J. OPERATOR THEORY
60:1(2008), 137–148

ON THE STRUCTURE OF THE SPECTRAL RADIUS ALGEBRAS

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Communicated by Kenneth R. Davidson

ABSTRACT. We study the spectral radius algebras associated to compact operators and we give some sufficient conditions for membership in them. Since it is known that each such algebra has an invariant subspace this leads to some new invariant subspace theorems. We will also compare our method with several well known approaches to the invariant subspace problem.

KEYWORDS: *Invariant subspaces, compact operators, spectral radius algebras.*

MSC (2000): Primary 47A65; Secondary 47A62, 47B49.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. If T is an operator in $\mathcal{L}(\mathcal{H})$ and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ then $\mathcal{M}$ is invariant for T if $T\mathcal{M} \subset \mathcal{M}$ and it is nontrivial if it is different from the zero subspace and $\mathcal{H}$. The question whether every operator in $\mathcal{L}(\mathcal{H})$ has a nontrivial invariant subspace (n. i. s.) is known as the invariant subspace problem.

If K is a compact operator, a celebrated result of Lomonosov [12] asserts that all operators that commute with K possess a common invariant subspace. In other words, the algebra $\{K\}'$, the commutant of K, has a n. i. s. In [10], using this result, we have shown that there exists a larger algebra $\mathcal{B}_K$ that has a n. i. s. We call such an algebra a *spectral radius algebra* or an SR-algebra. An SR-algebra can be associated to any operator A but in what follows we will consider only compact operators K. The algebra $\mathcal{B}_K$ always contains the commutant of K, with the inclusion proper whenever K is not quasinilpotent. The case when the spectral radius $r(K) = 0$ remains open, although we have shown in [1] that the inclusion is proper when K is the Volterra operator on $L^2(0,1)$. Since $\mathcal{B}_K$ is in many cases strictly bigger than $\{K\}'$ and it has a n. i. s. it is of interest to find which operators belong to it. As we have shown in [10], if λ is a complex number and $|\lambda| \leqslant 1$, then every operator T that satisfies $KT = \lambda TK$ belongs to $\mathcal{B}_K$. Following the terminology established in [10] and subsequent papers [1], [3], [2], we call such an operator T an *extended eigenvector* of K corresponding to the *extended eigenvalue* λ. It is sometimes convenient to consider only the nontrivial case, i.e., when $\lambda \neq 1$. In this paper, though, we will consider $\lambda = 1$ as an extended eigenvalue. The fact that extended eigenvectors of a compact operator have n. i. s. was originally proved in [4]. (Of course, when $\lambda = 1$ it is contained in the Lomonosov's theorem.) Our approach, however, has as an immediate consequence that if $X_1, X_2, \ldots, X_n$ are extended eigenvectors of K corresponding to extended eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, then $X_1 + X_2 + \cdots + X_n \in \mathcal{B}_K$ and, hence, has a n. i. s. On the other hand, a rather elementary example will show that $\mathcal{B}_K$ is not a span of extended eigenvectors. Therefore, it is of interest to learn more about the structure of $\mathcal{B}_K$ and, in particular, decide which operators (if any) do not belong to any SR-algebra associated to a compact operator.

Although the study of $\mathcal{B}_A$ can be used to establish the existence of invariant subspaces when A is not compact (cf. [2]), in this paper we will concentrate on the case of SR-algebras associated to a compact operator. Nevertheless, many of the results remain true when K is not compact. In addition, it will be convenient to assume that K is a *quasi-affinity* (meaning, an operator with a zero kernel and a dense range). In this framework we will establish some sufficient conditions for membership in $\mathcal{B}_K$ and the relationship between these conditions. In particular, given a compact operator K and a complex number λ, $|\lambda| \leqslant 1$, we will consider the map F_λ defined on $\mathcal{L}(\mathcal{H})$ by $F_\lambda(T) = KT - \lambda TK$. Clearly, if $T \in \operatorname{Ker} F_\lambda$ then T is an extended eigenvector for K, so $T \in \mathcal{B}_K$. Our main result (Theorem 3.1 below) is that, if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct complex numbers with $|\lambda_i| \leqslant 1$, $1 \leqslant i \leqslant n$, and if $T \in \operatorname{Ker}\,(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n})$ then $T \in \mathcal{B}_K$. In addition, we will explore the connection between the membership of T in $\operatorname{Ker}\,(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n})$ and T being a sum of extended eigenvectors of K (see Theorem 3.8 below).

