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REMOVING SOURCES FROM HIGHER-RANK GRAPHS

CYNTHIA FARTHING

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ABSTRACT. For a higher-rank graph Λ with sources we detail a construction that creates a higher-rank graph $\overline{\Lambda}$ that does not have sources and contains Λ as a subgraph. Furthermore, when Λ is row-finite the Cuntz-Krieger algebra of Λ, $C^*(\Lambda)$ is a full corner of $C^*(\overline{\Lambda})$, the Cuntz-Krieger algebra of $\overline{\Lambda}$.

KEYWORDS: *C^*-algebras, higher-rank graphs, desingularization.*

MSC (2000): 46L05.

INTRODUCTION

Higher-rank graphs are generalizations of directed graphs that were introduced by Kumjian and Pask in [7] who were motivated by the C^*-algebras of buildings that were studied by Robertson and Steger in [15], [16], [17]. In this paper, we extend a higher-rank graph with sources to another higher-rank graph that has no sources. We will do this in such a way that the C^*-algebras of the graphs are strongly Morita equivalent, thereby removing one of the technical difficulties encountered when working with higher-rank graphs.

A higher-rank graph can be viewed as a union of k directed graphs with the same vertex set, where the edges of the different graphs are painted with k different colors. A higher-rank graph also includes a *factorization property* that dictates how the edges of different colors fit together to form paths. More precisely, a higher-rank graph Λ is a countable category together with a degree functor $d : \Lambda \to \mathbb{N}^k$ which satisfies the factorization property: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. The rank of Λ is k, and therefore, Λ is also called a k-graph. The C^*-algebras of higher-rank graphs include the C^*-algebras associated to directed graphs which have been the focus of much attention in recent years. (See [10] for a detailed account of graph C^*-algebras. We will use the conventions established in [10] when discussing directed graphs.)

The development of the C^*-algebras of higher-rank graphs has progressed in a manner similar to that of the C^*-algebras associated with directed graphs. The C^*-algebras of directed graphs were first defined in terms of groupoids [8]. Next, in [2], the graph C^*-algebra is realized as the universal C^*-algebra generated by a collection of projections and partial isometries satisfying certain relations. Both of these methods required that the directed graphs be *row-finite*, that is, each vertex has finitely many edges pointing toward it. The groupoid techniques also required that the directed graphs did not have any *sources*. (A source is a vertex that does not have any edges pointing toward it.) In [6], the C^*-algebra of an arbitrary directed graph was defined as a universal C^*-algebra. Using a method similar to that used in [8] for directed graphs, Kumjian and Pask realized the C^*-algebra of a higher-rank graph to be the C^*-algebra of a groupoid associated to the higher-rank graph. Therefore, they also required that the higher-rank graphs be row-finite and have no sources (Definitions 1.3 and 1.4). Raeburn, Sims and Yeend in [11] defined, in a universal way, the C^*-algebras for a class of higher-rank graphs known as *locally convex* k-graphs. Later, they extended their definition to include the C^*-algebras of *finitely aligned* k-graphs in [12]. Finitely aligned k-graphs allow for vertices to receive infinitely many edges and appear to be the most general class of k-graphs to which a C^*-algebra can be associated.

One of the main accomplishments of Drinen and Tomforde in [3] is the development of the method known as *desingularization*. If E is a directed graph, possibly with sources and possibly not row-finite, a desingularization of F is a row-finite directed graph without sources that is obtained from E. Furthermore, the C^*-algebras associated with E and F, $C^*(E)$ and $C^*(F)$, respectively, are Morita equivalent. Therefore, when studying the C^*-algebras associated to directed graphs, it usually suffices to consider directed graphs that are row-finite and have no sources. The desingularization method, in addition to providing easier proofs for the uniqueness theorems of graph C^*-algebras, also led to the description of the ideal structure of graph algebras. (See also [1].)

The construction detailed in this paper, which "removes sources" from a higher-rank graph, will have similar effects on the study of higher-rank graph C^*-algebras. First of all, by transforming an arbitrary row-finite higher-rank graph into a locally convex graph, we will be able to use the Cuntz-Krieger relations from Definition 3.3 in [11] which are much simpler than those used to define the algebras of finitely aligned k-graphs (Definition 1.9). Also, the construction given here may allow for some of the results that exist for the C^*-algebras of row-finite higher-rank graphs without sources to be extended to more general higher-rank graph C^*-algebras. For example, in [4], Evans completely describes the K-theory of the C^*-algebras associated to row-finite k-graphs without sources when $k = 2$ and obtains some partial results for $k \geqslant 3$. Robertson and Sims give necessary and sufficient conditions describing when the C^*-algebra corresponding to a row-finite k-graph without sources is simple in [14]. Since ideal structure and K-theory is preserved under Morita equivalence, it is expected that these results will hold in the more general setting.

Our goal is to produce a desingularization method for higher-rank graphs that is analogous to the process used for directed graphs. If a vertex v is a source in a directed graph E, then the desingularization process "adds a head to v". This means we attach a graph of the form

$$v \xleftarrow{e_{v_1}} v_1 \xleftarrow{e_{v_2}} v_2 \xleftarrow{e_{v_3}} v_3 \longleftarrow \cdots \longleftarrow v_{n-1} \xleftarrow{e_{v_n}} v_n \longleftarrow \cdots$$

to v. This method was used by Bates, et. al. in [2] as well as by Drinen and Tomforde in [3].

In a directed graph, adding an edge to a vertex automatically creates another directed graph. Therefore, dealing with sources in a directed graph is a local problem. However, in a higher-rank graph, adding an edge of some degree to one vertex will require that several edges of different degrees be added to other vertices to ensure that the factorization property still holds. Hence, adding edges to a vertex in a higher-rank graph is a global issue. The method we develop here uses the so-called boundary paths of a higher-rank graph to identify the sources and then extends those boundary paths in the necessary directions.

This paper is designed as follows. In Section 2, we define the terminology necessary to discuss the C^*-algebra of a finitely aligned k-graph. In Section 3, given a row-finite higher-rank graph Λ, we construct a row-finite higher-rank graph $\overline{\Lambda}$ that is source free. We show that the C^*-algebra of the original k-graph sits naturally inside the C^*-algebra of the extended k-graph as a full corner. Section 4 includes examples of 2-graphs with sources and how they are extended to graphs without sources using the method in this paper.

1. PRELIMINARIES

DEFINITION 1.1. Given $k \in \mathbb{N}$, a *k-graph* (Λ, d) is a countable category Λ together with a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree functor*, which satisfies the *factorization property*: for every $\lambda \in \mathrm{Mor}(\Lambda)$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \mathrm{Mor}(\Lambda)$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$.

The factorization property enables the objects of Λ to be identified with the morphisms with degree zero. Therefore, we can talk only about the morphisms of Λ. We will write $\lambda \in \Lambda$ instead of $\lambda \in \mathrm{Mor}(\Lambda)$.

We will use the following notation throughout this paper. For $n \in \mathbb{N}^k$, let

$$\Lambda^n = \{\lambda \in \Lambda : d(\lambda) = n\}.$$

Thus, $\mathrm{Obj}(\Lambda)$ is identified with Λ^0. For $E \subseteq \Lambda$ and $\lambda \in \Lambda$, define

$$\lambda E = \{\lambda\mu : \mu \in E, r(\mu) = s(\lambda)\} \quad \text{and} \quad E\lambda = \{\mu\lambda : \mu \in E, s(\mu) = r(\lambda)\}.$$

We will use $e_1, e_2, \dots, e_k$ to denote the usual basis for $\mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we denote by $m \vee n$ the coordinate-wise maximum and the coordinate-wise minimum by $m \wedge n$. For $m, n \in \mathbb{N}^k$, $m \vee n$ is the least element in $\mathbb{N}^k$ that is greater than or equal to both m and n, and $m \wedge n$ is the greatest element in $\mathbb{N}^k$ that is less than or equal to both m and n.

We will use the convention that $\vee$ and $\wedge$ precede addition and subtraction in the order of operations; thus $m + n \wedge p = m + (n \wedge p)$ for $m, n, p \in \mathbb{N}^k$. For $m, n, p \in \mathbb{N}^k$, it is straightforward to show that $(m + p) \wedge (n + p) = (m \wedge n) + p$ and $(m + p) \vee (n + p) = (m \vee n) + p$.

Given a path $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ satisfying $0 \leqslant m \leqslant n \leqslant d(\lambda)$, the factorization property guarantees that there are unique paths λ_i, $i = 1, 2, 3$, such that $d(\lambda_1) = m$, $d(\lambda_2) = n - m$, $d(\lambda_3) = d(\lambda) - n$ and $\lambda = \lambda_1 \lambda_2 \lambda_3$. We shall write $\lambda(0, m)$ for λ_1, $\lambda(m, n)$ for λ_2 and $\lambda(n, d(\lambda))$ for λ_3.

EXAMPLES 1.2. (i) Let $E = (E^0, E^1, r, s)$ be a directed graph. Let E^* denote the category generated freely over all finite paths. Let $l : E^* \to \mathbb{N}$ give the length of a path. Then (E^*, l) is a 1-graph.

(ii) For $m \in (\mathbb{N} \cup \{\infty\})^k$, define $\Omega_{k,m}$ to be the k-graph with

$$\begin{aligned}
&\mathrm{Obj}(\Omega_{k,m}) = \{p \in \mathbb{N}^k : p \leqslant m\}, \\
&\mathrm{Mor}(\Omega_{k,m}) = \{(p, q) \in \mathrm{Obj}(\Omega_{k,m}) \times \mathrm{Obj}(\Omega_{k,m}) : p \leqslant q\}, \\
&r(p, q) = p, \quad s(p, q) = q, \quad d(p, q) = q - p.
\end{aligned}$$

Drawn below are $\Omega_{2,(\infty,\infty)}$ and $\Omega_{2,(1,2)}$. In the diagrams, edges of degree $(1, 0)$ are solid; edges of degree $(0, 1)$ are dashed. In each diagram $\lambda = ((0, 2), (1, 2))$ while $\mu = ((0, 0), (0, 1))$.

$$\begin{tikzcd}
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
\bullet \arrow[d, dashed] & \bullet \arrow[l, "\lambda"'] \arrow[d, dashed] & \bullet^{(2,2)} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
\bullet \arrow[d, dashed, "\mu"'] & \bullet \arrow[l] \arrow[d, dashed] & \bullet \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
(0,0)\bullet & \bullet \arrow[l] & \bullet \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
\qquad
\begin{tikzcd}
(0,2) \arrow[d, dashed] & (1,2) \arrow[l, "\lambda"'] \arrow[d, dashed] \\
(0,1) \arrow[d, dashed, "\mu"'] & (1,1) \arrow[l] \arrow[d, dashed] \\
(0,0) & (1,0) \arrow[l]
\end{tikzcd}$$

$\Omega_{2,(\infty,\infty)}$ $\qquad$ $\Omega_{2,(1,2)}$

The path $((0, 0), (1, 2))$ in either of the above graphs is viewed as the 1×2 rectangle from $(0, 0)$ to $(1, 2)$. The factorization property means that any of the ways that one can connect $(1, 2)$ to $(0, 0)$ using the segments shown are, in fact, the same path.

DEFINITION 1.3. A k-graph (Λ, d) is *row-finite* if $v\Lambda^n$ is at most finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

DEFINITION 1.4. A vertex $v \in \Lambda^0$ is a *source* if $v\Lambda^n = \emptyset$ for some $n \in \mathbb{N}^k$.

In Example 1.2 (ii), every vertex in $\Omega_{2,(1,2)}$ is a source. On the other hand, the 2-graph $\Omega_{2,(\infty,\infty)}$ has no sources. The "removing sources" construction in this paper will extend $\Omega_{2,(1,2)}$ to $\Omega_{2,(\infty,\infty)}$.

DEFINITION 1.5. For $\lambda, \mu \in \Lambda$, if there exist $\alpha, \beta \in \Lambda$ such that $\lambda\alpha = \mu\beta$ and $d(\lambda\alpha) = d(\lambda) \vee d(\mu)$, then $\lambda\alpha$ is called a *minimal common extension of* λ *and* μ. Define $\Lambda^{\min}(\lambda, \mu)$ to be the set

$$\Lambda^{\min}(\lambda, \mu) = \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}.$$

DEFINITION 1.6. A k-graph (Λ, d) is *finitely aligned* if $\Lambda^{\min}(\lambda, \mu)$ is at most finite for all $\lambda, \mu \in \Lambda$.

Definitions 1.4 and 1.6 highlight some key differences between 1-graphs and k-graphs for $k \geqslant 2$. First of all, in the directed graph setting, a source is a vertex v for which $v\Lambda^1 = \emptyset$, or equivalently, if $v\Lambda = \{v\}$. However, a vertex in a 1-graph is a source in the sense of Definition 1.4 if there exists a path $\lambda \in v\Lambda$ such that $s(\lambda)\Lambda = \{s(\lambda)\}$. This is not the case for arbitrary k-graphs. Consider the graph $\Omega_{2,(\infty,1)}$ drawn here:

$$\begin{tikzcd}
(0,1) \arrow[d, dashed] & (1,1) \arrow[l] \arrow[d, dashed] & (2,1) \arrow[l] \arrow[d, dashed] & (3,1) \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
(0,0) & (1,0) \arrow[l] & (2,0) \arrow[l] & (3,0) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Each of the vertices $(m, 1)$, $m \in \mathbb{N}$ is a source since $(m, 1)\Omega^{e_2}_{2,(\infty,1)} = \emptyset$. However, there is no vertex $v \in \Omega^0_{2,(\infty,1)}$ with $v\Omega_{2,(\infty,1)} = \{v\}$. The difference is that in a k-graph for $k \geqslant 2$, vertices can be sources in some directions, but not in all. Secondly, if Λ is a 1-graph and $\lambda, \mu \in \Lambda$, the only way two paths can have a minimal common extension is if one path is a subpath of the other. Therefore, the set $\Lambda^{\min}(\lambda, \mu)$ is either empty or a singleton. Consequently, any 1-graph is finitely aligned.

DEFINITION 1.7. Let (Λ, d) be a k-graph; let $v \in \Lambda^0$ and $E \subset v\Lambda$. We say that E is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all *finite exhaustive subsets* of Λ by $\mathcal{FE}(\Lambda)$.

EXAMPLES 1.8. (i) For all $m \in (\mathbb{N} \cup \{\infty\})^k$ and $v \in \Omega^0_{k,m}$, any finite (non-empty) subset of $v\Omega_{k,m}$ is a finite exhaustive set.

(ii) Consider the k-graph Λ below:

$$\begin{tikzcd}
\bullet & \bullet \arrow[l] \arrow[d, dashed, "\gamma"] & & \\
\bullet \arrow[u, leftarrow, dashed, "\alpha"] \arrow[d, dashed, "\lambda"'] & u \arrow[l] \arrow[d, dashed] & \bullet \arrow[l, "\eta"'] \arrow[d, dashed] & \\
v & \bullet \arrow[l, "\mu"] & w \arrow[l, "\beta"] & \bullet \arrow[l, Rightarrow, "\xi_i, i\in\mathbb{N}"]
\end{tikzcd}$$

Dashed edges represent edges of degree $(0,1)$ and solid edges represent edges of degree $(1,0)$. The edges ξ_i where $i \in \mathbb{N}$ each have degree $(1,0)$. Any finite exhaustive subset of $w\Lambda$ must contain w. The set $\{\mu\}$ is a finite exhaustive subset of $v\Lambda$, whereas $\{\lambda\}$ is not because $\Lambda^{\min}(\lambda, \mu\beta\xi_i) = \emptyset$ for any $i \in \mathbb{N}$.

