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DUALITY FOR CROSSED PRODUCTS OF HILBERT C^*-MODULES

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ABSTRACT. Let (A, G, α) be a C^*-dynamical system and let X be an A-Hilbert module with an α-compatible action η of G. Then it is shown that there exist a coaction δ_A of G on the reduced crossed product $A \times_{\alpha,r} G$ and a coaction δ_X of G on the reduced crossed product $X \times_{\eta,r} G$ such that $(X \times_{\eta,r} G) \times_{\delta_X} G \cong X \otimes \mathcal{C}(L^2(G))$, where $\mathcal{C}(L^2(G))$ denotes the C^*-algebra of all compact operators on $L^2(G)$. Furthermore, when A has a nondegenerate coaction δ_A of G on A and X is an A-Hilbert module with a nondegenerate δ_A-compatible coaction δ_X of G, it is shown that there exists a dual action $\widehat{\delta}_X$ of G on the crossed product $X \times_{\delta_X} G$ such that $(X \times_{\delta_X} G) \times_{\widehat{\delta}_X,r} G \cong X \otimes \mathcal{C}(L^2(G))$.

KEYWORDS: *Hilbert C^*-module, crossed product, duality.*

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1. INTRODUCTION

C^*-crossed products are most important objects and tools in the theory of C^*-algebras, in particular, in C^*-dynamical systems. When we deal with C^*-crossed products, the most important theorem is the duality for C^*-crossed products.

On the other hand, in recent research of C^*-algebras, Hilbert C^*-modules are getting to become a more important, standard tool in various research areas in C^*-algebras, for example, such as KK-theory, Morita equivalence and so on (see [9] for KK-theory and [15] for Morita equivalence). A successful use of Hilbert C^*-modules is in Morita equivalence for C^*-algebras, and then Hilbert C^*-modules work as an imprimitivity bimodule. Let (A, G, α) and (B, G, β) be C^*-dynamical systems and suppose that A and B are Morita equivalent. Then an outstanding problem in which we are very much interested is whether the C^*-crossed products $A \times_\alpha G$ and $B \times_\beta G$ are also Morita equivalent. In fact, it was shown in [3] and [4] that if there exists an $A - B$ imprimitivity bimodule X with an (α,β)-compatible action η of G, $A\times_\alpha G$ and $B\times_\beta G$ become Morita equivalent, and then an imprimitivity bimodule for those C^*-crossed products is regarded as a crossed product of X by η. In this paper, we will employ the construction of crossed products of Hilbert C^*-modules given by [5], which would be most understandable for us from viewpoint of analogy to C^*-crossed products of C^*-algebras.

Let (A,G,α) be a C^*-dynamical system. In Section 3, we discuss duality for crossed products of (right) Hilbert C^*-modules. First of all, we will define the reduced crossed product $X\times_{\eta,r} G$ of X by an α-compatible action η of G, and we will show that there exists a dual coaction δ_A of G on the reduced crossed product $A\times_{\alpha,r} G$ and a dual coaction δ_X of G on the reduced crossed product $X\times_{\eta,r} G$ such that the $((A\times_{\alpha,r} G)\times_{\delta_A} G)$-Hilbert module $(X\times_{\eta,r} G)\times_{\delta_X} G$ is isomorphic to the $(A\otimes\mathcal{C}(L^2(G)))$-Hilbert module $X\otimes\mathcal{C}(L^2(G))$. Applying this duality to abelian group action case, we can obtain that $(X\times_\eta G)\times_{\widehat{\eta}}\widehat{G}$ is isomorphic to $X\otimes\mathcal{C}(L^2(G))$, where $\widehat{\eta}$ is the dual action of the dual group $\widehat{G}$ of G on $X\times_\eta G$ (see [10] for duality for crossed products of imprimitivity bimodules by abelian group actions).

In Section 4, we discuss duality of crossed products of Hilbert C^*-modules by coactions of locally compact groups G. Let A be a C^*-algebra and let δ_A be a nondegenerate coaction of G on A. Suppose that X is a Hilbert A-module with a nondegenerate δ_A-compatible coaction δ_X of G on X. We then show that there exists a dual action $\widehat{\delta}_X$ of G on the crossed product $X\times_{\delta_X} G$ such that the $((A\times_{\delta_A} G)\times_{\widehat{\delta}_A,r} G)$-Hilbert module $(X\times_{\delta_X} G)\times_{\widehat{\delta}_X,r} G$ is isomorphic (as a Hilbert C^*-module) to the $(A\otimes\mathcal{C}(L^2(G)))$-Hilbert module $X\otimes\mathcal{C}(L^2(G))$.

Finally it would be significant to mention the common strategy to show two kinds of duality theorems for crossed products of Hilbert C^*-modules. For simplicity, consider the case of a C^*-dynamical system (A,G,α) with a locally compact group G and a right A-Hilbert module X with an α-compatible action η of G. Regarding X as a left $\mathcal{K}(X)$- and right A-Hilbert module, we consider the linking algebra $\mathcal{L}(X)$ for X which is a C^*-algebra and we obtain the canonical C^*-dynamical system $(\mathcal{L}(X),G,\theta)$ where θ is the canonical action of G associated with α and η. Then by Imai-Takai's duality, we see that

$$\mathcal{L}((X\times_{\eta,r}G)\times_{\delta_X}G)=(\mathcal{L}(X)\times_{\theta,r}G)\times_{\delta_{\mathcal{L}}}G\cong\mathcal{L}(X)\otimes\mathcal{C}(L^2(G))=\mathcal{L}(X\otimes\mathcal{C}(L^2(G))).$$

Taking the right upper corners of those linking algebras, then it would be shown that $(X\times_{\eta,r} G)\times_{\delta_X} G$ is isomorphic to $X\otimes\mathcal{C}(L^2(G))$. However, it is not necessarily obvious that the duality isomorphism between $(\mathcal{L}(X)\times_{\theta,r} G)\times_{\delta_{\mathcal{L}}} G$ and $\mathcal{L}(X)\otimes\mathcal{C}(L^2(G))$ gives the isomorphism as a Hilbert C^*-module between $(X\times_{\eta,r} G)\times_{\delta_X} G$ and $X\otimes\mathcal{C}(L^2(G))$. Hence what we have to do is to clarify this point. In fact, for example, let $\mathbb{C}$ denote the field of all complex numbers and consider the one-dimensional $\mathbb{C}$-$\mathbb{C}$-imprimitivity bimodule X. Then the linking algebra $\mathcal{L}(X)$ for X is the 2×2 matrix algebra $M_2(\mathbb{C})$, and the automorphism $\mathrm{Ad}u$ on $M_2(\mathbb{C})$

defined by the unitary matrix $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ sends $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$. Thus we see that an isomorphism between linking algebras does not necessarily give a bijective correspondence between the right upper corners of those linking algebras.

Although we discuss duality for crossed products of *right* Hilbert C^*-modules, by symmetry duality for crossed products of *left* Hilbert C^*-modules also can be obtained.

2. NOTATION AND PRELIMINARIES

Recall the definition of a Hilbert C^*-module. Let A be a C^*-algebra. By a *left A-Hilbert module* (or a *left Hilbert A-module*), we mean a left A-module X equipped with an A-valued pairing $\langle\cdot,\cdot\rangle$, called an A-valued inner product, satisfying the following conditions:

(H1) $\langle\cdot,\cdot\rangle$ is sesquilinear. (We make the convention that $\langle\cdot,\cdot\rangle$ is linear in the first variable and is conjugate-linear in the second variable.)

(H2) $\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y \in X$.

(H3) $\langle ax, y\rangle = a\langle x, y\rangle$ for all $x, y \in X$ and $a \in A$.

(H4) $\langle x, x\rangle \geqslant 0$ for all $x \in X$, and $\langle x, x\rangle = 0$ implies that $x = 0$.

(H5) X is a Banach space with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$.

Let B be a C^*-algebra. Right B-Hilbert modules are defined similarly except that we require that B should act on the right of X, that the B-valued inner product $\langle\cdot,\cdot\rangle$ should be conjugate-linear in the first variable, and that $\langle x, yb\rangle = \langle x, y\rangle b$ for all $x, y \in X$ and $b \in B$. Here note that the action of a C^*-algebra on X is automatically nondegenerate (Proposition 1.7 in [1]).

Let A and B be C^*-algebras. We denote by ${}_A\langle\cdot,\cdot\rangle$ the A-valued inner product on the left A-Hilbert module and by $\langle\cdot,\cdot\rangle_B$ the B-valued inner product on the right B-Hilbert module, respectively. By an *$A - B$ Hilbert bimodule*, we mean a left A-Hilbert module and a right B-Hilbert module X satisfying the following condition:

(H6) ${}_A\langle x, y\rangle\cdot z = x\cdot\langle y, z\rangle_B$ for all $x, y, z \in X$.

Note that an $A - B$ Hilbert bimodule automatically satisfies the following condition:

(H7) ${}_A\langle xb, y\rangle = {}_A\langle x, yb^*\rangle$ and $\langle ax, y\rangle_B = \langle x, a^*y\rangle_B$ for all $x, y \in X, a \in A$ and $b \in B$.

Let X be an $A - B$ Hilbert bimodule. Following [5], we say that a *representation* of X as an $A - B$ Hilbert bimodule is a triple (π_A, π_X, π_B) consisting of nondegenerate representations π_A and π_B of A and B on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, together with a linear map $\pi_X : X \to \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that:

(R1) $\pi_X(ax) = \pi_A(a)\pi_X(x)$,

(R2) $\pi_X(xb) = \pi_X(x)\pi_B(b)$,

(R3) $\pi_A({}_A\langle x, y\rangle) = \pi_X(x)\pi_X(y)^*$, and

(R4) $\pi_B(\langle x,y\rangle_B) = \pi_X(x)^*\pi_X(y)$

for all $a \in A, x, y \in X$, and $b \in B$, where $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ denotes the set of all bounded linear operators from $\mathcal{H}_B$ into $\mathcal{H}_A$.

Now we suppose that X is a right A-Hilbert module with the A-inner product $\langle\cdot,\cdot\rangle$. We define a linear operator $\Theta_{x,y}$ on X by

$$\Theta_{x,y}(z) = x\cdot\langle y,z\rangle$$

for all $x, y, z \in X$. We denote by $\mathcal{K}(X)$ the C^*-algebra generated by the set $\{\Theta_{x,y} : x\, y \in X\}$ (see Proposition 2.21 and Lemma 2.25 in [15]). Then X is a full left $\mathcal{K}(X)$-Hilbert module with respect to the natural left action defined by $t \cdot x = t(x)$ for $t \in \mathcal{K}(X)$, $x \in X$, and the inner product ${}_{\mathcal{K}(X)}\langle x,y\rangle \equiv \Theta_{x,y}$. Thus X is a $\mathcal{K}(X) - A$ Hilbert bimodule. If a representation $(\pi_A, \mathcal{H}_A)$ of A is given, then we can concretely construct $\pi_{\mathcal{K}(X)}$ and π_X (see Example 2.8 in [5]). For ease of notation, we will usually write $\pi_{\mathcal{K}}$ for $\pi_{\mathcal{K}(X)}$ unless otherwise confused.

LEMMA 2.1. *Let A be a C^*-algebra and let X be a right A-Hilbert module. Suppose that $(\pi_{\mathcal{K}}, \pi_X, \pi_A)$ is a representation of X as a $\mathcal{K}(X) - A$ Hilbert bimodule. If π_A is a faithful representation of A, then the representation $\pi_{\mathcal{K}}$ of $\mathcal{K}(X)$ is also faithful.*

Proof. This easily follows from (R1) and (R4). ∎

Let A be a C^*-algebra and let X be a $\mathcal{K}(X) - A$ Hilbert bimodule. We denote by $\widetilde{X}$ the dual Hilbert module of X, which is the set X with the left A-action and the right $\mathcal{K}(X)$-action defined by

$$a\cdot\widetilde{x} = \widetilde{(x\cdot a^*)}, \quad \widetilde{x}\cdot t = \widetilde{(t^*\cdot x)} \quad \text{for } t \in \mathcal{K}(X) \text{ and } a \in A,$$

where we write $\widetilde{x}$ if we view $x \in X$ as an element of $\widetilde{X}$. In addition, $\widetilde{X}$ is an $A - \mathcal{K}(X)$ Hilbert bimodule equipped with the A- and $\mathcal{K}(X)$-valued inner products given by

$${}_A\langle\widetilde{x},\widetilde{y}\rangle = \langle x,y\rangle_A, \quad \langle\widetilde{x},\widetilde{y}\rangle_{\mathcal{K}(X)} = {}_{\mathcal{K}(X)}\langle x,y\rangle$$

for $x, y \in X$ (see page 49 in [15] for the details of dual Hilbert C^*-modules). Put

$$\mathcal{L}(X) = \left\{ \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} : t \in \mathcal{K}(X), a \in A, x, y \in X \right\}.$$

For $L = \left(\begin{smallmatrix} t & x \\ \widetilde{y} & a \end{smallmatrix}\right)$, the adjoint L^* of L is defined by

$$L^* = \begin{pmatrix} t^* & y \\ \widetilde{x} & a^* \end{pmatrix}.$$

Addition and scalar multiplication on $\mathcal{L}(X)$ are defined by the usual formulas for matrices, and in addition, product in $\mathcal{L}(X)$ is given by

$$\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\begin{pmatrix} t' & x' \\ \widetilde{y'} & a' \end{pmatrix} = \begin{pmatrix} tt' + {}_{\mathcal{K}(X)}\langle x,y'\rangle & t\cdot x' + x\cdot a' \\ \widetilde{y}\cdot t' + a\cdot\widetilde{y'} & \langle y,x'\rangle_A + aa' \end{pmatrix}.$$

Then $\mathcal{L}(X)$ becomes a C^*-algebra (Proposition 2.3 in [1]). We call the C^*-algebra $\mathcal{L}(X)$ the *linking algebra* for X. For ease of notation, we write

$$\mathcal{L}(X) = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix}$$

for the linking algebra for X (see page 50 in [15] for more details of linking algebras).