Finally, we recall the full strength of the Lomonosov's result. We will say that an operator T has the *Lomonosov property* relative to K (notation: $T \in \operatorname{Lom}(K)$) if there is an operator A that is not a scalar multiple of the identity and such that A commutes with both T and K. Lomonosov's theorem asserts that if $T \in \operatorname{Lom}(K)$ for a nonzero compact operator K then T has a n. i. s. We will show that $\operatorname{Lom}(K)$ cannot be compared with $\mathcal{B}_K$. Namely, neither one need be a subset of the other. Therefore, the study of SR-algebras is not merely a recycling of the Lomonosov's theorem, but a genuinely different approach to the invariant subspace problem.

2. THE SPECTRAL RADIUS ALGEBRA

In this section we will briefly review some basic facts about spectral radius algebras. Interested readers can find more details in [10].

If $A \in \mathcal{L}(\mathcal{H})$ and $m \geqslant 1$, we define

$$R_m(A) = R_m := \Big(\sum_{n=0}^{\infty} d_m^{2n} A^{*n} A^n \Big)^{1/2} \quad \text{where } d_m = \frac{1}{\frac{1}{m} + r(A)}. \tag{2.1}$$

Since $d_m \uparrow 1/r(A)$ (or $d_m \to \infty$ if $r(A) = 0$), the sum in (2.1) is norm convergent and the operators R_m are well defined, positive, and invertible. The spectral radius algebra $\mathcal{B}_A$ consists of all operators $T \in \mathcal{L}(\mathcal{H})$ with the property that $\sup_{m \in \mathbb{N}} \|R_m T R_m^{-1}\| < \infty$. The name comes from the fact that was proved in [7]: $\lim_{m\to\infty} \|R_m A R_m^{-1}\| = r(A)$. The following result from [10] justifies our interest in SR-algebras.

THEOREM 2.1. *Let K be a compact operator. Then $\mathcal{B}_K$ has a* n. i. s.

In general, it is very hard to verify that an operator belongs to an SR-algebra using the definition above. In most cases both R_m and, especially, R_m^{-1} are very hard to compute. Instead, we will be using the following test that has appeared in the proof of Proposition 2.3 in [10].

PROPOSITION 2.2. *Let A be an operator in $\mathcal{L}(\mathcal{H})$. Then $T \in \mathcal{B}_A$ if and only if there exists $M > 0$ such that, for all $x \in \mathcal{H}$ and $m \in \mathbb{N}$, $\sum_{n\geqslant 0} d_m^{2n} \|A^n T x\|^2 \leqslant M \sum_{n\geqslant 0} d_m^{2n} \|A^n x\|^2$.*

As a consequence, if $\|A^n T x\| \leqslant M\|A^n x\|$ for all n and x then $T \in \mathcal{B}_A$. In particular, this is the case when T commutes with A or if T is an extended eigenvector of A corresponding to an extended eigenvalue of modulus at most 1. Since $\mathcal{B}_A$ is an algebra, if $AX_i = \lambda_i X A_i$ and $|\lambda_i| \leqslant 1$, $1 \leqslant i \leqslant n$, then $X_1 + X_2 + \cdots + X_n \in \mathcal{B}_A$. This immediately raises the question whether every operator in $\mathcal{B}_A$ can be represented as a finite or infinite sum of extended eigenvectors. The following two examples show that the answer can be negative as well as affirmative, even when A is compact.

EXAMPLE 2.3. Let $\mathcal{H} = \mathbb{C} \oplus L^2(0,1)$, and let K be an operator on $\mathcal{H}$ defined (relative to this decomposition) by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & V \end{smallmatrix}\right)$, where V is the Volterra operator. (Any compact operator with empty point spectrum will do.) It is easy to see that, if $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $\lambda \in \mathbb{C}$, then $KX = \lambda XK$ implies that the operators VC and λC (mapping $\mathbb{C}$ to $L^2(0,1)$) must be equal. Since $V - \lambda$ is injective for all $\lambda \in \mathbb{C}$, C must map every complex number to the zero function in $L^2(0,1)$. In particular, if $\{e_n\}_{n=1}^{\infty}$ is any orthonormal basis for $L^2(0,1)$, relative to that basis the matrix of C (consisting of a single column) must have all entries equal to 0. Therefore, if $C = (1,0,0,\dots)^{\mathrm{tr}}$ and $T = \left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right)$, then T cannot be in the span of extended eigenvectors. On the other hand, $T \in \mathcal{B}_K$. Indeed, in view of Proposition 2.2 it suffices to show that, for all x and n, $\|K^n T x\| \leqslant \|K^n x\|$. If we write $x = \alpha \oplus f$ (with $\alpha \in \mathbb{C}$

and $f \in L^2(0,1)$) then the last inequality becomes $\|V^n C\alpha\| \leqslant \|\alpha \oplus V^n f\|$ which is obvious.