DEFINITION 1.9. Let (Λ, d) be a finitely aligned k-graph. A *Toeplitz-Cuntz-Krieger Λ-family* in a C^*-algebra B is a family of partial isometries $\{t_\lambda : \lambda \in \Lambda\}$ satisfying the *Toeplitz-Cuntz-Krieger relations*:

(TCK1) $\{t_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
(TCK2) $t_{\lambda\mu} = t_\lambda t_\mu$ for all $\lambda, \mu \in \Lambda$ with $s(\lambda) = r(\mu)$;
(TCK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$ for all $\lambda, \mu \in \Lambda$.

A *Cuntz-Krieger Λ-family* in a C^*-algebra B is a Toeplitz-Cuntz-Krieger Λ-family that also satisfies

(CK) $\prod_{\lambda\in E} (t_v - t_\lambda t_\lambda^*) = 0$ for all $v \in \Lambda^0$ and $E \in v\,\mathcal{FE}(\Lambda)$.

Of course, the hypothesis that (Λ, d) is finitely aligned guarantees that the sums in Definition 1.9 are finite sums and hence make sense in any C^*-algebra.

DEFINITION 1.10. Let (Λ, d) be a finitely aligned k-graph. The *C^*-algebra of Λ*, denoted $C^*(\Lambda)$, is the C^*-algebra generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$ which is universal if the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in a C^*-algebra B, then there exists a C^*-homomorphism $\pi : C^*(\Lambda) \to B$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$.

We also call $C^*(\Lambda)$ the Cuntz-Krieger algebra of Λ.

DEFINITION 1.11. A C^*-subalgebra B of a C^*-algebra A is called a *corner* if there is a projection p in $M(A)$ the multiplier algebra of A, such that $B = pAp$. A corner is called *full* if it is not contained in any proper closed two-sided ideal.

Two C^*-algebras A and B are *Morita equivalent* if there exists an $A - B$ imprimitivity bimodule X. If B is a full corner of A, then one can show that an $A - B$ imprimitivity bimodule exists, and hence the two algebras are Morita equivalent. Many C^*-algebraic properties are invariant under Morita equivalence, including ideal structure (simplicity, in particular), being AF, pure infiniteness, real rank zero, stable rank one, primitive ideal space, K-theory, Ext, KK-theory and E theory. For this reason, the notion of Morita equivalence is extremely useful in classifying C^*-algebras. See Chapter 3 of [13] for more details.

DEFINITION 1.12. Let (Λ_1, d_1) and (Λ_2, d_2) be k-graphs. A *graph morphism* is a functor $F : \Lambda_1 \to \Lambda_2$ such that $d_2(F(\lambda)) = d_1(\lambda)$ for all $\lambda \in \Lambda$.

DEFINITION 1.13. Let (Λ, d) be a k-graph. Define the *path space of* Λ to be the set

$$X_\Lambda = \{x : \Omega_{k,m} \to \Lambda : m \in (\mathbb{N} \cup \{\infty\})^k \text{ and } x \text{ is a graph morphism}\}.$$

We extend the range and degree maps of Λ to X_Λ by defining, for $x : \Omega_{k,m} \to \Lambda$, $r(x) = x(0)$ and $d(x) = m$.

REMARKS 1.14. (i) The factorization property of k-graphs implies that each $x \in X_\Lambda$ is completely determined by $\{x(0,p) : p \leqslant m\}$: if $l \leqslant n \leqslant p$ and $x(0,p) = \lambda_p$, then $x(l,n) = \lambda_p(l,n)$. If $m_i < \infty$ for all $i \in \{1, 2, \ldots, k\}$, then $x(0,m)$ completely determines x.

(ii) For each $\lambda \in \Lambda$, define $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ by $x_\lambda(0, d(\lambda)) = \lambda$. Then the map $\lambda \mapsto x_\lambda$ embeds Λ in X_Λ.

Let $x : \Omega_{k,m} \to \Lambda$ be a graph morphism. For $p \leqslant m$, let $\sigma^p x : \Omega_{k,m-p} \to \Lambda$ be defined by $\sigma^p x(a,b) = x(a+p, b+p)$ for $a, b \in \mathbb{N}^k$ such that $a \leqslant b \leqslant m - p$. If λ is a path such that $s(\lambda) = x(0)$ define $\lambda x : \Omega_{k,m+d(\lambda)} \to \Lambda$ by $(\lambda x)(0, d(\lambda)) = \lambda$ and $(\lambda x)(0,p) = \lambda x(0, p - d(\lambda))$ for $p \in \mathbb{N}^k$ such that $d(\lambda) \leqslant p \leqslant d(x) + d(\lambda)$.

DEFINITION 1.15. A k-graph (Λ, d) is *locally convex* if whenever $\lambda \in v\Lambda^{e_i}$ and $\mu \in v\Lambda^{e_j}$ for some $v \in \Lambda^0$ and $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, there exists $\xi \in s(\lambda)\Lambda^{e_j}$ and $\eta \in s(\mu)\Lambda^{e_i}$.

DEFINITION 1.16. Let (Λ, d) be a k-graph. For $q \in \mathbb{N}^k$, define

$$\Lambda^{\leqslant q} = \{\lambda \in \Lambda : d(\lambda) \leqslant q, \text{ and } s(\lambda)\Lambda^{e_i} = \emptyset \text{ when } d(\lambda) + e_i \leqslant q\}.$$

EXAMPLES 1.17. (i) For any $m \in (\mathbb{N} \cup \{\infty\})^k$, $\Omega_{k,m}$ is locally convex. More generally, if Λ has no sources, then Λ is locally convex since $v\Lambda^{e_i}$ is nonempty for all $v \in \Lambda^0$ and $i \in \{1, 2, \ldots, k\}$.

(ii) The 2-graph in Example 1.8 (ii) is not locally convex. For the vertex u, we have $\eta \in u\Lambda^{e_1}$ and $\gamma \in u\Lambda^{e_2}$. However, $s(\eta)\Lambda^{e_2}$ and $s(\gamma)\Lambda^{e_1}$ are both empty.

REMARK 1.18. Condition (CK) of Definition 1.9 replaced earlier versions of the Cuntz-Krieger condition that were used for row-finite k-graphs with no sources [7] and for locally convex k-graphs [11]. The condition from [11] is

$$\text{(CK}'\text{)} \quad t_v = \sum_{\lambda \in \Lambda^{\leqslant m}} t_\lambda t_\lambda^* \text{ for all } v \in \Lambda^0 \text{ and } m \in \mathbb{N}^k.$$

It is shown in Appendix B of [12] that the conditions in Definition 1.9 are equivalent to those in [11] when the k-graph is locally convex.

DEFINITION 1.19. Let (Λ, d) be a k-graph; let $x : \Omega_{k,m} \to \Lambda$ be a graph morphism. Then x is a *boundary path* if there exists $n_x \in \mathbb{N}^k$ such that $n_x \leqslant m$ and for $p \in \mathbb{N}^k$

$$(n_x \leqslant p \leqslant m, \text{ and } p_i = m_i) \Rightarrow x(p)\Lambda^{e_i} = \emptyset.$$

We write $\Lambda^{\leqslant\infty}$ for the collection of all boundary paths of Λ.

Boundary paths are essential to the construction detailed in the next section. Examples of boundary paths are given in Section 3. We will use the following results about boundary paths.

LEMMA 1.20. *Let Λ be a finitely aligned k-graph.*

(i) *(*[12], *Lemma* 2.10*) If $x \in \Lambda^{\leqslant\infty}$, then $\sigma^p x$ and λx are elements of $\Lambda^{\leqslant\infty}$ for any $p \leqslant d(x)$ and $\lambda \in x(0)\Lambda$.*

(ii) *(*[12], *Lemma* 2.11*) For any $v \in \Lambda^0$, the set $v\Lambda^{\leqslant\infty}$ is nonempty.*

2. REMOVING SOURCES

In this section, we will develop a method that extends a k-graph with sources, named Λ, to a k-graph without sources, $\overline{\Lambda}$. When Λ is row-finite, $C^*(\Lambda)$ is Morita equivalent to $C^*(\overline{\Lambda})$. The following theorem is the goal of this section.

THEOREM 2.1. *Let (Λ, d) be a row-finite k-graph. Then there exists a row-finite k-graph $(\overline{\Lambda}, \overline{d})$ without sources and an isomorphism ι of Λ onto a subgraph of $\overline{\Lambda}$ such that the C^*-subalgebra of $C^*(\overline{\Lambda})$ generated by $\{s_\lambda : \lambda \in \iota\Lambda\}$ is a full corner of $C^*(\overline{\Lambda})$ and is canonically isomorphic to $C^*(\Lambda)$.*

We will spend the rest of the section constructing $\overline{\Lambda}$ and proving Theorem 2.1. We begin by defining two equivalence relations $\sim$ and $\approx$. The equivalence classes given by $\sim$ correspond to the paths that will be added to Λ, and the equivalence classes of $\approx$ correspond to the new vertices.

DEFINITION 2.2. Let $V_\Lambda = \{(x; m) : x \in \Lambda^{\leqslant\infty} \text{ and } m \nleqslant d(x)\}$.

The set V_Λ extends each element of $\Lambda^{\leqslant\infty}$ in the proper directions. Notice that the set V_Λ is disjoint from Λ^0 because every vertex in Λ can be written as $x(m)$ for some $x \in \Lambda^{\leqslant\infty}$ and $m \leqslant d(x)$. However, extending each boundary path separately adds many more vertices to Λ than necessary because boundary paths can overlap. An example of such overlap would occur when x and y are paths in $\Lambda^{\leqslant\infty}$ such that $y = \sigma^p x$ for some $p \leqslant d(x)$. To take possible overlap into account, we define the following relation on V_Λ.

DEFINITION 2.3. Define a relation $\approx$ on V_Λ by: $(x; m) \approx (y; p)$ if

(V1) $x(m \wedge d(x)) = y(p \wedge d(y))$;

(V2) $m - m \wedge d(x) = p - p \wedge d(y)$.

Condition (V1) ensures that two new vertices are related if they project down onto the same vertex in Λ. Condition (V2) relates two vertices in V_Λ if they are the same "distance" from Λ.

The proof of the next proposition is clear.

PROPOSITION 2.4. *The relation $\approx$ on V_Λ is an equivalence relation.*

DEFINITION 2.5. Let $P_\Lambda = \{(x;(m,n)) : x \in \Lambda^{\leqslant\infty}, n \not\leqslant d(x), \text{ and } m \leqslant n\}$.

Recall the definition of $\Omega_{k,m}$ in Example 1.2 (ii) where paths were denoted by pairs of vertices. Definition 2.5 uses an analogous way to describe the paths that extend the original k-graph. Since in Definition 2.5 $n \not\leqslant d(x)$ but m may or may not be less than or equal to $d(x)$, we are requiring that the additional paths start (have source) outside of the original k-graph but may or may not end (have range) in the original k-graph. Again, the elements of P_Λ are paths extending each boundary path, and therefore, the overlapping of boundary paths must be taken into account.

DEFINITION 2.6. Define a relation $\sim$ on P_Λ by $(x;(m,n)) \sim (y;(p.q))$ if

(P1) $x(m \wedge d(x), n \wedge d(x)) = y(p \wedge d(y), q \wedge d(y))$;
(P2) $m - m \wedge d(x) = p - p \wedge d(y)$;
(P3) $n - m = q - p$.

PROPOSITION 2.7. *The relation $\sim$ on P_Λ is an equivalence relation.*

Let $\widetilde{P}_\Lambda = P_\Lambda/\sim$ and $\widetilde{V}_\Lambda = V_\Lambda/\approx$. The equivalence classes of $\widetilde{P}_\Lambda$ will be denoted $[x;(m,n)]$, and the equivalence classes of $\widetilde{V}_\Lambda$ will be denoted $[x;m]$.

As mentioned earlier, our goal is to define a new category $\overline{\Lambda}$ that extends Λ. The elements of $\widetilde{V}_\Lambda$ will become the additional objects joined to Λ, and the new morphisms will be the elements of $\widetilde{P}_\Lambda$. We now proceed by defining the range and source maps as well as the composition ($\circ$) and identity (id) functions on $\widetilde{P}_\Lambda$ that will be used to define the new category.

DEFINITION 2.8. Define $\widetilde{r} : \widetilde{P}_\Lambda \to (\widetilde{V}_\Lambda \cup \Lambda^0)$ and $\widetilde{s} : \widetilde{P}_\Lambda \to \widetilde{V}_\Lambda$ as follows:

$$\widetilde{r}([x;(m,n)]) = \begin{cases} x(m) & \text{if } m \leqslant d(x), \\ [x;m] & \text{if } m \not\leqslant d(x); \end{cases} \quad \widetilde{s}([x;(m,n)]) = [x;n].$$

PROPOSITION 2.9. *The maps $\widetilde{r}$ and $\widetilde{s}$ are well defined.*

Proof. Suppose $(x;(m,n)) \sim (y;(p,q))$. Then (P1) of Definition 2.6 implies that $n \wedge d(x) - m \wedge d(x) = q \wedge d(y) - p \wedge d(y)$. Subtracting this from the equation in (P3) gives

$$\begin{aligned} & n - m + m \wedge d(x) - n \wedge d(x) = q - p + p \wedge d(y) - q \wedge d(y) \\ \Leftrightarrow\ & n - n \wedge d(x) - (m - m \wedge d(x)) = q - q \wedge d(y) - (p - p \wedge d(y)) \\ \Leftrightarrow\ & n - n \wedge d(x) = q - q \wedge d(y) \quad \text{using (P2).} \end{aligned}$$

Since (P1) gives $x(n \wedge d(x)) = y(q \wedge d(y))$, it follows that $(x;n) \approx (y;q)$. Therefore, $\widetilde{s}$ is well defined.

To show $\widetilde{r}$ is well-defined, first consider the case where $m \leqslant d(x)$. Then $m \wedge d(x) = m$. Therefore, $m - m \wedge d(x) = 0$, and (P2) implies that $p \wedge d(y) = p$. Hence, $x(m) = y(p)$ by (P1).

If $m \nleqslant d(x)$, then (P1) of Definition 2.6 implies $x(m \wedge d(x)) = y(p \wedge d(y))$, and thus condition (V1) of Definition 2.3 is satisfied. Condition (P2) of Definition 2.6 is precisely (V2) of Definition 2.3. Therefore, $(x;m) \approx (y;p)$, and $\widetilde{r}$ is well defined. ∎

PROPOSITION 2.10. *Suppose $x, y, \in \Lambda^{\leqslant\infty}$, and suppose $p, q \in \mathbb{N}^k$ are such that $p \leqslant d(x)$, $q \leqslant d(y)$ and $\sigma^p x = \sigma^q y$. For all $a, b \in \mathbb{N}^k$, if $a \leqslant b$ and $b + p \nleqslant d(x)$, then $b + q \nleqslant d(y)$ and $[x;(a+p, b+p)] = [y;(a+q, b+q)]$.*

Proof. By definition, $d(\sigma^p x) = d(x) - p$ and $d(\sigma^q y) = d(y) - q$. Therefore

$$d(x) = d(\sigma^p x) + p \quad \text{and} \quad d(y) = d(\sigma^q y) + q. \tag{2.1}$$

Suppose $a, b \in \mathbb{N}^k$ are such that $a \leqslant b$ and $b + p \nleqslant d(x)$. Then

$$\begin{aligned} b + p \nleqslant d(x) \Leftrightarrow b + p \nleqslant d(\sigma^p x) + p \Leftrightarrow b \nleqslant d(\sigma^p x) \Leftrightarrow b \nleqslant d(\sigma^q y) \\ \Leftrightarrow b \nleqslant d(y) - q \Leftrightarrow b + q \nleqslant d(y). \end{aligned}$$

Thus $[x;(a+p, b+p)]$ and $[y;(a+q, b+q)]$ are elements in $\widetilde{P}_\Lambda$. To show that $[x;(a+p, b+p)] = [y;(a+q, b+q)]$, consider

$$\begin{aligned} &x((a+p) \wedge d(x), (b+p) \wedge d(x)) \\ &\quad = x(a \wedge d(\sigma^p x) + p, b \wedge d(\sigma^p x) + p) = \sigma^p x(a \wedge d(\sigma^p x), b \wedge d(\sigma^p x)) \\ &\quad = \sigma^q y(a \wedge d(\sigma^q y), b \wedge d(\sigma^q y)) = y((a+q) \wedge d(y), (b+q) \wedge d(y)). \end{aligned}$$

Thus condition (P1) of Definition 2.6 is satisfied. To show condition (P2), we have

$$\begin{aligned} a + p - (a+p) \wedge d(x) = a + p - (a \wedge d(\sigma^p x) + p) = a - a \wedge d(\sigma^p x) \\ = a - a \wedge d(\sigma^q y) = a + q - (a+q) \wedge d(y). \end{aligned}$$

Condition (P3) is clear. Hence, $[x;(a+p, b+p)] = [y;(a+q, b+q)]$. ∎

If $p = 0$, then $x = \sigma^q y$, and Proposition 2.10 implies that for all $b \nleqslant d(x)$, we have $[x;(a,b)] = [y;(a+q, b+q)]$.

