J. OPERATOR THEORY
60:1(2008), 113–124

TOPOLOGICAL STRUCTURE OF THE UNITARY GROUP OF CERTAIN C^*-ALGEBRAS

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Communicated by Şerban Strătilă

ABSTRACT. Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras where A is a purely infinite simple C^*-algebra and B is an essential ideal of E. In the case B is the compacts or a nonunital purely infinite simple C^*-algebra we completely determine the homotopy groups of the unitary group of E in terms of K-theory. The result can be viewed as a generalization of the well-known Kuiper's theorem to a new class of C^*-algebras (including certain separable C^*-algebras).

KEYWORDS: *C^*-algebras, unitary group, homotopy groups.*

MSC (2000): Primary 46L05; Secondary 46L80, 55Q52.

INTRODUCTION

Let A be a unital C^*-algebra. It is known that the unitary group of A denoted by $U(A)$ carries important information about the internal structure of the C^*-algebra A. $U(A)$ is a topological group with the norm topology inherited from A. A long standing open problem in topology is to compute the homotopy groups of the unitary group of $\mathcal{M}_n(C)$ [1]. One important result in this direction was obtained by Kuiper in [9] where he proved that the unitary group of the algebra of bounded operators on an infinite Hilbert space is contractible (i.e. $\pi_n(B(H)) = 0$ for every n).

A generalization of Kuiper's result was obtained by Mingo [11] and then generalized by Cuntz and Higson [7]: The unitary group of $\mathcal{L}(H_A)$, the C^*-algebra of A-linear endomorphisms of the countably generated trivial Hilbert A-module H_A which have an adjoint, is contractible. The homotopy groups of the unitary group were computed for other classes of algebras: von Neumann Algebras [3], [4], [13]; $\mathcal{A}_\theta$, a noncommutative irrational torus [12]; tensor products of any C^*-algebra with an infinite dimensional simple AF-algebra or a Cuntz algebra $\mathcal{O}_n$ [14]; purely infinite simple C^*-algebras [19], nonelementary simple C^*-algebras with real rank zero and topological rank one [17].

In all the above results the formula for the homotopy groups of the unitary group is given by the K-theory of the algebra:

$$\pi_n(U(A)) = \begin{cases} K_0((A)) & n \text{ odd}, \\ K_1((A)) & n \text{ even}. \end{cases} \tag{0.1}$$

In this article we will extend Kuiper's result to new classes of C^*-algebras , namely:

(1) essential extensions of purely infinite simple C^*-algebras by the compacts;
(2) extensions of purely infinite simple C^*-algebras by purely infinite simple C^*-algebras with real rank zero.

For these C^*-algebras we are going to show that formula (0.1) still holds.

If A is a nonunital C^*-algebra let A^+ be the unital C^*-algebra obtained by joining a unit to A, and let $A^+ = A$ if A is unital. For a nonunital C^*-algebra $U(A)$ is defined to be $U(A^+)$. Let $\mathcal{K}$ be the C^*-algebra of compact operators on a separable infinite Hilbert space H. Let $U_n(A)$ be the unitary group of $M_n(A)$. We are going to denote by $U_\infty(A)$ the inductive limit of $U_n(A)$ where the inclusion map is $U_n(a) \ni u \to \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in U_{n+1}(A)$.

If p is a projection in A then $U_p(A)$ represents the unitary group of pAp. Note that if A is a nonunital C^*-algebra then $pAp = pA^+p$ and $((1-p)A(1-p))^+ = (1-p)A^+(1-p)$. We view $U_p(A)$ as a subgroup of $U(A)$,

$$U_p(A) = \Big\{ u \in U(A) : u \text{ has the matrix form } \begin{pmatrix} u_0 & 0 \\ 0 & 1-p \end{pmatrix}, \\ \text{with respect to the decomposition } 1 = p + 1 - p \Big\}.$$

Two *projections* p and q in A are said to be *equivalent*, denoted by $p \sim q$, if there exists a partial isometry v in A such that $vv^* = p$ and $v^*v = q$. The equivalence class of p is denoted by $[p]$. Two *projections* p and q in A are said to be *unitarily equivalent*, denoted by $p \simeq q$, if there exists a unitary u in A^+ such that $upu^* = q$. Here p and q are said to be homotopic equivalent, denoted by $p \approx q$, if they are in the same path component of $\mathrm{Proj}(A)$. It is known that $p \approx q \Rightarrow p \simeq q \Rightarrow p \sim q$, but the converses are not true in general.