EXAMPLE 2.4. Let $\mathcal{H} = \mathbb{C}^2$ and let $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$. Then $\mathcal{B}_A$ consists of all upper triangular matrices in $\mathcal{L}(\mathcal{H})$. On the other hand, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$,$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ are all extended eigenvectors of A: the first two commute with A while the last corresponds to the extended eigenvalue $\lambda = 1/2$. Thus, $\mathcal{B}_A$ is the span of extended eigenvectors of A.

As mentioned in the introduction, a spectral radius algebra $\mathcal{B}_K$ can be quite different from the collection $\mathrm{Lom}\,(K)$ of operators that possess the Lomonosov property relative to K. In fact, we will also consider a strengthening of the Lomonosov's theorem (cf. [13]). We start with an example that shows that it is possible for an operator to be in $\mathrm{Lom}\,(K)$ but not in $\mathcal{B}_K$, for a given compact operator K.

EXAMPLE 2.5. Let $\mathcal{G}$ be a proper, infinite dimensional subspace of $\mathcal{H}$ and write $\mathcal{H} = \mathcal{G} \oplus \mathcal{G}^\perp$. Relative to this decomposition let $A = 1 \oplus 0$ and let K be any compact operator that commutes with A. Then $K = K_1 \oplus K_2$, where K_1 and K_2 are compact operators. Suppose that $r(K_1) \geqslant r(K_2)$. (The case $r(K_1) \leqslant r(K_2)$ can be handled in a similar way.) Since K_1 is compact, $\mathcal{B}_{K_1}$ has a n. i. s. so $\mathcal{B}_{K_1} \neq \mathcal{L}(\mathcal{G})$. Let Z be an operator in $\mathcal{L}(\mathcal{G})$ that is not in $\mathcal{B}_{K_1}$, and define $T = Z \oplus 0$. Since $r(K_1) \geqslant r(K_2)$, $d_m(K_1) = d_m(K)$, and it is not hard to see, using Proposition 2.2, that $T \notin \mathcal{B}_K$. On the other hand A clearly commutes with both K and T, so $T \in \mathrm{Lom}\,(K)$.

In order to state the mentioned improvement of the Lomonosov's theorem we recall that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ has the Pearcy-Salinas Property (PS) if there is a net $\{A_\alpha\}$ in $\mathcal{A}$ such that $A_\alpha \to A \neq 0$ in the weak operator topology and such that $\|\pi(A_\alpha)\| \to 0$. (Here, π denotes the projection onto the Calkin algebra.) The result of [13] can be summarized in the following form.

THEOREM 2.6. *If $\mathcal{A}$ is a weakly closed proper subalgebra of $\mathcal{L}(\mathcal{H})$ and if $\mathcal{A}$ has the PS Property then $\mathcal{A}$ has a* n. i. s.

Clearly, if $\mathcal{A}$ contains a compact operator K then it has the PS Property (just take $A_\alpha = K$). In particular, $\mathcal{B}_K$ always has the PS Property. However, the hard part is demonstrating that it is not weakly dense. Therefore, Theorem 2.6 is most effective when $\mathcal{A}$ is known to be a proper subalgebra of $\mathcal{L}(\mathcal{H})$, which naturally leads to $\mathcal{A} = \{A\}'$. Some sufficient conditions for $\{A\}'$ to possess the PS Property can be found in [13]. We will write $T \in PS(A)$ if T commutes with A and $\{A\}'$ has the PS Property. Once again, Example 2.5 demonstrates that it is possible that $\{A\}'$ has the PS Property (just take $A_\alpha = K$) and $T \in \{A\}'$ but $T \notin \mathcal{B}_K$. On the other hand, there are operators that do not satisfy the hypotheses of either of the two theorems of Lomonosov. In particular, if T is a so-called quasi-analytic shift (cf. [15]) then it neither belongs to $\mathrm{Lom}\,(K)$ for any compact K nor it commutes with any operator A such that $\{A\}'$ has the PS Property. The former was proved in [9], the latter in [8]. Yet, every weighted shift T satisfies $KT = \lambda TK$ if $K = \operatorname{diag}(1, \lambda, \lambda^2, \dots)$. When $|\lambda| < 1$, K is compact and $T \in \mathcal{B}_K$.

This discussion shows that, regarding our understanding of these classes, there is a lack of symmetry. On one hand, every quasi-analytic weighted shift belongs to some SR-algebra $\mathcal{B}_K$ but not to any $\operatorname{Lom}(K)$ nor any $PS(A)$. On the other hand, we were unable to find an example of the other type, i.e., an operator T such that either $T \in \operatorname{Lom}(K)$ for some compact operator K or $T \in PS(A)$ for some A but $T \notin \mathcal{B}_K$ for any compact operator K.

