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EVOLUTION INTEGRALS

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ABSTRACT. A framework analogous to path integrals in quantum physics is set up for dynamical systems in a W^*-algebraic setting. We consider spaces of evolutions, defined in a specific way, of a W^*-algebra A as an analogue of spaces of classical paths, and show how integrals over such spaces, which we call "evolution integrals", lead to dynamics in a Hilbert space on a "higher level" which is viewed as an analogue of quantum dynamics obtained from path integrals. The measures with respect to which these integrals are performed are projection valued.

KEYWORDS: *Dynamical systems, W^*-algebras, spectral measures.*

MSC (2000): 46L55.

1. INTRODUCTION

Path integrals in quantum physics essentially express the dynamics of a quantum system as an integral over a space of paths in the configuration space (or in the phase space) of the corresponding classical system; see for example [7], or many other standard texts on quantum physics. Also see [3] for the early work on this topic. In this paper the goal is to set up an analogous framework for abstract dynamical systems, where the dynamics of a system on a "higher level" is expressed in terms of an integral over a space of evolutions of a system on a "lower level". The "lower level" will be given by a W^*-algebra while the "higher level" will be expressed in terms of a Hilbert space. We will call such integrals "evolution integrals", and they will be defined in Section 4.

By an abstract dynamical system we mean a pair (A, α) where A is a C^*-algebra and α is a representation $G \to \mathrm{Aut}(A) : g \mapsto \alpha_g$ of some group G in the automorphism group $\mathrm{Aut}(A)$ of A. We can refer to α as the *evolution* of the system. But for a given G, we can have different evolutions of A, namely different representations of G in $\mathrm{Aut}(A)$.

All of these evolutions by definition have group properties. However, in path integrals not all the paths in the space of paths over which we integrate can be expected to be a segment from a possibly longer path which has group properties. For example a path may intersect itself in phase space. Therefore we will also allow more general evolutions in the case of abstract dynamical systems, where the group structure does not play a role anymore. First write α as $(\alpha_g)_{g\in G}$. Note that at every point in "time", $g \in G$, an element $a \in A$ at "time" $e \in G$ (the identity element) will have evolved to $\alpha_g(a)$ by means of the $*$-automorphism α_g. If we want to retain this fact, but ignore the group structure, we can generalize evolutions by allowing $(\alpha_g)_{g\in G}$ to be viewed as an evolution even when $G \ni g \mapsto \alpha_g \in \text{Aut}(A)$ is an arbitrary function. This is a very wide class of functions, but because of the methods we employ in this paper (see Section 3) we will not restrict ourselves to smaller classes, for example functions which are continuous in some specified topologies on G and $\text{Aut}(A)$. Of course, we are now no longer working with group representations, and in fact we will not make use of the group structure of G or $\text{Aut}(A)$ at all in this paper. Therefore we will replace G by an arbitrary set T. In Section 4 we will describe how a grouplike structure emerges on the "higher level" when we consider certain collections of sets T in a measure space $(\mathfrak{T}, \Sigma, \mu)$.

We will in fact have to generalize even further, again because of the methods we use, and replace $\text{Aut}(A)$ by its closure $\overline{\text{Aut}(A)}$, which will be compact, in an appropriate topology on the space of bounded linear operators on A. In order to be able to do this, we will take A to be a W^*-algebra, rather than just a C^*-algebra. The evolutions of A that we will consider will therefore be all functions of the form $T \ni t \mapsto \alpha_t \in \overline{\text{Aut}(A)}$. We discuss this space of evolutions, namely $\overline{\text{Aut}(A)}^T$, in more detail in Section 2.

As will be explained in Section 4, the "evolution integrals" over $\overline{\text{Aut}(A)}^T$ will differ from path integrals in an important respect. In the former case we will use a projection valued measure obtained by spectral theory and the representation theory of C^*-algebras applied to $C(\overline{\text{Aut}(A)}^T)$, which we discuss in Section 3, but this measure will not be unique. We will in fact have many different measures, all of them natural (or canonical) in a sense to be explained, and all of them being unitary transformations of one another. Another difference is that the system we start off with (the "lower level") need not abelian, that is to say classical. In other words A can be a noncommutative W^*-algebra. The analogy between path integrals and integrals over $\overline{\text{Aut}(A)}^T$ is clarified in Section 5.

