S implies that c is a nonzero complex number. If a sequence of vectors (x_n) tends to zero and $Tx_n \to u$, then by (3.1), $u = 0$, and hence T is bounded. The same is true for S. Moreover, $S^*T = cI$. From here we conclude that $\phi(xy^*) = cTxy^*T^{-1}$ for every nilpotent xy^* of rank one. By linearity we have $\phi(A) = cTAT^{-1}$ for every $A \in F_0(H)$.

Similarly, if we have the fourth possibility above and if we assume in addition that T and S are bijective, then $\phi(A) = cRA^{\mathrm{t}}R^{-1}$ for every $A \in F_0(H)$. Here, c is a nonzero complex number, A^{t} denotes the transpose of A with respect to some fixed orthonormal basis, and R is a bounded bijective linear operator. Indeed, choose an orthonormal basis (e_k) in H and define a conjugate-linear involution $J : H \to H$ by $J(\sum_{k=1}^{\infty} \lambda_k e_k) = \sum\limits_{k=1}^{\infty} \overline{\lambda}_k e_k$. Then $(xy^*)^{\mathrm{t}} = (Jy)(Jx)^*$, where the transpose is defined with respect to the chosen orthonormal basis. Thus, we have $\phi((xy^*)^{\mathrm{t}}) = \phi((Jy)(Jx)^*) = (TJx)(SJy)^*$ for every rank one nilpotent xy^*. Hence, the map $A \mapsto \phi(A^{\mathrm{t}})$ is of the third type above with linear maps TJ and SJ instead of T and S, respectively. Therefore, we have $\phi(A^{\mathrm{t}}) = cRAR^{-1}$ for every $A \in F_0(H)$. Here, $R = TJ$ is a bounded linear bijective operator and c is a nonzero complex number. It follows that $\phi(A) = cRA^{\mathrm{t}}R^{-1}$, $A \in F_0(H)$.

Proof of Proposition 3.1. For every pair of nonzero vectors $x, y \in H$ define $L_x = \{xu^* : u \in H \text{ and } u^*x = 0\}$ and $R_y = \{zy^* : z \in H \text{ and } y^*z = 0\}$. Each of L_x and R_y is a linear subspace of $B(H)$ consisting of nilpotents of rank at most one. Note that if a sum of two rank one operators $u_1w_1^* + u_2w_2^*$ is of rank at most one then u_1 and u_2 are linearly dependent, or w_1 and w_2 are linearly dependent. Thus, every linear space of nilpotent operators of rank at most one is contained either in L_x for some nonzero $x \in H$, or in R_y for some nonzero $y \in H$. This implies that for every nonzero $x \in H$, $\phi(L_x)$ is contained in L_z for some nonzero $z \in H$ or in R_y for some nonzero $y \in H$.

Choose a nonzero $x \in H$. Then we have either $\phi(L_x) \subset L_z$ for some nonzero $z \in H$, or $\phi(L_x) \subset R_y$ for some nonzero $y \in H$. We will consider only one of these two cases, say the second one. We will prove that for every nonzero $u \in H$ there exists a nonzero $w \in H$ such that $\phi(L_u) \subset R_w$. Assume that there is a nonzero $u \in H$ such that this is not true. Then u and x are linearly independent and $\phi(L_u) \subset L_v$ for some nonzero $v \in H$. We can find linearly independent vectors $z_1, z_2 \in H$ such that $z_k^*x = z_k^*u = 0$, $k = 1, 2$. We know that there exist nonzero vectors $s, t, p, q \in H$ such that

$$\phi(xz_1^*) = sy^*, \quad \phi(xz_2^*) = ty^*, \quad \text{and} \quad \phi(uz_1^*) = vp^*, \quad \phi(uz_2^*) = vq^*.$$

Moreover, by injectivity of ϕ, s and t are linearly independent and p and q are linearly independent. Since $xz_1^* + uz_1^*$ is a nilpotent of rank one, the operator $sy^* + vp^*$ is of rank one. Consequently, s and v are linearly dependent or y and p are linearly dependent. Similarly, y and q are linearly dependent or t and v are linearly dependent. Assume that s and v are linearly dependent (the case when t and v are linearly dependent can be treated in the same way). Then, after absorbing a constant in vp^* and vq^* we may assume that $v = s$. Because s and t are linearly independent, we have necessarily that y and q are linearly dependent. Consequently, $\phi(uz_2^*) = \lambda sy^*$ for some nonzero scalar λ, contradicting the fact that xz_1^* and uz_2^* are linearly independent. Hence, s and v must be linearly independent, and the same is true for t and v. But then both pairs p, y and q, y are linearly dependent, contradicting the fact that p and q are linearly independent.