LEMMA 2.2. *Let A be a C^*-algebra and let X be a right A-Hilbert module. Suppose that $(\pi_A, \mathcal{H}_A)$ is a faithful representation of A. Then there exist a representation $(\pi_{\mathcal{K}}, \pi_X, \pi_A)$ of X and a representation $(\pi_A, \pi_{\widetilde{X}}, \pi_{\mathcal{K}})$ of $\widetilde{X}$ such that $\pi_X, \pi_{\widetilde{X}}$ and $(\pi_{\mathcal{K}}, \mathcal{H}_{\mathcal{K}})$ are faithful. Define the representation $(\pi_{\mathcal{L}}, \mathcal{H})$ of the linking algebra $\mathcal{L}(X)$ for X by*

$$\pi_{\mathcal{L}}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) = \begin{pmatrix} \pi_{\mathcal{K}}(t) & \pi_X(x) \\ \pi_{\widetilde{X}}(\widetilde{y}) & \pi_A(a) \end{pmatrix},$$

where $\mathcal{H} = \mathcal{H}_{\mathcal{K}} \oplus \mathcal{H}_A$, $\pi_X(x) \in \mathcal{B}(\mathcal{H}_A, \mathcal{H}_{\mathcal{K}})$, $\pi_{\widetilde{X}}(\widetilde{y}) \in \mathcal{B}(\mathcal{H}_{\mathcal{K}}, \mathcal{H}_A)$. Then $(\pi_{\mathcal{L}}, \mathcal{H})$ is a faithful representation of $\mathcal{L}(X)$.

Proof. The proof is straightforward (see Example 2.8 in [5] for the existence of $\pi_{\mathcal{K}}, \pi_X$ and $\pi_{\widetilde{X}}$). ∎

Let X be a right B-Hilbert module over a C^*-algebra B and let A be a C^*-algebra. We denote by $X \otimes A$ the external tensor product of X and A (see 1.2.4 in [9] or 3.4 in [15] for the detail), where we regard A as a right A-Hilbert module in a canonical way. We remark that $X \otimes A$ becomes a right $(B \otimes_{\min} A)$-Hilbert module equipped with the $(B \otimes_{\min} A)$-valued inner product by

$$\langle\langle x_1 \otimes a_1, x_2 \otimes a_2 \rangle\rangle = \langle x_1, x_2 \rangle_B \otimes \langle a_1, a_2 \rangle_A = \langle x_1, x_2 \rangle_B \otimes a_1^* a_2$$

for $x_i \in X$ and $a_i \in A$. From now on, throughout this paper any tensor product of Hilbert C^*-modules always means the external tensor product.

The following result is essentially Remark 1.50 in [6] and will be repeatedly used without comment.

LEMMA 2.3. *Let A and B be C^*-algebras and let X be a right B-Hilbert module. We regard the external tensor product $X \otimes A$ as a $(\mathcal{K}(X) \otimes_{\min} A) - (B \otimes_{\min} A)$ Hilbert bimodule. We define a homomorphism Ψ from the injective C^*-tensor product $\mathcal{L}(X) \otimes_{\min} A$ into the linking algebra $\mathcal{L}(X \otimes A)$ for $X \otimes A$ by*

$$\Psi : \begin{pmatrix} t & x \\ \widetilde{y} & b \end{pmatrix} \otimes a \to \begin{pmatrix} t \otimes a & x \otimes a \\ (y \otimes a)^{\sim} & b \otimes a \end{pmatrix}$$

where $a \in A$. Then Ψ is an isomorphism. Thus we can identify $\mathcal{L}(X) \otimes_{\min} A$ with $\mathcal{L}(X \otimes A)$ as a C^*-algebra.

DEFINITION 2.4. Let A and B be C^*-algebras. For convenience of notation, we assume that X is a right A-Hilbert module and that Y is a right B-Hilbert module. We say that a linear map ρ from X into Y is a *homomorphism* (as a Hilbert C^*-module) if there exist a homomorphism π from A into B such that

$$\pi(\langle x,y\rangle_A) = \langle \rho(x), \rho(y)\rangle_B \tag{2.1}$$

for all $x, y \in X$.

We remark that it follows from (2.1) above that

$$\rho(xa) = \rho(x)\pi(a) \tag{2.2}$$

for all $x \in X, a \in A$. In particular, in the above, we say that X is *isomorphic* (as a Hilbert C^*-module) to Y if ρ from X into Y is surjective and π from A into B is bijective. In this case, such a map ρ from X onto Y is automatically *injective*, hence bijective. In fact, this follows easily from condition (2.1).

DEFINITION 2.5. Let A and B be C^*-algebras. Let X and Y be a $\mathcal{K}(X) - A$ Hilbert bimodule and a $\mathcal{K}(Y) - B$ Hilbert bimodule, respectively. We say that a homomorphism Ψ from $\mathcal{L}(X)$ into $\mathcal{L}(Y)$ is *componentwise* if there exist homomorphisms $\rho_1 : \mathcal{K}(X) \to \mathcal{K}(Y)$, $\rho_2 : X \to Y$, $\rho_3 : \widetilde{X} \to \widetilde{Y}$ and $\rho_4 : A \to B$ such that

$$\Psi\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) = \begin{pmatrix} \rho_1(t) & \rho_2(x) \\ \rho_3(\widetilde{y}) & \rho_4(a) \end{pmatrix}.$$

We call each homomorphism ρ_1, ρ_2, ρ_3 and ρ_4 a *component* of Ψ.

The following result plays an important role when we derive a Hilbert C^*-module isomorphism from an isomorphism between linking algebras.

LEMMA 2.6. *Let X be a $\mathcal{K}(X) - A$ Hilbert bimodule and let Y be a $\mathcal{K}(Y) - B$ Hilbert bimodule. Suppose that a homomorphism Ψ from $\mathcal{L}(X)$into $\mathcal{L}(Y)$ has a form of $\Psi = \left(\begin{smallmatrix} \gamma & \rho \\ \rho' & \pi \end{smallmatrix}\right)$ where $\rho : X \to Y$ and $\rho' : \widetilde{X} \to \widetilde{Y}$ are linear mappings, $\pi : A \to B$ and $\gamma : \mathcal{K}(X) \to \mathcal{K}(Y)$ are homomorphisms. Then we have*

$$\pi(\langle x,y\rangle_A) = \langle \rho(x), \rho(y)\rangle_B$$

for all x, y in X, that is, ρ is a homomorphism from X into Y. Hence, if Ψ is an isomorphism, ρ is an isomorphism as a Hilbert C^*-module from X onto Y.

Proof. Take $\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & y \\ 0 & 0 \end{smallmatrix}\right) \in \mathcal{L}(X)$. Then we have

$$\begin{aligned}\Psi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right) &= \Psi\left(\begin{pmatrix} 0 & 0 \\ \widetilde{x} & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right) \\ &= \Psi\left(\begin{pmatrix} 0 & 0 \\ 0 & \langle x,y\rangle_A \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & \pi(\langle x,y\rangle_A) \end{pmatrix}.\end{aligned}$$

On the other hand, we have

$$\Psi\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^*\right)\Psi\left(\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ \widetilde{\rho(x)} & 0 \end{pmatrix}\begin{pmatrix} 0 & \rho(y) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \langle \rho(x), \rho(y)\rangle_B \end{pmatrix}.$$

Thus we obtain that $\pi(\langle x,y\rangle_A) = \langle \rho(x), \rho(y)\rangle_B$ for all x, y in X. ∎

Let (A, G, α) be a C^*-dynamical system. By a *C^*-dynamical system,* we mean a triple (A, G, α) consisting of a C^*-algebra A, a locally compact group G with left invariant Haar measure ds and a group homomorphism α from G into the automorphism group of A such that $G \ni t \to \alpha_t(x)$ is continuous for each x in A in the norm topology. Denote by $K(A, G)$ the linear space of all continuous functions from G into A with compact support and by $L^1(A, G)$ the completion of $K(A, G)$ by the L^1-norm (see 7.6 in [13] for the Banach*-algebra structure of $L^1(A, G)$). Then the C^*-crossed product $A \times_\alpha G$ of A by G is the enveloping C^*-algebra of $L^1(A, G)$.

Recall that for any covariant representation $(\pi, u, \mathcal{H})$, the representation $(\pi \times u, \mathcal{H})$ of $A \times_\alpha G$ is defined by

$$(\pi \times u)(x) = \int_G \pi(x(t))u_t \mathrm{d}t, \quad x \in L^1(A, G).$$

Throughout this paper, for a given representation $(\pi, \mathcal{H})$ of A, we always denote by $\widetilde{\pi}$ the representation of A on the Hilbert space $L^2(\mathcal{H}, G)$ defined by

$$(\widetilde{\pi}(a)\xi)(t) = \pi(\alpha_{t^{-1}}(a))\xi(t)$$

for $a \in A, \xi \in L^2(\mathcal{H}, G)$, where $L^2(\mathcal{H}, G)$ is the Hilbert space of all square integrable functions from G into $\mathcal{H}$. Define a unitary representation λ^A on $L^2(\mathcal{H}, G)$ by

$$(\lambda^A{}_s\xi)(t) = \xi(s^{-1}t).$$

Then $(\widetilde{\pi}, \lambda^A, L^2(\mathcal{H}, G))$ is a covariant representation of A. If π is faithful, then $(\widetilde{\pi} \times \lambda^A)(A \times_\alpha G)$ is called the reduced C^*-crossed product of A by G and we denote it by $A \times_{\alpha,r} G$.

Let (A, G, α) and (B, G, β) be C^*-dynamical systems and let X be an $A - B$ Hilbert bimodule. Suppose that there exists an *α-compatible action* (respectively a *β-compatible action*) η of G on X, that is, a group homomorphism from G into the group of invertible linear transformations on X such that

(E1) $\eta_t(a \cdot x) = \alpha_t(a)\eta_t(x)$ (respectively $\eta_t(x \cdot b) = \eta_t(x)\beta_t(b)$),

(E2) ${}_A\langle \eta_t(x), \eta_t(y)\rangle = \alpha_t({}_A\langle x, y\rangle)$ (respectively $\langle \eta_t(x), \eta_t(y)\rangle_B = \beta_t(\langle x, y\rangle_B)$),

for each $t \in G, a \in A, b \in B, x, y \in X$; and such that $t \to \eta_t(x)$ is continuous from G into X for each $x \in X$ in norm. The combination of these two compatibility conditions will be simply called (α, β)-compatible. Then there exists an $(A \times_\alpha G) - (B \times_\beta G)$ Hilbert bimodule $X \times_\eta G$ containing a dense subspace $K(X, G)$ such that

$$(f \cdot x)(s) = \int_G f(t)\eta_t(x(t^{-1}s))\mathrm{d}t, \qquad (x \cdot g)(s) = \int_G x(t)\beta_t(g(t^{-1}s))\mathrm{d}t,$$

$${}_{A\times_\alpha G}\langle x, y\rangle(s) = \int_G {}_A\langle x(st^{-1}), \eta_s(y(t^{-1}))\rangle \mathrm{d}t, \quad \langle x, y\rangle_{B\times_\beta G}(s) = \int_G \beta_{t^{-1}}(\langle x(t), y(ts)\rangle_B)\mathrm{d}t,$$

for $f \in K(A,G), x,y \in K(X,G)$, and $g \in K(B,G)$ (see Proposition 3.5 in [5]). We call $X \times_\eta G$ the (*full*) *crossed product* of X by G. Here $K(X,G)$ (respectively $K(A,G)$ and $K(B,G)$) denotes the set of continuous functions from G into X (respectively A and B) with compact support.

DEFINITION 2.7. Let (A,G,α) and (B,G,β) be C^*-dynamical systems. Let X be an $A-B$ Hilbert bimodule with an (α,β)-compatible action η of G. Suppose that $(\pi_A, u, \mathcal{H}_A)$ and $(\pi_B, v, \mathcal{H}_B)$ are covariant representations of A and B, respectively. Then we say that a representation $(\pi_A, \pi_X, \pi_B, u, v)$ of X into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ is *covariant* if

$$\pi_X(\eta_t(x)) = u_t \pi_X(x) v_t^* \quad \text{for all } x \in X, t \in G.$$

Then we define the representation $\pi_X \times v$ of $X \times_\eta G$ into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ by the following, for $x \in K(X,G)$:

$$(\pi_X \times v)(x) = \int_G \pi_X(x(s)) v_s \mathrm{d}s.$$

LEMMA 2.8. *Let $(\pi_A, \pi_X, \pi_B, u, v)$ be as above. Then $(\pi_A \times u, \pi_X \times v, \pi_B \times v)$ is a representation of $X \times_\eta G$ into $\mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$.*

Proof. We have only to verify that, for $x, y \in K(X,G), f \in K(A,G), g \in K(B,G)$,

$$\begin{aligned}
&(\pi_X \times v)(fxg) = (\pi_A \times u)(f)(\pi_X \times v)(x)(\pi_B \times v)(g);\\
&(\pi_A \times u)({}_{A\times_\alpha G}\langle x,y\rangle) = (\pi_X \times v)(x)(\pi_X \times v)(y)^*;\\
&(\pi_B \times v)(\langle x,y\rangle_{B\times_\beta G}) = (\pi_X \times v)(x)^*(\pi_X \times v)(y).
\end{aligned}$$

For $g \in K(B,G)$ and $x \in K(X,G)$, we have

$$\begin{aligned}
(\pi_X \times v)(xg) &= \int_G \pi_X((xg)(s)) v_s \mathrm{d}s = \int_G \int_G \pi_X(x(t)\beta_t(g(t^{-1}s))) v_s \mathrm{d}s\mathrm{d}t\\
&= \int_G\int_G \pi_X(x(t))\pi_B(\beta_t(g(t^{-1}s))) v_s \mathrm{d}s\mathrm{d}t = \int_G\int_G \pi_X(x(t)) v_t \pi_B(g(t^{-1}s)) v_t^* v_s \mathrm{d}s\mathrm{d}t\\
&= \int_G \pi_X(x(t)) v_t (\pi_B \times v)(g)\mathrm{d}t = (\pi_X \times v)(x)(\pi_B \times v)(g).
\end{aligned}$$

For $x, y \in K(X,G)$, we have

$$\begin{aligned}
(\pi_B \times v)(\langle x,y\rangle_{B\times_\beta G}) &= \int_G \pi_B(\langle x,y\rangle_{B\times_\beta G}(s)) v_s \mathrm{d}s = \int_G\int_G \pi_B(\beta_{t^{-1}}(\langle x(t), y(ts)\rangle_B)) v_s \mathrm{d}s\mathrm{d}t\\
&= \int_G\int_G v_t^* \pi_B(\langle x(t), y(ts)\rangle_B) v_t v_s \mathrm{d}s\mathrm{d}t = \int_G\int_G v_t^* \pi_X(x(t))^* \pi_X(y(ts)) v_{ts} \mathrm{d}s\mathrm{d}t\\
&= \int_G v_t^* \pi_X(x(t))^* (\pi_X \times v)(y)\mathrm{d}t = (\pi_X \times v)(x)^*(\pi_X \times v)(y).
\end{aligned}$$

Similarly we can prove that $(\pi_X \times v)(fx) = (\pi_A \times u)(f)(\pi_X \times v)(x)$ and that $(\pi_A \times u)({}_{A\times_\alpha G}\langle x, y\rangle) = (\pi_X \times v)(x)(\pi_X \times v)(y)^*$. So we will leave the detail to the reader. Thus we complete the proof. ∎

Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Let η be an (α, β)-compatible action of G on an $A - B$ Hilbert bimodule X. Consider a representation (π_A, π_X, π_B) of X, where $(\pi_A, \mathcal{H}_A)$ (respectively $(\pi_B, \mathcal{H}_B)$) is a representation of A (respectively B). Then we obtain the covariant representations $(\widetilde{\pi}_A, \lambda^A, L^2(\mathcal{H}_A, G))$ and $(\widetilde{\pi}_B, \lambda^B, L^2(\mathcal{H}_B, G))$ of A and B, respectively. Define the representation $\widetilde{\pi}_X$ of X into $\mathcal{B}(L^2(\mathcal{H}_B, G), L^2(\mathcal{H}_A, G))$ by

$$(\widetilde{\pi}_X(x)\xi)(t) = \pi_X(\eta_{t^{-1}}(x))\xi(t)$$

for all $x \in X$, $t \in G$ and $\xi \in L^2(\mathcal{H}_B, G)$. Then we have the following.