E is an *extension* of A by B if the C^*-algebras $A, E,$ and B form a short exact sequence $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$. E is an *essential extension* if B is an essential ideal of E ($B \cap I \neq 0 \ \forall$ nonzero I ideal of E). An essential extension of A by B can be viewed as a subalgebra of $\mathcal{M}(A)$ ([10], 5.2).

A C^*-algebra $\mathcal{A}$ is said to be *purely infinite* if for any $x \in \mathcal{A}$ the hereditary subalgebra $\overline{xAx}$ of $\mathcal{A}$ contains an infinite projection. A C^*-algebra A has *real rank zero* if the set of invertible self-adjoint elements of $\mathcal{A}^+$ is dense in $\mathcal{A}^+_{\mathrm{sa}}$ (the self adjoint part of $\mathcal{A}$). This is equivalent with the *FS Property*: the set of all self-adjoint elements with finite spectrum is norm dense in the set of all self-adjoint elements. It was proven by S. Zhang that a simple C^*-algebra $\mathcal{A}$ is purely infinite if and only if each projection of $\mathcal{A}$ is infinite and A has real rank zero.

In this paper we will extend the results obtained by S. Zhang in [19] for purely infinite simple C^*-algebras and some of the techniques used here are similar to the ones used by S. Zhang.

1. MAIN RESULT

In this section, we state our main result and corollaries.

THEOREM 1.1. *Let* $0 \to B \to E \to A \to 0$ *be an exact sequence of* C^**-algebras, where* A *is a purely infinite simple* C^**-algebra,* B *is an essential ideal of* E*.*

If $B = \mathcal{K}$ *or* B *is a purely infinite simple* C^**-algebra and every projection in* A *lifts to a projection in* E*, then*

$$\pi_n(U(E)) = \pi_n(U(pEp)) = \begin{cases} K_0(E) & n \text{ odd}, \\ K_1(E) & n \text{ even}, \end{cases} \quad \forall p \in \mathrm{Proj}(E), \pi(p) \neq 0, 1.$$

If B is the C^*-algebra of compact operators, E is $B(H)$ and A is the Calkin algebra, we obtain:

COROLLARY 1.2 ([9], Theorem (3)). *If* H *is a separable Hilbert space then* $U(B(H))$ *is contractible.*

An example of separable purely infinite C^*-algebras is the Cuntz algebra O_n. The Cuntz algebra O_n is the C^*-algebra generated by n isometries $S_1, \ldots, S_n$ on a separable Hilbert space $\mathcal{H}$ such that $S_i S_i^* = 1$, $\sum\limits_{i=1}^{n} S_i^* S_i = 1$.

If $E_n = C^*(S_1, \ldots, S_n)$, $S_i S_i^* = 1$, $\sum\limits_{i=1}^{n} S_i^* S_i < 1$ then we have the following short exact sequence:

$$0 \to \mathcal{K} \to E_n \to O_n \to 0$$

and the above theorem determines the homotopy groups of the unitary group of these separable C^*-algebras.

COROLLARY 1.3. *Let* E_n *be the extension of the Cuntz algebra* O_n*. Then*

$$\pi_n(U(E_n)) = \begin{cases} \mathbb{Z} & n \text{ odd}, \\ 0 & n \text{ even}. \end{cases}$$

REMARK. Theorem 2.1 extends Kuiper's result to some separable purely infinite C^*-algebras that are not necessarily simple.

2. PROOF OF THE MAIN RESULT

Let E be an essential extension of a purely infinite simple C^*-algebra A by a C^*-algebra B, where B is either a purely infinite simple C^*-algebra or the C^*-algebra of compact operators. In the first part of this section we will study some properties of projections of E and in the second part we will focus on the unitary group of E.