The following proposition will be used to compose two paths in $\widetilde{P}_\Lambda$.

PROPOSITION 2.11. *Let $[x;(m,n)], [y;(p,q)] \in \widetilde{P}_\Lambda$ be such that $[x;n] = [y;p]$. Define $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Then*

(i) *$z \in \Lambda^{\leqslant\infty}$;*

(ii) *$m \wedge d(x) = m \wedge d(z)$ and $n \wedge d(x) = n \wedge d(z)$;*

(iii) *$x(m \wedge d(x), n \wedge d(x)) = z(m \wedge d(z), n \wedge d(z))$ and $y(p \wedge d(y), q \wedge d(y)) = z(n \wedge d(z), (n+q-p) \wedge d(z))$.*

Proof. (i) Since $y \in \Lambda^{\leqslant\infty}$, the path z belongs to $\Lambda^{\leqslant\infty}$ by Lemmas 2.10 and 2.11 of [12].

(ii) We will show that $m \wedge d(x) = m \wedge d(z)$ and $n \wedge d(x) = n \wedge d(z)$ on a coordinate by coordinate basis. Let $i \in \{1, 2, \ldots, k\}$. Since $[x; n] = [y; p]$, it follows that $n - (n \wedge d(x)) = p - (p \wedge d(y))$. Therefore,

$$d(z) = n \wedge d(x) + d(y) - p \wedge d(y) = d(y) + n - p.$$

Furthermore, since $n - n \wedge d(x) = p - p \wedge d(y)$, $n_i \leqslant d(x)_i$ if and only if $p_i \leqslant d(y)_i$.

Case 1. Suppose $d(y)_i = \infty$. Then $p_i < d(y)_i$, and so $m_i \leqslant n_i \leqslant d(x)_i$. Moreover, $d(z)_i = \infty$ by definition, so $(m \wedge d(x))_i = m_i = (m \wedge d(z))_i$ and $(n \wedge d(x))_i = n_i = (n \wedge d(z))_i$.

Case 2. Suppose $d(y)_i < \infty$. We have

$$d(z)_i = d(y)_i + n_i - p_i = d(y)_i + (n \wedge d(x))_i - (p \wedge d(y))_i < \infty.$$

Suppose $p_i \leqslant d(y)_i$. Then, as before, $m_i \leqslant n_i \leqslant d(x)_i$. Also $d(y)_i - p_i \geqslant 0$. This implies $m_i \leqslant n_i \leqslant n_i + d(y)_i - p_i = d(z)_i$. Thus $(m \wedge d(x))_i = m_i = (m \wedge d(z))_i$ and $(n \wedge d(x))_i = n_i = (n \wedge d(z))_i$.

If instead $p_i > d(y)_i$, then $n_i > d(x)_i$ as well, and in this case

$$d(z)_i = (n \wedge d(x))_i + d(y)_i - (p \wedge d(y))_i = d(x)_i + d(y)_i - d(y)_i = d(x)_i.$$

Consequently $(m \wedge d(x))_i = (m \wedge d(z))_i$ and $(n \wedge d(x))_i = (n \wedge d(z))_i$.

So in either case, we have that both $(m \wedge d(x))_i = (m \wedge d(z))_i$ and $(n \wedge d(x))_i = (n \wedge d(z))_i$. Since i was arbitrarily chosen, this proves (ii).

(iii) Notice that (ii) implies that $m - m \wedge d(x) = m - m \wedge d(z)$ and that

$$z(m \wedge d(z), n \wedge d(z)) = z(m \wedge d(x), n \wedge d(x)) = x(m \wedge d(x), n \wedge d(x))$$

because $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Thus $[x; (m, n)] = [z; (m, n)]$.

To show $[z; (n, n + q - p)] = [y; (p, q)]$, we have that $\sigma^{n \wedge d(x)} z = \sigma^{p \wedge d(y)} y$. By (ii), we have $n \wedge d(z) = n \wedge d(x)$, and since $[x; n] = [y; p]$, it follows that $n - n \wedge d(z) = p - p \wedge d(y)$. Then

$$\begin{aligned}
&[z; (n, n + q - p)] \\
&= [z; (n - n \wedge d(z) + n \wedge d(z), n + q - p - n \wedge d(z) + n \wedge d(z))] \\
&= [y; (n - n \wedge d(z) + p \wedge d(y), n + q - p - n \wedge d(z) + p \wedge d(y))] \quad \text{(by Prop. 2.10)} \\
&= [y; (p - p \wedge d(y) + p \wedge d(y), p + q - p - p \wedge d(y) + p \wedge d(y))] = [y; (p, q)].
\end{aligned}$$

This proves (iii). ∎

DEFINITION 2.12. Let $\widetilde{P}_\Lambda \times_{\widetilde{V}_\Lambda} \widetilde{P}_\Lambda$ be the set

$$\{([x; (m, n)], [y; (p, q)]) \in \widetilde{P}_\Lambda \times \widetilde{P}_\Lambda : \widetilde{s}([x; (m, n)]) = \widetilde{r}([y; (p, q)])\}.$$

For $([x; (m, n)], [y; (p, q)]) \in \widetilde{P}_\Lambda \times_{\widetilde{V}_\Lambda} \widetilde{P}_\Lambda$, let $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$. Define

$$[x; (m, n)] \circ [y; (p, q)] = [z; (m, n + q - p)].$$

PROPOSITION 2.13. *The composition defined on* $\widetilde{P}_\Lambda \times_{\widetilde{V}_\Lambda} \widetilde{P}_\Lambda$ *given in Definition* 2.12 *is well-defined.*

Proof. This follows from Proposition 2.11. ∎

PROPOSITION 2.14. *For* $\lambda \in \Lambda$ *and* $(x;(m,n)) \in P_\Lambda$ *with* $s(\lambda) = x(m)$, *let* $z = \lambda\sigma^m x$. *Then*

(i) $z \in \Lambda^{\leqslant\infty}$, *and*

(ii) $[z;(d(\lambda), n-m+d(\lambda))] = [x;(m,n)]$.

Proof. Since $x \in \Lambda^{\leqslant\infty}$, (i) follows from Lemmas 2.10 and 2.11 of [12]. Using the fact that $\sigma^{d(\lambda)}z = \sigma^m x$, Proposition 2.10 implies that

$$[z;(d(\lambda), n-m+d(\lambda))] = [x;(m, n-m+m)] = [x;(m,n)].$$

Thus (ii) follows. ∎

DEFINITION 2.15. Let $\Lambda \times_{\Lambda^0} \widetilde{P}_\Lambda$ be the set

$$\{(\lambda, [x;(m,n)]) \in \Lambda \times \widetilde{P}_\Lambda : s(\lambda) = \widetilde{r}([x;(m,n)])\}$$

For $(\lambda, [x;(m,n)]) \in \Lambda \times_{\Lambda^0} \widetilde{P}_\Lambda$, let $z = \lambda\sigma^m x$. Define

$$\lambda \circ [x;(m,n)] = [z;(0, d(\lambda)+n-m)].$$

The proof of the following is a direct consequence of Proposition 2.14.

PROPOSITION 2.16. *The composition defined on* $\Lambda \times_{\Lambda^0} \widetilde{P}_\Lambda$ *given in Definition* 2.15 *is well-defined.*

REMARK 2.17. If $[x;(m,n)]$, $[y;(p,q)]$, and $[z;(m, n+q-p)]$ are as in Definition 2.12, notice that $[z;(m,n)] \circ [z;(n, n+q-p)] = [z;(m, n+q-p)]$ as well. Thus Proposition 2.13 and Proposition 2.9 imply that

$$\widetilde{r}([z;(m,n+q-p)]) = \widetilde{r}([z;(m,n)]) = \widetilde{r}([x;(m,n)]),$$
$$\widetilde{s}([z;(m,n+q-p)]) = \widetilde{s}([z;(n,n+q-p)]) = \widetilde{s}([y;(p,q)]).$$

Similarly, if λ, $[x;(m,n)]$, and $[z;(d(\lambda), n-m+d(\lambda))]$ are as in Definition 2.15, it follows that

$$\widetilde{r}([z;(d(\lambda), n-m+d(\lambda))]) = r(\lambda),$$
$$\widetilde{s}([z;(d(\lambda), n-m+d(\lambda))]) = \widetilde{s}([x;(m,n)]).$$

We are now ready to define the k-graph $\overline{\Lambda}$ mentioned in Theorem 2.1. The objects of $\overline{\Lambda}$ consist of the objects of Λ together with the elements of $\widetilde{V}_\Lambda$; the morphisms of $\overline{\Lambda}$ are the morphisms of Λ and the elements of $\widetilde{P}_\Lambda$, and Definitions 2.12 and 2.15 describe the composition in $\overline{\Lambda}$.

DEFINITION 2.18. Define $\overline{\Lambda}$ by

$$\mathrm{Obj}(\overline{\Lambda}) = \mathrm{Obj}(\Lambda) \cup \widetilde{V}_\Lambda = \Lambda^0 \cup \widetilde{V}_\Lambda, \quad \mathrm{Mor}(\overline{\Lambda}) = \mathrm{Mor}(\Lambda) \cup \widetilde{P}_\Lambda = \Lambda \cup \widetilde{P}_\Lambda,$$

with $\overline{r}$ and $\overline{s}$ defined as follows:

$$\overline{r} : \mathrm{Mor}(\overline{\Lambda}) \to \mathrm{Obj}(\overline{\Lambda}); \quad \overline{r}\,|_{\mathrm{Mor}(\Lambda)} = r, \quad \text{and} \quad \overline{r}\,|_{\widetilde{P}_\Lambda} = \widetilde{r};$$

and

$$\overline{s} : \mathrm{Mor}(\overline{\Lambda}) \to \mathrm{Obj}(\overline{\Lambda}); \quad \overline{s}\,|_{\mathrm{Mor}(\Lambda)} = s, \quad \text{and} \quad \overline{s}\,|_{\widetilde{P}_\Lambda} = \widetilde{s}.$$

Define $\circ : \mathrm{Mor}(\overline{\Lambda}) \times_{\mathrm{Obj}(\overline{\Lambda})} \mathrm{Mor}(\overline{\Lambda}) \to \mathrm{Mor}(\overline{\Lambda})$ as follows. For an element $(\lambda, [x;(m,n)]) \in \Lambda \times_{\Lambda^0} \widetilde{P}_\Lambda$ define

$$\lambda \circ [x;(m,n)] = [\lambda\sigma^m x;(0, d(\lambda) + n - m)].$$

For $([x;(m,n)], [y;(p,q)]) \in \widetilde{P}_\Lambda \times_{\widetilde{V}_\Lambda} \widetilde{P}_\Lambda$, let $z = x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y$, and define

$$[x;(m,n)] \circ [y;(p,q)] = [z;(m, n+q-p)].$$

For $\lambda, \mu \in \Lambda$ define $\lambda \circ \mu$ as in Λ.

Define $\mathrm{id}_{[x;m]} = [x;(m,m)]$ for $[x;m] \in \widetilde{V}_\Lambda$, and define id_v as in Λ for $v \in \Lambda^0$; that is, $\mathrm{id}_v = v$ for $v \in \Lambda^0$.

LEMMA 2.19. *With the definitions given above, $\overline{\Lambda}$ is a category.*

Proof. Using the axioms for a category detailed in Section I.2 of [9], it must be shown that:

(i) $\overline{r}(\mathrm{id}_c) = c = \overline{s}(\mathrm{id}_c)$ for all $c \in \mathrm{Obj}(\overline{\Lambda})$;

(ii) $\overline{s}(f \circ g) = \overline{s}(g)$ and $\overline{r}(f \circ g) = \overline{r}(f)$ for all $f, g \in \mathrm{Mor}(\overline{\Lambda})$;

(iii) $(f \circ g) \circ h = f \circ (g \circ h)$ for all $f, g, h \in \mathrm{Mor}(\overline{\Lambda})$;

(iv) $f \circ \mathrm{id}_c = f$ and $\mathrm{id}_c \circ g = g$ for all $c \in \mathrm{Obj}(\overline{\Lambda})$ and $f, g \in \mathrm{Mor}(\overline{\Lambda})$ such that $\overline{s}(f) = c = \overline{r}(g)$.

(i) Since $\overline{r} = r$ and $\overline{s} = s$ on Λ, (i) holds for $v \in \Lambda^0$ because Λ is a category. If $[x;m] \in \widetilde{V}_\Lambda$, then $m \not\leq d(x)$. Therefore,

$$\overline{r}(\mathrm{id}_{[x;m]}) = \overline{r}([x;(m,m)]) = [x;m] = \overline{s}([x;(m,m)]) = \overline{s}(\mathrm{id}_{[x;m]}).$$

Thus (i) is true for all $c \in \mathrm{Obj}(\overline{\Lambda})$.

(ii) Suppose $\lambda, \mu \in \Lambda \subseteq \mathrm{Mor}(\overline{\Lambda})$. Then (ii) follows because Λ is a category and $\overline{s}$ agrees with s on Λ. If $\lambda \in \Lambda$ and $[x;(m,n)] \in \widetilde{P}_\Lambda \subseteq \mathrm{Mor}(\overline{\Lambda})$, then $\overline{s}([\lambda\sigma^{m \wedge d(x)} x;(0, d(\lambda)+n-m)]) = [\lambda\sigma^{m \wedge d(x)} x; d(\lambda)+n-m]$. Thus (ii) is true because $[\lambda\sigma^m x; d(\lambda)+n-m] = [\sigma^m x; n-m] = [x;n]$ by Proposition 2.10 (applied twice). To show that (ii) holds for $[x;(m,n)], [y;(p,q)] \in \widetilde{P}_\Lambda$, the definition of composition in $\overline{\Lambda}$ yields $[x;(m,n)] \circ [y;(p,q)] = [x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y;(m, n+q-p)]$. Therefore,

$$\overline{s}([x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y;(m, n+q-p)]) = [x(0, n \wedge d(x))\sigma^{p \wedge d(y)} y; n+q-p],$$

and

$$\begin{aligned}[x(0,n\wedge d(x))\sigma^{p\wedge d(y)}y;n+q-p] &= [\sigma^{p\wedge d(y)}y;n+q-p-n\wedge d(x)]\\ &= [y;n+q-p-n\wedge d(x)+p\wedge d(y)]\\ &\qquad\text{(by Proposition 2.10)}\\ &= [y;n-n\wedge d(x)+q-(p-p\wedge d(x))] = [y;q]\end{aligned}$$

since $[x;n]=[y;p]$ implies that $n-n\wedge d(x)=p-p\wedge d(y)$.

Showing that $\overline{r}(f\circ g)=\overline{r}(f)$ follows in a similar manner.

(iii) There are four cases to consider.