LEMMA 2.9. *Let X be an $A - B$ Hilbert bimodule, and let $(\widetilde{\pi}_A, \lambda^A, L^2(\mathcal{H}_A, G))$ and $(\widetilde{\pi}_B, \lambda^B, L^2(\mathcal{H}_B, G))$ be as above. Then $(\widetilde{\pi}_A, \widetilde{\pi}_X, \widetilde{\pi}_B, \lambda^A, \lambda^B)$ is a representation of X into $\mathcal{B}(L^2(\mathcal{H}_B, G), L^2(\mathcal{H}_A, G))$, and we have, for $s, t \in G$ and $\xi \in L^2(\mathcal{H}_B, G)$,*

$$(\widetilde{\pi}_X(\eta_s(x))\xi)(t) = (({\lambda^A}_s \widetilde{\pi}_X(x){\lambda^B}_s^*)\xi)(t).$$

Proof. The proof is straightforward. So the detail is left to the reader. ∎

DEFINITION 2.10. Let (A, G, α) and (B, G, β) be C^*-dynamical systems. Let η be a (β, α)-compatible action of G on a $B - A$ Hilbert bimodule X. Consider a representation (π_B, π_X, π_A) of X, where $(\pi_A, \mathcal{H}_A)$ and $(\pi_B, \mathcal{H}_B)$ are faithful representations of A and B, respectively. Then π_X is automatically faithful. Consider the representation $\widetilde{\pi}_X \times \lambda^A$ of $X \times_\eta G$ into $\mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_B, G))$. Then we say that $(\widetilde{\pi}_X \times \lambda^A)(X \times_\eta G)$ is the *reduced crossed product* of X by G, and we denote it by $X \times_{\eta,r} G$. It is easy to verify that $X \times_{\eta,r} G$ is a $(B \times_{\beta,r} G) - (A \times_{\alpha,r} G)$ Hilbert bimodule. We will see later that it follows from Proposition 2.11 that $X \times_{\eta,r} G$ does not depend on the choice of a pair of faithful representations π_A and π_B of A and B.

From now on, we suppose that X is a right A-Hilbert module. Note that $\mathcal{K}(X)$ has a canonical action $\mathrm{Ad}\eta$ of G which is defined by

$$\mathrm{Ad}\eta_s(t) = \eta_s \cdot t \cdot \eta_{s^{-1}}, \quad t \in \mathcal{K}(X).$$

Then η is an $(\mathrm{Ad}\eta, \alpha)$-compatible action of G on X. Define the action $\widetilde{\eta}$ of G on $\widetilde{X}$ by

$$\widetilde{\eta}_s(\widetilde{x}) = \widetilde{\eta_s(x)}$$

for $\widetilde{x} \in \widetilde{X}$ and $s \in G$. We thus define an action θ of G on the linking algebra $\mathcal{L}(X)$ for X by

$$\theta_s\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) = \begin{pmatrix} \mathrm{Ad}\eta_s(t) & \eta_s(x) \\ \widetilde{\eta}_s(\widetilde{y}) & \alpha_s(a) \end{pmatrix}.$$

Then we denote θ_s simply by

$$\theta_s = \begin{pmatrix} \mathrm{Ad}\eta_s & \eta_s \\ \widetilde{\eta}_s & \alpha_s \end{pmatrix}.$$

Thus we obtain the C^*-dynamical system $(\mathcal{L}(X), G, \theta)$. Here we remark that $\mathcal{L}(X) \times_\theta G$ is canonically identified with the linking algebra $\mathcal{L}(X \times_\eta G)$ (cf. the proof of Theorem 4.1 in [5]). This fact plays a crucial role throughout this paper and we will use it repeatedly. We remark that $\widetilde{X} \times_{\widetilde{\eta},r} G$ is canonically identified with $(X \times_{\eta,r} G)\widetilde{}$. The following result is essentially Lemma 3.3 in [6].

PROPOSITION 2.11. *Let X be a $\mathcal{K}(X) - A$ Hilbert bimodule and let $\mathcal{L}(X)$ be the linking algebra for X. Suppose that $(\mathcal{L}(X), G, \theta)$ is the C^*-dynamical system above. We define a representation $\pi_{\mathcal{L}}$ of $\mathcal{L}(X)$ by*

$$\pi_{\mathcal{L}} = \begin{pmatrix} \pi_{\mathcal{K}} & \pi_X \\ \pi_{\widetilde{X}} & \pi_A \end{pmatrix},$$

where $(\pi_{\mathcal{K}}, \pi_X, \pi_A)$ and $(\pi_A, \pi_{\widetilde{X}}, \pi_{\mathcal{K}})$ are representations of X and $\widetilde{X}$ respectively, and define the unitary representation $\lambda^{\mathcal{L}}$ of G by $\lambda^{\mathcal{L}} = \lambda^{\mathcal{K}} \oplus \lambda^A$ on $L^2(\mathcal{H}_{\mathcal{K}}, G) \oplus L^2(\mathcal{H}_A, G)$. Then we have

$$(\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}})(\mathcal{L}(X) \times_\theta G) = \begin{pmatrix} (\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}})(\mathcal{K}(X) \times_{\mathrm{Ad}\eta} G) & (\widetilde{\pi}_X \times \lambda^A)(X \times_\eta G) \\ (\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}})(\widetilde{X} \times_{\widetilde{\eta}} G) & (\widetilde{\pi}_A \times \lambda^A)(A \times_\alpha G) \end{pmatrix}.$$

Hence we obtain that

$$\mathcal{L}(X) \times_{\theta,r} G = \begin{pmatrix} \mathcal{K}(X) \times_{\mathrm{Ad}\eta,r} G & X \times_{\eta,r} G \\ \widetilde{X} \times_{\widetilde{\eta},r} G & A \times_{\alpha,r} G \end{pmatrix}.$$

Here we need to give a remark about the above proposition, which will be used later to prove Lemma 4.5. It is not necessarily essential in Proposition 2.11 that the action of G on $\mathcal{K}(X)$ is $\mathrm{Ad}\eta$. In fact, if $\mathcal{K}(X)$ admits another action κ of G and if η on X is κ-compatible, we can define an action θ of G on $\mathcal{L}(X)$ by

$$\theta_s = \begin{pmatrix} \kappa_s & \eta_s \\ \widetilde{\eta}_s & \alpha_s \end{pmatrix}.$$

Then it is not hard to verify that we can obtain the result similar to Proposition 2.11 above, that is,

$$(\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}})(\mathcal{L}(X) \times_\theta G) = \begin{pmatrix} (\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}})(\mathcal{K}(X) \times_\kappa G) & (\widetilde{\pi}_X \times \lambda^A)(X \times_\eta G) \\ (\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}})(\widetilde{X} \times_{\widetilde{\eta}} G) & (\widetilde{\pi}_A \times \lambda^A)(A \times_\alpha G) \end{pmatrix}.$$

Now as a corollary to Proposition 2.11, we obtain the following.

PROPOSITION 2.12. *Under the notation in Lemma 2.9, suppose that $\widetilde{\pi}_A$ is a faithful representation of A. Then $\widetilde{\pi}_X \times \lambda^A$ is a faithful representation of $X \times_{\eta,r} G$.*

Proof. Since $\widetilde{\pi}_A$ is faithful, so is also $\widetilde{\pi}_X$. Since we see that $\widetilde{\pi}_{\mathcal{L}} = \begin{pmatrix} \widetilde{\pi}_{\mathcal{K}} & \widetilde{\pi}_X \\ \widetilde{\pi}_{\widetilde{X}} & \widetilde{\pi}_A \end{pmatrix}$ and since it then follows from Lemma 2.1 that $\widetilde{\pi}_{\mathcal{K}}$ is faithful, we see that so is also

$\widetilde{\pi}_{\mathcal{L}}$. Thus $\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}$ is a faithful representation of $\mathcal{L}(X) \times_{\theta,r} G$ (cf. 7.7.5 in [13]). Then it follows from Proposition 2.11 that every component of $\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}$, in particular, $\widetilde{\pi}_X \times \lambda^A$ is faithful. ∎

PROPOSITION 2.13. *Let (A, G, α) be a C^*-dynamical system and let X be a right A-Hilbert module. Suppose that there exists an α-compatible action η of G on X. If G is amenable, then $X \times_\eta G$ is isomorphic to $X \times_{\eta,r} G$.*

Proof. Consider the C^*-crossed product $\mathcal{L}(X) \times_\theta G$ of $\mathcal{L}(X)$ by θ as in Proposition 2.11. Then $\mathcal{L}(X) \times_\theta G$ is canonically identified with the linking algebra $\mathcal{L}(X \times_\eta G)$ (cf. the proof of Theorem 4.1 in [5]). Since G is amenable, $\mathcal{L}(X) \times_\theta G$ is isomorphic to the reduced C^*-crossed product $\mathcal{L}(X) \times_{\theta,r} G$ which is canonically identified with $\mathcal{L}(X \times_{\eta,r} G)$ by Proposition 2.11. Since such an isomorphism is componentwise, taking the right upper corners of those linking algebras, the desired result follows from Lemma 2.6. ∎

Now the reader is referred to the proof of 7.7.12 in [13] for the following discussion. We always denote by $C_0(G)$ the set of all continuous functions on G vanishing at infinity which is a C^*-algebra in a canonical way and let τ be the left translation on $C_0(G)$, that is,

$$\tau_s(f)(t) = f(s^{-1}t)$$

for $f \in C_0(G)$. Then we obtain a C^*-dynamical system $(C_0(G), G, \tau)$.

Let $(\mathcal{L}(X), G, \theta)$ be as above, and we denote by $C_0(\mathcal{L}(X), G)$ the C^*-algebra of all continuous functions vanishing at infinity from G into $\mathcal{L}(X)$, which is isomorphic to $\mathcal{L}(X) \otimes C_0(G)$. Let ρ_A be a faithful representation of A on a Hilbert space $\mathcal{H}_A$. Take faithful representations $\rho_{\mathcal{K}}$ of $\mathcal{K}(X)$ on $\mathcal{H}_{\mathcal{K}}$, ρ_X of X into $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_{\mathcal{K}})$, and $\rho_{\widetilde{X}}$ of $\widetilde{X}$ into $\mathcal{B}(\mathcal{H}_{\mathcal{K}}, \mathcal{H}_A)$, as in Lemma 2.2. We define a faithful representation π_X of the $C_0(\mathcal{K}(X), G) - C_0(A, G)$ Hilbert bimodule $C_0(X, G) (\cong X \otimes C_0(G))$ into $\mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_{\mathcal{K}}, G))$ by

$$(\pi_X(z)\xi)(t) = \rho_X(z(t))\xi(t)$$

for $z \in C_0(X, G), \xi \in L^2(\mathcal{H}_A, G)$. Then we obtain the representation $\widetilde{\pi}_X \times \lambda^A$ of $(X \otimes C_0(G)) \times_{\eta\otimes\tau,r} G$ into $\mathcal{B}(L^2(\mathcal{H}_A, G \times G), L^2(\mathcal{H}_{\mathcal{K}}, G \times G))$. Representations $\pi_{\mathcal{K}}$ of $C_0(\mathcal{K}(X), G)$, $\pi_{\widetilde{X}}$ of $C_0(\widetilde{X}, G)$ and π_A of $C_0(A, G)$ are also defined in a similar way, and similarly we obtain $\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}}, \widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}}$ and $\widetilde{\pi}_A \times \lambda^A$. Let $\pi_{\mathcal{L}}$ be the faithful representation of $\mathcal{L}(X)$ on $\mathcal{H}$ $(= \mathcal{H}_{\mathcal{K}} \oplus \mathcal{H}_A)$ defined by $\pi_{\mathcal{L}} = \begin{pmatrix} \rho_{\mathcal{K}} & \rho_X \\ \rho_{\widetilde{X}} & \rho_A \end{pmatrix}$. Define a faithful representation π of $C_0(\mathcal{L}(X), G)$ on $L^2(\mathcal{H}, G)$ by

$$(\pi(z)\xi)(t) = \pi_{\mathcal{L}}(z(t))\xi(t)$$

for $z \in K(\mathcal{L}(X), G), \xi \in L^2(\mathcal{H}, G)$, and put

$$\gamma_s(z)(t) = \theta_s(z(s^{-1}t))$$

for $s, t \in G$. Then we obtain a faithful representation $(\widetilde{\pi} \times \lambda, L^2(\mathcal{H}, G \times G))$ of $C_0(\mathcal{L}(X), G) \times_{\gamma,r} G$. Define a unitary operator w on $L^2(\mathcal{H}, G \times G)$ by

$$(w\xi)(s,t) = \Delta(t)^{1/2}\xi(st,t) = \begin{pmatrix} \Delta(t)^{1/2}\xi_1(st,t) \\ \Delta(t)^{1/2}\xi_2(st,t) \end{pmatrix} \equiv \begin{pmatrix} (w_1\xi_1)(s,t) \\ (w_2\xi_2)(s,t) \end{pmatrix}$$

for $\xi = \xi_1 \oplus \xi_2 \in L^2(\mathcal{H}, G \times G)$ with $\xi_1 \in L^2(\mathcal{H}_{\mathcal{K}}, G \times G)$ and $\xi_2 \in L^2(\mathcal{H}_A, G \times G)$. Take any $z = \left(\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right) \in K(K(\mathcal{L}(X), G), G)$ $(= K(\mathcal{L}(X), G \times G))$. Then we have

$$\begin{aligned}
&(w^*(\widetilde{\pi} \times \lambda)(z)w\xi)(s,t) \\
&= \int_G \pi_{\mathcal{L}}(\theta_{ts^{-1}}(z(r,s)))\xi(r^{-1}s,t)\mathrm{d}r \\
&= \int_G \begin{pmatrix} \rho_{\mathcal{K}}(\mathrm{Ad}\eta_{ts^{-1}}(z_1(r,s))) & \rho_X(\eta_{ts^{-1}}(z_2(r,s))) \\ \rho_{\widetilde{X}}(\widetilde{\eta}_{ts^{-1}}(z_3(r,s))) & \rho_A(\alpha_{ts^{-1}}(z_4(r,s))) \end{pmatrix} \begin{pmatrix} \xi_1(r^{-1}s,t) \\ \xi_2(r^{-1}s,t) \end{pmatrix} \mathrm{d}r \\
&= \begin{pmatrix} \int_G \rho_{\mathcal{K}}(\mathrm{Ad}\eta_{ts^{-1}}(z_1(r,s)))\xi_1(r^{-1}s,t)\mathrm{d}r + \int_G \rho_X(\eta_{ts^{-1}}(z_2(r,s)))\xi_2(r^{-1}s,t)\mathrm{d}r \\ \int_G \rho_{\widetilde{X}}(\widetilde{\eta}_{ts^{-1}}(z_3(r,s)))\xi_1(r^{-1}s,t)\mathrm{d}r + \int_G \rho_A(\alpha_{ts^{-1}}(z_4(r,s)))\xi_2(r^{-1}s,t)\mathrm{d}r \end{pmatrix} \\
&= \begin{pmatrix} (w_1^*(\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}})(z_1)w_1\xi_1)(s,t) + (w_1^*(\widetilde{\pi}_X \times \lambda^A)(z_2)w_2\xi_2)(s,t) \\ (w_2^*(\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}})(z_3)w_1\xi_1)(s,t) + (w_2^*(\widetilde{\pi}_A \times \lambda^A)(z_4)w_2\xi_2)(s,t) \end{pmatrix}.
\end{aligned}$$