We have proved that for every nonzero $x \in H$ there is a nonzero $y \in H$ such that $\phi(L_x) \subset R_y$. If there exists a nonzero $y \in H$ such that $\phi(L_x) \subset R_y$ for every nonzero $x \in H$, then because every finite rank trace zero operator is a linear combination of nilpotents of rank one, we have $\phi(F_0(H)) \subset R_y$. In this case the second possibility in our proposition holds true.

So, we may assume that there exist nonzero vectors x_1 and x_2 such that $\phi(L_{x_1}) \subset R_{y_1}$ and $\phi(L_{x_2}) \subset R_{y_2}$ and y_1 and y_2 are linearly independent. It then follows that for every pair of linearly independent vectors $u_1, u_2 \in H$ we have $\phi(L_{u_1}) \subset R_{z_1}$ and $\phi(L_{u_2}) \subset R_{z_2}$, where z_1 and z_2 are linearly independent. Indeed, assume on the contrary that $\phi(L_{u_1}) \subset R_z$ and $\phi(L_{u_2}) \subset R_z$ for some nonzero $z \in H$. Consider first the case that x_1, x_2, u_1, and u_2 are linearly independent. Choose a nonzero $v \in \{x_1, x_2, u_1, u_2\}^\perp$. We have $\phi(x_1v^*) = w_1y_1^*$, $\phi(x_2v^*) = w_2y_2^*$, $\phi(u_1v^*) = w_3z^*$, and $\phi(u_2v^*) = w_4z^*$. Because $(x_1 + x_2)v^*$ is a nilpotent of rank one, $w_1y_1^* + w_2y_2^*$ is of rank one, and therefore, w_1 and w_2 are linearly dependent. Also, $(x_1 + u_1)v^*$ is a nilpotent of rank one, and consequently, $w_1y_1^* + w_3z^*$ is of rank one. Similarly, $w_2y_2^* + w_4z^*$ is of rank one. If z and y_1 are linearly dependent, then z and y_2 must be linearly independent which yields that w_4 and w_2 are linearly dependent. It follows that $\phi(u_2v^*) = \mu w_1y_1^*$ for some nonzero $\mu \in \mathbb{C}$. Thus, the rank one nilpotent $(x_1 - \mu^{-1}u_2)v^*$ is mapped into the zero operator, a contradiction. In the same way we prove that z and y_2 are linearly independent. Hence, all vectors w_1, w_2, w_3, w_4 belong to the same one-dimensional subspace of H. In particular $w_4 = \tau w_3$ for some nonzero $\tau \in \mathbb{C}$, and consequently, $\phi((\tau u_1 - u_2)v^*) = 0$, a contradiction. If x_1, x_2, u_1, and u_2 are linearly dependent, then we can find u_3 and u_4 such that x_1, x_2, u_3, and u_4 are linearly independent and u_1, u_2, u_3, and u_4 are linearly independent. Now we use the previous argument first for the vectors x_1, x_2, u_3, and u_4 and then for the vectors u_1, u_2, u_3, and u_4 to conclude that $\phi(L_{u_1}) \subset R_{z_1}$ and $\phi(L_{u_2}) \subset R_{z_2}$ with z_1 and z_2 being linearly independent.

For every nonzero $x \in H$ there exists a nonzero $y \in H$ and a mapping $T_x : x^\perp \to y^\perp$ such that $\phi(xv^*) = (T_xv)y^*$. From linearity of ϕ it follows that T_x is conjugate-linear. Moreover, T_x is injective.

Let x and z be two arbitrary nonzero vectors from H. We will show that the restrictions of T_x and T_z to $\{x,z\}^\perp$ differ only in a multiplicative constant. There is nothing to prove if x and z are linearly dependent. So, assume that they are linearly independent. We have $\phi(xv^*) = (T_x v)y^*$ and $\phi(zv^*) = (T_z v)u^*$, $v \in \{x,z\}^\perp$. Here, y and u are linearly independent vectors. Since $(x+z)v^*$ is a rank one nilpotent, $v \in \{x,z\}^\perp$, $T_x v$ and $T_z v$ are linearly dependent for every $v \in \{x,z\}^\perp$. Because T_x and T_z are injective, we have $T_x v = \xi T_z v$, $v \in \{x,z\}^\perp$, for some nonzero $\xi \in \mathbb{C}$.