Case 1. Suppose $\lambda,\mu,\nu\in\Lambda\subseteq\mathrm{Mor}(\overline{\Lambda})$. Condition (iii) holds in this case because Λ is a category and composition in $\overline{\Lambda}$ on $\Lambda\subseteq\mathrm{Mor}(\overline{\Lambda})$ agrees with the composition in Λ.

Case 2. Suppose $\lambda,\mu\in\Lambda$ and $[x;(m,n)]\in\widetilde{P}_\Lambda\subseteq\mathrm{Mor}(\overline{\Lambda})$. Then

$$\begin{aligned}(\lambda\circ\mu)\circ[x;(m,n)] &= (\lambda\mu)\circ[x;(m,n)] = [(\lambda\mu)\sigma^m x;(0,n-m+d(\lambda\mu))]\\ &= [\lambda(\mu\sigma^m x);(0,n-m+d(\lambda)+d(\mu))]\\ &\qquad\text{(because composition in } \Lambda \text{ is associative)}\\ &= \lambda\circ[\mu\sigma^m x;(0,n-m+d(\mu))] = \lambda\circ(\mu\circ[x;(m,n)]).\end{aligned}$$

Case 3. Suppose $\lambda\in\Lambda$ and $[x;(m,n)],[y;(p,q)]\in\widetilde{P}_\Lambda$. Then

$$\begin{aligned}(\lambda\circ[x;(m,n)])\circ[y;(p,q)] &= [\lambda\sigma^m x;(0,n-m+d(\lambda))]\circ[y;(p,q)]\\ &= [z;(0,n-m+d(\lambda)+q-p)]\end{aligned}$$

where $z=(\lambda\sigma^m x)(0,(n-m+d(\lambda))\wedge d(\lambda\sigma^m x))\sigma^{p\wedge d(y)}y$.

On the other hand,

$$\begin{aligned}\lambda\circ([x;(m,n)]\circ[y;(p,q)]) &= \lambda\circ[w;(m,n+q-p)]\\ &= [\lambda\sigma^m w;(0,n-m+q-p+d(\lambda))]\end{aligned}$$

where $w=x(0,n\wedge d(x))\sigma^{p\wedge d(y)}y$.

To show

$$[z;(0,n-m+d(\lambda)+q-p)]=[\lambda\sigma^m w;(0,n-m+q-p+d(\lambda))], \tag{2.2}$$

notice that

$$\begin{aligned}&z\,(0\wedge d(z),(n-m+q-p+d(\lambda))\wedge d(z))\\ &\quad= z\,(0,(n-m+d(\lambda))\wedge d(z))\circ\\ &\qquad\quad\circ z\,((n-m+d(\lambda))\wedge d(z),(n-m+d(\lambda)+q-p)\wedge d(z))\\ &\quad= (\lambda\sigma^m x)\,(0,(n-m+d(\lambda))\wedge d(\lambda\sigma^m x))y\,(p\wedge d(y),q\wedge d(y))\end{aligned} \tag{2.3}$$

by Proposition 2.11 (iii).

Since $d(\lambda\sigma^m x)=d(\lambda)-m+d(x)$, we have that

$$(n-m+d(\lambda))\wedge d(\lambda\sigma^m x)=(n-m+d(\lambda))\wedge(d(x)-m+d(\lambda))=d(\lambda)-m+n\wedge d(x)$$

because addition in $\mathbb{N}^k$ distributes over $\wedge$. Thus we can continue with the calculation:

$$\begin{aligned}(2.4)\qquad (\lambda\sigma^m x)&(0,(n-m+d(\lambda))\wedge d(\lambda\sigma^m x))y(p\wedge d(y),q\wedge d(y))\\ &=(\lambda\sigma^m x)(0,d(\lambda)-m+n\wedge d(x))y(p\wedge d(y),q\wedge d(y))\\ &=\lambda(\sigma^m x)(0,-m+n\wedge d(x))y(p\wedge d(y),q\wedge d(y))\\ &=\lambda x(m,n\wedge d(x))y(p\wedge d(y),q\wedge d(y)).\end{aligned}$$

Equations (2.3) and (2.4) show that

$$(2.5)\quad z\,(0\wedge d(z),(n-m+q-p+d(\lambda))\wedge d(z))=\lambda x(m,n\wedge d(x))y(p\wedge d(y),q\wedge d(y)).$$

Similarly, it can be shown using Proposition 2.11 that the right hand side of Equation (2.5) is equal to $(\lambda\sigma^m w)(0,(n-m+q-p+d(\lambda))\wedge d(\lambda\sigma^m w))$ as well. Therefore, Condition (P1) of Definition 2.6 is satisfied. Condition (P2) holds by Proposition 2.11 (ii). Clearly, Condition (P3) holds; therefore Equation (2.2) holds.

Case 4. Suppose $[x;(m,n)],[y;(p,q)],[z;(t,u)]\in\widetilde{P}_\Lambda\subseteq\operatorname{Mor}(\overline{\Lambda})$. We must show that

$$([x;(m,n)]\circ[y;(p,q)])\circ[z;(t,u)]=[x;(m,n)]\circ([y;(p,q)]\circ[z;(t,u)]).$$

Let $W_1=x(0,n\wedge d(x))\sigma^{p\wedge d(y)}y$. Then $[x;(m,n)]\circ[y;(p,q)]=[W_1;(m,q-p+n)]$. Next, define Z_1 to be the path $W_1(0,(q-p+n)\wedge d(W_1))\sigma^{t\wedge d(z)}z$. Then,

$$\begin{aligned}([x;(m,n)]\circ[y;(p,q)])\circ(z,t,u)&=[W_1;(m,q-p+n)]\circ[z;(t,u)]\\ &=[Z_1;(m,u-t+q-p+n)].\end{aligned}$$

On the other hand, for the graph morphisms W_2 and Z_2 defined as

$$W_2=y(0,q\wedge d(y))\sigma^{t\wedge d(z)}z,\quad Z_2=x(0,n\wedge d(x))\sigma^{p\wedge d(W_2)}W_2$$

we see that

$$\begin{aligned}[x;(m,n)]\circ([y;(p,q)]\circ[z;(t,u)])&=[x;(m,n)]\circ[W_2;(p,u-t+q)]\\ &=[Z_2;(m,u-t+q-p+n)].\end{aligned}$$

We must show $[Z_1;(m,u-t+q-p+n)]=[Z_2;(m,u-t+q-p+n)]$.

Using Proposition 2.11 (ii) again, we see that

$$m\wedge d(Z_1)=m\wedge d(x)=m\wedge d(Z_2).$$

Therefore $m-m\wedge d(Z_1)=m-m\wedge d(Z_2)$, and so Condition (P2) of Definition 2.6 is satisfied. It is clear that Condition (P3) holds. To show that Condition (P1) is met, it can be shown using Proposition 2.11 repeatedly that

$$\begin{aligned}Z_1(m\wedge d(Z_1),&(u-t+q-p+n)\wedge d(Z_1))\\ &=x(m\wedge d(x),n\wedge d(x))y(p\wedge d(y),q\wedge d(y))z(t\wedge d(z),u\wedge d(z))\\ &=Z_2(m\wedge d(Z_2),(u-t+q-p+n)\wedge d(Z_2)).\end{aligned}$$

Thus the equation

$$[Z_1; (m, u - t + q - p + n)] = [Z_2; (m, u - t + q - p + n)],$$

holds, and composition is associative in this case.

Hence, $\overline{\Lambda}$ satisfies (iii).

(iv) Suppose v is an element of Λ^0. Then (iv) follows for all $f, g \in \Lambda$ such that $s(f) = v = r(g)$ because Λ is a category. There does not exist an $f \in \widetilde{P}_\Lambda$ such that $\overline{s}(f) = v$. However, for a path $g \in \widetilde{P}_\Lambda$ such that $\overline{r}(g) = v$, we have that $g = [x; (m, n)]$ for some $x \in \Lambda^{\leqslant\infty}$ with $x(m) = v$. Therefore,

$$\begin{aligned} v \circ [x; (m, n)] &= [v\sigma^m x; (0, n - m)] = [\sigma^m x; (0, n - m)] \\ &= [x; (m, n)] \quad \text{(by Proposition 2.10).} \end{aligned}$$

Next suppose $[x; m] \in \widetilde{V}_\Lambda \subseteq \mathrm{Obj}(\overline{\Lambda})$. There is no path $f \in \Lambda$ such that $\overline{r}(f) = [x; m]$ or $\overline{s}(f) = [x; m]$; thus suppose $f \in \mathrm{Mor}(\overline{\Lambda})$ is a path such that $\overline{s}(f) = [x; m]$. Then we can write $f = [y; (p, q)]$ for some $[y; (p, q)] \in \widetilde{P}_\Lambda$ such that $[x; m] = [y; q]$. By definition of composition in $\overline{\Lambda}$, we have

$$\begin{aligned} [y; (p, q)] \circ \mathrm{id}_{[x;m]} &= [y; (p, q)] \circ [x; (m, m)] = [y(0, q \wedge d(y))\sigma^{m \wedge d(x)} x; (p, q + m - m)] \\ &= [y(0, q \wedge d(y))\sigma^{m \wedge d(x)} x; (p, q)] \\ &= [y; (p, q)] \quad \text{(by Proposition 2.11 (iii)).} \end{aligned}$$

It is shown similarly that if $[z; (t, u)]$ is an element of $\widetilde{P}_\Lambda$ with $[x; m] = [z; t]$, the equality $\mathrm{id}_{[x;m]} \circ [z; (t, u)] = [z; (t, u)]$ holds. We have shown that (iv) holds.

Thus $\overline{\Lambda}$ is a category. ∎

From now on, we will write $\lambda\mu$ instead of $\lambda \circ \mu$ for all $\lambda, \mu \in \mathrm{Mor}(\overline{\Lambda})$.

We will view $\mathbb{N}^k$ as a category with one object $(\star)$, a morphism set equal to $\mathbb{N}^k$ and composition determined by addition in $\mathbb{N}^k$.

DEFINITION 2.20. Define $\overline{d} : \overline{\Lambda} \to \mathbb{N}^k$ as follows. For all $c \in \mathrm{Obj}(\overline{\Lambda})$, let $\overline{d}(c) = \star$. Furthermore, define

$$\overline{d} \,|_{\mathrm{Mor}(\Lambda)} = d, \quad \text{and} \quad \overline{d}([x; (m, n)]) = n - m, \quad \text{for } [x; (m, n)] \in \widetilde{P}_\Lambda.$$

It is straightforward to show that $\overline{d}$ defines a functor. Together $\overline{\Lambda}$ and $\overline{d}$ form a k-graph. The key to proving this is the next lemma, which shows that factorization property of Definition 1.1 holds.

LEMMA 2.21. *The category $\overline{\Lambda}$ with the functor $\overline{d}$ defined in Definition* 2.20 *satisfies the factorization property. That is, for $f \in \mathrm{Mor}(\overline{\Lambda})$ with $\overline{d}(f) = a + b$, there exist unique elements $g, h \in \mathrm{Mor}(\overline{\Lambda})$ such that $f = g \circ h$ with $\overline{d}(g) = a$ and $\overline{d}(h) = b$.*

Proof. If $f \in \Lambda \subseteq \mathrm{Mor}(\overline{\Lambda})$, then since Λ has the factorization property and $\overline{d}$ agrees with d on Λ, the required elements exist and are unique.

Suppose that $[x;(m,n)] \in \widetilde{P}_\Lambda \subseteq \mathrm{Mor}(\overline{\Lambda})$. Then $\overline{d}([x;(m,n)]) = n - m$. Suppose that $n - m = a + b$. There are three cases to consider: $m \not\leqslant d(x)$; $m \leqslant d(x)$ while $m + a \not\leqslant d(x)$; and $m \leqslant m + a \leqslant d(x)$.

Case 1. Suppose $m \not\leqslant d(x)$. By definition of composition in $\widetilde{P}_\Lambda$ and Remark 2.17, the necessary elements exist, namely $[x;(m,m+a)]$ and $[x;(m+a,n)]$. For uniqueness, suppose that $[x;(m,n)] = [x;(m,m+a)][x;(m+a,n)]$ as well as $[x;(m,n)] = [y;(p,q)][z;(t,u)]$ with $q - p = a$ and $u - t = b$. Using the definition of composition in $\widetilde{P}_\Lambda$, $[y;(p,q)][z;(t,u)] = [w;(p,q+u-t)]$ where $w = y(0, q \wedge d(y))\sigma^{t \wedge d(z)}z$. Since $\overline{\Lambda}$ is a category, it follows that

$$[x;m] = \overline{r}([x;(m,n)]) = \overline{r}([y;(p,q)]) = [y;p] \text{ and}$$
$$[x;n] = \overline{s}([x;(m,n)]) = \overline{s}([z;(t,u)]) = [z;u].$$

Also, since $\overline{s}([y;(p,q)]) = \overline{r}([z;(t,u)])$, it follows that $[y;q] = [z;t]$. Therefore, Condition (V2) of Definition 2.3 gives the following equalities:

$$m - m \wedge d(x) = p - p \wedge d(y), \tag{2.6}$$
$$n - n \wedge d(x) = u - u \wedge d(z), \tag{2.7}$$
$$q - q \wedge d(y) = t - t \wedge d(z). \tag{2.8}$$

Furthermore, since $[x;(m,n)]$ equals both $[x;(m,m+a)][x;(m+a,n)]$ and $[y;(p,q)][z;(t,u)]$, Condition (P1) of Definition 2.6 implies that

$$\begin{aligned} x(m \wedge d(x), n \wedge d(x)) &= x(m \wedge d(x), (m+a) \wedge d(x))x((m+a) \wedge d(x), n \wedge d(x)) \\ &= y(p \wedge d(y), q \wedge d(y))z(t \wedge d(z), u \wedge d(z)). \end{aligned}$$

The equality of the above paths implies that

$$n \wedge d(x) - m \wedge d(x) = q \wedge d(y) - p \wedge d(y) + u \wedge d(z) - t \wedge d(z). \tag{2.9}$$

If $q \wedge d(y) - p \wedge d(y) = (m+a) \wedge d(x) - m \wedge d(x)$, then the factorization property of Λ will imply that $x(m \wedge d(x), (m+a) \wedge d(x)) = y(p \wedge d(y), q \wedge d(y))$. Then by (2.6) and the fact that $a = (m+a) - m = q - p$, it will follow that $[x;(m,m+a)] = [y;(p,q)]$. Consequently, we will have $[x;(m+a,n)] = [z;(t,u)]$. Thus, we must show that $q \wedge d(y) - p \wedge d(y) = (m+a) \wedge d(x) - m \wedge d(x)$. This will be done on a coordinate by coordinate basis; i.e., for all $i \in \{1,2,\ldots,k\}$, we will show that $(q \wedge d(y))_i - (p \wedge d(y))_i = ((m+a) \wedge d(x))_i - (m \wedge d(x))_i$.

Fix $i \in \{1,2,\ldots,k\}$. Then (2.6) implies that $m_i \leqslant d(x)_i$ precisely when $p_i \leqslant d(y)_i$. Similarly, by (2.7), $n_i \leqslant d(x)_i$ if and only if $u_i \leqslant d(z)_i$, and (2.8) ensures $q_i \leqslant d(y)_i$ if and only if $t_i \leqslant d(z)_i$. Therefore there are 5 subcases to consider:

(1-i) $p_i \leqslant q_i \leqslant d(y)_i$ and $m_i \leqslant m_i + a_i \leqslant d(x)_i$;
(1-ii) $p_i \leqslant q_i \leqslant d(y)_i$ and $m_i \leqslant d(x)_i < m_i + a_i$;
(1-iii) $p_i \leqslant d(y)_i < q_i$ and $m_i \leqslant m_i + a_i \leqslant d(x)_i$;
(1-iv) $p_i \leqslant d(y)_i < q_i$ and $m_i \leqslant d(x)_i < m_i + a_i$;
(1-v) $d(y)_i < p_i \leqslant q_i$ and $d(x)_i < m_i \leqslant m_i + a_i$.