Thus we obtain that

$$\begin{aligned}
w^*(\widetilde{\pi} \times \lambda)(z)w &= \begin{pmatrix} w_1^*(\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}})(z_1)w_1 & w_1^*(\widetilde{\pi}_X \times \lambda^A)(z_2)w_2 \\ w_2^*(\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}})(z_3)w_1 & w_2^*(\widetilde{\pi}_A \times \lambda^A)(z_4)w_2 \end{pmatrix} \\
&\in \mathcal{L}(X) \otimes \mathcal{C}(L^2(G)) = \begin{pmatrix} \mathcal{K}(X) \otimes \mathcal{C}(L^2(G)) & X \otimes \mathcal{C}(L^2(G)) \\ \widetilde{X} \otimes \mathcal{C}(L^2(G)) & A \otimes \mathcal{C}(L^2(G)) \end{pmatrix},
\end{aligned}$$

which shows that the isomorphism $z \to w^*(\widetilde{\pi} \times \lambda)(z)w$ is componentwise from $C_0(\mathcal{L}(X), G) \times_{\gamma,r} G$ onto $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$. Furthermore, if we denote by ρ the right regular representation of G on $L^2(G)$ and if we define

$$\widetilde{\rho}_t(z)(r,s) = z(r,st) = \begin{pmatrix} z_1(r,st) & z_2(r,st) \\ z_3(r,st) & z_4(r,st) \end{pmatrix},$$

then we have

$$w^*(\widetilde{\pi} \times \lambda)(\widetilde{\rho}_t(z))w = (\theta \otimes \mathrm{Ad}\rho)_t(w^*(\widetilde{\pi} \times \lambda)(z)w)$$

(see the proof of 7.7.12 in [13]), which shows that the isomorphism $z \to w^*(\widetilde{\pi} \times \lambda)(z)w$ is G-equivariant. If we identify $C_0(\mathcal{L}(X), G)$ with $\mathcal{L}(X) \otimes C_0(G)$, we obtain

$$\gamma_s = \theta_s \otimes \tau_s.$$

Then $C_0(\mathcal{L}(X), G) \times_{\gamma,r} G$ can be regarded as a linking algebra. In fact, it follows from Lemma 2.3 and the remark preceding Proposition 2.11 that

$$(\mathcal{L}(X) \otimes C_0(G)) \times_{\theta\otimes\tau,r} G = \mathcal{L}(X \otimes C_0(G)) \times_{\theta\otimes\tau,r} G = \mathcal{L}((X \otimes C_0(G)) \times_{\eta\otimes\tau,r} G).$$

Now we define the isomorphism $\Psi : C_0(\mathcal{L}(X), G) \times_{\gamma,r} G \to \mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ by $\Psi(z) = w^*(\widetilde{\pi} \times \lambda)(z)w$, and we obtain the following lemma.

LEMMA 2.14. *The isomorphism* $\Psi : C_0(\mathcal{L}(X), G) \times_{\gamma,r} G \to \mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ *is componentwise. Furthermore* Ψ *carries* $\widetilde{\rho}$ *to* $\theta \otimes \mathrm{Ad}\rho$.

3. DUALITY FOR CROSSED PRODUCTS BY GROUP ACTIONS

In this section, we shall prove the duality theorem for crossed products of Hilbert C^*-modules by actions of groups. Throughout this section, if necessary, without comment we suppose that a C^*-algebra is concretely represented on the universal Hilbert space.

First of all we briefly review the definition of the crossed products by coactions. Let G be a locally compact group with left invariant Haar measure $\mathrm{d}s$. We denote by λ the left regular representation of G on $L^2(G)$. We define the representation $\widetilde{\lambda}$ of $L^1(G)$ on $L^2(G)$ by

$$\widetilde{\lambda}(f) = \int_G f(s)\lambda_s \mathrm{d}s$$

for $f \in L^1(G)$. Then the reduced group C^*-algebra $C^*_{\mathrm{r}}(G)$ of G is defined as the norm closure of $\widetilde{\lambda}(L^1(G))$ in the set of all bounded linear operators on $L^2(G)$. If no confusion is possible, we write $\lambda(f)$ for $\widetilde{\lambda}(f)$ above. In the definition of a coaction of G, the references ([7], [8], [12], [14]) of the duality theorems adopt the use of $C^*_{\mathrm{r}}(G)$. Therefore we prefer the use of $C^*_{\mathrm{r}}(G)$ to that of the full group C^*-algebra $C^*(G)$ for the convenience of the reader.

Let A be a C^*-algebra and denote by $M(A \otimes_{\min} C^*_{\mathrm{r}}(G))$ the multiplier algebra of the injective C^*-tensor product $A \otimes_{\min} C^*_{\mathrm{r}}(G)$. We then define the C^*-subalgebra $\widetilde{M}(A \otimes_{\min} C^*_{\mathrm{r}}(G))$ of $M(A \otimes_{\min} C^*_{\mathrm{r}}(G))$ by

$$\widetilde{M}(A \otimes_{\min} C^*_{\mathrm{r}}(G)) = \{m \in M(A \otimes_{\min} C^*_{\mathrm{r}}(G)) : m(1 \otimes x), (1 \otimes x)m \in A \otimes_{\min} C^*_{\mathrm{r}}(G) \text{ for all } x \in C^*_{\mathrm{r}}(G)\}.$$

We denote by W_G the unitary operator on $L^2(G \times G)$ defined by

$$(W_G\xi)(s,t) = \xi(s, s^{-1}t) \quad \text{for } \xi \in L^2(G \times G) \text{ and } s, t \in G.$$

Define the homomorphism δ_G from $C^*_{\mathrm{r}}(G)$ into $\widetilde{M}(C^*_{\mathrm{r}}(G) \otimes_{\min} C^*_{\mathrm{r}}(G))$ by

$$\delta_G(\lambda(f)) = W_G(\lambda(f) \otimes 1)W_G{}^* \quad \text{for } f \in L^1(G).$$

We say that an injective homomorphism δ from A into $\widetilde{M}(A \otimes_{\min} C^*_{\mathrm{r}}(G))$ is a *coaction* of a locally compact group G on A if δ satisfies:

(C1) there is an approximate identity $\{e_i\}$ for A such that $\delta(e_i) \to 1$ strictly in $\widetilde{M}(A \otimes_{\min} C_r^*(G))$;
(C2) $(\delta \otimes \mathrm{id})(\delta(a)) = (\mathrm{id} \otimes \delta_G)(\delta(a))$ for all $a \in A$, where we always denote by *id* the identity map on each considered set.

Furthermore, the coaction δ is said to be *nondegenerate* if it satisfies the additional condition:

(C3) for every nonzero $\varphi \in A^*$, there exists $\psi \in C_r^*(G)^*$ such that $(\varphi \otimes \psi) \circ \delta \neq 0$.

This is equivalent to the condition that the closed linear span of $\delta(A)(1_A \otimes C_r^*(G))$ be equal to $A \otimes_{\min} C_r^*(G)$ (see, for example, 2.2 in [14]), where 1_A is the identity of the multiplier algebra $M(A)$ for A. (In (C2) and (C3), we implicitly extended δ to $\widetilde{M}(A \otimes_{\min} C_r^*(G))$, which is ensured by (C1).) Throughout this paper, we always denote by the same symbol δ the extension of δ to $\widetilde{M}(A \otimes_{\min} C_r^*(G))$.

Let δ be a coaction of a locally compact group G on A and let $C_0(G)$ be the set of all continuous functions on G vanishing at infinity. We denote by M_f the multiplication operator on $L^2(G)$ given by $f \in C_0(G)$ which is defined by

$$(M_f \xi)(t) = f(t)\xi(t)$$

for all $\xi \in L^2(G)$. Then the *crossed product* $A \times_\delta G$ of A by δ is the C^*-subalgebra of $M(A \otimes \mathcal{C}(L^2(G)))$ generated by the set $\{\delta(a)(1 \otimes M_f) : a \in A, f \in C_0(G)\}$, where $\mathcal{C}(L^2(G))$ denotes the C^*-algebra of all compact linear operators on $L^2(G)$.

We denote by $M(X)$ the set of all multipliers of a right A-Hilbert module X. Here we must remark that we do not require X to be full as a right A-Hilbert module. In fact, even though X is not full, it is possible to define a multiplier of a $\mathcal{K}(X) - A$ Hilbert bimodule X. But we need to require X to satisfy condition (H6) in Section 2, and we leave checking to need (H6) in [7] to the reader. Following Section 1 in [7], we refer to $M(X)$ as the *multiplier bimodule* of X, and note that $M(X)$ is an $M(\mathcal{K}(X)) - M(A)$ Hilbert bimodule, where $M(\mathcal{K}(X))$ and $M(A)$ are the multiplier algebras for $\mathcal{K}(X)$ and A, respectively. The reader is referred to [7] for the further details of multiplier modules of Hilbert C^*-modules.

Let $\delta_A : A \to \widetilde{M}(A \otimes_{\min} C_r^*(G))$ be a coaction of a locally compact group G on the C^*-algebra A and let $\delta_B : B \to \widetilde{M}(B \otimes_{\min} C_r^*(G))$ be a coaction of G on the C^*-algebra B. Suppose that X is a $B - A$ Hilbert bimodule. We say that a linear map $\delta_X : X \to M(X \otimes C_r^*(G))$ is a *δ_A-compatible coaction* (respectively a *δ_B-compatible coaction*) of G on X if δ_X satisfies the following conditions:

(D1) $\delta_X(x)(1_A \otimes z)$ lies in $X \otimes C_r^*(G)$ for all $x \in X$ and $z \in C_r^*(G)$;
(respectively (D1)′ $(1_B \otimes z)\delta_X(x)$ lies in $X \otimes C_r^*(G)$ for all $x \in X$ and $z \in C_r^*(G)$;)
(D2) $\delta_X(x \cdot a) = \delta_X(x) \cdot \delta_A(a)$ for all $x \in X$ and $a \in A$;
(respectively (D2)′ $\delta_X(b \cdot x) = \delta_B(b) \cdot \delta_X(x)$ for all $x \in X$ and $b \in B$;)
(D3) $\delta_A(\langle x, y\rangle_A) = \langle \delta_X(x), \delta_X(y)\rangle_{M(A \otimes_{\min} C_r^*(G))}$;
(respectively (D3)′ $\delta_B({}_B\langle x, y\rangle) = {}_{M(B \otimes_{\min} C_r^*(G))}\langle \delta_X(x), \delta_X(y)\rangle$;)

(D4) $\quad (\delta_X \otimes \mathrm{id}) \circ \delta_X = (\mathrm{id} \otimes \delta_G) \circ \delta_X$.

(In (D1) and (D2) (respectively in (D1)′ and (D2)′), we implicitly extended the module actions on the $(B \otimes_{\min} C^*_{\mathrm{r}}(G)) - (A \otimes_{\min} C^*_{\mathrm{r}}(G))$ Hilbert bimodule $X \otimes C^*_{\mathrm{r}}(G)$ to actions of the multiplier algebras on the multiplier bimodule; in (D3) (respectively (D3)′) we extended the inner products to $M(X \otimes C^*_{\mathrm{r}}(G))$; and in (D4), we used the strictly continuous extensions of $\delta_X \otimes \mathrm{id}$ and $\mathrm{id} \otimes \delta_G$ to make sense of the compositions.) The combination of these two compatibility conditions will be simply called (δ_B, δ_A)-compatible.

Furthermore, we say that δ_X is *nondegenerate* if δ_X satisfies the following additional conditions:

(D5) $\quad$ the closed linear span of $\delta_X(X)(1_A \otimes C^*_{\mathrm{r}}(G))$ is equal to $X \otimes C^*_{\mathrm{r}}(G)$;
(D5)′ $\quad$ the closed linear span of $(1_B \otimes C^*_{\mathrm{r}}(G))\delta_X(X)$ is equal to $X \otimes C^*_{\mathrm{r}}(G)$.