Choose linearly independent $x, z \in H$. We have $\phi(xv^*) = (T_x v)y^*$, $v \in x^\perp$, and $\phi(zv^*) = (T_z v)u^*$, $v \in z^\perp$. By absorbing a constant we may assume that $T_x = T_z$ on $\{x,z\}^\perp$. Define a conjugate-linear map $T : H \to H$ by

$$Tv = \begin{cases} T_x v & : \quad v \in x^\perp, \\ T_z v & : \quad v \in z^\perp \end{cases}.$$

Let $u \in H$ be any nonzero vector. By absorbing a constant we may and we will assume that the restrictions of T_u and T_x to the subspace $\{u,x\}^\perp$ coincide. Hence, the restriction of T_u to the subspace $\{u,x\}^\perp$ coincides with the restriction of T to this subspace.

We want to prove that T_u is the restriction of T to $u^\perp$. There is nothing to prove if u and x are linearly dependent. So, assume they are not.

We start with the case when $u \notin \operatorname{span}\{x,z\}$. Then we have $u^\perp = \{u,x\}^\perp + \{u,z\}^\perp$. We know that the restriction of T_u to the subspace $\{u,z\}^\perp$ coincides with the restriction of ηT to this subspace. Here η is some nonzero complex number and all we have to do is to show that $\eta = 1$. To this end we choose a nonzero v orthogonal to u, x, z. We have $T_u v = Tv$ and $T_u v = \eta T v$. Hence, $\eta = 1$, as desired.

In the case when $u \in \operatorname{span}\{x,z\}$ we choose $w \in H$ such that x, z, w as well as x, u, w are linearly independent. By the previous step we know that T_w is the restriction of T to $w^\perp$. As before we prove that the restrictions of T_u and T_w to the orthogonal complement of u, w coincide. This yields the desired conclusion that T_u is the restriction of T to $u^\perp$.

We can now conclude that for every nonzero $x \in H$ there exists a nonzero $y \in H$ such that

$$\phi(xv^*) = (Tv)y^*, \quad v \in x^\perp.$$

The mapping $x \mapsto y$ is obviously conjugate-linear and injective. Setting $Sx = y$ we complete the proof. ∎

4. PROOF OF THE MAIN RESULT

This section is devoted to the proof of our main theorem. So, assume that $\phi : B(H) \to B(H)$ is a bijective linear map such that $\phi(A) \sim \phi(B)$ whenever $A \sim B$. Let us first prove that $\phi(I) = \mu I$ for some nonzero $\mu \in \mathbb{C}$. By surjectivity there exists $A \in B(H)$ such that $\phi(A) = I$. We have to show that A is a scalar operator.

If not, then it is easy to find $B \neq A$ such that $A \sim B$. Then $\phi(A) = I \sim \phi(B)$, and consequently, $\phi(B) = I$, contradicting the injectivity of ϕ.

Next we will show that ϕ maps nilpotents of rank one into nilpotents of rank one. Choose a rank one operator $B \in B(H)$ and let $A \in B(H)$ be the operator with $\phi(A) = B$. Clearly, A is not a scalar operator. Thus, we can find a vector x such that x and Ax are linearly independent. Consequently, there exists $y \in H$ such that $y^*x = 0$ and $y^*Ax = 1$. Set $N = xy^*$. Then $N^2 = 0$ and $NAN = N$. For every $\lambda \in \mathbb{C}$ we have $\phi((I+\lambda N)A(I-\lambda N)) = R_\lambda B R_\lambda^{-1}$ for some invertible $R_\lambda \in B(H)$. Thus

$$\phi(A - (I+\lambda N)A(I-\lambda N)) - B$$

is of rank one for every complex λ. Hence

$$\operatorname{rank}(\lambda^2\phi(N) + \lambda\phi(AN - NA) - B) = 1.$$

Dividing by λ^2, sending λ to infinity, and applying the fact that the set of all operators of rank at most one is closed, we arrive at $\operatorname{rank}\phi(N) = 1$. Moreover, $N \sim 2N$, and therefore, $\phi(N) \sim 2\phi(N)$. As $\phi(N)$ is of rank one, it has to be nilpotent. Since all nilpotents of rank one are similar, we conclude that ϕ maps nilpotents of rank one into nilpotents of rank one. It follows that