PROBLEM 2.7. Is there an operator T such that either $T \in \operatorname{Lom}(K)$ or $T \in PS(A)$ but T belongs to no $\mathcal{B}_K$?

In fact, it is an open question whether there is an operator T that belongs to no $\mathcal{B}_K$. Until this is settled it is possible that Theorem 2.1 contains an affirmative answer to the invariant subspace problem.

PROBLEM 2.8. Does every operator belong to an SR-algebra associated to a compact operator?

3. SOME CONDITIONS FOR THE MEMBERSHIP IN $\mathcal{B}_K$

As mentioned in the introduction, we define $F_\lambda(T) = KT - \lambda TK$, as in [3]. One knows that if $T \in \operatorname{Ker} F_\lambda$ then T is in $\mathcal{B}_K$. The main result of this paper is a generalization of this fact.

THEOREM 3.1. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct complex numbers such that $|\lambda_i| \leqslant 1$, $1 \leqslant i \leqslant n$. Suppose that K is a compact operator such that $r(K) > 0$. If $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ then $T \in \mathcal{B}_K$. Consequently, T has a* n. i. s.

Proof. The idea of the proof can be clearly seen in the case $n = 2$ so we prove this case. Let $|\lambda|, |\mu| \leqslant 1$ and $F_\lambda \circ F_\mu(T) = 0$. A calculation shows that $K^2T = (\lambda + \mu)KTK - \lambda\mu TK^2$. Furthermore, if $\alpha_1 = \lambda + \mu$, $\beta_1 = -\lambda\mu$, and if we define recursively $\alpha_{n+1} = \alpha_1\alpha_n + \beta_n$ and $\beta_{n+1} = \beta_1\alpha_n$ then $K^{n+1}T = \alpha_n KTK^n + \beta_n TK^{n+1}$. Moreover, it is not hard to find that α_n is of the form $C_1\lambda^n + C_2\mu^n$ so both sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are bounded. It follows that there is $M > 0$ such that, for all n and x, $\|K^{n+1}Tx\| \leqslant M\|K^n x\|$. Consequently,

$$\begin{aligned}\sum_{n\geqslant 0} d_m^{2n}\|K^nTx\|^2 &= \|Tx\|^2 + \sum_{n\geqslant 1} d_m^{2n}\|K^nTx\|^2 = \|Tx\|^2 + \sum_{n\geqslant 0} d_m^{2n+2}\|K^{n+1}Tx\|^2 \\ &\leqslant \|Tx\|^2 + \sum_{n\geqslant 0} d_m^{2n}d_m^2 M\|K^nx\|^2 = \|Tx\|^2 + Md_m^2\sum_{n\geqslant 0} d_m^{2n}\|K^nx\|^2.\end{aligned}$$

Given that $r(K) > 0$ we have that $d_m < 1/r(K)$ so

$$\sum_{n\geqslant 0} d_m^{2n}\|K^nTx\|^2 \leqslant \Big(1 + \frac{M}{r(K)^2}\Big)\sum_{n\geqslant 0} d_m^{2n}\|K^nx\|^2,$$

and the proof is complete. ∎

In the proof above, if $\lambda = \mu$ then $\alpha_n = (C_1 + C_2 n)\lambda^n$ which is still bounded when $|\lambda| < 1$. When $\lambda = \mu$ and $|\lambda| = 1$ the theorem is not true any more, as the following example shows.

EXAMPLE 3.2. Let $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^2$ and let $|\lambda| = 1$. Define operators T and K in $\mathcal{L}(\mathcal{H})$ as $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $K = \left(\begin{smallmatrix} K_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where $K_1 = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$. It is not hard to see that, if $x = (0,0,0,1)^{\mathrm{tr}}$, then $\|K^n T x\| = \sqrt{n^2+1}$ and $\|K^n x\| = 1$. If T belonged to $\mathcal{B}_K$, Proposition 2.2 would imply that there is $M > 0$ so that $\sum\limits_n d_m^{2n}(n^2+1) \leqslant M \sum\limits_n d_m^{2n}$ for all m. Since the left hand side dominates $\sum\limits_n d_m^{2n} n = d_m^2/(1-d_m^2)^2$ and the right hand side is $1/(1-d_m^2)$, and since $d_m \to 1/r(K) = 1$, it is easy to see that $T \notin \mathcal{B}_K$.

However, $F_\lambda \circ F_\lambda(T) = 0$. Indeed, a calculation shows that $F_\lambda \circ F_\lambda(T) = \left(\begin{smallmatrix} 0 & K_1^2 - 2\lambda K_1 + \lambda^2 \\ 0 & 0 \end{smallmatrix}\right)$ and $K_1^2 - 2\lambda K_1 + \lambda^2 = (K_1 - \lambda)^2 = 0$.