For a Hilbert A-module X with a coaction δ_X of G, we define a coaction $\delta_{\widetilde{X}}$ of G associated with δ_X on the dual Hilbert A-module $\widetilde{X}$ by

$$\delta_{\widetilde{X}}(\widetilde{x}) = \widetilde{\delta_X(x)} \quad \text{for } \widetilde{x} \in \widetilde{X}.$$

Let δ_A be a coaction of G on a C^*-algebra A and let X be a right A-Hilbert module throughout this section, and we suppose that A and $\mathcal{K}(X)$ are concretely represented on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_{\mathcal{K}}$, respectively. Given a δ_A-compatible coaction δ_X of G on X, the *crossed product* $X \times_{\delta_X} G$ of X by δ_X is the right $(A \times_{\delta_A} G)$-Hilbert closed submodule of $M(X \otimes \mathcal{C}(L^2(G))) \subset \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_{\mathcal{K}}, G))$ generated by the set $\{\delta_X(x)(1_A \otimes M_f) : x \in X, f \in C_0(G)\}$. Then the inner product on $X \times_{\delta_X} G$ is given in terms of the usual operator adjoint $* : \mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_{\mathcal{K}}, G)) \to \mathcal{B}(L^2(\mathcal{H}_{\mathcal{K}}, G), L^2(\mathcal{H}_A, G))$ by

$$\langle x, y \rangle_{A \times_{\delta_A} G} = x^* y \quad \text{for } x, y \in X \times_{\delta_X} G$$

(see Theorem 3.2 in [7] for the detail).

Let (A, G, α) be a C^*-dynamical system and let $A \times_{\alpha,r} G$ be the reduced C^*-crossed product of A by G. If π is a faithful representation of A on a Hilbert space $\mathcal{H}$, there is a faithful representation $(\widetilde{\pi} \times \lambda, \mathcal{H} \otimes L^2(G))$ of $A \times_{\alpha,r} G$. Then

$$((\widetilde{\pi} \times \lambda) \otimes \mathrm{id})(\delta(x)) = (1_A \otimes W_G)((\widetilde{\pi} \times \lambda) \otimes \mathrm{id})(x \otimes 1)(1_A \otimes W_G{}^*)$$

for $x \in A \times_{\alpha,r} G$ defines a nondegenerate coaction δ of G on $A \times_{\alpha,r} G$, which is called the *dual coaction* (cf. 2.3(1) in [14]). The duality that $(A \times_{\alpha,r} G) \times_\delta G$ is isomorphic to $A \otimes \mathcal{C}(L^2(G))$ is referred to as Imai-Takai's duality [8].

DEFINITION 3.1. Let (A, G, α) be a C^*-dynamical system and let X be a right A-Hilbert module with an α-compatible action η of G. We then regard X as a $\mathcal{K}(X) - A$ Hilbert bimodule and let $(\pi_{\mathcal{K}}, \pi_X, \pi_A)$ be a representation of X, where $(\pi_{\mathcal{K}}, \mathcal{H}_{\mathcal{K}})$ and $(\pi_A, \mathcal{H}_A)$ are representations of $\mathcal{K}(X)$ and A, respectively. If π_X is a faithful representation of X into $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_{\mathcal{K}})$, then there is a faithful representation $\widetilde{\pi}_X \times \lambda^A$ of $X \times_{\eta,r} G$ (see Proposition 2.12). Denote by $1_{\mathcal{K}}$ the identity of the multiplier algebra $M(\mathcal{K}(X))$ for $\mathcal{K}(X)$. Then the *dual coaction* δ_X of G on $X \times_{\eta,r} G$ is defined by

$$((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta_X(x)) = (1_{\mathcal{K}} \otimes W_G)((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(x \otimes 1)(1_A \otimes W_G{}^*)$$

for $x \in X \times_{\eta,r} G$. Similarly, we define also the *dual coaction* $\delta_{\widetilde{X}}$ of G on $\widetilde{X} \times_{\widetilde{\eta},r} G$. We remark that if we canonically identify $\widetilde{X} \times_{\widetilde{\eta},r} G$ with $(X \times_{\eta,r} G)\widetilde{}$, then $\delta_{\widetilde{X}}(\widetilde{x}) = \widetilde{\delta_X(x)}$.

For a representation $(\pi_{\mathcal{K}}, \pi_X, \pi_A)$ of X, as in Lemma 2.2 we define a representation $\pi_{\mathcal{L}}$ of the linking algebra $\mathcal{L}(X)$ for X by

$$\pi_{\mathcal{L}} = \begin{pmatrix} \pi_{\mathcal{K}} & \pi_X \\ \pi_{\widetilde{X}} & \pi_A \end{pmatrix},$$

where the representation $\pi_{\widetilde{X}}$ of $\widetilde{X}$ is defined by $\pi_{\widetilde{X}}(\widetilde{x}) = \widetilde{\pi_X(x)}$. From now on, we will consider only such a form as a representation $\pi_{\mathcal{L}}$ of $\mathcal{L}(X)$.

We recall that $\mathcal{K}(X)$ has a canonical action $\mathrm{Ad}\eta$ of G, and so η is an $(\mathrm{Ad}\eta, \alpha)$-compatible action of G on X. Denote again by $\widetilde{\eta}$ the action of G on $\widetilde{X}$ defined by $\widetilde{\eta}_s(\widetilde{x}) = \widetilde{\eta_s(x)}$ for $\widetilde{x} \in \widetilde{X}$ and $s \in G$. As in Section 2, we denote again by θ the action of G on the linking algebra $\mathcal{L}(X)$ of X given by

$$\theta_s\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) = \begin{pmatrix} \mathrm{Ad}\eta_s(t) & \eta_s(x) \\ \widetilde{\eta}_s(\widetilde{y}) & \alpha_s(a) \end{pmatrix}.$$

From now on, we use this notation for $(\mathcal{L}(X), G, \theta)$ without comment.

PROPOSITION 3.2. *Let $(\mathcal{L}(X), G, \theta)$ be the C^*-dynamical system above and let $\delta_{\mathcal{L}}$ be the dual coaction of G on $\mathcal{L}(X) \times_{\theta,r} G$. Then we have $\delta_{\mathcal{L}} = \begin{pmatrix} \delta_{\mathcal{K}} & \delta_X \\ \delta_{\widetilde{X}} & \delta_A \end{pmatrix}$, where δ_X and $\delta_{\widetilde{X}}$ are as in Definition 3.1, $\delta_{\mathcal{K}}$ and δ_A are the dual coactions of $A \times_{\alpha,r} G$ and $\mathcal{K}(X) \times_{\mathrm{Ad}\eta,r} G$, respectively.*

Proof. Take a faithful representation $\pi_{\mathcal{L}} = \begin{pmatrix} \pi_{\mathcal{K}} & \pi_X \\ \pi_{\widetilde{X}} & \pi_A \end{pmatrix}$ of $\mathcal{L}(X)$. Put

$$\delta = \begin{pmatrix} \delta_{\mathcal{K}} & \delta_X \\ \delta_{\widetilde{X}} & \delta_A \end{pmatrix}.$$

In order to show that $\delta_{\mathcal{L}} = \delta$, it suffices to verify that δ satisfies

$$((\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}) \otimes \mathrm{id})(\delta(z)) = (1_{\mathcal{L}(X)} \otimes W_G)((\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}) \otimes \mathrm{id})(z \otimes 1)(1_{\mathcal{L}(X)} \otimes W_G{}^*)$$

for $z = \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \in \mathcal{L}(X) \times_{\theta,r} G \; (= \mathcal{L}(X \times_{\eta,r} G))$.

Since $(\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}) \otimes \mathrm{id} = \begin{pmatrix} (\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id} & (\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id} \\ (\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id} & (\widetilde{\pi}_A \times \lambda^A) \otimes \mathrm{id} \end{pmatrix}$ (see Proposition 2.11) and since we have

$$1_{\mathcal{L}(X)} \otimes W_G = \begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 1_A \end{pmatrix} \otimes W_G = \begin{pmatrix} 1_{\mathcal{K}} \otimes W_G & 0 \\ 0 & 1_A \otimes W_G \end{pmatrix},$$

we have

$$
\begin{aligned}
&(1_{\mathcal{L}(X)} \otimes W_G)((\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}) \otimes \mathrm{id})(z \otimes 1)(1_{\mathcal{L}(X)} \otimes W_G{}^*) \\
&= \begin{pmatrix} (1_{\mathcal{K}} \otimes W_G)((\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id})(t \otimes 1)(1_{\mathcal{K}} \otimes W_G{}^*) & (1_{\mathcal{K}} \otimes W_G)((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(x \otimes 1)(1_A \otimes W_G{}^*) \\ (1_A \otimes W_G)((\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id})(\widetilde{y} \otimes 1)(1_{\mathcal{K}} \otimes W_G{}^*) & (1_A \otimes W_G)((\widetilde{\pi}_A \times \lambda^A) \otimes \mathrm{id})(a \otimes 1)(1_A \otimes W_G{}^*) \end{pmatrix} \\
&= \begin{pmatrix} ((\widetilde{\pi}_{\mathcal{K}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id})(\delta_{\mathcal{K}}(t)) & ((\widetilde{\pi}_X \times \lambda^A) \otimes \mathrm{id})(\delta_X(x)) \\ ((\widetilde{\pi}_{\widetilde{X}} \times \lambda^{\mathcal{K}}) \otimes \mathrm{id})(\delta_{\widetilde{X}}(\widetilde{y})) & ((\widetilde{\pi}_A \times \lambda^A) \otimes \mathrm{id})(\delta_A(a)) \end{pmatrix} \\
&= ((\widetilde{\pi}_{\mathcal{L}} \times \lambda^{\mathcal{L}}) \otimes \mathrm{id}) \left(\begin{pmatrix} \delta_{\mathcal{K}}(t) & \delta_X(x) \\ \delta_{\widetilde{X}}(\widetilde{y}) & \delta_A(a) \end{pmatrix} \right).
\end{aligned}
$$

This shows that $\delta_{\mathcal{L}} = \delta$. Thus we complete the proof. ∎

The following result is essentially Proposition 3.5 in [6]. So we omit the proof.

LEMMA 3.3. *With notation as in Proposition* 3.2, *δ_X is a $(\delta_{\mathcal{K}}, \delta_A)$-compatible coaction and $\delta_{\widetilde{X}}$ is a $(\delta_A, \delta_{\mathcal{K}})$-compatible coaction. Furthermore, δ_X and $\delta_{\widetilde{X}}$ are nondegenerate.*

LEMMA 3.4. *Let $(\mathcal{L}(X), G, \theta)$ be as above and let $\delta_{\mathcal{L}}$ be the dual coaction of G on $\mathcal{L}(X) \times_{\theta,r} G$. Then*

$$
\begin{aligned}
(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G &= \mathcal{L}(X \times_{\eta,r} G) \times_{\delta_{\mathcal{L}}} G \\
&= \begin{pmatrix} (\mathcal{K}(X) \times_{\mathrm{Ad}\eta,r} G) \times_{\delta_{\mathcal{K}}} G & (X \times_{\eta,r} G) \times_{\delta_X} G \\ (\widetilde{X} \times_{\widetilde{\eta},r} G) \times_{\delta_{\widetilde{X}}} G & (A \times_{\alpha,r} G) \times_{\delta_A} G \end{pmatrix}.
\end{aligned}
$$

Proof. Since $\mathcal{L}(X) \times_{\theta,r} G = \mathcal{L}(X \times_{\eta,r} G)$, the first equality is trivial. By Appendix: Remarks (4) in [7], we can identify $(\mathcal{K}(X) \times_{\mathrm{Ad}\eta,r} G) \times_{\delta_{\mathcal{K}}} G, (X \times_{\eta,r} G) \times_{\delta_X} G, (\widetilde{X} \times_{\widetilde{\eta},r} G) \times_{\delta_{\widetilde{X}}} G$ and $(A \times_{\alpha,r} G) \times_{\delta_A} G$ with the corresponding corners in the crossed product $\mathcal{L}(X \times_{\eta,r} G) \times_{\delta_{\mathcal{L}}} G$, respectively. Thus we complete the proof. ∎

From now on, we denote as usual by ρ the right regular representation of G on $L^2(G)$, that is,

$$(\rho_s \xi)(t) = \Delta(s)^{1/2} \xi(ts)$$

for $s, t \in G$, where Δ is the modular function of G with respect to left invariant Haar measure ds. Let $(C_0(G), G, \tau)$ be a C^*-dynamical system, where τ is the left translation on $C_0(G)$, that is,

$$\tau_s(f)(t) = f(s^{-1}t)$$

for $f \in C_0(G)$. Here we employ the result that there is an isomorphism Φ from $(A \times_{\alpha,r} G) \times_{\delta} G$ onto $(A \otimes C_0(G)) \times_{\alpha \otimes \tau, r} G$ which carries the *dual action* $\widehat{\delta}_s$ ($\equiv$

$\mathrm{Ad}(1_A \otimes 1 \otimes \rho_s))$ to $\mathrm{Ad}(1_A \otimes \rho_s \otimes 1)$ (see Lemma 6.1 in [14], or Proposition 3.1 in [8]). In fact, the isomorphism Φ is defined by

$$\begin{aligned}
&\Phi(\delta(\widetilde{\pi}(a))) = (\pi \otimes M)\widetilde{\ }(a \otimes 1), \quad a \in A; \\
&\Phi(\delta(1_A \otimes \lambda(f))) = 1_A \otimes 1 \otimes \lambda(f), \quad f \in L^1(G); \\
&\Phi(1_A \otimes 1 \otimes M_g) = (\pi \otimes M)\widetilde{\ }(1_A \otimes g), \quad g \in C_0(G);
\end{aligned}$$

where π is a faithful representation of A, and it satisfies that

$$\Phi(1_A \otimes 1 \otimes \rho_s) = 1_A \otimes \rho_s \otimes 1, \quad s \in G.$$

In the following lemma, we use this isomorphism Φ for $(\mathcal{L}(X), G, \theta)$ and we keep the notation in Definition 3.1 and Proposition 3.2.