Cases (1-i) and (1-v) are shown by a simple calculation.

For Case (1-iv), since $n_i \geqslant m_i + a_i > d(x)_i$, it follows that $u_i > d(z)_i$. Furthermore, the fact that $q_i > d(y)_i$ gives the inequality $t_i > d(z)_i$. Substituting into (2.9), we obtain

$$d(x)_i - m_i = d(y)_i - p_i + d(z)_i - d(z)_i = d(y_i) - p_i,$$

which also equals $((m+a) \wedge d(x))_i - (m \wedge d(x))_i = (q \wedge d(y))_i - (p \wedge d(y))_i$.

Equations (2.6), (2.7), (2.8) and (2.9) can be used to show that the remaining subcases do not, in fact, occur.

Case 2. Suppose that $m \leqslant d(x)$ and $m + a \nleqslant d(x)$. By definition of composition in $\widetilde{P}_\Lambda$ and Remark 2.17, we have $[x;(m,n)] = [x;(m,m+a)][x;(m+a,n)]$. For uniqueness, suppose that $[x;(m,n)]$ also equals $[y;(p,q)][z;(t,u)]$ where $q - p = a$ and $u - t = b$. As in Case 1, since $[x;(m,n)] = [y;(p,q)][z;(t,u)]$, we know

$$x(m) = \overline{r}([x;(m,n)]) = \overline{r}([y;(p,q)]) = y(p) \text{ and}$$
$$[x;n] = \overline{s}([x;(m,n)]) = \overline{s}([z;(t,u)]) = [z;u].$$

Condition (P1) of Definition 2.6 implies that

$$\begin{aligned} x(m, n \wedge d(x)) &= x(m,(m+a) \wedge d(x))x((m+a) \wedge d(x), n \wedge d(x)) \\ &= y(p, q \wedge d(y))z(t \wedge d(z), u \wedge d(z)), \end{aligned}$$

and therefore in this case, equation (2.9) is replaced with

$$n \wedge d(x) - m = q \wedge d(y) - p + u \wedge d(z) - t \wedge d(z). \tag{2.10}$$

Since $m \leqslant d(x)$, it follows that $p \leqslant d(y)$, and equations (2.7) and (2.8) still hold.

The factorization property of Λ will give the uniqueness provided that

$$(m+a) \wedge d(x) - m = q \wedge d(y) - p.$$

Again, this will be done on a coordinate by coordinate basis. Fix $i \in \{1, 2, \ldots, k\}$. This time there are four subcases to consider:

(2-i) $p_i \leqslant q_i \leqslant d(y)_i$ and $m_i \leqslant m_i + a_i \leqslant d(x)_i$;
(2-ii) $p_i \leqslant q_i \leqslant d(y)_i$ and $m_i \leqslant d(x)_i < m_i + a_i$;
(2-iii) $p_i \leqslant d(y)_i < q_i$ and $m_i \leqslant m_i + a_i \leqslant d(x)_i$;
(2-iv) $p_i \leqslant d(y)_i < q_i$ and $m_i \leqslant d(x)_i < m_i + a_i$.

Subcases (2-i) is a simple calculation. The same argument used to prove Subcase (1-iv) proves Subcase (2-iv). Similar to Case 1, the Subcases (2-ii) and (2-iii) do not occur.

Case 3. Suppose that $m \leqslant d(x)$ and $m + a \leqslant d(x)$. Then using the definition of composition in $\overline{\Lambda}$, we have $[x;(m,n)] = x(m,m+a)[x;(m+a,n)]$. To show uniqueness, suppose that $[x;(m,n)]$ can be written as $\lambda[y;(p,q)]$ for some $\lambda \in \Lambda$ and $[y;(p,q)] \in \widetilde{P}_\Lambda$, with $\overline{d}(\lambda) = a$ and $q - p = b = n - (m+a)$. Then by Condition (P1) of Definition 2.6,

$$x(m, n \wedge d(x)) = x(m, m+a)x(m+a, n \wedge d(x)) = \lambda y(p, q \wedge d(y)).$$

The factorization property of Λ gives that $x(m, m+a) = \lambda$. Consequently, the equality $x(m+a, n \wedge d(x)) = y(p, q \wedge d(y))$ holds. So, $[x; (m+a, n)] = [y; (p, q)]$ which gives uniqueness in this case. ∎

THEOREM 2.22. *Let (Λ, d) be a k-graph. Then the extension of this k-graph given by the pair $(\overline{\Lambda}, \overline{d})$ of Definition* 2.18 *is a k-graph with no sources.*

Proof. The fact that $\overline{\Lambda}$ is a k-graph follows from Lemmas 2.19 and 2.21.

We will show that $v\overline{\Lambda}^{e_i}$ is nonempty for all $v \in \overline{\Lambda}^0$ and all $i \in \{1, 2, \ldots, k\}$. If $v \in \overline{\Lambda}^0 \backslash \Lambda^0$, then $v = [x; m]$ for some $x \in \Lambda^{\leqslant\infty}$ and $m \nleqslant d(x)$. Then for each $i \in \{1, 2, \ldots, k\}$, the path $[x; (m, m+e_i)]$ is an element of $v\overline{\Lambda}^{e_i}$. If $v \in \Lambda^0$, choose $x \in v\Lambda^{\leqslant\infty}$, which is nonempty by Lemma 2.11 of [12]. Fix $i \in \{1, 2, \ldots, k\}$. If $d(x)_i > 0$, then $x(0, e_i) \in v\Lambda^{e_i} \subseteq v\overline{\Lambda}^{e_i}$. If $d(x)_i = 0$, then $[x; (0, e_i)] \in v\overline{\Lambda}^{e_i}$. Hence, for all $v \in \overline{\Lambda}^0$ and $i \in \{1, 2, \ldots, k\}$, $v\overline{\Lambda}^{e_i} \neq \emptyset$. Therefore, $\overline{\Lambda}$ is a k-graph without sources. ∎

Notice that Definition 2.18 provides a way to extend *any* k-graph to a larger k-graph without sources. We will show next that if Λ is finitely aligned or row-finite, then the extension $\overline{\Lambda}$ will have the same property.

LEMMA 2.23. *Let (Λ, d) be a finitely aligned k-graph and let $(\overline{\Lambda}, \overline{d})$ be the k-graph given in Definition* 2.18. *For $\lambda, \mu \in \Lambda$, we have $\Lambda^{\min}(\lambda, \mu) = \overline{\Lambda}^{\min}(\lambda, \mu)$.*

Proof. Of course $\Lambda^{\min}(\lambda, \mu) \subseteq \overline{\Lambda}^{\min}(\lambda, \mu)$ because $\Lambda \subseteq \overline{\Lambda}$. To show the other containment, suppose there is a pair $([x; (m, n)], [y; (p, q)]) \in \overline{\Lambda}^{\min}(\lambda, \mu)$ which is not an element of $\Lambda^{\min}(\lambda, \mu)$. Then

$$[\lambda\sigma^m x; (0, n-m+d(\lambda))] = \lambda[x; (m, n)] = \mu[y; (p, q)] = [\mu\sigma^p y; (0, q-p+d(\mu))] \tag{2.11}$$

where $\overline{d}(\lambda[x; (m, n)]) = d(\lambda) \vee d(\mu)$. Therefore $n - m = d(\lambda) \vee d(\mu) - d(\lambda)$. But (2.11) and Condition (P1) of Definition 2.6 imply

$$\lambda x(m, n \wedge d(x)) = \mu y(p, q \wedge d(y)).$$

Both λ and μ are subpaths of $\lambda x(m, n \wedge d(x)$, which means that the inequality $d(\lambda) + n \wedge d(x) - m \geqslant d(\lambda) \vee d(\mu)$ holds. Hence,

$$n \wedge d(x) - m \geqslant d(\lambda) \vee d(\mu) - d(\lambda) = n - m.$$

It follows that $n \wedge d(x) = n$, and so $n \leqslant d(x)$, contradicting our assumption that the path $[x; (m, n)]$ is not an element of Λ. Thus the set $\overline{\Lambda}^{\min}(\lambda, \mu)$ is a subset of $\Lambda^{\min}(\lambda, \mu)$, and the proof is complete. ∎

Let λ and μ be two paths in a k-graph Λ. Recall from Definition 1.5, that if (α, β) is an element of $\Lambda^{\min}(\lambda, \mu)$, then the path $\lambda\alpha = \mu\beta$ is a minimal common extension of λ and μ. We denote the set of all minimal common extensions of λ and μ by $\mathrm{MCE}(\lambda, \mu)$. Therefore Λ is finitely aligned if and only if $\mathrm{MCE}(\lambda, \mu)$ is finite for all $\lambda, \mu \in \Lambda$.

THEOREM 2.24. *Let (Λ, d) be a k-graph and let $(\overline{\Lambda}, \overline{d})$ be the k-graph given in Definition* 2.18*. If Λ is finitely aligned, the extension $\overline{\Lambda}$ is also finitely aligned. If Λ is row-finite, then so is $\overline{\Lambda}$.*

Proof. Suppose that Λ is finitely aligned. To show $\overline{\Lambda}$ is finitely aligned, we will show that $|\operatorname{MCE}(\lambda, \mu)| < \infty$ for all $\lambda, \mu \in \overline{\Lambda}$. Fix two paths λ and μ in $\overline{\Lambda}$. Let $L = \overline{d}(\lambda) \vee \overline{d}(\mu)$.

First, if $\overline{r}(\lambda) \neq \overline{r}(\mu)$, then $\operatorname{MCE}(\lambda, \mu) = \emptyset$.

Next, suppose $\lambda = [x; (m, n)]$ and $\mu = [y; (p, q)]$ are elements of $\overline{\Lambda} \backslash \Lambda$ such that $\overline{r}(\lambda) = \overline{r}(\mu)$. Any element of $\operatorname{MCE}(\lambda, \mu)$ is of the form $[z; (a_z, a_z + L)]$ for some $z \in \Lambda^{\leqslant\infty}$ and $a_z \in \mathbb{N}^k$ with $a_z + L \not\leqslant d(z)$. Also $[z; (a_z, a_z + \overline{d}(\lambda))] = \lambda$ and $[z; (a_z, a_z + \overline{d}(\mu))] = \mu$. Let $\xi_z = z(a_z \wedge d(z), (a_z + L) \wedge d(z))$. Then $\xi_z \in \Lambda$, and by Proposition 2.11, we have that

$$\begin{aligned}\xi_z &= z(a_z \wedge d(z), (a_z + \overline{d}(\lambda)) \wedge d(z)) z((a_z + \overline{d}(\lambda)) \wedge d(z), (a_z + L) \wedge d(z)) \\ &= x(m \wedge d(x), n \wedge d(x)) z((a_z + \overline{d}(\lambda)) \wedge d(z), (a_z + L) \wedge d(z)).\end{aligned}$$

Also the degree of $x(m \wedge d(x), n \wedge d(x))$ is

$$\begin{aligned}n \wedge d(x) - m \wedge d(x) &= (n - m) \wedge d(x) \\ &= (n - m) \wedge d(z) \quad \text{(by Proposition 2.11 (ii))} \\ &= \overline{d}(\lambda) \wedge d(z).\end{aligned}$$

On the other hand, we have

$$\xi_z = y(p \wedge d(y), q \wedge d(y)) z((a_z + \overline{d}(\mu)) \wedge d(z), (a_z + L) \wedge d(z)),$$

where $q \wedge d(y) - p \wedge d(y) = \overline{d}(\mu) \wedge d(z)$. The properties of $\wedge$ and $\vee$ show that

$$(\overline{d}(\lambda) \wedge d(z)) \vee (\overline{d}(\mu) \wedge d(z)) = (\overline{d}(\lambda) \vee \overline{d}(\mu)) \wedge d(z) = d(\xi_z).$$

Therefore, the path ξ_z is a minimal common extension of $x(m \wedge d(x), n \wedge d(x))$ and $y(p \wedge d(y), q \wedge d(y))$.

If $[z; (a_z, a_z + L)]$ and $[w; (a_w, a_w + L)]$ are two distinct elements of the set $\operatorname{MCE}(\lambda, \mu)$, then Condition (P1) of Definition 2.6 is not satisfied. This implies that ξ_z and ξ_w (as defined above) are two minimal common extensions of the paths $x(m \wedge d(x), n \wedge d(x))$ and $y(p \wedge d(y), q \wedge d(y))$ in Λ. It follows that

$$|\operatorname{MCE}(\lambda, \mu)| = |\operatorname{MCE}(x(m \wedge d(x), n \wedge d(x)), y(p \wedge d(y), q \wedge d(y)))|,$$

which is finite because Λ is finitely aligned.

If $\lambda \notin \Lambda$ and $\mu \in \Lambda$ such that $\overline{r}(\lambda) = \overline{r}(\mu)$, then λ may be written as $[x; (0, n)]$ for some $x \in \Lambda^{\leqslant\infty}$ with $x(0) = r(\mu)$. In this case every element in $\operatorname{MCE}(\lambda, \mu)$ is of the form $[z; (0, L)]$ and $z(0, L \wedge d(z)) \in \operatorname{MCE}(x(0, n \wedge d(x)), \mu)$. An argument similar to the previous case shows that $|\operatorname{MCE}([x; (0, n)], \mu)|$ is the same as $|\operatorname{MCE}(x(0, n \wedge d(x)), \mu)|$, and the latter is finite because Λ is finitely aligned.

For $\lambda, \mu \in \Lambda$, Proposition 2.23 implies that $\overline{\Lambda}^{\min}(\lambda, \mu) = \Lambda^{\min}(\lambda, \mu)$. Thus $\overline{\Lambda}^{\min}(\lambda, \mu)$ is finite since Λ is finitely aligned.

Therefore if Λ is finitely aligned, $\mathrm{MCE}(\lambda,\mu)$ is finite for all $\lambda,\mu\in\overline{\Lambda}$, showing that $\overline{\Lambda}$ is finitely aligned.

Now, suppose that Λ is row-finite; fix $v\in\overline{\Lambda}^0$ and $i\in\{1,2,\dots,k\}$. Since $\Lambda\subseteq\overline{\Lambda}$ is row finite, the set $v\Lambda^{e_i}$ is at most finite. Let $P=v\overline{\Lambda}^{e_i}\backslash\Lambda$. Any element in P is of the form $[x;(m_x,m_x+e_i)]$ for some boundary path $x\in\Lambda^{\leqslant\infty}$ and $m_x\in\mathbb{N}^k$. Two elements $[x;(m_x,m_x+e_i)],[y;(m_y,m_y+e_i)]\in P$, are distinct if and only if

$$x(m_x\wedge d(x),(m_x+e_i)\wedge d(x))\neq y(m_y\wedge d(y),(m_y+e_i)\wedge d(y)).$$

Hence, $|P|$ is equal to

$$|\{x(m_x\wedge d(x),(m_x+e_i)\wedge d(x)):[x;m_x,m_x+e_i]\in P\}|.$$

Because Λ is row-finite and P is a subset of $\{w\}\cup w\Lambda^{e_i}$ (where $w=x(m\wedge d(x))$ for any $[x;(m_x,m_x+e_i)]\in P$), P is a finite set. Thus the k-graph $\overline{\Lambda}$ is row-finite. ∎

If $\{t_\lambda:\lambda\in\overline{\Lambda}\}$ is a Cuntz-Krieger $\overline{\Lambda}$-family, we will show that $\{t_\lambda:\lambda\in\Lambda\}$ is a Cuntz-Krieger Λ-family. The key elements to show this are Lemma 2.23, which proves that $\overline{\Lambda}^{\min}(\lambda,\mu)$ equals $\Lambda^{\min}(\lambda,\mu)$ for paths $\lambda,\mu\in\Lambda$, and the following lemma that shows any finite exhaustive subset E of Λ is exhaustive in $\overline{\Lambda}$.