One knows that, if K is a compact operator and $KT = \lambda TK$, then T has a n. i. s. regardless of the complex number λ. From the viewpoint of the spectral algebras this is a consequence of the fact that, if $|\lambda| \leqslant 1$, then $T \in \mathcal{B}_K$ while, if $|\lambda| \geqslant 1$, then $T^* \in \mathcal{B}_{K^*}$. A similar argument leads to the following corollary.

COROLLARY 3.3. *Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be distinct complex numbers with the property that $|\lambda_i| \geqslant 1$, $1 \leqslant i \leqslant n$. Suppose that K is a compact operator such that $r(K) > 0$. If $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ then $T^* \in \mathcal{B}_{K^*}$. Consequently, T has a* n. i. s.

When the complex numbers $\lambda_1, \lambda_2, \dots$ are not all on the same side of the unit circle, Theorem 3.1 ceases to be true. More precisely, it is possible for an operator T to satisfy $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ with neither $T \in \mathcal{B}_K$ nor $T^* \in \mathcal{B}_{K^*}$.

EXAMPLE 3.4. Let $K = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ acting on $\mathbb{C}^2$, and let $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. then $KX = (1/2)XK$, $KY = 2YK$, and it is easy to see that, if $T = X + Y$, $F_2 \circ F_{1/2}(T) = 0$. On the other hand, $T^* = T \notin \mathcal{B}_K = \mathcal{B}_{K^*}$. Indeed, the only invariant subspaces of K are $\mathbb{C} \oplus (0)$ and $(0) \oplus \mathbb{C}$, none of which is invariant for T.

The assumption $r(K) > 0$ in Theorem 3.1 is essential. Of course, $\operatorname{Ker} F_\lambda \subset \mathcal{B}_K$ if $|\lambda| \leqslant 1$ but, if $n \geqslant 2$, the theorem is no longer true. The following example shows why.

EXAMPLE 3.5. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}$ and let $K = e_3 \otimes e_1$, $T = e_1 \otimes e_2$. Then, $K^2 = TK = 0$ so, for any complex numbers λ and μ, $F_\lambda \circ F_\mu(T) = 0$. However, $T \notin \mathcal{B}_K$. Indeed, $K^2 = 0$ so $R_m^2 = 1 + m^2 e_1 \otimes e_1$. Relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$, where $\mathcal{H}_1$ is the one dimensional subspace spanned by e_1,

$$R_m = \begin{pmatrix} \sqrt{1+m^2} & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix},$$

where $Z = (1\,0\,0\,\cdots)$. An easy calculation shows that $\|R_m T R_m^{-1}\| = \sqrt{1+m^2} \to \infty$ so $T \notin \mathcal{B}_K$.

Theorem 3.1 shows that, if $F_\lambda(T)$ belongs to $\operatorname{Ker} F_\mu$ for some $\mu \neq \lambda$ and $|\lambda|, |\mu| \leqslant 1$, then $T \in \mathcal{B}_K$. In other words, if $F_\lambda(T)$ is an extended eigenvector for K then $T \in \mathcal{B}_K$. Since $\mathcal{B}_K$ contains more than just extended eigenvectors it is natural to ask the following question.

PROBLEM 3.6. Is it true that if $|\lambda| < 1$ and $F_\lambda(T) \in \mathcal{B}_K$ then $T \in \mathcal{B}_K$?

It is easy to see that, if T is a sum of extended eigenvectors of K, then T satisfies the hypothesis of Theorem 3.1. Example 2.3 shows that not every member of $\mathcal{B}_K$ need be of such form. This leads to a natural question: which operators can be written as finite sums of extended eigenvectors of K? Before we can answer that, we need a technical result.

PROPOSITION 3.7. *Let A be an operator in $\mathcal{L}(\mathcal{H})$ and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be complex numbers. For any T,*

$$F_{\lambda_n} \circ F_{\lambda_{n-1}} \circ \cdots \circ F_{\lambda_1}(T) = K^n T + \beta_1 K^{n-1} T K + \cdots + \beta_{n-1} K T K^{n-1} + \beta_n T K^n$$

where the roots of the polynomial $p_n(z) = z^n + \beta_1 z^{n-1} + \cdots + \beta_{n-1} z + \beta_n$ are precisely $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Proof. We will prove the result by induction on n. The case $n = 1$ is obvious. Suppose that the proposition is true for $n - 1$. Then