LEMMA 3.5. *Let $(\mathcal{L}(X), G, \theta)$ be the C^*-dynamical system as in Lemma 3.4. Then the above Φ:$(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G \to (\mathcal{L}(X) \otimes C_0(G)) \times_{\theta\otimes\tau,r} G$ is a componentwise isomorphism which carries $\widehat{\delta}_{\mathcal{L}}$ to $\mathrm{Ad}(1_{\mathcal{L}(X)} \otimes \rho_s \otimes 1)$.*

Proof. We have only to show that Φ is componentwise. Take any $z = \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \in \mathcal{L}(X) = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix}$, any $f \in L^1(G)$ and $g \in C_0(G)$. Since we see that $\widetilde{\pi}_{\mathcal{L}} = \begin{pmatrix} \widetilde{\pi}_{\mathcal{K}} & \widetilde{\pi}_X \\ \widetilde{\pi}_{\widetilde{X}} & \widetilde{\pi}_A \end{pmatrix}$, we have

$$\begin{aligned}
&\Phi\left(\begin{pmatrix} \delta_{\mathcal{K}}(\widetilde{\pi}_{\mathcal{K}}(t)) & \delta_X(\widetilde{\pi}_X(x)) \\ \delta_{\widetilde{X}}(\widetilde{\pi}_{\widetilde{X}}(\widetilde{y})) & \delta_A(\widetilde{\pi}_A(a)) \end{pmatrix}\right) \\
&= \Phi\left(\delta_{\mathcal{L}}\left(\begin{pmatrix} \widetilde{\pi}_{\mathcal{K}}(t) & \widetilde{\pi}_X(x) \\ \widetilde{\pi}_{\widetilde{X}}(\widetilde{y}) & \widetilde{\pi}_A(a) \end{pmatrix}\right)\right) = \Phi(\delta_{\mathcal{L}}(\widetilde{\pi}_{\mathcal{L}}(z))) \\
&= (\pi_{\mathcal{L}} \otimes M)\widetilde{\ }(z \otimes 1) = \begin{pmatrix} (\pi_{\mathcal{K}} \otimes M)\widetilde{\ }(t \otimes 1) & (\pi_X \otimes M)\widetilde{\ }(x \otimes 1) \\ (\pi_{\widetilde{X}} \otimes M)\widetilde{\ }(\widetilde{y} \otimes 1) & (\pi_A \otimes M)\widetilde{\ }(a \otimes 1) \end{pmatrix};
\end{aligned}$$

$$\begin{aligned}
&\Phi\left(\begin{pmatrix} \delta_{\mathcal{K}}(1_{\mathcal{K}} \otimes \lambda(f)) & 0 \\ 0 & \delta_A(1_A \otimes \lambda(f)) \end{pmatrix}\right) \\
&= \Phi\left(\delta_{\mathcal{L}}\left(\begin{pmatrix} 1_{\mathcal{K}} & 0 \\ 0 & 1_A \end{pmatrix} \otimes \lambda(f)\right)\right) = \Phi(\delta_{\mathcal{L}}(1_{\mathcal{L}(X)} \otimes \lambda(f))) \\
&= 1_{\mathcal{L}(X)} \otimes 1 \otimes \lambda(f) = \begin{pmatrix} 1_{\mathcal{K}} \otimes 1 \otimes \lambda(f) & 0 \\ 0 & 1_A \otimes 1 \otimes \lambda(f) \end{pmatrix};
\end{aligned}$$

$$\begin{aligned}
&\Phi\left(\begin{pmatrix} 1_{\mathcal{K}} \otimes 1 \otimes M_g & 0 \\ 0 & 1_A \otimes 1 \otimes M_g \end{pmatrix}\right) = \Phi(1_{\mathcal{L}(X)} \otimes 1 \otimes M_g) \\
&= (\pi_{\mathcal{L}} \otimes M)\widetilde{\ }(1_{\mathcal{L}(X)} \otimes g) = (\pi_{\mathcal{L}} \otimes M)\widetilde{\ }\left(\begin{pmatrix} 1_{\mathcal{K}} \otimes g & 0 \\ 0 & 1_A \otimes g \end{pmatrix}\right) \\
&= \begin{pmatrix} (\pi_{\mathcal{K}} \otimes M)\widetilde{\ }(1_{\mathcal{K}} \otimes g) & 0 \\ 0 & (\pi_A \otimes M)\widetilde{\ }(1_A \otimes g) \end{pmatrix}.
\end{aligned}$$

Hence Φ is a componentwise isomorphism. ∎

Consider the C^*-dynamical system $(C_0(G), G, \tau)$, where τ is the left translation on $C_0(G)$. Then it is well known, as the Stone-von Neumann theorem, that there exists an isomorphism from the reduced C^*-crossed product $C_0(G) \times_{\tau,r} G$ onto $\mathcal{C}(L^2(G))$ (see Theorem C.34 in [15]). Define an action $\widetilde{\rho}$ of G on $C_0(G) \times_{\tau,r} G$ by

$$\widetilde{\rho}_s(x)(t) = x(ts)$$

for $x \in L^1(C_0(G), G)$ (see Lemma 2.14). Then the above isomorphism carries $\widetilde{\rho}_s$ on $C_0(G) \times_{\tau,r} G$ to $\mathrm{Ad}\rho_s$ on $\mathcal{C}(L^2(G))$. Hence the canonical isomorphism carries the action $\theta \otimes \widetilde{\rho}$ of G on $\mathcal{L}(X) \otimes (C_0(G) \times_{\tau,r} G)$ to the action $\theta \otimes \mathrm{Ad}\rho$ of G on $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ (cf. 7.7.12 in [13]). With the canonical identification $(\mathcal{L}(X) \otimes C_0(G)) \times_{\iota\otimes\tau,r} G = \mathcal{L}(X) \otimes (C_0(G) \times_{\tau,r} G)$, we can identify $\mathrm{Ad}(1_{\mathcal{L}(X)} \otimes \rho \otimes 1)$ with $\theta \otimes \widetilde{\rho}$. Now it only remains to apply Imai-Takai's duality to the C^*-dynamical system $(\mathcal{L}(X), G, \theta)$. Then the duality isomorphism carries the dual action $\widehat{\delta}_{\mathcal{L}}$ of G on $(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G$ to the action $\theta \otimes \mathrm{Ad}\rho$ of G on $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$, (see Theorem 6.3 in [14] for the detail).

PROPOSITION 3.6. *Let $(\mathcal{L}(X), G, \theta)$ be the above C^*-dynamical system where G is a locally compact group. Then $(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G$ is componentwisely isomorphic to $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$, where $\delta_{\mathcal{L}}$ is the dual coaction of G on $\mathcal{L}(X) \times_{\theta,r} G$. Furthermore the isomorphism carries the dual action $\widehat{\delta}_{\mathcal{L}}$ to $\theta \otimes \mathrm{Ad}\rho$, where ρ is the right regular representation of G on $L^2(G)$.*

Proof. We remark that, as is well known, $C_0(\mathcal{L}(X), G) \times_{\gamma,r} G$ is canonically identified with $(\mathcal{L}(X) \otimes C_0(G)) \times_{\theta\otimes\tau,r} G$ and the identification map is a componentwise isomorphism. Consider the isomorphisms $\Phi : (\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G \to (\mathcal{L}(X) \otimes C_0(G)) \times_{\theta\otimes\tau} G$ in Lemma 3.5 and $\Psi : C_0(\mathcal{L}(X), G) \times_{\gamma,r} G \to \mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ in Lemma 2.14. Then $\Psi \circ \Phi$ gives a desired isomorphism. ∎

Now we define the *dual action* $\widehat{\delta}_X$ of G on $(X \times_{\eta,r} G) \times_{\delta_X} G$ by

$$\widehat{\delta}_{Xs}(z) = (1_{\mathcal{K}} \otimes 1 \otimes \rho_s) z (1_A \otimes 1 \otimes \rho_s)^*, \quad z \in (X \times_{\eta,r} G) \times_{\delta_X} G.$$

LEMMA 3.7. *Under the notation in Proposition 3.6, let $\widehat{\delta}_{\mathcal{L}}$ be the dual action of G on $(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G$. Then $\widehat{\delta}_{\mathcal{L}}$ is componentwise, in fact, we have*

$$\widehat{\delta}_{\mathcal{L}} = \begin{pmatrix} \widehat{\delta}_{\mathcal{K}} & \widehat{\delta}_X \\ \widehat{\delta}_{\widetilde{X}} & \widehat{\delta}_A \end{pmatrix}.$$

Proof. By Lemma 3.4, we have

$$(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G = \begin{pmatrix} (\mathcal{K}(X) \times_{\mathrm{Ad}\eta,r} G) \times_{\delta_{\mathcal{K}}} G & (X \times_{\eta,r} G) \times_{\delta_X} G \\ (\widetilde{X} \times_{\widetilde{\eta},r} G) \times_{\delta_{\widetilde{X}}} G & (A \times_{\alpha,r} G) \times_{\delta_A} G \end{pmatrix}.$$

Take any $\left(\begin{smallmatrix} t & x \\ \widetilde{y} & a \end{smallmatrix}\right) \in \mathcal{L}((X \times_{\eta,r} G) \times_{\delta_X} G)$. Then we have

$$\begin{aligned}
&\widehat{\delta}_{\mathcal{L}s}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) \\
&= \mathrm{Ad}(1_{\mathcal{L}(X)} \otimes 1 \otimes \rho_s)\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) \\
&= \begin{pmatrix} 1_{\mathcal{K}} \otimes 1 \otimes \rho_s & 0 \\ 0 & 1_A \otimes 1 \otimes \rho_s \end{pmatrix} \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \begin{pmatrix} (1_{\mathcal{K}} \otimes 1 \otimes \rho_s)^* & 0 \\ 0 & (1_A \otimes 1 \otimes \rho_s)^* \end{pmatrix} \\
&= \begin{pmatrix} (1_{\mathcal{K}} \otimes 1 \otimes \rho_s)t(1_{\mathcal{K}} \otimes 1 \otimes \rho_s)^* & (1_{\mathcal{K}} \otimes 1 \otimes \rho_s)x(1_A \otimes 1 \otimes \rho_s)^* \\ (1_A \otimes 1 \otimes \rho_s)\widetilde{y}(1_{\mathcal{K}} \otimes 1 \otimes \rho_s)^* & (1_A \otimes 1 \otimes \rho_s)a(1_A \otimes 1 \otimes \rho_s)^* \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\delta}_{\mathcal{K}s}(t) & \widehat{\delta}_{Xs}(x) \\ \widehat{\delta}_{\widetilde{X}s}(\widetilde{y}) & \widehat{\delta}_{As}(a) \end{pmatrix} = \begin{pmatrix} \widehat{\delta}_{\mathcal{K}s} & \widehat{\delta}_{Xs} \\ \widehat{\delta}_{\widetilde{X}s} & \widehat{\delta}_{As} \end{pmatrix} \left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right),
\end{aligned}$$

which shows the desired result. ∎

Now we are in a position to establish the main result in this section.

THEOREM 3.8 (Duality). *Let (A, G, α) be a C^*-dynamical system where G is a locally compact group, and let X be a Hilbert A-module. Suppose that η is an α-compatible action of G on X. Then there exist a coaction δ_A of G on $A \times_{\alpha,r} G$ and a coaction δ_X of G on $X \times_{\eta,r} G$ such that the $((A \times_{\alpha,r} G) \times_{\delta_A} G)$-Hilbert module $(X \times_{\eta,r} G) \times_{\delta_X} G$ is isomorphic to the $(A \otimes \mathcal{C}(L^2(G)))$-Hilbert module $X \otimes \mathcal{C}(L^2(G))$. Furthermore the isomorphism carries the dual action $\widehat{\delta}_X$ to $\eta \otimes \mathrm{Ad}\rho$.*

Proof. Since $(\mathcal{L}(X) \times_{\theta,r} G) \times_{\delta_{\mathcal{L}}} G$ is componentwisely isomorphic to $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ by Proposition 3.6, taking the right upper corners of the linking algebras $\mathcal{L}((X \times_{\eta,r} G) \times_{\delta_X} G)$ and $\mathcal{L}(X \otimes \mathcal{C}(L^2(G)))$, it follows from Lemma 2.6 that $(X \times_{\eta,r} G) \times_{\delta_X} G$ is isomorphic as a Hilbert C^*-module to $X \otimes \mathcal{C}(L^2(G))$. Since $\widehat{\delta}_{\mathcal{L}}$ is componentwise by Lemma 3.7 and since so is also $\theta \otimes \mathrm{Ad}\rho$ by definition, it is easy to verify that the duality isomorphism carries the dual action $\widehat{\delta}_X$ to $\eta \otimes \mathrm{Ad}\rho$. ∎

4. DUALITY FOR CROSSED PRODUCTS BY COACTIONS

In this section, we shall prove the duality theorem for crossed products of Hilbert C^*-modules by coactions of locally compact groups.

Let A be a C^*-algebra and let δ_A be a coaction of a locally compact group G on A. Suppose that X is a right A-Hilbert module with a nondegenerate δ_A-compatible coaction δ_X of G. First we need to establish a canonical coaction of G on $\mathcal{K}(X)$ associated with δ_X. Recall that for given $x, y \in X$, the operator $\Theta_{x,y}$ on X is defined by $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A$ for $z \in X$, and that $\mathcal{K}(X)$ is the C^*-algebra generated by those operators $\Theta_{x,y}$. Given a coaction δ_X of G on X, we define a linear map $\delta_{\mathcal{K}}$ on $\mathcal{K}(X)$ by

$$\delta_{\mathcal{K}}(\Theta_{x,y}) = \Theta_{\delta_X(x),\delta_X(y)}$$

for all $x, y \in X$. Lemma 4.1 below is shown in Proposition 2.8 in [2] in a spatial form of $\delta_{\mathcal{K}}$ based upon representation theory of Hilbert C^*-modules. Of course, it is possible to give its direct proof without use of the representation theory of Hilbert C^*-modules. But the direct proof is long and is a little bit complicated.

LEMMA 4.1. *Suppose that δ_X is nondegenerate. Then $\delta_{\mathcal{K}}$ above is a nondegenerate coaction of G on $\mathcal{K}(X)$. Furthermore, δ_X is $\delta_{\mathcal{K}}$-compatible.*

Let X be a (right) A-Hilbert module. From now on, we regard X as a $\mathcal{K}(X) - A$ Hilbert bimodule and we consider only $\delta_{\mathcal{K}}$ above as a coaction of G on $\mathcal{K}(X)$. Then the linking algebra $\mathcal{L}(X)$ for X is given by $\mathcal{L}(X) = \begin{pmatrix} \mathcal{K}(X) & X \\ \widetilde{X} & A \end{pmatrix}$. The following result is Lemma 2.22 in [6].

LEMMA 4.2. *Let δ_A be a nondegenerate coaction of G on A and let δ_X be a nondegenerate δ_A-compatible coaction of G on X. Then $\delta_{\mathcal{L}} = \begin{pmatrix} \delta_{\mathcal{K}} & \delta_X \\ \delta_{\widetilde{X}} & \delta_A \end{pmatrix}$ is a nondegenerate coaction of G on $\mathcal{L}(X)$, where $\delta_{\widetilde{X}}$ is defined by $\delta_{\widetilde{X}}(\widetilde{x}) = \widetilde{\delta_X(x)}$ for $\widetilde{x} \in \widetilde{X}$.*

As in Section 3, we denote again by ρ the right regular representation of G on $L^2(G)$, that is, $(\rho_s\xi)(t) = \Delta(s)^{1/2}\xi(ts)$ for $s, t \in G$, where Δ is the modular function of G with respect to left invariant Haar measure ds. For each $f \in L^1(G)$, we set

$$\widetilde{\rho}(f) = \int_G f(s)\rho_s \mathrm{d}s.$$

If no confusion is possible, we write $\rho(f)$ for $\widetilde{\rho}(f)$. Let δ_A be a nondegenerate coaction of G on a C^*-algebra A. Without loss of generality, we can assume that the C^*-algebra A is concretely represented on a Hilbert space $\mathcal{H}_A$, and also we denote by 1_A the identity of the multiplier algebra $M(A)$ of A. We define $\widehat{\delta}_{As} = \mathrm{Ad}(1_A \otimes \rho_s)$ which gives an action of G on $A \times_{\delta_A} G$. Then we define a faithful representation $\widetilde{\pi}$ of $A \times_{\delta_A} G$ on $L^2(L^2(\mathcal{H}_A, G), G)$ $(= L^2(\mathcal{H}_A, G) \otimes L^2(G) = (\mathcal{H}_A \otimes L^2(G)) \otimes L^2(G)$) by

$$(\widetilde{\pi}(z)\xi)(s) = \widehat{\delta}_{As}{}^{-1}(z)(\xi(s))$$

for $z \in A \times_{\delta_A} G$ and $\xi \in L^2(L^2(\mathcal{H}_A, G), G)$.