LEMMA 2.1. *Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras, B a C^*-algebra with real rank zero. Let $p, q \in \mathrm{Proj}(E)$ such that $[\pi(p)] \leqslant [\pi(q)]$, i.e. there exists $\overline{q}_0 \in \mathrm{Proj}(A)$ such that $\pi(p) \sim \overline{q}_0 \leqslant \pi(q)$. Then there exists a lifting projection q_0 of $\overline{q}_0$ and a projection p_0 in B such that $[p - p_0] = [q_0]$ and $q_0 \leqslant q$.*

Proof. Let $\overline{v} \in A$ be a partial isometry such that $\overline{v}\overline{v}^* = \overline{q}_0$ and $\overline{v}^*\overline{v} = \pi(p)$. Let $v \in \pi^{-1}(\overline{v})$. Since $\pi(qvp) = \overline{qvp} = \overline{v}$ we can assume that $vp = v$ and $qv = v$. Let $b := p - v^*v \in pBp$. Since $RR(B) = 0$, then the hereditary subalgebra pBp has an approximate unit consisting of projections. Therefore, there exists a projection $p_0 \in pBp$ such that $\|b - bp_0\| < 1$ which implies that:

$$\begin{aligned}\|(p-p_0)(p-p_0-v^*v)(p-p_0)\| &= \|(p-p_0)(b-p_0)(p-p_0)\| \\ &= \|(p-p_0)b(p-p_0)\| = \|b - bp_0\| < 1.\end{aligned}$$

Then $(p-p_0)v^*v(p-p_0)$ is invertible in $(p-p_0)E(p-p_0)$. Consider now $x = [(p-p_0)v^*v(p-p_0)]^{-1}$, i.e.

$$(p-p_0)v^*v(p-p_0)x = x(p-p_0)v^*v(p-p_0) = p - p_0.$$

Consider $w = x^{1/2}v^*$. Then $w\,w^* = x^{1/2}v^*vx^{1/2} = x^{1/2}(p-p_0)v^*v(p-p_0)x^{1/2} = p - p_0$. Therefore, w is a partial isometry in E.

Since

$$\begin{aligned}\pi(x) &= \pi(xp) = \pi(x)\pi(p)^3 = \pi(x)\pi((p-p_0)v^*v(p-p_0)) \\ &= \pi(x(p-p_0)v^*v(p-p_0)) = \pi(p-p_0) = \pi(p),\end{aligned}$$

then $\pi(w) = \pi(x^{1/2})\pi(v^*) = \overline{v}^*$. Hence, w is a lift of $\overline{v}^*$. Let $q_0 = w^*w$. Then $q_0 \sim p - p_0$. Since $wq = x^{1/2}v^*q = x^{1/2}(qv)^* = x^{1/2}v^* = w$, then $q_0 \leqslant q$. ∎

LEMMA 2.2. *Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras with $B \subset E$ an essential ideal. If either B is isomorphic to the compacts or B is a purely infinite simple C^*-algebra, then $[q] < [p]$ for any $q \in \mathrm{Proj}(B)$ and for any $p \in \mathrm{Proj}(E) \backslash B$.*

Proof. First notice that $pBp \neq 0$. Otherwise, $\|px\|^2 = \|pxx^*p\| = 0$ for any $x \in B$ contradicting the essentiality of B.

Case 1. $B = K$.

Since pKp is a hereditary subalgebra of K, pKp is isomorphic to either M_n or K .

If $pKp \simeq M_n$ let p_n be the unit in pKp . Then $p - p_n \in \mathrm{Proj}(E)\backslash K$ and $(p-p_n)K(p-p_n) = ((p-p_n)p)K(p(p-p_n)) = (p-p_n)(pKp)(p-p_n) = 0$. Therefore, $pKp \simeq K$. Choose $q' \in \mathrm{Proj}(pKp)$ to have the same dimension as q.

Case 2. B is a purely infinite simple C^*-algebra.