$$\begin{aligned}
&F_{\lambda_n} \circ F_{\lambda_{n-1}} \circ \cdots \circ F_{\lambda_1}(T) \\
&\quad = K[F_{\lambda_{n-1}} \circ \cdots \circ F_{\lambda_1}(T)] - \lambda_n [F_{\lambda_{n-1}} \circ \cdots \circ F_{\lambda_1}(T)]K \\
&\quad = K[K^{n-1}T + \beta_1 K^{n-2} T K + \cdots + \beta_{n-2} K T K^{n-2} + \beta_{n-1} T K^{n-1}] \\
&\qquad\quad - \lambda_n [K^{n-1}T + \beta_1 K^{n-2} T K + \cdots + \beta_{n-2} K T K^{n-2} + \beta_{n-1} T K^{n-1}]K \\
&\quad = K^n T + (\beta_1 - \lambda_n) K^{n-1} T K + \cdots + (\beta_{n-1} - \lambda_n \beta_{n-2}) K T K^{n-1} - \lambda_n \beta_{n-1} T K^n.
\end{aligned}$$

So, it remains to prove that the zeros of $q(z) = z^n + (\beta_1 - \lambda_n) z^{n-1} + \cdots + (\beta_{n-1} - \lambda_n \beta_{n-2}) z - \lambda_n \beta_{n-1}$ are precisely the numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. Clearly, $q(z) = z p_{n-1}(z) - \lambda_n p_{n-1}(z) = (z - \lambda_n) p_{n-1}(z)$ and the assertion follows by induction. ∎

Now we can establish a necessary and sufficient condition for an operator to be a finite sum of extended eigenvectors of K. In order to state this result we introduce classes $\mathcal{S}_n$. Given a positive integer n and a compact operator K, we will say that $T \in \mathcal{S}_n$ if there is an operator $S = S_n$ such that $K^n T = S K^n$.

THEOREM 3.8. *Let K be a compact quasi-affinity, let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ be distinct, and let $T \in \mathcal{L}(\mathcal{H})$. Then there exist operators $X_i \in \operatorname{Ker} F_{\lambda_i}$ such that $T = \sum\limits_i X_i$ if and only if $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ and $T \in \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-1}$.*

Proof. If $T = \sum_i X_i$ then $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ is obvious. Also, $T \in \mathcal{S}_m$ for all m, with $S_m = \sum_i \lambda_i^m X_i$. Therefore, we concentrate on the opposite implication.

Suppose that $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$. It follows from Proposition 3.7 that $K^n T + \beta_1 K^{n-1} TK + \beta_2 K^{n-2} TK^2 + \cdots + \beta_n TK^n = 0$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the roots of the polynomial $p_n(z) = z^n + \beta_1 z^{n-1} + \beta_2 z^{n-2} + \cdots + \beta_n$.

Let $V = (v_{ij})$ be the $n \times n$ Vandermonde matrix, $v_{ij} = \lambda_j^i$, which is clearly invertible. If $K^j T = S_j K^j$ we define operators $X_1, X_2, \ldots, X_n$ by

$$V \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_{n-1} \\ -\beta_1 S_{n-1} - \beta_2 S_{n-2} - \cdots - \beta_{n-1} S_1 - \beta_n T \end{pmatrix}. \tag{3.1}$$

In order to show that $T = \sum_i X_i$ we multiply (3.1) from the left by an elementary matrix that induces the row operation which replaces R_n (row n) by $R_n + \beta_1 R_{n-1} + \cdots + \beta_{n-1} R_1$. We obtain

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ f(\lambda_1) & f(\lambda_1) & \ldots & f(\lambda_n) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ \vdots \\ S_{n-1} \\ -\beta_n T \end{pmatrix}$$

where $f(z) = p(z) - \beta_n$, so $f(\lambda_i) = -\beta_n$. It follows that $\sum_i X_i = T$.

Next, we notice that $\sum_i \lambda_i^j X_i = S_j$, $1 \leqslant j \leqslant n-1$, so $\sum_i \lambda_i^j X_i K^j = S_j K^j = K^j T = K^j \sum_i X_i$ holds for $1 \leqslant j \leqslant n-1$. In fact, it holds for $j = n$ as well. Indeed, $K^n \sum_i X_i = K^n T = -\beta_1 K^{n-1} TK - \beta_2 K^{n-2} TK^2 - \cdots - \beta_n TK^n = -\beta_1 S_{n-1} K^{n-1} - \beta_2 S_{n-2} K^n - \cdots - \beta_n TK^n$ and it is easy to see that this expression equals $\sum_i \lambda_i^n X_i K^n$, by considering the bottom rows in (3.1). Now, let $1 \leqslant j \leqslant n-1$. Then $K \sum_i \lambda_i^j X_i K^j = K^{j+1} \sum_i X_i = \sum_i \lambda_i^{j+1} X_i K^{j+1}$. Moreover, the equality $K \sum_i \lambda_i^j X_i K^j = \sum \lambda_i^{j+1} X_i K^{j+1}$ is true when $j = 0$, since it becomes $KTK^n = S_1 K^{n-1}$, and $KT = S_1 K$. Using the fact that K has a dense range, these equalities can be organized in a matricial form as