DEFINITION 4.3. Let δ_A be a nondegenerate coaction of G on a C^*-algebra A and let δ_X be a nondegenerate δ_A-compatible coaction of G on a Hilbert A-module X. We assume that A and $\mathcal{K}(X)$ are concretely represented on a Hilbert space $\mathcal{H}_A$ and on a Hilbert space $\mathcal{H}_{\mathcal{K}}$, respectively. Then X can be concretely represented into $\mathcal{B}(\mathcal{H}_A, \mathcal{H}_{\mathcal{K}})$. Here we remark that the $(\mathcal{K}(X) \times_{\delta_{\mathcal{K}}} G) - (A \times_{\delta_A} G)$ Hilbert bimodule $X \times_{\delta_X} G$ is concretely represented into $\mathcal{B}(L^2(\mathcal{H}_A, G), L^2(\mathcal{H}_{\mathcal{K}}, G))$ and that the inner products are then given by

$$ {}_{\mathcal{K}(X)\times_{\delta_{\mathcal{K}}}G}\langle x,y\rangle = xy^* \quad \text{and} \quad \langle x,y\rangle_{A\times_{\delta_A}G} = x^*y $$

for $x, y \in X \times_{\delta_X} G$. Then we define the *dual action* $\widehat{\delta}_X$ of G on $X \times_{\delta_X} G$ by

$$ \widehat{\delta}_{Xs}(\cdot) = (1_{\mathcal{K}} \otimes \rho_s)(\cdot)(1_A \otimes \rho_s)^* $$

for $s \in G$ which gives an action of G on $X \times_{\delta_X} G$. We define a representation $\widetilde{\pi}_X$ of $X \times_{\delta_X} G$ into $\mathcal{B}(L^2(\mathcal{H}_A, G) \otimes L^2(G), L^2(\mathcal{H}_{\mathcal{K}}, G) \otimes L^2(G))$ by

$$ (\widetilde{\pi}_X(z)\xi)(s) = \widehat{\delta}_{Xs}^{-1}(z)(\xi(s)) $$

for $z \in X \times_{\delta_X} G$ and $\xi \in L^2(L^2(\mathcal{H}_A, G), G)$. Similarly, we define the *dual action* $\widehat{\delta}_{\widetilde{X}}$ of G on $\widetilde{X} \times_{\delta_{\widetilde{X}}} G$ and a representation $\widetilde{\pi}_{\widetilde{X}}$ of $\widetilde{X} \times_{\delta_{\widetilde{X}}} G$.

LEMMA 4.4. *Let $\widehat{\delta}_X$ and $\widehat{\delta}_{\widetilde{X}}$ be as above. Then $\widehat{\delta}_X$ is $(\widehat{\delta}_{\mathcal{K}}, \widehat{\delta}_A)$-compatible and $\widehat{\delta}_{\widetilde{X}}$ is $(\widehat{\delta}_A, \widehat{\delta}_{\mathcal{K}})$-compatible.*

Proof. By symmetry, if we show that $\widehat{\delta}_X$ is $(\widehat{\delta}_{\mathcal{K}}, \widehat{\delta}_A)$-compatible, then $(\widehat{\delta}_A, \widehat{\delta}_{\mathcal{K}})$-compatibility of $\widehat{\delta}_{\widetilde{X}}$ follows. Hence we will show only that $\widehat{\delta}_X$ is $(\widehat{\delta}_{\mathcal{K}}, \widehat{\delta}_A)$-compatible. In fact, for $x, y \in X \times_{\delta_X} G$ and $a \in A \times_{\delta_A} G$, we have

$$ \begin{aligned} \widehat{\delta}_{Xs}(xa) &= (1_{\mathcal{K}} \otimes \rho_s)xa(1_A \otimes \rho_s)^* \\ &= (1_{\mathcal{K}} \otimes \rho_s)x(1_A \otimes \rho_s)^*(1_A \otimes \rho_s)a(1_A \otimes \rho_s)^* = \widehat{\delta}_{Xs}(x)\widehat{\delta}_{As}(a); \\ \langle \widehat{\delta}_{Xs}(x), \widehat{\delta}_{Xs}(y)\rangle_{A\times_{\delta_X}G} &\equiv \widehat{\delta}_{Xs}(x)^*\widehat{\delta}_{Xs}(y) \\ &= ((1_{\mathcal{K}} \otimes \rho_s)x(1_A \otimes \rho_s)^*)^*((1_{\mathcal{K}} \otimes \rho_s)y(1_A \otimes \rho_s)^*) \\ &= (1_A \otimes \rho_s)x^*y(1_A \otimes \rho_s)^* = \widehat{\delta}_{As}(\langle x,y\rangle_{A\times_{\delta_X}G}). \end{aligned} $$

Thus we see that $\widehat{\delta}_X$ is $\widehat{\delta}_A$-compatible. Similarly $\widehat{\delta}_{\mathcal{K}}$-compatibility of $\widehat{\delta}_X$ can be also shown. ∎

From now on, as a nondegenerate coaction $\delta_{\mathcal{L}}$ of G on $\mathcal{L}(X)$, we consider only

$$ \delta_{\mathcal{L}} = \begin{pmatrix} \delta_{\mathcal{K}} & \delta_X \\ \delta_{\widetilde{X}} & \delta_A \end{pmatrix}. $$

LEMMA 4.5. *Let $\delta_{\mathcal{L}}$ be the above nondegenerate coaction of G on $\mathcal{L}(X)$. Then we see that $\widehat{\delta}_{\mathcal{L}} = \begin{pmatrix} \widehat{\delta}_{\mathcal{K}} & \widehat{\delta}_X \\ \widehat{\delta}_{\widetilde{X}} & \widehat{\delta}_A \end{pmatrix}$, and we have*

$$ \begin{aligned} (\mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G) \times_{\widehat{\delta}_{\mathcal{L}},r} G &= \mathcal{L}(X \times_{\delta_X} G) \times_{\widehat{\delta}_{\mathcal{L}},r} G = \mathcal{L}((X \times_{\delta_X} G) \times_{\widehat{\delta}_X,r} G) \\ &= \begin{pmatrix} (\mathcal{K}(X) \times_{\delta_{\mathcal{K}}} G) \times_{\widehat{\delta}_{\mathcal{K}},r} G & (X \times_{\delta_X} G) \times_{\widehat{\delta}_X,r} G \\ (\widetilde{X} \times_{\delta_{\widetilde{X}}} G) \times_{\widehat{\delta}_{\widetilde{X}},r} G & (A \times_{\delta_A} G) \times_{\widehat{\delta}_A,r} G \end{pmatrix}. \end{aligned} $$

Proof. Since $\mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G = \mathcal{L}(X \times_{\delta_X} G)$ by Appendix: Remarks (4) in [7], the first equality in the second assertion follows.

Now we show the first assertion. Take any $\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \in \mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G$. Then we have

$$\begin{aligned}
\widehat{\delta}_{\mathcal{L}s}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right) &= (1_{\mathcal{L}(X)} \otimes \rho_s) \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} (1_{\mathcal{L}(X)} \otimes \rho_s)^* \\
&= \begin{pmatrix} 1_{\mathcal{K}} \otimes \rho_s & 0 \\ 0 & 1_A \otimes \rho_s \end{pmatrix} \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \begin{pmatrix} (1_{\mathcal{K}} \otimes \rho_s)^* & 0 \\ 0 & (1_A \otimes \rho_s)^* \end{pmatrix} \\
&= \begin{pmatrix} (1_{\mathcal{K}} \otimes \rho_s) t (1_{\mathcal{K}} \otimes \rho_s)^* & (1_{\mathcal{K}} \otimes \rho_s) x (1_A \otimes \rho_s)^* \\ (1_A \otimes \rho_s) \widetilde{y} (1_{\mathcal{K}} \otimes \rho_s)^* & (1_A \otimes \rho_s) a (1_A \otimes \rho_s)^* \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\delta}_{\mathcal{K}s}(t) & \widehat{\delta}_{Xs}(x) \\ \widehat{\delta}_{\widetilde{X}s}(\widetilde{y}) & \widehat{\delta}_{As}(a) \end{pmatrix} = \begin{pmatrix} \widehat{\delta}_{\mathcal{K}s} & \widehat{\delta}_{Xs} \\ \widehat{\delta}_{\widetilde{X}s} & \widehat{\delta}_{As} \end{pmatrix} \left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right),
\end{aligned}$$

which shows the first assertion.

The second equality in the second assertion follows from the remark following Proposition 2.11. ∎

Let A be a C^*-algebra with a nondegenerate coaction δ_A of G. Here we employ again the notation in the paragraph preceding Definition 4.3. Note that there is an isomorphism Φ from $A \otimes \mathcal{C}(L^2(G))$ onto $(A \times_{\delta_A} G) \times_{\widehat{\delta}_A, r} G$ (for example, see page 768 in [14]), and that Φ is given by the correspondences between generators:

$$\begin{aligned}
&\Phi(\delta_A(a)) = \widetilde{\pi}(\delta_A(a)), \quad a \in A; \\
&\Phi(1_A \otimes M_f) = \widetilde{\pi}(1_A \otimes M_f), \quad f \in C_0(G); \\
&\Phi(1_A \otimes \rho(g)) = 1_A \otimes 1 \otimes \lambda(g), \quad g \in L^1(G).
\end{aligned}$$

We define a dual coaction $\widehat{\widehat{\delta}}_A$ of G on $(A \times_{\delta_A} G) \times_{\widehat{\delta}_A, r} G$ by

$$\widehat{\widehat{\delta}}_A(z) = (1_A \otimes 1 \otimes W_G)(z \otimes 1)(1_A \otimes 1 \otimes W_G{}^*)$$

for $z \in (A \times_{\delta_A} G) \times_{\widehat{\delta}_A, r} G$, which is a nondegenerate coaction of G (see [12]). Furthermore we define also a coaction $\widetilde{\delta}$ of G on $A \otimes \mathcal{C}(L^2(G))$ by

$$\widetilde{\delta}(z) = (1_A \otimes W_G{}^*)((\mathrm{id} \otimes \sigma) \circ (\delta_A \otimes \mathrm{id}))(z)(1_A \otimes W_G)$$

for $z \in A \otimes \mathcal{C}(L^2(G))$, where σ is the flip map from $C^*_{\mathrm{r}}(G) \otimes \mathcal{C}(L^2(G))$ onto $\mathcal{C}(L^2(G)) \otimes C^*_{\mathrm{r}}(G)$. Then the inverse Φ^{-1} of Φ carries the dual coaction $\widehat{\widehat{\delta}}_A$ of G on $(A \times_{\delta_A} G) \times_{\widehat{\delta}_A, r} G$ to the coaction $\widetilde{\delta}$ of G on $A \otimes \mathcal{C}(L^2(G))$ (see Theorem 8 in [12]). Now we apply this fact to $\mathcal{L}(X)$ with a nondegenerate coaction $\delta_{\mathcal{L}}$ of G.

LEMMA 4.6. *Let $\mathcal{L}(X)$ be the linking algebra for a Hilbert A-module X. Suppose that $\delta_{\mathcal{L}}$ is the above nondegenerate coaction of G on $\mathcal{L}(X)$. Let Φ be the isomorphism above from $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ onto $(\mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G) \times_{\widehat{\delta}_{\mathcal{L}}, r} G$. Then Φ is a componentwise* *isomorphism from* $\mathcal{L}(X \otimes \mathcal{C}(L^2(G)))$ *onto* $\mathcal{L}((X \times_{\delta_X} G) \times_{\widehat{\delta}_X, r} G)$. *Furthermore,* Φ^{-1} *carries the dual coaction* $\widehat{\widehat{\delta}}_{\mathcal{L}}$ *of* G *on* $(\mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G) \times_{\widehat{\delta}_{\mathcal{L}}, r} G$ *to the coaction* $\widetilde{\delta}_{\mathcal{L}}$ *of* G *on* $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$, *that is,* $(\Phi^{-1} \otimes \mathrm{id}) \circ \widehat{\widehat{\delta}}_{\mathcal{L}} = \widetilde{\delta}_{\mathcal{L}} \circ \Phi^{-1}$.