$$\phi(F_0(H)) \subset F_0(H).$$

Using the result of Davidson and Marcoux we will prove now that if $A \in B(H)$ is not of the form scalar plus compact, then $\phi(A)$ is not of the form scalar plus compact as well. Indeed, assume on the contrary that $\phi(A) = \lambda I + K$ for some scalar λ and some compact operator K. Let $B \in B(H)$ be any operator. We then know that $B = \mu_1 B_1 + \cdots + \mu_6 B_6$ for some scalars $\mu_1, \ldots, \mu_6$ and some operators $B_1, \ldots, B_6$ that are all similar to A. It follows that $\phi(B)$ has the form scalar plus compact. As $B \in B(H)$ was an arbitrary operator, this contradicts the surjectivity of ϕ.

Next we will prove an analogous statement for finite rank operators. More precisely, if $A \in B(H)$ is not of the form scalar plus finite rank, then $\phi(A)$ is not of the form scalar plus finite rank as well. Assume on the contrary that there exists $A \in B(H)$ such that $A \notin \mathbb{C}I + F(H)$ and $\phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{C}$ and some finite rank operator F. Then, by Lemma 2.5, there exists an integer M such that for every finite rank square-zero operator $B \in B(H)$ we have $\operatorname{rank}\phi(B) \leqslant M$. We know that ϕ maps the set of nilpotents of rank one into itself. Thus, we can apply Proposition 3.1. Assume that we have the third possibility. Using the Jordan canonical form we see that every square-zero operator B of rank m, where m is any positive integer larger than M, can be written as $B = \sum_{k=1}^{m} x_k y_k^*$ where $x_1, \ldots, x_k, y_1, \ldots, y_k$ are linearly independent and $y_k^* x_k = 0$, $k = 1, \ldots, m$. But then, since T and S are injective, the operator $\phi(B) = \sum_{k=1}^{m} (Tx_k)(Sy_k)^*$ is of rank

$m > M$, a contradiction. In a similar way we prove that the fourth possibility cannot occur. So, we have either the first, or the second possibility. We will consider just the first one since the proof in the second case goes through in almost the same way. Thus, there are a nonzero $x \in H$ and a conjugate-linear map $\tau : F_0(H) \to x^\perp$ such that $\phi(C) = x(\tau(C))^*$ for every $C \in F_0(H)$. Let $z \in H$ be a vector linearly independent of x. Choose a nonzero $v \in H$. By surjectivity, there exists $D \in B(H)$ such that $\phi(D) = zv^*$. According to the previous step, D must be a scalar plus compact and of course, D is not a scalar operator. Applying Lemma 2.2 we can find linearly independent $N_1, N_2 \in F_0(H)$ such that both $D + N_1$ and $D + N_2$ are similar to D. It follows that $zv^* + x(\tau(N_1))^* \sim zv^*$. Now, since $zv^* + x(\tau(N_1))^*$ is of rank one and because z and x are linearly independent, the vectors v and $\tau(N_1)$ must be linearly dependent. In the same way we prove that v and $\tau(N_2)$ are linearly dependent. But then $\phi(N_1)$ and $\phi(N_2)$ are linearly dependent, contradicting the bijectivity of ϕ. We have proved that the set of operators that are not of the form scalar plus finite rank is invariant under ϕ.

We have

$$\phi(F_0(H)) \subset F_0(H) \subset F(H) \subset F(H) + \mathbb{C}I \subset \phi(F(H) + \mathbb{C}I),$$

the last inclusion being just a reformulation of the previous step. Let P be any projection of rank one. Then $F(H) = F_0(H) \oplus \mathbb{C}P$. Indeed, let $C \in F(H)$. Then $C = (\operatorname{tr} C)P + (C - (\operatorname{tr} C)P)$ and $(C - (\operatorname{tr} C)P)$ is a trace zero operator. Hence, $F_0(H)$ is a subspace of codimension 1 in $F(H)$. Also, $F(H)$ is of codimension 1 in $F(H) + \mathbb{C}I$. By bijectivity, $\phi(F_0(H))$ is of codimension 2 in $\phi(F(H) + \mathbb{C}I)$. It follows that

$$\phi(F_0(H)) = F_0(H) \quad \text{and} \quad \phi(F(H) + \mathbb{C}I) = F(H) + \mathbb{C}I.$$