Since $pBp \neq 0$ we can choose a projection $p_0 \in B, p_0 \neq 0$ such that $p_0 \leqslant p$. Since B is purely infinite then $[q] < [p_0]$. This completes the proof. ∎

LEMMA 2.3. *Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras, A be a purely infinite simple C^*-algebra, $B \subset E$ essential ideal. Assume that either B is isomorphic to the compacts or B is a purely infinite simple C^*-algebra. If p, q $\in \mathrm{Proj}(E)\backslash B$ then $[p] < [q]$.*

Proof. Since A is a purely infinite simple C^*-algebra then $[\pi(p)] < [\pi(q)]$, i.e. there exists $\overline{q}_0 \in \mathrm{Proj}(A)$ such that $\pi(p) \sim \overline{q}_0 \leqslant \pi(q)$, $\pi(q) - \overline{q}_0 \neq 0$. From Lemma 2.1, there exists $p_0 \in \mathrm{Proj}(B)$, $q_0' \in \mathrm{Proj}(E)$ such that $p - p_0 \sim q_0' \leqslant q$. Then from Lemma 2.2, there exists a projection $p_0' \in \mathrm{Proj}(B)$ such that $p_0 \sim p_0' < q - q_0'$. For $q_0 = q_0' + p_0'$ we have $p \sim q_0 \leqslant q$. ∎

COROLLARY 2.4. *Let $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ be a short exact sequence of C^*-algebras, A a purely infinite simple C^*-algebra, $B \subset E$ essential ideal. Assume that either B is isomorphic to the compacts or B is a purely infinite simple C^*-algebra. Then every projection of $E\backslash B$ is infinite.*

REMARK. The assumption that B is an essential ideal of E is a necessary condition in Lemma 2.2.

In our setting the above assumption is equivalent to the nontriviality of the extension E ($E \neq B \oplus A$). Indeed if B is a nonessential ideal of E then there exists a nonzero ideal I of A such that $B \cap I = 0$. Then $\pi(I)$ is a nonzero ideal of A. Since A is simple then $\pi(I) = B$ and therefore $E = B \oplus A$.

From now on we will use three well known lemmas about the Murray von Neumann equivalence of projections and the homotopic equivalence of projections:

LEMMA 2.5. *If A is a C^*-algebra and if p, q are two nonzero equivalent projections of A such that $pq = 0$ then $p \approx q$.*

LEMMA 2.6 ([7], Lemma 1). *If A is a C^*-algebra and p, q are two nonzero equivalent projections of A such that $\|pqp\| < 1$, then $q \approx q_0 \leqslant 1 - p$ and $p \approx p_0 \leqslant 1 - q$.*

LEMMA 2.7 ([8], Lemma 2.1). *If A is a C^*-algebra and p, q are two nonzero projections of A such that $\|p - pqp\| \leqslant \beta < 1/4$, then there exists a unitary u in A^+, such that*

$$upu^* \leqslant q, \quad \|u - 1\| \leqslant 6\beta^{1/2}.$$

We will prove next that in the above setting two projections p and q of E are Murray von Neumann equivalent if and only if p and q are homotopic equivalent.

LEMMA 2.8. *Let* $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ *be a short exact sequence of* C^*-*algebras, and* A *be a purely infinite simple* C^*-*algebra. Assume that* $B \subset E$ *either is an essential ideal,* B *is isomorphic to the compacts or is a purely infinite simple* C^*-*algebra. If* $p, q \in \mathrm{Proj}(E)$, $\pi(p) \neq 1 \neq \pi(q)$, *and* $p \sim q$ *then* p, q *are homotopic equivalent.*

Proof. If $p \in B$ then $q = vpv^* \in B$. Then it is known that p and q are homotopic equivalent, since B is isomorphic to the compacts or B is a nonunital purely infinite simple C^*-algebra. When $p, q \notin B$ we consider two cases.

Case 1. E is unital.

Since $RR(E) = 0$ by Lemma 3.2 of [15] we have that there exists two projections p_1, p_2 in E such that $p \approx p_1 + p_2$, $p_1 \leqslant q$, $p_2 \leqslant 1 - q$.

Then $\pi(q) \neq 0, 1$ implies that $\pi(p_1) \neq 1 \neq \pi(p_2)$.

Then $\pi(p) \neq 1$ implies that $\pi(p_1) \neq \pi(q)$ or $\pi(p_2) \neq \pi(1-q)$.

If $\pi(p_1) \neq \pi(q)$ then $q - p_1 \notin K$ and using Lemma 2.3 we can find $p_2' \sim p_2$, $p_2' \leqslant q - p_1$.

Since p_2' and p_2 are orthogonal, then by Lemma 2.5 they are homotopic and the homotopy can be chosen in $(1-p_1)E(1-p_1)$. So $p_1 + p_2 \approx p_1 + p_2'$.