The mathematical results of this paper are concentrated in Sections 2 and 3 where we develop the mathematical tools that will enable us to set up the basic framework of evolution integrals in Section 4. Sections 4 and 5 consist out of definitions, discussion and some simple results describing evolution integrals and their meaning. We will not look at any applications of evolution integrals in this paper.

2. THE EVOLUTION SPACE

In this section, A will denote an arbitrary W^*-algebra, in other words a C^*-algebra which when viewed as a Banach space has some Banach space as a predual; see for example [9]. It is known that this predual of A is unique, and we will denote it by A_*. Let $\mathrm{Aut}(A)$ denote the set of all $*$-automorphisms of A.

We will use the following notation: For any normed spaces X and Y, let $B(X \times Y)$ denote the Banach space of all bounded bilinear mappings $X \times Y \to \mathbb{C}$, and $L(X,Y)$ the normed space of all bounded linear mappings $X \to Y$. Furthermore we set $L(X) := L(X,X)$. A unit ball will be indicated by $X_1 := \{x \in X : \|x\| \leqslant 1\}$. When X and Y are Banach spaces, we will denote their projective tensor product (see for example [8]) by $X\widehat{\otimes}_\pi Y$.

THEOREM 2.1. *The space $L(A)$ has a Banach space predual $A\widehat{\otimes}_\pi A_*$, which gives a weak* topology on $L(A)$ in which the closure $\overline{\mathrm{Aut}(A)}$ of $\mathrm{Aut}(A)$ is a compact Hausdorff space.*

Proof. We have canonical isometric isomorphisms (which we can also call Banach space isomorphisms)

$$(A\widehat{\otimes}_\pi A_*)^* \cong B(A \times A_*) \cong L(A,(A_*)^*) = L(A) \tag{2.1}$$

so $L(A)$ has $A\widehat{\otimes}_\pi A_*$ as a predual. By Alaoglu's theorem the unit ball $L(A)_1$ is compact in the weak* topology thus obtained on $L(A)$, and of course the weak* topology on $L(A)$ is Hausdorff, since a predual separates the points of a space. In particular then, $L(A)_1$ is weak* closed in $L(A)$. By definition any $\alpha \in \mathrm{Aut}(A)$ is a linear mapping $A \to A$, but since it is a $*$-isomorphism and A a C^*-algebra, we also know that it is norm preserving, so $\|\alpha\| = 1$. Hence $\mathrm{Aut}(A) \subset L(A)_1$. Since $L(A)_1$ is weak* closed, we have $\overline{\mathrm{Aut}(A)} \subset L(A)_1$ for the weak* closure of $\mathrm{Aut}(A)$ in $L(A)$, and therefore $\overline{\mathrm{Aut}(A)}$ is compact and Hausdorff. ∎

By Tychonoff's theorem we then immediately have:

COROLLARY 2.2. *For any set T, the product space $\overline{\mathrm{Aut}(A)}^T := \prod_{t\in T} \overline{\mathrm{Aut}(A)}$ is compact and Hausdorff.*

Because A_* is the unique predual of A, the predual $A\widehat{\otimes}_\pi A_*$ is canonical in the sense that the Banach space isomorphisms in (2.1) are canonical. In this sense we can view the weak* topology in which the closure $\overline{\mathrm{Aut}(A)}$ was taken as a natural weak* topology on $L(A)$. It is however unclear whether $A\widehat{\otimes}_\pi A_*$ is the unique predual of $L(A)$; for example this type of problem has been studied in [4], but assuming the Radon-Nikodým property for A and A_*, which unfortunately we do not have in general [2], [1].

We will call $\overline{\mathrm{Aut}(A)}^T$ in Corollary 2.2 the *evolution space* of A over the set T.

Theorem 2.1 is the only place in this paper where we use the properties of $*$-automorphisms of A. From now on we will only use the fact that $\overline{\mathrm{Aut}(A)}^T$ is a compact Hausdorff space.