We apply Proposition 3.1 once again. Because $\phi(F_0(H)) = F_0(H)$ we have either the third or the fourth possibility with T and S bijective. Thus, by the remark following Proposition 3.1, we have either $\phi(A) = cTAT^{-1}$, $A \in F_0(H)$, or $\phi(A) = cTA^{\mathrm{t}}T^{-1}$, $A \in F_0(H)$. Composing ϕ with a similarity transformation and the transposition, if necessary, and then multiplying it by c^{-1}, we may assume that

$$\phi(A) = A$$

for every finite rank trace zero operator A.

An operator $A \in B(H)$ is called an involution if $A^2 = I$. If A is an involution then $H = \operatorname{Ker}(A - I) \oplus \operatorname{Ker}(A + I)$. We will say that an involution is infinite if both the eigenspaces corresponding to the eigenvalues 1 and -1 are infinite-dimensional. Any two infinite involutions are similar. In the next step we will show that for every infinite involution A we have $\phi(A) = A + \lambda I$ for some $\lambda \in \mathbb{C}$.

Assume for a moment that we have already done this. By the result of Davidson and Marcoux every $A \in B(H)$ is a linear combination of at most 6 infinite involutions. It follows that for every $A \in B(H)$ we have $\phi(A) = A + f(A)I$, where $f : B(H) \to \mathbb{C}$ is a linear functional. The Hilbert space H can be identified with $H \oplus H$. Let $N \in B(H \oplus H)$ be a square-zero operator

$$N = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}I & 0 \\ 0 & I \end{bmatrix} = 2N.$$

Thus, $\phi(N) = N + f(N)I \sim 2N + 2f(N)I = \phi(2N)$, and consequently, $f(N) = 0$, which further implies that $\phi(N) = N$. Applying the result of Davidson and Marcoux once again we see that every $A \in B(H)$ is a linear combination of at most 6 operators similar to N. It follows that $\phi(A) = A$ for every $A \in B(H)$, as desired.

So, it remains to prove that every infinite involution A is mapped by ϕ into a sum of itself and a scalar operator. Denote by H_1 the eigenspace of A corresponding to the eigenvalue 1 and by H_{-1} the eigenspace of A corresponding to the eigenvalue -1. Clearly, H is a (not necessarily orthogonal) direct sum of H_1 and H_{-1}. We will first prove that both H_1 and H_{-1} are invariant under $\phi(A)$. In order to see that H_1 is invariant under $\phi(A)$ we choose nonzero vectors $x \in H_1$ and $y \in H_1^{\perp}$. We have to show that $y^*\phi(A)x = 0$. Clearly, xy^* is a nilpotent of rank one. Further, $xy^*A = -xy^*$. Indeed, for $z \in H_1$ we have $(xy^*A)z = (xy^*)z = (y^*z)x = 0 = (-xy^*)z$. And if $z \in H_{-1}$, then $(xy^*)Az = -(xy^*)z$, as desired. Obviously, $Axy^* = xy^*$. Consequently,

$$\left(I + \frac{1}{2}\alpha xy^*\right)\left(A + \alpha xy^*\right)\left(I - \frac{1}{2}\alpha xy^*\right) = A$$

for every $\alpha \in \mathbb{C}$. Thus, $\phi(A) \sim \phi(A) + \alpha xy^*$, $\alpha \in \mathbb{C}$. Assume that $y^*\phi(A)x \neq 0$. Then the function

$$\lambda \mapsto y^*(\lambda I - \phi(A))^{-1}x = \frac{1}{\lambda^2}y^*\phi(A)x + \frac{1}{\lambda^3}y^*\phi(A)^2x + \cdots, \quad |\lambda| > \|\phi(A)\|,$$

is not identically equal to zero (note that we have used the fact that $y^*x = 0$). Hence, we can find $\lambda \in \mathbb{C}$, $|\lambda| > \|\phi(A)\|$, such that