LEMMA 2.25. *Let (Λ,d) be a finitely aligned k-graph and let $(\overline{\Lambda},\overline{d})$ be the k-graph given in Definition* 2.18. *Suppose $v\in\Lambda^0$ and $E\subseteq v\Lambda$ is a finite exhaustive subset of Λ. Then E is also finite exhaustive subset of $\overline{\Lambda}$.*

Proof. Since E is a finite exhaustive subset of Λ, for every $\lambda\in\Lambda$ such that $r(\lambda)=v$, there exists $\mu\in E$ with $\Lambda^{\min}(\lambda,\mu)\neq\emptyset$. Therefore, it remains to show the same holds for paths in $v\overline{\Lambda}\backslash\Lambda$.

Fix $[x;(m,n)]\in\overline{\Lambda}$ with $\overline{r}([x;(m,n)])=x(m)=v$. We may assume, without loss of generality, that $m=0$ because $[x;(m,n)]=[\sigma^m x;(0,n-m)]$ by Proposition 2.10.

Since $x\in\Lambda^{\leqslant\infty}$, by definition there exists $n_x\in\mathbb{N}^k$ such that $n_x\leqslant d(x)$ and such that if $p\in\mathbb{N}^k$, with $n_x\leqslant p\leqslant d(x)$ and $p_i=d(x)_i$, then $x(p)\Lambda^{e_i}=\emptyset$. Define

$$\lambda=x(0,(n\wedge d(x))\vee n_x),\quad \xi=x(0,n\wedge d(x)),\quad \eta=x(n\wedge d(x),(n\wedge d(x))\vee n_x).$$

Notice that if $(n\wedge d(x))_i=d(x)_i$ for some i, then $((n\wedge d(x))\vee n_x)_i=d(x)_i$. This implies that $d(\eta)_i=0$ and that $s(\eta)\Lambda^{e_i}=s(\lambda)\Lambda^{e_i}=\emptyset$. Hence,

$$(\star)\qquad \text{for any path } \zeta\in\Lambda \text{ such that } r(\zeta)=s(\lambda)=s(\eta),\quad d(\zeta)_i=0 \text{ if } (n\wedge d(x))_i=d(x)_i.$$

There exists $\mu\in E$ and $(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)$ because E is a finite exhaustive subset of Λ. Thus

$$\lambda\alpha=\mu\beta \text{ and } d(\lambda\alpha)=d(\lambda)\vee d(\mu)=((n\wedge d(x))\vee n_x)\vee d(\mu).$$

Since $\lambda = \xi\eta$, and $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$, it follows that $\Lambda^{\min}(\xi, \mu) \neq \emptyset$. In particular, let

$$\nu = (\lambda\alpha)(d(\xi), d(\xi) \vee d(\mu)) = (\eta\alpha)(0, (n \wedge d(x)) \vee d(\mu) - n \wedge d(x)) \text{ and}$$
$$\omega = (\lambda\alpha)(d(\mu), (n \wedge d(x)) \vee d(\mu)).$$

Then $(\nu, \omega) \in \Lambda^{\min}(\xi, \mu)$. Moreover, for all $i \in \{1, 2, \ldots, k\}$ satisfying the equality $(n \wedge d(x))_i = d(x)_i$, then $0 = d(\eta)_i = d(\alpha)_i$, giving $d(\nu)_i = 0$ and $d(\xi\nu)_i = d(\xi)_i$.

There exists $y \in \Lambda^{\leqslant\infty}$ such that $y(0, d(\eta\alpha)) = \eta\alpha$ by Lemmas 2.10 and 2.11 of [12]. Also, if $(n \wedge d(x))_i = d(x)_i$, then $d(y)_i = 0$ by $(\star)$.

Claim 1. Consider $n - n \wedge d(x)$. We claim that $n - n \wedge d(x) \not\leqslant d(y)$.

Proof of Claim 1. To see this, note that because $n \not\leqslant d(x)$ there exists i such that $n_i > d(x)_i \geqslant 0$. For this i, $(\star)$ implies that $d(y)_i = 0$. Therefore, $n_i - (n \wedge d(x))_i = n_i - d(x)_i > 0 = d(y)_i$, giving $(n - n \wedge d(x))_i \not\leqslant d(y)_i$. This proves Claim 1.

Claim 1 establishes that the vertex $[y; n - n \wedge d(x)]$ is an element of $\overline{\Lambda}^0$ and that $[y; (n - n \wedge d(x), n - n \wedge d(x) + d(\nu))]$ is a path in $\overline{\Lambda}$.

Claim 2. The vertices $[y; n - n \wedge d(x)]$ and $[x; n]$ are equal.

Proof of Claim 2. In the case where $(n \wedge d(x))_i = n_i$, then $(n - n \wedge d(x))_i = 0$, and $((n - n \wedge d(x)) \wedge d(y))_i = 0$. If, instead, $(n \wedge d(x))_i = d(x)_i$, then $d(y)_i = 0$ by $(\star)$, and $((n - n \wedge d(x)) \wedge d(y))_i = 0$. Therefore $(n - n \wedge d(x)) \wedge d(y) = 0$. It follows that

$$y((n - n \wedge d(x)) \wedge d(y)) = y(0) = r(\eta) = x(n \wedge d(x)).$$

Also $n - n \wedge d(x) - ((n - n \wedge d(x)) \wedge d(y)) = n - n \wedge d(x)$, which implies that $[y; n - n \wedge d(x)] = [x; n]$. This proves Claim 2.

Claim 2 implies that $[x; (0, n)]$ and $[y; (n - n \wedge d(x), n - n \wedge d(x) + d(\nu))]$ are composable in $\overline{\Lambda}$. Composing them produces $[x(0, n \wedge d(x))y; (0, n + d(\nu))]$.

Claim 3. We claim

$$\overline{d}([x(0, n \wedge d(x))y; (0, n + d(\nu))]) = n + d(\nu) = n \vee d(\mu).$$

Proof of Claim 3. Since $d(\nu) = (n \wedge d(x)) \vee d(\mu) - n \wedge d(x)$, we have

$$\begin{aligned} n + d(\nu) &= n + (n \wedge d(x)) \vee d(\mu) - n \wedge d(x) \\ &= n \vee (n - n \wedge d(x) + d(\mu)) \quad \text{(distributing over } \vee\text{)}. \end{aligned}$$

When $(n \wedge d(x))_i = n_i$, the equation above shows that $(n + d(\nu))_i = (n \vee d(\mu))_i$.

On the other hand, when $(n \wedge d(x))_i = d(x)_i$, then $d(\nu)_i = 0$ by $(\star)$. Therefore, $(n + d(\nu))_i = n_i$. Distributing over $\vee$ shows

$$d(\nu) = (n \wedge d(x)) \vee d(\mu) - n \wedge d(x) = 0 \vee (d(\mu) - n \wedge d(x)),$$

and since $d(\nu)_i = 0$, we see that

$$0 \geqslant (d(\mu) - n \wedge d(x))_i = d(\mu)_i - d(x)_i. \tag{2.12}$$

Equation (2.12) implies

$$n_i \geqslant d(\mu)_i + n_i - d(x)_i \geqslant d(\mu)_i \quad \text{(since } n_i \geqslant d(x)_i\text{)}.$$

Consequently, $(n \vee d(\mu))_i = n_i = (n + d(\nu))_i$ when $(n \wedge d(x))_i = d(x)_i$ as well, establishing Claim 3.

Recall that $y(0, d(\eta\alpha)) = \eta\alpha$. This gives

$$x(0, n \wedge d(x))y = \xi y = (\xi\eta\alpha)\sigma^{d(\eta\alpha)}y = (\lambda\alpha)\sigma^{d(\eta\alpha)}y = (\mu\beta)\sigma^{d(\eta\alpha)}y$$

because $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$. Hence

$$\begin{aligned}[x(0, n \wedge d(x))y; (0, n + d(\nu))] &= [\mu\beta\sigma^{d(\eta\alpha)}y; (0, n + d(\nu))] \\ &= \mu[\beta\sigma^{d(\eta\alpha)}y; (d(\mu), n + d(\nu) - d(\mu))].\end{aligned}$$

By Claim 2, we may also write

$$[x(0, n \wedge d(x))y; (0, n + d(\nu))] = [x; (0, n)][y; (n - n \wedge d(x), n - n \wedge d(x) + d(\nu))].$$

Claim 3 shows that $\overline{d}([x(0, n \wedge d(x))y; (0, n + d(\nu))]) = n \vee d(\mu)$. Therefore, the path $[x(0, n \wedge d(x))y; (0, n + d(\nu))]$ is a minimal common extension of $[x; (0, n)]$ and μ. The pair

$$([y; (n - n \wedge d(x), n - n \wedge d(x) + d(\nu))], [\beta\sigma^{d(\eta\alpha)}y; (d(\mu), n + d(\nu) - d(\mu))])$$

is an element of $\Lambda^{\min}([x; (0, n)], \mu)$, showing that E is a finite exhaustive subset of $\overline{\Lambda}$. ∎

The proof of the next theorem follows easily from Lemmas 2.23 and 2.25.

THEOREM 2.26. *Let (Λ, d) be a finitely aligned k-graph and let $(\overline{\Lambda}, \overline{d})$ be the k-graph given in Definition 2.18. If $\{t_\lambda : \lambda \in \overline{\Lambda}\}$ is a Cuntz-Krieger $\overline{\Lambda}$-family, then the restriction of this set to the elements generated by the subgraph Λ, $\{t_\lambda : \lambda \in \Lambda\}$, is a Cuntz-Krieger Λ-family.*

Proof. Conditions (TCK1) and (TCK2) of Definition 1.9 follow because the set $\{t_\lambda : \lambda \in \overline{\Lambda}\}$ is a Cuntz-Krieger $\overline{\Lambda}$-family. Condition (TCK3) is satisfied because Lemma 2.23 implies that $\overline{\Lambda}^{\min}(\lambda, \mu) \subseteq \Lambda$. Lemma 2.25 gives that any finite exhaustive subset of Λ is a finite exhaustive subset of $\overline{\Lambda}$. Therefore, the fact that $\{t_\lambda : \lambda \in \overline{\Lambda}\}$ is a Cuntz-Krieger $\overline{\Lambda}$-family implies that Condition (CK) of Definition 1.9 is satisfied, proving the result. ∎

In the next theorem, we show that $C^*(\Lambda)$ is naturally isomorphic to a subalgebra of $C^*(\overline{\Lambda})$. The isomorphism is natural in the sense that $C^*(\Lambda)$ is isomorphic to the C^*-algebra generated by the set of elements of the form t_λ where λ is a path in the original k-graph, Λ. Furthermore, the isomorphism maps generators to elements in the canonical way.

THEOREM 2.27. *Let (Λ, d) be a finitely aligned k-graph and let $(\overline{\Lambda}, \overline{d})$ be the k-graph given in Definition* 2.18. *Let $\{t_\lambda : \lambda \in \overline{\Lambda}\}$ be a Cuntz-Krieger $\overline{\Lambda}$ family. Then $C^*(\Lambda)$ is isomorphic to the subalgebra of $C^*(\overline{\Lambda})$ generated by the set $\{t_\lambda : \lambda \in \Lambda\}$.*

Proof. Let $C^*(\overline{\Lambda})$ be generated by $\{t_\lambda : \lambda \in \overline{\Lambda}\}$, and let $C^*(\Lambda)$ be generated by the Cuntz-Krieger Λ-family $\{s_\lambda : \lambda \in \Lambda\}$. Let $A = C^*(\{t_\lambda : \lambda \in \Lambda\})$ in $C^*(\overline{\Lambda})$. By Theorem 2.26, $\{t_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family; thus the universal property of $C^*(\Lambda)$ gives a $*$-homomorphism $\pi : C^*(\Lambda) \to C^*(\overline{\Lambda})$ such that $\pi(s_\lambda) = t_\lambda$ for all $\lambda \in \Lambda$. Because π maps the generators of $C^*(\Lambda)$ onto the set of generators of A, we have $\pi(C^*(\Lambda)) = A$. Since $t_v \neq 0$ for all $v \in \Lambda^0 \subseteq \overline{\Lambda}^0$, it follows that $\pi(s_v) = t_v \neq 0$ for all $v \in \Lambda^0$.

Let $\theta : \mathbb{T}^k \to \mathrm{Aut}(C^*(\overline{\Lambda}))$ and $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\Lambda))$ denote the gauge actions on $C^*(\overline{\Lambda})$ and $C^*(\Lambda)$, respectively. For all $z \in \mathbb{T}^k$ and $\lambda, \mu \in \Lambda$,

$$(\theta_z \circ \pi)(s_\lambda s_\mu^*) = \theta_z(t_\lambda t_\mu^*) = z^{d(\lambda)-d(\mu)} t_\lambda t_\mu^* = \pi(z^{d(\lambda)-d(\mu)} s_\lambda s_\mu^*) = (\pi \circ \gamma_z)(s_\lambda s_\mu^*).$$

It follows then that $\theta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}^k$ because $C^*(\Lambda)$ is spanned by elements of the form $s_\lambda s_\mu^*$ with $s(\lambda) = s(\mu)$ by Condition (TCK3) of Definition 1.9. Therefore by Theorem 4.2 of [12], π is injective. The previous paragraph shows that π maps $C^*(\Lambda)$ surjectively onto A. Thus $C^*(\Lambda) \cong A$. ∎

THEOREM 2.28. *Let (Λ, d) be a row-finite k-graph and let $(\overline{\Lambda}, \overline{d})$ be the k-graph given in Definition* 2.18. *Then $C^*(\Lambda)$ is a full corner of $C^*(\overline{\Lambda})$.*

It is in the following proof that the row-finite condition of Λ is necessary. The row-finiteness of Λ implies that its extension $\overline{\Lambda}$ is also row-finite and does not have any sources. Condition (CK) of Definition 1.9 is equivalent to Condition (CK′) in Remark 1.18 when the k-graph is row-finite and has no sources. Both conditions are used in the proof of Theorem 2.28

Proof of Theorem 2.28. Suppose $C^*(\overline{\Lambda})$ is generated by $\{t_\lambda : \lambda \in \overline{\Lambda}\}$. Let $A = C^*(\{t_\lambda : \lambda \in \Lambda\}) \subseteq C^*(\overline{\Lambda})$. Then $A \cong C^*(\Lambda)$ by Theorem 2.27. We will show that A is a full corner of $C^*(\overline{\Lambda})$.

Using an argument like that in Lemma 1.29(c) of [2], the sum $\sum_{v \in \Lambda^0} t_v$ converges strictly in $M(C^*(\overline{\Lambda}))$ to a projection p satisfying

$$p t_\lambda t_\mu^* p = \begin{cases} t_\lambda t_\mu^* & \text{if } \overline{r}(\lambda), \overline{r}(\mu) \in \Lambda^0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for all $\lambda, \mu \in \Lambda$, $t_\lambda t_\mu^* = p t_\lambda t_\mu^* p \in pC^*(\overline{\Lambda})p$. Hence $A \subseteq pC^*(\overline{\Lambda})p$ because A is spanned by elements of the form $t_\lambda t_\mu^*$ where λ and μ are paths in Λ with $s(\lambda) = s(\mu)$.

For the reverse conclusion, we must show that $p t_\lambda t_\mu^* p \in A$ for all paths λ and μ in $\overline{\Lambda}$ with $\overline{s}(\lambda) = \overline{s}(\mu)$ (again because $pC^*(\overline{\Lambda})p$ is spanned by elements of this form).

If either $\overline{r}(\lambda)$ or $\overline{r}(\mu)$ is in $\overline{\Lambda}^0\backslash\Lambda$, then $pt_\lambda t_\mu^* p = 0 \in A$. This leaves the case when λ and μ are elements in $\overline{\Lambda}$ such that $\overline{r}(\lambda), \overline{r}(\mu) \in \Lambda^0$ and $\overline{s}(\lambda) = \overline{s}(\mu)$. Thus, we must prove the following claim.