3. NATURAL SPECTRAL MEASURES

In path integrals the measure can be heuristically viewed as an infinite dimensional Lebesgue measure. In particular we want such a measure to be natural in a certain sense, and intuitively we want it to assign the same weight to each path. In the case of an abstract dynamical system we would similarly like to obtain measures on $\overline{\mathrm{Aut}(A)}^T$ which are in some way natural and intuitively assign the same weight to each evolution. We will approach this problem using the representation theory of C^*-algebras, in particular for the C^*-algebra $C(K)$ of continuous functions $K \to \mathbb{C}$ with K a compact Hausdorff space. We will in fact obtain projection valued measures, and this will be done using the following result, which is the reason why compact Hausdorff spaces play an important role in this paper:

If K is a compact Hausdorff space, H a Hilbert space, and $\varphi : C(K) \to L(H)$ a unital $*$-homomorphism, then there is a unique spectral measure E relative to (K, H) such that

$$\varphi(f) = \int_K f \mathrm{d}E \tag{3.1}$$

for all $f \in C(K)$, where by a *spectral measure* relative to (K, H) we mean a map E from the σ-algebra of Borel sets of K to the set of projections in $L(H)$ such that $E(\varnothing) = 0$, $E(K) = 1$, $E(V_1 \cap V_2) = E(V_1)E(V_2)$ for all Borel $V_1, V_2 \subset K$, and for all $x, y \in H$ the function $E_{x,y} : V \mapsto \langle x, E(V)y \rangle$ is a regular complex Borel measure on K; see for example [6]. We will refer to E as the *spectral resolution* of φ, and will sometimes denote it by E_φ. Note that the integral $\int_K f \mathrm{d}E$ is defined for all bounded complex-valued Borel functions f on K, the space of such functions being denoted by $B_\infty(K)$ which with the sup-norm is a C^*-algebra, by demanding that $\left\langle x, \left(\int_K f \mathrm{d}E\right) y \right\rangle = \int_K f \mathrm{d}E_{x,y}$ for all $x, y \in H$. Besides being well defined, this integral in fact also allows us to use (3.1) to naturally extend φ to a unital $*$-homomorphism $\widetilde{\varphi} : B_\infty(K) \to L(H) : f \mapsto \int_K f \mathrm{d}E$. We will from now on consistently use this notation to denote the extension $B_\infty(K) \to L(H)$ of a unital $*$-homomorphism $C(K) \to L(H)$ given by its spectral resolution.

However, we will require a φ which is in some way a canonical representation of $C(K)$. Since $C(K)$ is an abelian C^*-algebra, and K is a compact Hausdorff space, the set of pure states of $C(K)$ can be identified with K via

$$\omega_x(f) := f(x)$$

which defines a pure state ω_x on $C(K)$ corresponding to $x \in K$. Let (H_x, π_x, Ω_x) be the GNS representation of $C(K)$ associated to ω_x, namely H_x is a Hilbert space (which in this case happens to be one dimensional), $\pi_x : C(K) \to L(H_x)$ is a $*$-homomorphism, and $\Omega_x \in H_x$ is a cyclic vector for π_x such that $\omega_x(f) = \langle \Omega_x, \pi_x(f)\Omega_x \rangle$ for all $f \in C(K)$. Now consider the direct sum of all such representations, namely

$$H := \bigoplus_{x \in K} H_x \quad \text{and} \quad \pi := \bigoplus_{x \in K} \pi_x$$

which we will call a *pure representation* (H, π) of $C(K)$. This of course is a faithful representation by the standard representation theory of C^*-algebras (see for example [5]), and it is straightforward to see that it is unital, i.e. $\pi(1) = 1 \in L(H)$. We can view this representation as being canonical, since no pure state is given preference over another. The spectral resolution E_π can in this sense then be viewed as a natural spectral measure defined on the Borel σ-algebra of K. Note however that such a pure representation is not quite unique, since the GNS representation is only unique up to unitary equivalence, where this unitary operator can be from one Hilbert space to another. For example, if $U : H \to H$ is unitary, then $U^*\pi(\cdot)U$ is also a pure representation of $C(K)$ obtained out of unitary transformations of the GNS representations above from the H_x's to other subspaces of H. Hence the spectral measure that we obtain is also not unique, despite being natural. This lack of uniqueness has a role to play, as we will briefly discuss in Section 4.