Using Lemma 2.3 we can find a subprojection q' of $1 - q$ such that $q' \sim p_2' + p_1$. So $q' \approx p_2' + p_1$. Therefore $p \approx q$.

If $\pi(p_2) \neq \pi(1-q)$ the same proof as above holds.

Case 2. E is nonunital.

$RR(E) = 0$ implies that there exists $p_0 \in \mathrm{Proj}(E)$ such that $\|(1-p_0)q\| < 1$. By Lemma 2.7 there exists a unitary element u in E^+ such that $\|u - 1\| < 6\varepsilon^{1/2}$ and $p \leqslant up_0u^* = p_0'$. Then $\|p - p_0'\| \leqslant 2\|u-1\| \leqslant 12\varepsilon^{1/2}$ hence $\|(1-p_0')q\| \leqslant \|(1-p_0)q\| + \|(p_0 - p_0')q\| \leqslant \varepsilon + 12\varepsilon^{1/2}$.

By Lemma 2.7 there exists a unitary v in E^+ with $\|v - 1\|$ small if ε is small enough, and $vqv^* \leqslant p_0'$. Since $q \approx vqv^* < p_0'$ and $p \approx u^*pu < p_0'$ we can work in $p_0'Ep_0'$ where the proof from Case 1 will apply. ∎

LEMMA 2.9. *Let* $0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0$ *be a short exact sequence of* C^*-*algebras,* $RR(E) = 0$ *and* A *be a purely infinite simple* C^*-*algebra. If* $x_1, x_2, \ldots, x_n$ *are elements in* E *and* e, f *are two fixed projections in* E *but not in* B *then for any positive number* ε *there exists two projections in* E *but not in* B $e_0 < e, f_0 < f$ *such that* $\sup\limits_{1 \leqslant i \leqslant n} \|f_0 x_i e_0 t\| < \varepsilon$.

Proof. Since E is a C^*-algebra with real rank zero and the set $\{p \in \mathrm{Proj}(E) \backslash B\}$ is a set with no minimal projections one can use the proof of Lemma 2.2 in [19] to get the result. ∎

For the rest of this article we are going to consider the following short exact sequence:

$$0 \to B \xrightarrow{i} E \xrightarrow{\pi} A \to 0, \quad \text{with } B \text{ an essential ideal of } E.$$

In our study we assume that A is a purely infinite simple C^*-algebra, and B is the C^*-algebra of compact operators or a purely infinite simple C^*-algebra. In the following lemmas we also assume that E has real rank zero.

If B has the first K-theory group trivial then it follows that $RR(E) = 0$. So if $B = \mathcal{K}$ then $RR(E) = 0$. In general $RR(E) = 0$ if and only if $RR(A) = 0, RR(B) = 0$, and every projection of A can be lifted to a projection of E.

LEMMA 2.10. *Let E be an essential extension of A by B. Let X be a compact topological space. If $f(\cdot)$ is a unitary in $(C(X) \otimes E)^+$ and p is any fixed projection in E, $\pi(p) \neq 0, 1$, then the continuous map $p(\cdot) := f(\cdot)pf(\cdot)^*$ from X to* $\mathrm{Proj}(E)$ *is homotopic to the constant map p.*

Proof. The following proof is a modification of the proof of Lemma 2.4 in [19]. With respect to the decomposition $1 = p + (1 - p)$, we write:

$$f(\cdot) = \begin{pmatrix} a(\cdot) & b(\cdot) \\ c(\cdot) & d(\cdot) \end{pmatrix},$$

where $a(\cdot) = pf(\cdot)p$, $b(\cdot) = pf(\cdot)(1 - p)$, $c(\cdot) = (1 - p)f(\cdot)p$, $d(\cdot) = (1 - p)f(\cdot)(1 - p)$.