$$y^*(\lambda I - \phi(A))^{-1}x = \frac{1}{\alpha}$$

for some nonzero complex α. It follows that

$$(\phi(A) + \alpha xy^* - \lambda I)(\lambda I - \phi(A))^{-1}x = -x + \alpha x(y^*(\lambda I - \phi(A))^{-1}x) = 0,$$

and consequently, λ is an eigenvalue of $\phi(A) + \alpha xy^*$. But then it has to be an eigenvalue of $\phi(A)$, contradicting the fact that $|\lambda| > \|\phi(A)\|$. In almost the same way we prove that H_{-1} is invariant under $\phi(A)$ as well. Hence, with respect to the direct sum decomposition $H = H_1 \oplus H_{-1}$, the operator $\phi(A)$ has the matrix representation

$$\phi(A) = \begin{bmatrix} \phi(A)_1 & 0 \\ 0 & \phi(A)_{-1} \end{bmatrix}.$$

We know that

$$\phi\left(\begin{bmatrix} -2P & 0 \\ 0 & 2Q \end{bmatrix}\right) = \begin{bmatrix} -2P & 0 \\ 0 & 2Q \end{bmatrix}$$

whenever $P \in B(H_1)$ and $Q \in B(H_{-1})$ are idempotent operators of rank one. Obviously,

$$A + \begin{bmatrix} -2P & 0 \\ 0 & 2Q \end{bmatrix} \sim A.$$

It follows that

$$\begin{bmatrix} \phi(A)_1 - 2P & 0 \\ 0 & \phi(A)_{-1} + 2Q \end{bmatrix} \sim \begin{bmatrix} \phi(A)_1 & 0 \\ 0 & \phi(A)_{-1} \end{bmatrix}$$

whenever $P \in B(H_1)$ and $Q \in B(H_{-1})$ are idempotent operators of rank one. According to Lemma 2.3 both $\phi(A)_1$ and $\phi(A)_{-1}$ are scalars operators. From

$$\phi(A) = \begin{bmatrix} \lambda I & 0 \\ 0 & \mu I \end{bmatrix} \sim \begin{bmatrix} \lambda I - 2P & 0 \\ 0 & \mu I + 2Q \end{bmatrix}$$

we get that $\lambda - 2 = \mu$. Hence

$$\phi(A) = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} + (\lambda - 1)I.$$

This completes the proof.

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REFERENCES

[1] S.A. AMITSUR, Generalized polynomial identities and pivotal monomials, *Trans. Amer. Math. Soc.* **114**(1965), 210–226.

[2] M. BREŠAR, P. ŠEMRL, On locally linearly dependent operators and derivations, *Trans. Amer. Math. Soc.* **351**(1999), 1257–1275.

[3] K.R. DAVIDSON, L.W. MARCOUX, Linear spans of unitary and similarity orbits of a Hilbert space operator, *J. Operator Theory* **52**(2004), 113–132.

[4] F. HIAI, Similarity preserving linear maps on matrices, *Linear Algebra Appl.* **97**(1987), 127–139.

[5] G.X. JI, Similarity-preserving linear maps on $B(H)$, *Linear Algebra Appl.* **360**(2003), 249–257.

[6] G.X. JI, Asymptotic similarity-preserving linear maps on $B(H)$, *Linear Algebra Appl.* **368**(2003), 371–378.

[7] G.X. JI, H.K. DU, Similarity-invariant subspaces and similarity-preserving linear maps, *Acta Math. Sinica* **18**(2002), 489–498.

[8] C.-K. LI, S. PIERCE, Linear preserver problems, *Amer. Math. Monthly* **108**(2001), 591–605.

[9] C.-K. LI, N.-K. TSING, Linear preserver problem: A brief introduction and some special techniques, *Linear Algebra Appl.* **162-164**(1992), 217–235.

[10] M.H. LIM, A note on similarity preserving linear maps on matrices, *Linear Algebra Appl.* **190**(1993), 229–233.

[11] B.S. MITYAGIN, I.S. EDELHSTEIN, Homotopy type of linear groups for two classes of Banach algebras [Russian], *Funktsional. Anal. i Prilozhen.* **4**(1970), 61–72.

[12] T. PETEK, Linear mappings preserving similarity on $B(H)$, *Studia Math.* **161**(2004), 177–186.

[13] S. PIERCE, General introduction. A survey of linear preserver problems, *Linear Multilinear Algebra* **33**(1992), 1–129.

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