The unit vector Ω_x can also be viewed as an element of H in the obvious canonical way: $\Omega'_x := (\Omega'_{x,y})_{y \in K}$ with $\Omega'_{x,y} = \Omega_x$ when $y = x$, and $\Omega'_{x,y} = 0$ otherwise. Hence Ω'_x represents ω_x as a vector in H, namely $\omega_x(f) = \langle \Omega_x, \pi_x(f)\Omega_x \rangle = \langle \Omega'_x, \pi(f)\Omega'_x \rangle$ for all $f \in C(K)$. The vectors Ω'_x, $x \in K$, are orthonormal in H. Furthermore, since $C(K)$ is an abelian C^*-algebra and ω_x is a pure state, it follows that H_x is one dimensional for every $x \in K$. This means that $\{\Omega'_x\}_{x \in K}$ is a total orthonormal set in H, i.e. it is an orthonormal basis for H. This gives us a simple interpretation for H, namely it has an orthonormal basis $\{\Omega'_x\}_{x \in K}$ whose elements represent the points of K.

We now want to argue that $E := E_\pi$ attaches the same "weight" to each of K 's elements, and more generally that if the Borel sets $V_1, V_2 \subset K$ are in some intuitive sense "equally big", then the projections $E(V_1)$ and $E(V_2)$ are "equally big". We do this via the following theorem, in which $\text{span}M$ denotes the space of finite linear combinations of elements of the set M in a vector space, and $\overline{\text{span}M}$ its norm closure in case of a normed space:

THEOREM 3.1. *Let (H, π) be a pure representation of $C(K)$ where K is a compact Hausdorff space, and let E be the spectral resolution of π. Let (H_x, π_x, Ω_x) for $x \in K$*

be the GNS *representations of which* (H, π) *is the direct sum (as above). Represent each* Ω_x *in the canonical way as an element of* H*, and still denote it by* Ω_x*. It then follows that* $E(V)$ *is the projection of* H *onto the subspace*

$$\overline{\mathrm{span}\{\Omega_x : x \in V\}}$$

for any Borel $V \subset K$*.*

Proof. Consider any $x \in K$ and define the positive Borel measure E_x on K by $E_x(V) = \langle \Omega_x, E(V)\Omega_x \rangle$. The first thing to notice is that from $E(K) = 1$ and $\|\Omega_x\| = 1$ we have $E_x(K) = 1$. Since E_x is regular as mentioned above, we can in particular approximate $E_x(\{x\})$ from above, namely for any $\varepsilon > 0$ there exists an open set $V_0 \subset K$ containing x such that

$$E_x(V_0) < E_x(\{x\}) + \varepsilon. \tag{3.2}$$

Since $\{x\}$ is closed and K is normal, we know by Urysohn's lemma that there is a continuous $f : K \to [0,1]$ such that $f(x) = 1$ and $f|_{K\backslash V_0} = 0$. Furthermore, since $f \in C(K)$, it follows that

$$\int_K f \mathrm{d}E_x = \langle \Omega_x, \pi(f)\Omega_x \rangle = f(x).$$

Also note from $\chi_{\{x\}} \leqslant f \leqslant \chi_{V_0}$, where χ denotes characteristic functions, that $\int_K \chi_{\{x\}} \mathrm{d}E_x \leqslant \int_K f \mathrm{d}E_x \leqslant \int_K \chi_{V_0} \mathrm{d}E_x$. Hence $E_x(\{x\}) \leqslant f(x) \leqslant E_x(V_0)$, which when combined with (3.2) gives

$$f(x) - \varepsilon < E_x(\{x\}) \leqslant f(x)$$

and since $\varepsilon > 0$ was arbitrary, this means that $E_x(\{x\}) = f(x) = 1$. Thus for any Borel $V \subset K$ containing x we have $1 = E_x(\{x\}) \leqslant E_x(V) \leqslant E_x(K) = 1$, so $\|E(V)\Omega_x\|^2 = E_x(V) = 1 = \|\Omega_x\|^2$ hence $E(V)\Omega_x = \Omega_x$, since $E(V)$ is a projection. For any Borel $V \subset K$ not containing x, on the other hand, it follows that $\|E(V)\Omega_x\|^2 = E_x(V) = 1 - E_x(K\backslash V) = 0$, so $E(V)\Omega_x = 0$. We have therefore shown that

$$E(V)\Omega_x = \begin{cases} \Omega_x & \text{if } x \in V, \\ 0 & \text{if } x \in K\backslash V, \end{cases}$$

for all Borel $V \subset K$. However, we have already argued that the vectors Ω_x, $x \in K$, form a total orthonormal set in H, which completes the proof. ∎

Now we argue intuitively as follows: By Theorem 3.1, $E(\{x\})$ is the projection onto $\mathbb{C}\Omega_x$ for every $x \in K$, hence it seems clear that E attaches an equal "weight" to every point of K, namely projections of equal size, in this case one-dimensional. More generally, if the Borel sets $V_1, V_2 \subset K$ are "equally big", by which we intuitively mean they have "the same number of points", then Theorem 3.1 tells us that $E(V_1)H$ and $E(V_2)H$ are "equally big" in the sense that they are both spanned by "the same number of vectors" from the total orthonormal set

$\{\Omega_x : x \in K\}$, i.e. we can view the projections $E(V_1)$ and $E(V_2)$ as being equally big.