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_n \\ \ldots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} \begin{pmatrix} KX_1 - \lambda_1 X_1 K \\ KX_2 - \lambda_2 X_2 K \\ \\ KX_n - \lambda_n X_n K \end{pmatrix} = 0.$$

In view of the invertibility of the Vandermonde matrix on the left it follows that $KX_i = \lambda_i X_i K$, $1 \leqslant i \leqslant n$, and the theorem is proved. ∎

Once again the assumption that $\lambda_i \neq \lambda_j$ for $i \neq j$ is essential. Namely, Theorem 3.8 can be restated as

$$\bigvee_{i=1}^{n} \text{Ker } F_{\lambda_i} = \text{Ker } (F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}) \cap \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-1}. \tag{3.2}$$

When $\lambda_i = \lambda_j$ for some $i \neq j$ the equality fails. Without loss of generality let $\lambda_{n-1} = \lambda_n$. The left hand side of (3.2) collapses to $\bigvee_{i=1}^{n-1}$ Ker F_{λ_i} which, by Theorem 3.8, equals Ker $(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_{n-1}}) \cap \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-2}$. This is still a subset of the right hand side in (3.2) because Ker $(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_{n-1}}) \subset$ Ker $(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_{n-1}} \circ F_{\lambda_n})$ and, if $T \in$ Ker $(F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_{n-1}}) \cap \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-2}$, then $T \in \mathcal{S}_{n-1}$. Indeed, if $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_{n-1}}(T) = 0$ then, using Proposition 3.7, $K^{n-1}T = (-1) \sum_{i=1}^{n-1} \beta_i K^{n-1-i} T K^i = \sum_{i=1}^{n-1} (-\beta_i) S_{n-1-i} K^{n-1} = \left(\sum_{i=1}^{n-1} (-1\beta_i) S_{n-1-i} \right) K^{n-1}$ so $T \in \mathcal{S}_{n-1}$. However, the following example shows that the inclusion can be proper.

EXAMPLE 3.9. Let $\mathcal{H}$, T, and K be as in Example 3.2. Then $F_\lambda \circ F_\lambda(T) = 0$. Also, if $S = \begin{pmatrix} 0 & K_1 \\ 0 & 0 \end{pmatrix}$ then $KT = SK$ so $T \in \mathcal{S}_1$. Consequently, T belongs to the right side of (3.2). It does not belong to the left side, though, since $F_\lambda(T) \neq 0$.

In fact, a little more can be said. If T belongs to the right side of (3.2) and if T is a sum of extended eigenvectors of K, then the appropriate eigenvalues must be some of those that appear on the right side. In order to prove this we start with a result which may be of independent interest.

PROPOSITION 3.10. *Extended eigenvectors corresponding to different extended eigenvalues of an operator with a dense range are linearly independent.*

Proof. We prove this by induction on n. The case $n = 1$ is obvious. Let the statement be true for $n-1$, and suppose that, to the contrary, $X_n = \sum_{i=1}^{n-1} X_i$, where $KX_i = \lambda_i X_i K$, $1 \leqslant i \leqslant n$. Then $\lambda_n X_n K = KX_n = K \sum_{i=1}^{n-1} X_i = \sum_{i=1}^{n-1} \lambda_i X_i K$, so $\lambda_n X_n = \sum_{i=1}^{n-1} \lambda_i X_i$. Since $X_n = \sum_{i=1}^{n-1} X_i$, we obtain $\sum_{i=1}^{n-1} (\lambda_n - \lambda_i) X_i = 0$, and by the induction hypothesis $\lambda_n = \lambda_i$ which is a contradiction. ∎

Now we can prove the promised fact about the extended eigenvalues.