Claim. If $\lambda, \mu \in \overline{\Lambda}$ with $\overline{r}(\lambda), \overline{r}(\mu) \in \Lambda^0$ and $\overline{s}(\lambda) = \overline{s}(\mu) \notin \Lambda^0$, then $pt_\lambda t_\mu^* p$ is an element of A.

Proof of Claim. Since $\overline{s}(\lambda)$ and $\overline{s}(\mu)$ are not in Λ^0, the paths λ and μ are paths in $\overline{\Lambda}\backslash\Lambda$. Thus there exist $x, y \in \Lambda^{\leqslant\infty}$, and $m, q \in \mathbb{N}^k$ such that $m \nleqslant d(x)$, $q \nleqslant d(y)$, $\lambda = [x; (0, m)]$ and $\mu = [y; (0, q)]$. We will proceed by induction on m.

Suppose for an inductive hypothesis that for all $n < m$ the Claim holds for all paths ξ and η with $\overline{r}(\xi), \overline{r}(\eta) \in \Lambda^0$ and $\overline{s}(\xi) = \overline{s}(\eta) = [x; n]$.

Since $\overline{s}(\lambda) = \overline{s}(\mu)$, (V1) and (V2) of Definition 2.3 show that the paths $[x; (m \wedge d(x), m)]$ and $[y; (q \wedge d(y), q)]$ are equal. Let

$$\lambda' = x(0, m\wedge d(x)), \quad \mu' = y(0, q\wedge d(y)), \quad \nu = [x; (m\wedge d(x), m)] = [y; (q\wedge d(y), q)].$$

Then $\lambda = \lambda'\nu$ and $\mu = \mu'\nu$. There are two cases to consider.

Case 1. There exist $i_0, i_1 \in \{1, 2, \ldots, k\}$ such that $m_{i_j} \geqslant d(x)_{i_j} + 1$.

Let $a = m - e_{i_0}$. Then $m \wedge d(x) < a < m$, and $a \wedge d(x) = m \wedge d(x)$. Furthermore, $\nu = [x; (m \wedge d(x), a)][x; (a, m)]$. We will show that $\{[x; (a, m)]\}$ is a finite exhaustive subset of $[x; a]\overline{\Lambda}$. Suppose $[z; (t, u)] \in [x; a]\overline{\Lambda}$. Then the path $[z; (t, t + (m - a) \vee (u - t))]$ is a minimal common extension of $[z; (t, u)]$ and $[x; (a, m)]$. To see this, we must show that $[z; (t, t + m - a)] = [x; (a, m)]$. Since $[z; t] = \overline{r}([z; (t, u)]) = [x; a]$ it follows that

$$z(t \wedge d(z)) = x(a \wedge d(x)) \quad \text{and} \quad t - t \wedge d(z) = a - a \wedge d(x). \tag{2.13}$$

If $i \neq i_0$, we have $a_i = m_i$ and so $((t + m - a) \wedge d(z))_i = (t \wedge d(z))_i$. Since $m_{i_0} - a_{i_0} = 1$ and $m_{i_0} \geqslant d(x)_{i_0} + 1$, it follows that $a_{i_0} \geqslant d(x)_{i_0}$ which implies that $t_{i_0} \geqslant d(z)_{i_0}$ because of (2.13). Thus $d(z)_{i_0} = (t \wedge d(z))_{i_0} = ((t + m - a) \wedge d(z))_{i_0}$. Hence

$$z(t \wedge d(z), (t + m - a) \wedge d(z)) = z(t \wedge d(z), t \wedge d(z)) = x(a \wedge d(x), m \wedge d(x))$$

because $a \wedge d(x) = m \wedge d(x)$. This, together with (2.13) shows that the equality $[z; (t, t + m - a)] = [x; (a, m)]$ holds. Therefore, the pair

$$([z; (u, t + (m - a) \vee (u - t))], [z; (t + m - a, t + (m - a) \vee (u - t))])$$

is an element in $\overline{\Lambda}^{\min}([z; (t, u)], [x; (a, m)])$. Since $[z; (t, u)] \in [x; a]\overline{\Lambda}$ was arbitrary, this implies that $\{[x; (a, m)]\}$ is a finite exhaustive subset of $[x; a]\overline{\Lambda}$.

Let $\nu' = [x; (m \wedge d(x), a)]$. Then $\nu = \nu'[x; (a, m)]$, and $\overline{r}(\nu') = \overline{r}(\nu) = \overline{s}(\lambda')$. Furthermore

$$\begin{aligned} pt_\lambda t_\mu^* p &= pt_{\lambda'\nu'[x;(a,m)]} t^*_{\mu'\nu'[x;(a,m)]} p = pt_{\lambda'} t_{\nu'} t_{[x;(a,m)]} t^*_{[x;(a,m)]} t^*_{\nu'} t^*_{\mu'} p \\ &= pt_{\lambda'} t_{\nu'} t_{[x;a]} t^*_{\nu'} t^*_{\mu'} p \quad (\text{because } \{[x; (a, m)]\} \in [x; a]\, \mathcal{FE}(\overline{\Lambda})) \\ &= pt_{\lambda'\nu'} t^*_{\mu'\nu'} p \end{aligned}$$

which belongs to A by the inductive hypothesis since $\overline{s}(\nu') = [x;a]$ and $a < m$. This concludes Case 1.

Case 2. Suppose that $m = m \wedge d(x) + e_{i_0}$ for some $i_0 \in \{1, 2, \ldots, k\}$. Let u be the vertex $x(m \wedge d(x))$. We will show that $u\overline{\Lambda}^{e_{i_0}} \backslash \Lambda$ is the set $\{\nu\}$. Let $\xi \notin \Lambda$ be an element of $u\overline{\Lambda}^{e_{i_0}}$. Then $\xi = [z; (0, e_{i_0})]$ for some $z \in u\Lambda^{\leqslant\infty}$. We know that $e_{i_0} \nleqslant d(z)$. This along with the fact that $(e_{i_0})_j \leqslant d(z)_j$ for $j \neq i_0$ implies $d(z)_{i_0} = 0$. Then $e_{i_0} \wedge d(z) = 0$ and $0 - (e_{i_0} \wedge d(z)) = 0$; therefore $z(0, e_{i_0} \wedge d(z)) = z(0,0) = u$. It is then clear that $[z; (0, e_{i_0})] = [x; (m \wedge d(x), m)] = \nu$.

Let $E = u\overline{\Lambda}^{e_{i_0}} \cap \Lambda$. Then $E = u\overline{\Lambda}^{e_{i_0}} \backslash \{\nu\}$. Since $\overline{\Lambda}$ has no sources by Theorem 2.22 and is row-finite by Theorem 2.24, we have that $u\overline{\Lambda}^{\leqslant e_{i_0}} = u\overline{\Lambda}^{e_{i_0}}$. Then by Proposition B.1 of [12],

$$t_u = \sum_{\xi \in u\overline{\Lambda}^{\leqslant e_{i_0}}} t_\xi t_\xi^* = t_\nu t_\nu^* + \sum_{\lambda \in E} t_\xi t_\xi^*.$$

Thus

$$p t_\lambda t_\mu^* p = p t_{\lambda'\nu} t_{\mu'\nu}^* p = p t_{\lambda'} t_\nu t_\nu^* t_{\mu'}^* p = p t_{\lambda'} \Big(t_u - \sum_{\xi \in E} t_\xi t_\xi^* \Big) t_{\mu'}^* p$$

which belongs to A because $u \in \Lambda$ and $E \subseteq \Lambda$. This concludes Case 2, and proves the claim.

Therefore A is a corner of $C^*(\overline{\Lambda})p \subseteq A$, that is $A = pC^*(\overline{\Lambda})p$. To show that A is a full corner of $C^*(\overline{\Lambda})$, suppose that J is an ideal in $C^*(\overline{\Lambda})$ such that $A \subseteq J$. Of course $\{t_\lambda : \lambda \in \Lambda\} \subseteq J$ because this set generates A. Let v be a vertex in $\overline{\Lambda}^0 \backslash \Lambda$. Then $v = [x; m]$ for some $x \in \Lambda^{\leqslant\infty}$ and $m \nleqslant d(x)$. Let $\alpha = [x; (m \wedge d(x), m)]$. Then $\alpha \in \overline{\Lambda}$; $\overline{r}(\alpha) = x(m \wedge d(x)) \in \Lambda^0$, and $\overline{s}(\alpha) = v$. Thus $t_\alpha = t_{\overline{r}(\alpha)} t_\alpha \in J$ because $t_{\overline{r}(\alpha)} \in J$, and as a consequence, $t_v = t_\alpha^* t_\alpha \in J$. This shows that the set $\{t_v : v \in \overline{\Lambda}^0\}$ is contained in J. Next let $\lambda \in \overline{\Lambda} \backslash \Lambda$. Then $\overline{r}(\lambda) \in \overline{\Lambda}^0$ and $t_\lambda = t_{\overline{r}(\lambda)} t_\lambda \in J$. Hence $\{t_\lambda : \lambda \in \overline{\Lambda}\}$, the set of generators of $C^*(\overline{\Lambda})$ lies in J, which implies that $J = C^*(\overline{\Lambda})$. ∎

We now conclude the chapter with the proof of Theorem 2.1.

Proof of Theorem 2.1. The pair $(\overline{\Lambda}, \overline{d})$ of Definition 2.18 is a row-finite k-graph without sources by Theorems 2.24 and 2.22. By definition of $\overline{\Lambda}$, $\mathrm{Obj}(\Lambda) \subseteq \mathrm{Obj}(\overline{\Lambda})$, and $\mathrm{Mor}(\Lambda) \subseteq \mathrm{Mor}(\overline{\Lambda})$. Furthermore, $\overline{r}|_{\mathrm{Mor}(\Lambda)} = r$, $\overline{s}|_{\mathrm{Mor}(\Lambda)} = s$, and $\overline{d}|_\Lambda = d$. Thus the map $\iota : \Lambda \to \overline{\Lambda}$, where $\iota(\lambda) = \lambda$ for all $\lambda \in \Lambda$ is a k-graph isomorphism between Λ and $\iota\Lambda$. Therefore $A = \{t_\lambda : \lambda \in \iota\Lambda\}$ is isomorphic to $C^*(\Lambda)$ by Theorem 2.27, and is a full corner of $C^*(\overline{\Lambda})$ by Theorem 2.28. ∎

3. EXAMPLES

In this section, we will apply the construction of Section 2 to several examples of row-finite k-graphs. The examples include k-graphs that are and are not locally convex. The examples were chosen to illustrate how the conditions in Definitions 2.3 and 2.6 affect the construction as well as why they are necessary. For the diagrams in this chapter, edges of degree $(1,0)$ appearing in the original k-graph will be drawn with double solid arrows ($\Longrightarrow$); edges of degree $(0,1)$ in the original k-graph will be drawn with double dashed arrows ($=\!\Rightarrow$). Edges of degree $(1,0)$ or $(0,1)$ that appear in the extension will be represented, respectively, by solid arrows ($\longrightarrow$) and dashed arrows ($- \to$).

EXAMPLE 3.1. Let Λ be a row-finite 1-graph with sources. In [2] and [3] the method of "adding heads to sources" was used to create a row-finite 1-graph without sources that preserved the Morita equivalence class of $C^*(\Lambda)$. We will show that the method developed in Chapter 3 coincides with the previous construction of [2], [3].

Let $\Lambda_S = \{v \in \Lambda^0 : v\Lambda^1 = \emptyset\}$. Let $v \in \Lambda_S$. Then Λ_S is the set of sources as defined for a directed graph. In [2], *adding a head to* v means attaching the following graph to v:

$$v \xleftarrow{e_{v_1}} v_1 \xleftarrow{e_{v_2}} v_2 \xleftarrow{e_{v_3}} v_3 \longleftarrow \cdots \longleftarrow v_{n-1} \xleftarrow{e_{v_n}} v_n \longleftarrow \cdots$$

Let Γ denote the 1-graph that results from adding a head to each $v \in \Lambda_S$. Then any path in Γ is either a path in Λ or it is of the form $\lambda e_{v_1} e_{v_2} \cdots e_{v_n}$ for some $v \in \Lambda_S$, $\lambda \in \Lambda v$ and $n \in \mathbb{N}$ with $n \geqslant 1$.

Suppose $x : \Omega_{1,m} \to \Lambda$ is a graph morphism for some $m \in \mathbb{N}$ (so we are considering only finite paths). Then $x \in \Lambda^{\leqslant\infty}$ if and only if $x(m) = x(d(x)) \in \Lambda_S$. Thus,

$$V_\Lambda = \bigcup_{v \in \Lambda_S} \{(x;m) : x(d(x)) = v \text{ and } m > d(x)\} \text{ and}$$
$$P_\Lambda = \bigcup_{v \in \Lambda_S} \{(x;(m,n)) : x(d(x)) = v, m \leqslant n \text{ and } n > d(x)\}.$$

Suppose x and y are paths in $\Lambda^{\leqslant\infty}$ such that $d(x)$ and $d(y)$ are finite. Suppose further that $x(d(x)) = y(d(y)) = v$ for some source $v \in \Lambda_S$. Then it follows that $\sigma^{d(x)} x = \sigma^{d(y)} y = v$. Proposition 2.10 implies that for all $m \in \mathbb{N}$, $m \geqslant 1$

$$[x; d(x)+m] = [y; d(y)+m] = [v; m],$$
$$[x; (d(x)+m, d(x)+n)] = [y; (d(y)+m, d(y)+n)] = [v; (m,n)],$$

for all $m, n \in \mathbb{N}$ with $m \leqslant n$. For any $[x; (m,n)] \in \widetilde{P}_\Lambda$, let $v_x = x(d(x))$. Then $v_x \in \Lambda_S$ and we have that

$$[x; (m,n)] = \begin{cases} [v_x; (m - d(x), n - d(x))] & \text{if } m \geqslant d(x), \\ x(m, d(x))][v_x; (0, n - d(x))] & \text{if } m < d(x). \end{cases}$$

Therefore, the vertices and paths added to Λ to form $\overline{\Lambda}$ are

$$\widetilde{V}_\Lambda = \bigcup_{v \in \Lambda_S} \{[v; m] : m \geqslant 1\}, \text{ and}$$
$$\widetilde{P}_\Lambda = \bigcup_{v \in \Lambda_S} \{[v; (m,n)] : m, n \in \mathbb{N}, m \leqslant n\} \cup \{\lambda[v; (0,n)] : \lambda \in \Lambda v, n > 0\}.$$

The assignment $[v; m] \mapsto v_m$ and $[v; (m-1, m)] \mapsto e_{v_m}$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}$ with $m \geqslant 1$ creates a graph isomorphism between $\overline{\Lambda}$ and Γ when it is extended in a natural way to the entire category. That is, define $\Phi : \overline{\Lambda} \to \Gamma$ by

$$\Phi(\lambda) = \lambda \quad \text{for all } \lambda \in \Lambda;$$
$$\Phi([v; m]) = v_m \quad \text{for all } [v; m] \in \mathrm{Obj}(\overline{\Lambda});$$
$$\Phi([v; (m,n)]) = e_{v_{m+1}} e_{v_{m+2}} \cdots e_{v_n} \quad \text{for all } v \in \Lambda_S, m \leqslant n; \text{ and}$$
$$\Phi(\lambda[v; (0,n)]) = \lambda e_{v_1} e_{v_2} \cdots e_{v_n} \quad \text{for all } v \in \Lambda_S, \lambda \in v\Lambda, n \in \mathbb{N}.$$

Then Φ is a graph isomorphism, and so for 1-graphs, the desingularization developed in Section 2 is the same as the method used in [2], [3].