We will apply these ideas in the case where K is a product space of the form $\overline{\mathrm{Aut}(A)}^T$. Therefore we next study spectral measures as obtained above, in the case of topological product spaces.

THEOREM 3.2. *Let $\mathfrak{T}$ be any non-empty set, and K_t a compact Hausdorff space for every $t \in \mathfrak{T}$. Set*

$$K_T := \prod_{t\in T} K_t$$

with the product topology for every $T \subset \mathfrak{T}$, write $K := K_{\mathfrak{T}}$, let $\iota_T : K \to K_T : (x_t)_{t\in\mathfrak{T}} \mapsto (x_t)_{t\in T}$ and define $\psi_T : C(K_T) \to C(K) : f \mapsto f \circ \iota_T$. Let H be any Hilbert space and $\varphi : C(K) \to L(H)$ any unital $*$-homomorphism, and set $\varphi_T := \varphi \circ \psi_T$ for every $T \subset \mathfrak{T}$. Let E_T be the spectral resolution of φ_T and write $E := E_{\mathfrak{T}}$, i.e. E is the spectral resolution of φ. Then we have

$$E_T = E \circ \iota_T^{-1}$$

for all $T \subset \mathfrak{T}$ with $T \neq \varnothing$.

Proof. The case $T = \mathfrak{T}$ is trivial, so we will assume $T \neq \mathfrak{T}$. This will ensure that products written as $K_T \times K_{\mathfrak{T}\setminus T}$ are not trivial.

For any $x \in H$ we know from previous remarks that $E_{T,x} := (E_T)_{x,x}$ and $E_x := E_{x,x}$ are regular positive Borel measures. Since ι_T is continuous, we can similarly define

$$(E \circ \iota_T^{-1})_x(V) := \langle x, (E \circ \iota_T^{-1})(V)x\rangle = \langle x, E(\iota_T^{-1}(V))x\rangle = E_x \circ \iota_T^{-1}(V)$$

for every Borel $V \subset K_T$, from which it is clear that $F_{T,x} := (E \circ \iota_T^{-1})_x = E_x \circ \iota_T^{-1}$ is a positive Borel measure on K_T. Note that here we use the notation $F_T = E \circ \iota_T^{-1}$. We now firstly prove that $F_{T,x}$ is regular:

Consider any Borel $V \subset K_T$, then $\iota_T^{-1}(V)$ is a Borel set in K, but E_x is regular, so for any $\varepsilon > 0$ there exists a compact set $V_1 \subset K$ and an open set $V_0 \subset K$ such that $V_1 \subset \iota_T^{-1}(V) \subset V_0$ and

$$E_x(V_0) - \varepsilon < E_x(\iota_T^{-1}(V)) < E_x(V_1) + \varepsilon.$$

Since ι_T is continuous, $V_1' := \iota_T(V_1)$ is compact, and clearly we also have $V_1' \subset \iota_T(\iota_T^{-1}(V)) \subset V$ (in fact, the last inclusion is equality, since ι_T is a surjection) and $V_1 \subset \iota_T^{-1}(V_1')$ so $E_x(V_1) \leqslant F_{T,x}(V_1')$. With the set V_0 we need to be a bit more careful. We need an open $V_0' \subset K_T$ such that $V \subset V_0'$ and $\iota_T^{-1}(V_0') \subset V_0$. Since ι_T is a projection, we have $\iota_T^{-1}(V) = V \times K_{\mathfrak{T}\setminus T}$ while $K_{\mathfrak{T}\setminus T}$ is compact by Tychonoff's theorem. Hence, for every $v \in V$ we have $\{v\} \times K_{\mathfrak{T}\setminus T} \subset V_0$, and the tube lemma says that there is a "tube" $N_v \times K_{\mathfrak{T}\setminus T}$ with N_v an open neighbourhood of v in K_T such that $\{v\} \times K_{\mathfrak{T}\setminus T} \subset N_v \times K_{\mathfrak{T}\setminus T} \subset V_0$. Let $V_0' := \bigcup_{v\in V} N_v$, which is then an open set in K_T such that $V = \bigcup_{v\in V}\{v\} \subset V_0'$, and $\iota_T^{-1}(V_0') = \bigcup_{v\in V} N_v \times K_{\mathfrak{T}\setminus T} \subset V_0$ hence