PROPOSITION 3.11. *Suppose that K is a compact operator with a dense range, $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$, and T is a sum of extended eigenvectors X_i, $1 \leqslant i \leqslant m$,* *corresponding to distinct extended eigenvalues μ_i. Then the set $\{\mu_i\}_{i=1}^m$ is a subset of $\{\lambda_i\}_{i=1}^n$.*

Proof. By Proposition 3.7, $\sum \beta_i K^i T K^{n-i} = 0$ so $\sum \beta_i K^i (\sum X_j) K^{n-i} = 0$. Notice that $K^i X_j = \mu_j^i X_j K^i$ so $\sum\limits_{i,j} \beta_i \mu_j^i X_j K^n = 0$. Since K has a dense range we have that $\sum\limits_{i,j} \beta_i \mu_j^i X_j = 0$. By Proposition 3.10, $\sum\limits_{i} \beta_i \mu_j^i = 0$, $1 \leqslant j \leqslant m$, hence each μ_j is a root of $p(z)$. Consequently, each μ_j is in the set of roots of $p(z)$, $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. ∎

Using the theorem of Douglas in [5] it is easy to see that $T \in \mathcal{S}_n$ if and only if T^* leaves the range of K^{*n} invariant. Thus, these classes are all different. In the presence of the additional condition that T is in the kernel of $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}$ the situation is somewhat different. Namely, we have seen in the discussion following the proof of Theorem 3.8 that, if $T \in \text{Ker}\ (F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}) \cap \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-1}$, then $T \in \mathcal{S}_n$. Nevertheless, the following example shows that it is unlikely that the assumption $T \in \mathcal{S}_1 \cap \cdots \cap \mathcal{S}_{n-1}$ in Theorem 3.8 can be relaxed.

EXAMPLE 3.12. Let $\mathcal{H} = \mathbb{C}^2$, $\mathcal{H}_1 = \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, and $\mathcal{H}_2 = \mathcal{H}_1 \oplus \mathcal{H}_1$. We define operators $A, B, C, D \in \mathcal{L}(\mathcal{H})$ as $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $D = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, the operators $K_1, K_2 \in \mathcal{L}(\mathcal{H}_1)$ as

$$K_1 = \begin{pmatrix} 0 & C & 0 \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $K, T \in \mathcal{L}(\mathcal{H}_2)$ as $K = \left(\begin{smallmatrix} K_1 & 0 \\ 0 & K_2 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. First we notice that, if $Z \in \mathcal{L}(\mathcal{H}_1)$ is defined as

$$Z = \begin{pmatrix} C & 0 & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then $ZK_2 = K_1$ and $T \in \mathcal{S}_1$ with $S_1 = \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$. On the other hand, $K_2^2 = 0$ so $K^2 = \left(\begin{smallmatrix} K_1^2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Therefore, for any $R \in \mathcal{L}(\mathcal{H}_2)$, RK^2 is of the form $\left(\begin{smallmatrix} * & 0 \\ * & 0 \end{smallmatrix}\right)$. Since $K^2 T = \left(\begin{smallmatrix} 0 & K_1^2 \\ 0 & 0 \end{smallmatrix}\right)$ and $K_1^2 \neq 0$ it follows that $T \notin \mathcal{S}_2$. The verification that $T \in \text{Ker}\ (F_\lambda \circ F_\mu \circ F_\nu)$ for any complex numbers λ, μ, ν is based on Proposition 3.7, since $K^3 T = K^2 T K = K T K^2 = T K^3 = 0$.

Notice that the operator K in Example 3.12 is not a quasi-affinity. We leave open the question whether the presence of this additional hypothesis might force some of the classes $\mathcal{S}_n$ to be equal.

PROBLEM 3.13. Is it true that, if K is a compact quasi-affinity, $F_\lambda \circ F_\mu \circ F_\nu(T) = 0$, and $T \in \mathcal{S}_1$ implies $T \in \mathcal{S}_2$?

Finally, we turn our attention to the assumption that K is a quasi-affinity in Theorem 3.8. We chose to state it this way, but it is clear from the proof that we only need the range of K to be dense in $\mathcal{H}$. When the range is not dense, the situation is quite different. For example, if $\widetilde{K} = K \oplus 0$ and $\widetilde{T} = 0 \oplus T$ then $\widetilde{K}\widetilde{T} = \lambda\widetilde{T}\widetilde{K}(= 0)$ for any λ. Thus, one extended eigenvector may correspond to more than one extended eigenvalue. Since we were unable to settle this case, we leave it open.

PROBLEM 3.14. Does Theorem 3.8 remain true if the range of K is not dense?

Based on Theorem 3.8, it seems likely that there is an operator T such that $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$, but T is not a sum of extended eigenvectors of K. However we were unable to provide an example that would illustrate such a phenomenon. Thus, we leave open the following question.

PROBLEM 3.15. Are there a compact quasi-affinity K, distinct complex numbers $\lambda_1, \lambda_2, \dots, \lambda_n$, and an operator T that satisfies $F_{\lambda_1} \circ F_{\lambda_2} \circ \cdots \circ F_{\lambda_n}(T) = 0$ without being a sum of extended eigenvectors of K?

Acknowledgements. The author would like to express gratitude to the University of North Carolina Charlotte for its hospitality and partial support during the final stages of the work on this paper.

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Received March 20, 2006.