Since $f(X)$ is a compact subspace of $U(E^+)$, for any positive number $\varepsilon < 1/8$ there exists a subset $\{t_1, t_2, \ldots, t_n\}$ such that

$$\sup_{t \in X} \min_{1 \leqslant i \leqslant n} \|f(t) - f(t_i)\| < \varepsilon^{1/2}.$$

Applying Lemma 2.9 to the elements $c(t_1), c(t_2), \ldots, c(t_n)$ of E, we find $e_0 < p, f_0 < 1 - p$ $e_0, f_0 \in \mathrm{Proj}(E) \backslash B$ such that

$$\max_{1 \leqslant i \leqslant n} \|f_0 x_i e_0\| < \varepsilon^{1/2}.$$

From Lemma 2.3 there exists a partial isometry $v \in E$ such that $v^*v = p$ and $vv^* = p_0 < e_0$.

Since $\pi(p) \neq 1$ and $\pi(p_0) \neq 1$, we have, by Lemma 2.8, that p and p_0 are homotopic equivalent. Hence there exists a unitary $w \in U_0(E^+)$ such that $wpw^* = p_0$. So the constant map p and the constant map p_0 are homotopic and we have that $\max_{1 \leqslant i \leqslant n} \|f_0 x_i e_0\| < \varepsilon^{1/2}$.

Let $p_1(t) := f(t)wpw^*f(t)^* = f(t)p_0f(t)^* = \begin{pmatrix} a(t)p_0a(t)^* & a(t)p_0c(t)^* \\ c(t)p_0a(t)^* & c(t)p_0c(t)^* \end{pmatrix}$. Then $p(\cdot)$ is homotopic to $p_1(\cdot)$ as continuous maps from X to $\mathrm{Proj}(E)$. We have:

$$\begin{aligned}\sup_{t\in X}\left\|\begin{pmatrix} 0 & 0 \\ 0 & f_0\end{pmatrix} p_1(t)\begin{pmatrix} 0 & 0 \\ 0 & f_0\end{pmatrix}\right\| &= \sup_{t\in X}\|f_0c(t)p_0c(t)^*f_0\| \\ &\leqslant \Big[\sup_{t\in X}\min_{1\leqslant i\leqslant n}\|f_0(c(t)-c(t_i))p_0\| + \max\|f_0c(t_i)p_0\|\Big]^2 \\ &\leqslant \Big[\sup_{t\in X}\min_{1\leqslant i\leqslant n}\|f(t)-f(t_i)\| + \varepsilon^{1/2}\Big]^2 \leqslant 4\varepsilon \leqslant \frac{1}{2},\end{aligned}$$

where $\begin{pmatrix} 0 & 0 \\ 0 & f_0\end{pmatrix}$, $p_1(t)$ are two projections in $(C(X)\otimes E)^+$. Using Lemma 2.6 we find a unitary $v(\cdot)$ in $(C(X)\otimes E)^+$ homotopic to the identity such that

$$q(\cdot) := v(\cdot)p_1v(\cdot)^* \leqslant \begin{pmatrix} p & 0 \\ 0 & 1-p-f_0\end{pmatrix}.$$

Since $q(\cdot)\begin{pmatrix} 0 & 0 \\ 0 & f_0\end{pmatrix} = 0$ in $C(X)\otimes E$, using again Lemma 2.6 we find $u(\cdot) \in U_0((C(X)\otimes E)^+)$ such that $r(\cdot) := u(\cdot)q(\cdot)u(\cdot)^* \leqslant \begin{pmatrix} 0 & 0 \\ 0 & f_0\end{pmatrix}$. Thus $q(\cdot)$ and $r(\cdot)$ are orthogonal projections, and hence homotopic. Since $r(\cdot)$ and p are equivalent orthogonal projections in $C(X)\otimes E$ it follows that $r(\cdot)$ is homotopic to the constant map p.

Therefore, $p(\cdot) \approx p_1(\cdot) \approx q(\cdot) \approx r(\cdot) \approx p$. ∎

LEMMA 2.11. *Let E be an essential extension of a A by B. If $f(\cdot)$ is a unitary in $(C(X)\otimes E)^+$ and p is any fixed projection in E, $\pi(p) \neq 0, 1$, then $f(\cdot)$ is homotopic to a unitary in $(C(X)\otimes E)^+$ of the form:*

$$\begin{pmatrix} f_1(\cdot) & 0 \\ 0 & f_2(\cdot)\end{pmatrix} \quad \textit{(with respect to the decomposition } 1 = p + (1-p)\textit{)},$$

where $f_1(\cdot)$ is a unitary in $C(X)\otimes pEp$ and $f_2(\cdot)$ is a unitary in $(C(X)\otimes(1-p)E(1-p))^+$.