Proof. We may assume that A is concretely represented on some Hilbert space $\mathcal{H}_A$, and that $\mathcal{K}(X)$ is also concretely represented on some Hilbert space $\mathcal{H}_{\mathcal{K}}$. Then we can assume that $\mathcal{L}(X)$ is concretely represented on $\mathcal{H} = \mathcal{H}_{\mathcal{K}} \oplus \mathcal{H}_A$. Consider a (faithful) representation $\widetilde{\Pi}$ of $\mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G$ on $L^2(L^2(\mathcal{H}, G), G)$ $(= (\mathcal{H} \otimes L^2(G)) \otimes L^2(G))$ defined by

$$(\widetilde{\Pi}(z)\xi)(s) = \widehat{\delta}_{\mathcal{L}s}^{-1}(z)(\xi(s))$$

for $z \in \mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G$. The representation $\widetilde{\pi}_A$ of $A \times_{\delta_A} G$ on $L^2(L^2(\mathcal{H}_A, G), G)$ $(= (\mathcal{H}_A \otimes L^2(G)) \otimes L^2(G))$ is defined by

$$(\widetilde{\pi}_A(z)\xi)(s) = \widehat{\delta}_{As}^{-1}(z)(\xi(s))$$

for $z \in A \times_{\delta_A} G$ and the representation $\widetilde{\pi}_{\mathcal{K}}$ of $\mathcal{K}(X) \times_{\delta_{\mathcal{K}}} G$ is also similarly defined. Let $\widetilde{\pi}_X$ and $\widetilde{\pi}_{\widetilde{X}}$ be as in Definition 4.3. Then we claim that

$$\widetilde{\Pi} = \begin{pmatrix} \widetilde{\pi}_{\mathcal{K}} & \widetilde{\pi}_X \\ \widetilde{\pi}_{\widetilde{X}} & \widetilde{\pi}_A \end{pmatrix}.$$

For $\left(\begin{smallmatrix} t & x \\ \widetilde{y} & a \end{smallmatrix}\right) \in \mathcal{L}(X) \times_{\delta_{\mathcal{L}}} G$ and $\xi = \xi_{\mathcal{K}} \oplus \xi_A \in L^2(\mathcal{H}_{\mathcal{K}} \oplus \mathcal{H}_A, G)$, in fact, using Lemma 4.5, we have

$$\begin{aligned}\left(\widetilde{\Pi}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right)\xi\right)(s) &= \widehat{\delta}_{\mathcal{L}s}^{-1}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right)\xi(s) = \begin{pmatrix} \widehat{\delta}_{\mathcal{K}s}^{-1}(t) & \widehat{\delta}_{Xs}^{-1}(x) \\ \widehat{\delta}_{\widetilde{X}s}^{-1}(\widetilde{y}) & \widehat{\delta}_{As}^{-1}(a) \end{pmatrix}\xi(s) \\ &= \begin{pmatrix} (\widetilde{\pi}_{\mathcal{K}}(t)\xi_{\mathcal{K}})(s) \oplus (\widetilde{\pi}_X(x)\xi_A)(s) \\ (\widetilde{\pi}_{\widetilde{X}}(\widetilde{y})\xi_{\mathcal{K}})(s) \oplus (\widetilde{\pi}_A(a)\xi_A)(s) \end{pmatrix} = \left(\begin{pmatrix} \widetilde{\pi}_{\mathcal{K}}(t) & \widetilde{\pi}_X(x) \\ \widetilde{\pi}_{\widetilde{X}}(\widetilde{y}) & \widetilde{\pi}_A(a) \end{pmatrix}\xi\right)(s),\end{aligned}$$

which implies that $\widetilde{\Pi} = \left(\begin{smallmatrix} \widetilde{\pi}_{\mathcal{K}} & \widetilde{\pi}_X \\ \widetilde{\pi}_{\widetilde{X}} & \widetilde{\pi}_A \end{smallmatrix}\right)$.

The isomorphism Φ is given by the correspondences:

$$\begin{aligned}&\Phi(\delta_{\mathcal{L}}(z)) = \widetilde{\Pi}(\delta_{\mathcal{L}}(z)), \quad z \in \mathcal{L}(X);\\ &\Phi(1_{\mathcal{L}(X)} \otimes M_f) = \widetilde{\Pi}(1_{\mathcal{L}(X)} \otimes M_f);\\ &\Phi(1_{\mathcal{L}(X)} \otimes \rho(g)) = 1_{\mathcal{L}(X)} \otimes 1 \otimes \lambda(g).\end{aligned}$$

Recall again that we can identify $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G))$ with $\mathcal{L}(X \otimes \mathcal{C}(L^2(G)))$ by Lemma 2.3. Then we have

$$\begin{aligned}\Phi\left(\begin{pmatrix} \delta_{\mathcal{K}}(t) & \delta_X(x) \\ \delta_{\widetilde{X}}(\widetilde{y}) & \delta_A(a) \end{pmatrix}\right) &= \Phi\left(\delta_{\mathcal{L}}\left(\begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix}\right)\right) = \Phi(\delta_{\mathcal{L}}(z)) = \widetilde{\Pi}(\delta_{\mathcal{L}}(z)) \\ &= \widetilde{\Pi}\left(\begin{pmatrix} \delta_{\mathcal{K}}(t) & \delta_X(x) \\ \delta_{\widetilde{X}}(\widetilde{y}) & \delta_A(a) \end{pmatrix}\right) = \begin{pmatrix} \widetilde{\pi}_{\mathcal{K}}(\delta_{\mathcal{K}}(t)) & \widetilde{\pi}_X(\delta_X(x)) \\ \widetilde{\pi}_{\widetilde{X}}(\delta_{\widetilde{X}}(\widetilde{y})) & \widetilde{\pi}_A(\delta_A(a)) \end{pmatrix};\end{aligned}$$

$$
\begin{aligned}
&\Phi\left(\begin{pmatrix}1_{\mathcal{K}}\otimes M_f & 0\\ 0 & 1_A\otimes M_f\end{pmatrix}\right)\\
&\quad=\Phi\left(\begin{pmatrix}1_{\mathcal{K}} & 0\\ 0 & 1_A\end{pmatrix}\otimes M_f\right)=\Phi(1_{\mathcal{L}(X)}\otimes M_f)\\
&\quad=\widetilde{\Pi}(1_{\mathcal{L}(X)}\otimes M_f)=\widetilde{\Pi}\left(\begin{pmatrix}1_{\mathcal{K}} & 0\\ 0 & 1_A\end{pmatrix}\otimes M_f\right)\\
&\quad=\widetilde{\Pi}\left(\begin{pmatrix}1_{\mathcal{K}}\otimes M_f & 0\\ 0 & 1_A\otimes M_f\end{pmatrix}\right)=\begin{pmatrix}\widetilde{\pi}_{\mathcal{K}}(1_{\mathcal{K}}\otimes M_f) & 0\\ 0 & \widetilde{\pi}_A(1_A\otimes M_f)\end{pmatrix};
\end{aligned}
$$

$$
\begin{aligned}
\Phi\left(\begin{pmatrix}1_{\mathcal{K}}\otimes\rho(g) & 0\\ 0 & 1_A\otimes\rho(g)\end{pmatrix}\right)&=\Phi\left(\begin{pmatrix}1_{\mathcal{K}} & 0\\ 0 & 1_A\end{pmatrix}\otimes\rho(g)\right)=\Phi(1_{\mathcal{L}(X)}\otimes\rho(g))\\
&=1_{\mathcal{L}(X)}\otimes 1\otimes\lambda(g)=\begin{pmatrix}1_{\mathcal{K}}\otimes 1\otimes\lambda(g) & 0\\ 0 & 1_A\otimes 1\otimes\lambda(g)\end{pmatrix}.
\end{aligned}
$$

Hence Φ is a componentwise isomorphism. ∎

We define a (dual) coaction $\widehat{\widehat{\delta}}_X$ of G on $(X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G$ by

$$\widehat{\widehat{\delta}}_X(z)=(1_{\mathcal{K}}\otimes 1\otimes W_G)(z\otimes 1)(1_A\otimes 1\otimes {W_G}^*)$$

for $z\in(X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G$, which is a nondegenerate coaction of G. We define a coaction $\widetilde{\delta}_X$ of G on $X\otimes\mathcal{C}(L^2(G))$ by

$$\widetilde{\delta}_X(z)=(1_{\mathcal{K}}\otimes {W_G}^*)((\mathrm{id}\otimes\sigma)\circ(\delta_X\otimes\mathrm{id}))(z)(1_A\otimes W_G)$$

for $z\in X\otimes\mathcal{C}(L^2(G))$, where σ is the flip map from $C^*_{\mathrm{r}}(G)\otimes\mathcal{C}(L^2(G))$ onto $\mathcal{C}(L^2(G))\otimes C^*_{\mathrm{r}}(G)$.

Now we are in a position to establish duality for crossed products of Hilbert C^*-modules by coactions.

THEOREM 4.7 (Duality). *Let A be a C^*-algebra and let δ_A be a nondegenerate coaction of a locally compact group G on A. Suppose that X is a Hilbert A-module with a nondegenerate δ_A-compatible coaction δ_X of G. Then there exists a dual action $\widehat{\delta}_X$ of G on the crossed product $X\times_{\delta_X}G$ such that the $((A\times_{\delta_A}G)\times_{\widehat{\delta}_A,r}G)$-Hilbert module $(X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G$ is isomorphic to the $(A\otimes\mathcal{C}(L^2(G)))$-Hilbert module $X\otimes\mathcal{C}(L^2(G))$. Furthermore, the isomorphism carries $\widehat{\widehat{\delta}}_X$ to $\widetilde{\delta}_X$.*

Proof. By Lemma 4.5, we can identify $(\mathcal{L}(X)\times_{\delta_{\mathcal{L}}}G)\times_{\widehat{\delta}_{\mathcal{L}},r}G$ with

$$\mathcal{L}((X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G)=\begin{pmatrix}(\mathcal{K}(X)\times_{\delta_{\mathcal{K}}}G)\times_{\widehat{\delta}_{\mathcal{K}},r}G & (X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G\\ (\widetilde{X}\times_{\delta_{\widetilde{X}}}G)\times_{\widehat{\delta}_{\widetilde{X}},r}G & (A\times_{\delta_A}G)\times_{\widehat{\delta}_A,r}G\end{pmatrix}.$$

Since $\mathcal{L}(X\otimes\mathcal{C}(L^2(G)))$ is componentwisely isomorphic to $\mathcal{L}((X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G)$ by Lemma 4.6, taking the right upper corners of those linking algebras, it follows from Lemma 2.6 that $(X\times_{\delta_X}G)\times_{\widehat{\delta}_X,r}G$ is isomorphic to $X\otimes\mathcal{C}(L^2(G))$.

It remains to show that $\widehat{\widehat{\delta}}_X$ is carried to $\widetilde{\delta}_X$ by such an isomorphism. Let $\widehat{\widehat{\delta}}_{\mathcal{K}}$ and $\widehat{\widehat{\delta}}_A$ be the dual coactions of G on $(\mathcal{K}(X) \times_{\delta_{\mathcal{K}}} G) \times_{\widehat{\delta}_{\mathcal{K}},r} G$ and on $(A \times_{\delta_A} G) \times_{\widehat{\delta}_A,r} G$, respectively. Take any

$$z = \begin{pmatrix} t & x \\ \widetilde{y} & a \end{pmatrix} \in \mathcal{L}((X \times_{\delta_X} G) \times_{\widehat{\delta}_X,r} G).$$

Then we have

$$\begin{aligned}
\widehat{\widehat{\delta}}_{\mathcal{L}}(z) &= (1_{\mathcal{L}(X)} \otimes 1 \otimes W_G)(z \otimes 1)(1_{\mathcal{L}(X)} \otimes 1 \otimes {W_G}^*) \\
&= \mathrm{Ad}\begin{pmatrix} 1_{\mathcal{K}} \otimes 1 \otimes W_G & 0 \\ 0 & 1_A \otimes 1 \otimes W_G \end{pmatrix}\left(\begin{pmatrix} t \otimes 1 & x \otimes 1 \\ \widetilde{y} \otimes 1 & a \otimes 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} (1_{\mathcal{K}}\otimes 1\otimes W_G)(t\otimes 1)(1_{\mathcal{K}}\otimes 1\otimes {W_G}^*) & (1_{\mathcal{K}}\otimes 1\otimes W_G)(x\otimes 1)(1_A\otimes 1\otimes {W_G}^*) \\ (1_A\otimes 1\otimes W_G)(\widetilde{y}\otimes 1)(1_{\mathcal{K}}\otimes 1\otimes {W_G}^*) & (1_A\otimes 1\otimes W_G)(a\otimes 1)(1_A\otimes 1\otimes {W_G}^*) \end{pmatrix} \\
&= \begin{pmatrix} \widehat{\widehat{\delta}}_{\mathcal{K}}(t) & \widehat{\widehat{\delta}}_X(x) \\ \widehat{\widehat{\delta}}_{\widetilde{X}}(\widetilde{y}) & \widehat{\widehat{\delta}}_A(a) \end{pmatrix},
\end{aligned}$$

which shows that $\widehat{\widehat{\delta}}_{\mathcal{L}}$ is componentwise.

On the other hand, since we see that

$$\delta_{\mathcal{L}} \otimes \mathrm{id} = \begin{pmatrix} \delta_{\mathcal{K}} \otimes \mathrm{id} & \delta_X \otimes \mathrm{id} \\ \delta_{\widetilde{X}} \otimes \mathrm{id} & \delta_A \otimes \mathrm{id} \end{pmatrix}$$

on $\mathcal{L}(X) \otimes \mathcal{C}(L^2(G)) = \mathcal{L}(X \otimes \mathcal{C}(L^2(G)))$, $\delta_{\mathcal{L}} \otimes \mathrm{id}$ is componentwise. Furthermore it is easy to verify that $\mathrm{id} \otimes \sigma : \mathcal{L}(X) \otimes_{\min} (C^*_{\mathrm{r}}(G) \otimes \mathcal{C}(L^2(G))) \to \mathcal{L}(X) \otimes_{\min} (\mathcal{C}(L^2(G)) \otimes C^*_{\mathrm{r}}(G))$ is a componentwise isomorphism. Since $\mathrm{Ad}(1_{\mathcal{L}(X)} \otimes {W_G}^*)$ is a componentwise isomorphism on $\mathcal{L}(X) \otimes_{\min} (\mathcal{C}(L^2(G)) \otimes C^*_{\mathrm{r}}(G))$, we see that $\widetilde{\delta}_{\mathcal{L}}(z) = (1_{\mathcal{L}(X)} \otimes {W_G}^*)(\mathrm{id} \otimes \sigma)((\delta_{\mathcal{L}} \otimes \mathrm{id})(z))(1_{\mathcal{L}(X)} \otimes W_G)$ is also componentwise. Since Φ^{-1} is componentwise, so is $(\Phi^{-1} \otimes \mathrm{id}) \circ \widehat{\widehat{\delta}}_{\mathcal{L}} = \widetilde{\delta}_{\mathcal{L}} \circ \Phi^{-1}$. This means that the duality isomorphism carries $\widehat{\widehat{\delta}}_X$ to $\widetilde{\delta}_X$. ∎

REMARK 4.8. Let (A, G, α) be a C^*-dynamical system. Recall that every C^*-algebra A can be regarded as a Hilbert A-module in the usual way (see Example 3.5 in [15]). Then α is an α-compatible action of G on the Hilbert module A. Then Theorem 3.8 coincides with Imai-Takai's duality. For the coaction case, similarly we consider the Hilbert A-module A with a nondegenerate δ_A-compatible coaction δ_A of G. Then Theorem 4.7 coincides with Katayama's duality [12].

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ADDED IN PROOFS. As is mentioned in Section 1, by applying Theorem 3.8 (Duality) in this paper, we can give a proof of the duality theorem for crossed products of Hilbert C^*-modules by abelian group actions. Such a proof is shown in the author's paper entitled An alternative proof of the duality theorem for crossed products of Hilbert C^*-modules by abelian group actions, *Tech. Rep. Kansai Univ.* **48**(2006), 111–117.