EXAMPLE 3.2 ($\Omega_{k,m}$). Let Λ be the 2-graph $\Omega_{2,(1,1)}$ shown below:

$$\begin{tikzcd} v_2 \arrow[d, Rightarrow, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta"'] \arrow[d, Rightarrow, "\alpha"] \\ v_0 & v_1 \arrow[l, Rightarrow, "\lambda"] \end{tikzcd}$$

For this example, $\Lambda^{\leqslant\infty}$ consists of four elements:

$$\begin{array}{ll} w : \Omega_{2,(0,0)} \to \Lambda & x : \Omega_{2,(0,1)} \to \Lambda \\ w((0,0)) = v_3 & x((0,0),(0,1)) = \alpha \\ & \\ y : \Omega_{2,(1,0)} \to \Lambda & z : \Omega_{2,(1,1)} \to \Lambda \\ y((0,0),(1,0)) = \beta & z((0,0),(1,1)) = \lambda\alpha = \mu\beta. \end{array}$$

Since $d(w) = (0,0)$, the set $\{[w; m] : m \in \mathbb{N}^2, m > (0,0)\}$ lies in $\widetilde{V}_\Lambda$, and the set $\{[w; (m,n)] : m, n \in \mathbb{N}^2 \text{ and } m \leqslant n\}$ is a subset of $\widetilde{X}_\Lambda$. The figure that follows shows the 1-skeleton of these elements together with the original graph Λ. In this figure, $a_w = [w;(1,1)]$, $b_w = [w;(1,2)]$ and $\xi_w = [w;((1,1),(1,2))]$.

$$\begin{tikzcd}
 & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
 & \circ \arrow[d, dashed] & b_w \arrow[l] \arrow[d, dashed, "\xi_w"] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
 & \circ \arrow[d, dashed] & a_w \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_2 \arrow[d, Rightarrow, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta"'] \arrow[d, Rightarrow, "\alpha"] & \circ \arrow[l] & \circ \arrow[l] & \cdots \arrow[l] \\
v_0 & v_1 \arrow[l, Rightarrow, "\lambda"] & & &
\end{tikzcd}$$

From the boundary path x, we have $\{[x;m] : m \in \mathbb{N}^2, m \not\leqslant (0,1)\} \subset \widetilde{V}_\Lambda$ and $\{[x;(m,n)]) : m \leqslant n, n \not\leqslant (0,1)\} \subset \widetilde{P}_\Lambda$. Below, we see the 1-skeleton of these elements as well as Λ. In this case, $a_x = [x;(1,2)]$, $b_x = [x;(1,3)]$ and $\xi_x = [x;((1,2),(1,3))]$.

$$\begin{tikzcd}
 & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
 & \circ \arrow[d, dashed] & b_x \arrow[l] \arrow[d, dashed, "\xi_x"] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
 & \circ \arrow[d, dashed] & a_x \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_2 \arrow[d, Rightarrow, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta"'] \arrow[d, Rightarrow, "\alpha"] & \circ \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_0 & v_1 \arrow[l, Rightarrow] & \circ \arrow[l] & \circ \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

The elements of V_Λ and P_Λ resulting from the boundary paths y and z are similar. The next two figures show Λ together with the additional vertices and paths. In the first figure that follows, we have $a_y = [y;(2,1)]$, $b_y = [y;(2,2)]$ and $\xi_y = [y;((2,1),(2,2))]$, while in the second $a_z = [z;(2,2)]$, $b_z = [z;(2,3)]$ and $\xi_z = [z;((2,2),(2,3))]$.

$$
\begin{tikzcd}
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
\circ \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & b_y \arrow[l] \arrow[d, dashed, "\xi_y"] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
\circ \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & a_y \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_2 \arrow[d, Rightarrow, dashed, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta"'] \arrow[d, Rightarrow, dashed, "\alpha"] & \circ \arrow[l] & \circ \arrow[l] & \cdots \arrow[l] \\
v_0 & v_1 \arrow[l, Rightarrow, "\lambda"] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] & \\
\circ \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & b_z \arrow[l] \arrow[d, dashed, "\xi_z"] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
\circ \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & a_z \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_2 \arrow[d, Rightarrow, dashed, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta"'] \arrow[d, Rightarrow, dashed, "\alpha"] & \circ \arrow[l] \arrow[d, dashed] & \circ \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
v_0 & v_1 \arrow[l, Rightarrow, "\lambda"] & \circ \arrow[l] & \circ \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

Since $x = \sigma^{(1,0)}z$, Proposition 2.10 implies that $[x;m] = [z;m+(1,0)]$ for all $m \not\leqslant (1,0)$, and $[x;(m,n)] = [z;(m+(1,0), n+(1,0))]$ for all $m \leqslant n$, $n \not\leqslant (1,0)$. Similarly $y = \sigma^{(0,1)}z$ and $w = \sigma^{(1,1)}z$. Therefore by Proposition 2.10, we obtain the following equalities

$$
\begin{aligned}
[y;m] &= [z;m+(0,1)] && \text{for all } m \not\leqslant (0,1);\\
[y;(m,n)] &= [z;(m+(0,1), n+(0,1))] && \text{for all } m \leqslant n, n \not\leqslant (0,1);\\
[w;m] &= [z;m+(1,1)] && \text{for all } m > (0,0) \text{ and}\\
[w;(m,n)] &= [z;(m+(1,1), n+(1,1))] && \text{for all } m \leqslant n, n > (0,0).
\end{aligned}
$$

Thus,

$$
\widetilde{V}_\Lambda = \{[z;m] : m \not\leqslant (1,1)\}, \quad \text{and} \quad \widetilde{P}_\Lambda = \{[z;(m,n)] : m \leqslant n \text{ and } n \not\leqslant (1,1)\}.
$$

Therefore, $\overline{\Lambda}$ is $\Omega_{2,(\infty,\infty)}$.

It can be shown that $C^*(\Lambda) \cong M_4(\mathbb{C})$ and that $C^*(\overline{\Lambda}) \cong \mathcal{K}(\ell^2(\mathbb{N}^2))$. So we see that $C^*(\Lambda)$ is indeed a full corner of $C^*(\overline{\Lambda})$.

In general, if $\Lambda = \Omega_{k,m}$ for some $m \in (\mathbb{N} \cup \{\infty\})^k$, then $\overline{\Lambda} = \Omega_k$. This seems reasonable since Ω_k is the simplest k-graph without sources that contains $\Omega_{k,m}$ as a subgraph. In a sense, we are just "filling in the gaps" of $\Omega_{k,m}$ to extend it to Ω_k.

EXAMPLE 3.3 (A non-locally convex graph). Let Λ be the 2-graph shown below.

$$
\begin{tikzcd}
v_2 \arrow[d, Rightarrow, "\mu"'] & \\
v_0 & v_1 \arrow[l, Rightarrow, "\lambda"]
\end{tikzcd}
$$

While Λ is a subgraph of $\Omega_{2,(\infty,\infty)}$, the C^*-algebra of Λ is not isomorphic to a full corner of $C^*(\Omega_{2,(\infty,\infty)})$. According to [18], $C^*(\Lambda)$ will have two maximal ideals corresponding to the saturated and hereditary subsets of Λ which are $\{v_1\}$ and $\{v_2\}$. However, $C^*(\Omega_{2,(\infty,\infty)})$ is a simple C^*-algebra.

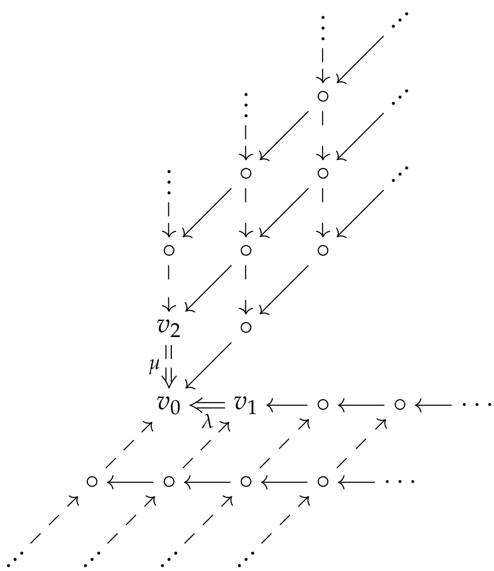
For this example, $\Lambda^{\leqslant\infty}$ consists of four boundary paths, but there are only two boundary paths that we must consider. All other elements of $\Lambda^{\leqslant\infty}$ are shifts of the paths x and y described below. As in the previous example, Proposition 2.10 implies that $\overline{\Lambda}$ is determined by these paths.

Define $x : \Omega_{2,(1,0)} \to \Lambda$ and $y : \Omega_{2,(0,1)} \to \Lambda$ to be the following graph morphisms:

$$\begin{array}{ll} x : \Omega_{2,(1,0)} \to \Lambda & \qquad y : \Omega_{2,(0,1)} \to \Lambda \\ x((0,0),(1,0)) = \lambda & \qquad y((0,0),(0,1)) = \mu. \end{array}$$

Both x and y extend to form a copy of $\Omega_{2,(\infty,\infty)}$ in $\overline{\Lambda}$. Now we must determine if the extensions of these paths are equivalent according to Definition 2.3 or Definition 2.6.

Let $[x;m]$ and $[y;p]$ be elements of $\overline{\Lambda}^0$. Suppose that $[x;m] = [y;p]$. Then because x and y agree only at $x((0,0)) = y((0,0)) = v_0$, we must have that $m \wedge d(x) = p \wedge d(y) = (0,0)$ by Condition (V1) of Definition 2.3. Therefore $m = (0, m_2)$ and $p = (p_1, 0)$ for $m_2, p_1 > 0$. But Condition (V2) would imply that $(0, m_2) = (p_1, 0)$, which is impossible. Hence, $[x;m] \neq [y;p]$ for all $m \not\leq d(x)$ and $p \not\leq d(y)$. Hence the two copies of $\Omega_{2,(\infty,\infty)}$ that these boundary paths contribute to $\overline{\Lambda}$ intersect only at v_0. The extension of Λ is drawn below. For this example $C^*(\Lambda) \cong M_2 \oplus M_2$ and $C^*(\overline{\Lambda}) \cong \mathcal{K}(\ell^2(\mathbb{N}^2)) \oplus \mathcal{K}(\ell^2(\mathbb{N}^2))$.



4. ADDITIONAL QUESTIONS

For directed graphs, the desingularization process developed in [3] takes any directed graph with sources and infinite receivers and builds a directed graph without these singular vertices while still preserving the Morita equivalence class of the graph C^*-algebras.

Consider the following directed graph E. This graph does not have any sources, but v receives infinitely many edges. Label the edges from w to v as α_i, $i \in \mathbb{N}$.

$$\cdots \longleftarrow \bullet \longleftarrow \bullet \longleftarrow v \xleftarrow{\infty} w \longleftarrow \bullet \longleftarrow \bullet \longleftarrow \cdots$$

The desingularization process will add a head to v and redistribute the edges to the new vertices. Let F denote the desingularization of E. The directed graph F is drawn below. There is a bijection between the set of all finite paths of E and the set of finite paths in F that have range and source in E. This bijection maps α_1 to f_1 and sends α_i, $i > 1$ to the path $e_{v_1} e_{v_2} \cdots e_{v_i - 1} f_i$.

$$\begin{tikzcd}
\cdots & \bullet \arrow[l] & \bullet \arrow[l] & v \arrow[l] & w \arrow[l, "f_1"'] \arrow[dl, "f_2"'] \arrow[ddl, "f_3"] & \bullet \arrow[l] & \bullet \arrow[l] & \cdots \arrow[l] \\
 & & & v_1 \arrow[u, "e_{v_1}"] & & & & \\
 & & & v_2 \arrow[u, "e_{v_2}"] & & & & \\
 & & & \vdots \arrow[u, "e_{v_3}"] & & & &
\end{tikzcd}$$

It remains to be seen if a desingularization process for infinite receivers in a higher-rank graph can be developed. The process outlined in this paper for dealing with sources in a higher-rank graph is analogous to the process of "adding a head to a source". When a head is attached to a source in a 1-graph, a copy of $\Omega_{1,\infty}$ is created in the 1-graph. The method developed in Section 2 extends a k-graph with sources in a way that creates a copy of $\Omega_{k,(\infty,\ldots,\infty)}$ in the extension. If the desingularization of a k-graph with infinite receivers is to remain analogous to what occurs in the 1-graph setting, then we must redistribute infinitely many edges of various degrees throughout a copy of $\Omega_{k,(\infty,\ldots,\infty)}$. Deciding how to do this is complicated by the fact that adding just one edge to a vertex often necessitates adding many edges to other vertices to ensure that the factorization property holds. Furthermore, there are many different ways that a vertex in a k-graph can receive infinitely many paths of a certain degree. For example, in the 2-graphs Λ_1 through Λ_4 below, the vertex v_0 receives infinitely many edges of degree $(1,1)$.

$$\begin{tikzcd} v_2 \arrow[d, Rightarrow, "\mu_i"', "\infty"] & v_3 \arrow[l, "\beta"'] \arrow[d, "\alpha"] \\ v_0 & v_1 \arrow[l, Leftarrow, "\infty"', "\lambda_i"] \end{tikzcd}$$
Λ_1
$(\lambda_i\alpha = \mu_i\beta)$

$$\begin{tikzcd} v_2 \arrow[d, Rightarrow, "\mu_i"', "\infty"] & v_3 \arrow[l, "\beta"'] \arrow[d, Rightarrow, "\infty"', "\alpha_i"] \\ v_0 & v_1 \arrow[l, "\lambda"] \end{tikzcd}$$
Λ_2
$(\lambda\alpha_i = \mu_i\beta)$

$$\begin{tikzcd} v_2 \arrow[d, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta_i"', "\infty"] \arrow[d, "\alpha"] \\ v_0 & v_1 \arrow[l, Rightarrow, "\infty"', "\lambda_i"] \end{tikzcd}$$
Λ_3
$(\lambda_i\alpha = \mu\beta_i)$

$$\begin{tikzcd} v_2 \arrow[d, "\mu"'] & v_3 \arrow[l, Rightarrow, "\beta_i"', "\infty"] \arrow[d, Rightarrow, "\infty"', "\alpha_i"] \\ v_0 & v_1 \arrow[l, "\lambda"] \end{tikzcd}$$
Λ_4
$(\lambda\alpha_i = \mu\beta_i)$

When Λ is a finitely aligned k-graph, the set $\Lambda^{\leqslant\infty}$ is used to create a non-degenerate Toeplitz-Cuntz-Krieger Λ-family in [11]. For locally convex, row-finite k-graphs, these paths are related to the sets $\Lambda^{\leqslant n}$, which appear in the Cuntz-Krieger relation (CK$'$) (Remark 1.18). The elements in $\Lambda^{\leqslant\infty}$, in a way, point out where the sources are in the k-graph and are crucial to the process developed in Section 2. In [19], a different set of boundary paths, the set $\partial\Lambda$, is introduced to study relative Cuntz-Krieger algebras of finitely aligned k-graphs. A graph morphism $x : \Omega_{k,m} \to \Lambda$ belongs to $\partial\Lambda$ if for every $n \leqslant m$ and every finite-exhaustive set $E \subseteq x(n)\Lambda$, there exists $\mu \in E$ such that $x(n, n + d(\mu)) = \mu$ ([19], Definition 4.4). The set $\partial\Lambda$ also plays a part in developing a groupoid model for finitely aligned k-graphs [5]. In general, the set $\Lambda^{\leqslant\infty}$ is a proper subset of $\partial\Lambda$, and in some sense, the paths of $\partial\Lambda$ are the limits of sequences of paths in $\Lambda^{\leqslant\infty}$. The elements in $\partial\Lambda$ identify which vertices in Λ are infinite receivers as well as sources. Perhaps a construction using these paths would lead to a desingularization of a k-graph with infinite receivers.

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