Proof. From the above lemma the map $p(\cdot) := f(\cdot)pf(\cdot)^*$ and the constant map p are homotopic. So there exists a path of unitaries $\{u(\cdot,s) : s\in[0,1]\}$ in $(C(X)\otimes E)^+$ such that $u(\cdot,0) = 1$ and $u(\cdot,1)^*pu(\cdot,1) = p(\cdot)$.

It follows that $u(\cdot,1)f(\cdot)p = pu(\cdot,1)f(\cdot)$ and hence,

$$u(\cdot,1)f(\cdot) = \begin{pmatrix} f_1(\cdot) & 0 \\ 0 & f_2(\cdot)\end{pmatrix}$$

where $f_1(\cdot)$ is a unitary in $C(X)\otimes pEp$ and $f_2(\cdot)$ is a unitary in $(C(X)\otimes(1-p)E(1-p))^+$. Since $u(\cdot,1)$ is homotopic to the identity, we draw the conclusion. ∎

LEMMA 2.12 ([18], 4.1). *Assume that B is a nonunital C^*-algebra with an approximate identity consisting of projections and p is a fixed projection of B. If u is a unitary of B^+, then there exists a projection $q > p$ in B (actually a member of an approximate identity of projections) such that u is homotopic to a unitary with the form $u_1 + (1 - q)$ in the unitary group of B, where u_1 is a unitary of qBq.*

LEMMA 2.13. *Let E be an essential extension of A by B. Let X be a compact topological space, p is any fixed projection in E, $\pi(p) \neq 0, 1$. Then every unitary $f(\cdot)$ in $(C(X) \otimes E)^+$ is homotopic to a unitary with the form:*

$$\begin{pmatrix} g(\cdot) & 0 \\ 0 & 1-p \end{pmatrix}, \quad \textit{where } g(\cdot) \textit{ is a unitary in } C(X) \otimes pEp.$$

Proof. From Lemma 2.11 it follows that $f(\cdot)$ is homotopic to a unitary with the form $\begin{pmatrix} f_1(\cdot) & 0 \\ 0 & f_2(\cdot) \end{pmatrix}$.

Case 1. E is unital.

Using Lemma 2.3 and Lemma 2.8 for any projection $p_1 < p$ we find a unitary $u \in E$ connected to the identity such that $uu^* = p_1$ and $u^*u = 1 - p$.

Then the unitary $\begin{pmatrix} p & 0 \\ 0 & f_2(\cdot) \end{pmatrix}$ is homotopic to the unitary $u \begin{pmatrix} p & 0 \\ 0 & f_2(\cdot) \end{pmatrix} u^*$ which has the matricial form $\begin{pmatrix} g_1(\cdot) & 0 \\ 0 & 1-p \end{pmatrix}$ with respect to the decomposition $1 = p + (1-p)$ where $g_1(\cdot) = \begin{pmatrix} uf_2(\cdot)u^* & 0 \\ 0 & p - p_1 \end{pmatrix}$ with respect to the decomposition $p = p_1 + (p - p_1)$.

It follows that:

$$\begin{pmatrix} f_1(\cdot) & 0 \\ 0 & f_2(\cdot) \end{pmatrix} = \begin{pmatrix} p & 0 \\ 0 & f_2(\cdot) \end{pmatrix} \begin{pmatrix} f_1(\cdot) & 0 \\ 0 & 1-p \end{pmatrix}$$

is homotopic to

$$\begin{pmatrix} g_1(\cdot) & 0 \\ 0 & 1-p \end{pmatrix} \begin{pmatrix} f_1(\cdot) & 0 \\ 0 & 1-p \end{pmatrix} := \begin{pmatrix} g(\cdot) & 0 \\ 0 & 1-p \end{pmatrix},$$

where $g(\cdot) = g_1(\cdot)f_2(\cdot)$

By the construction above $g(\cdot)$ is a unitary of $C(X) \otimes pEp$.

Case 2. E is nonunital.

Since E has real rank zero, E has an approximate identity consisting of projections, say $\{e_\lambda\}$. Then $\{1 \otimes e_\lambda\}$ is an approximate identity for $C(X) \otimes E$. By Lemma 2.12, there exists a projection $q = 1 \otimes e_\lambda$ such that $f(\cdot)$ is homotopic to a unitary $f_0(\cdot) + (1 - q)$ in the unitary group of $(C(X) \otimes E)^+$, where $f_0(\cdot)$ is a unitary in $C(X) \otimes e_\lambda E e_\lambda$.

From Lemma 2.3 and Lemma 2.8, it follows that there is a unitary w in E^+ homotopic to the identity such that $wpw^* = p_0 < e_\lambda$. Working now with the unital C^*-algebra $e_\lambda E e_\lambda$, we obtain the conclusion. ∎

LEMMA 2.14. *Let E be an essential extension of a purely infinite simple C^*-algebra A by compacts. Let X be a compact topological space. Let p be a fixed projection in E such that $\pi(p) \neq 0, 1$. If two continuous maps f_1, f_2 from X to U_p are homotopic as maps from X to $U_p(E)$ then they are homotopic as maps from X to $U(E)$.*

Proof. Using Lemma 2.3 and Lemma 2.8 we can make slight adjustments in the proof of Lemma 2.9 in [19] to draw the conclusion. ∎

REMARK. Let $(X, x_0), (Y, y_0)$ be two pointed topological spaces. $[X, Y]$ is the standard notation for the set of homotopy classes of continuous functions $f : X \to Y$, $f(x_0) = y_0$. By definition $\pi_n(U(E)) = [S^n, U(a)]$. The base point s_0 can be chosen arbitrarily since S^n is path connected. Since two path components of $U(E)$ are homeomorphic topological spaces, then any point in $U(E)$ can be chosen as base point. We choose here the base point for $U(E)$ to be the unit of E^+.

Lemma 2.14 shows that two maps from X to $U_p(E)$ are homotopic if they are homotopic as maps from X to $U(E)$. Therefore Lemma 2.13 gives an isomorphism between $[X, U_p(E)]$ and $[X, U(E)]$.

Since the C^*-algebra of compact operators is a nuclear C^*-algebra we have the following exact sequence:

$$0 \to B \otimes \mathcal{K} \xrightarrow{i} E \otimes \mathcal{K} \xrightarrow{\pi} A \otimes \mathcal{K} \to 0$$

and we have that $[X, U_p(E)]$ is isomorphic to $[X, U_\infty(E)]$.

As a conclusion of the above remark we have the following lemma:

LEMMA 2.15. *Let E be an essential extension of A by B. Let X be a compact topological space. Let p be a fixed projection in E, $\pi(p) \neq 0, 1$. Then*

$$[X, U_\infty] \simeq [X, U_p] \simeq [X, U(E)].$$

Replacing X by the m-th sphere we have, for any m,

$$\pi_m(U_\infty(E)) = \pi_m(U_p(E)) = \pi_m(U(E)).$$

Proof of Theorem 1.1. By definition of the K-theory we have that $K_1(E) = \pi_0(U_\infty(E))$. It follows by Bott periodicity [2] that

$$\pi_0(U_\infty(E)) = K_1(E) \quad \text{and} \quad \pi_m(U_\infty(E)) = \pi_{m-2}(U_\infty(E)), \quad \text{for any } m \geqslant 2.$$

Thus, $\pi_{2m+1}(U_\infty(E)) = K_0(E)$ and $\pi_{2m}(U_\infty(E)) = K_1(E)$.

Applying Lemma 2.15, one has, for any m,

$$\pi_m(U_\infty(E)) = \pi_m(U_p(E)) = \pi_m(U(E)).$$

Hence

$$\pi_{2m+1}(U(E)) = K_0(E) \quad \text{and} \quad \pi_{2m}(U(E)) = K_1(E).$$

Any projection $p \in E$ with the property that $\pi(p) \neq 0, 1$ is a full projection. It is known from 2.8 of [5] that $K_*(E) = K_*(pEp)$ for any full projection p.

It follows that $\pi_{2m+1}(U(pEp)) = K_0(E)$ and $\pi_{2m}(U(pEp)) = K_1(E)$. ∎

Acknowledgements. The author was partially supported by Taft Graduate Enrichment Award.

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Received March 6, 2006.