$F_{T,x}(V_0') \leqslant E_x(V_0)$. To summarize, we have found a compact $V_1' \subset K_T$ and an open $V_0' \subset K_T$ such that $V_1' \subset V \subset V_0'$ and

$$F_{T,x}(V_0') - \varepsilon < F_{T,x}(V) < F_{T,x}(V_1') + \varepsilon$$

which means that $F_{T,x}$ is regular.

Now we can prove the theorem. For any $f \in C(K_T)$ we have

$$\int_{K_T} f \mathrm{d}F_{T,x} = \int_K (f \circ \iota_T) \mathrm{d}E_x = \langle x, \varphi_T(f)x \rangle = \int_{K_T} f \mathrm{d}E_{T,x}$$

and since both $E_{T,x}$ and $F_{T,x}$ are regular, we then know from Riesz's representation theorem that $E_{T,x} = F_{T,x}$. In other words

$$\langle x, E_T(V)x \rangle = \langle x, F_T(V)x \rangle$$

for all $x \in H$ and all Borel $V \subset K_T$, hence by the polarization identity $E_T = F_T$. ∎

COROLLARY 3.3. *Extend φ_T in Theorem 3.2 to a unital $*$-homomorphism $\widetilde{\varphi}_T : B_\infty(K_T) \to L(H)$ defined by*

$$\widetilde{\varphi}_T(f) := \int_{K_T} f \mathrm{d}E_T$$

for all $T \subset \mathfrak{T}$, and write $\widetilde{\varphi} := \widetilde{\varphi}_{\mathfrak{T}}$ which is therefore the extension of φ given by E. Set $\widetilde{\psi}_T : B_\infty(K_T) \to B_\infty(K) : f \mapsto f \circ \iota_T$. Then $\widetilde{\varphi}_T = \widetilde{\varphi} \circ \widetilde{\psi}_T$ for all non-empty $T \subset \mathfrak{T}$.

Proof. Note that $\widetilde{\psi}_T$ is well defined, since ι_T is continuous. For any $x \in H$ we have using the notation of the previous proof that

$$\langle x, \widetilde{\varphi}_T(f)x \rangle = \int_{K_T} f \mathrm{d}E_{T,x} = \int_{K_T} f \mathrm{d}(E_x \circ \iota_T^{-1}) = \int_K (f \circ \iota_T) \mathrm{d}E_x = \langle x, \widetilde{\varphi} \circ \widetilde{\psi}_T(f)x \rangle$$

for all $f \in B_\infty(K_T)$. Hence by the polarization identity $\widetilde{\varphi}_T = \widetilde{\varphi} \circ \widetilde{\psi}_T$. ∎

We also have the following simple proposition regarding spectral resolutions:

PROPOSITION 3.4. *Let K be a compact Hausdorff space, H a Hilbert space, $\varphi : C(K) \to L(H)$ a unital $*$-homomorphism and E its spectral resolution. For any unitary $U : H \to H$ set $\varphi' := U^*\varphi(\cdot)U$ and let E' be its spectral resolution. Then $E' = U^*E(\cdot)U$. Furthermore, for the situation in Theorem 3.2 and Corollary 3.3, and with E_T' the spectral resolution of $\varphi_T' := \varphi' \circ \psi_T$, we have $E_T' = U^*E_T(\cdot)U$ and hence $\widetilde{\varphi}_T' = U^*\widetilde{\varphi}_T(\cdot)U$ with $\widetilde{\varphi}_T'$ the extension of φ_T' to $B_\infty(K_T)$ given by E_T'.*

Proof. We use a similar argument as in the proof of Theorem 3.2. Let $F := U^*E(\cdot)U$; then $F_{x,x} := \langle x, F(\cdot)x \rangle = E_{Ux,Ux}$ is a regular positive Borel measure on K for all $x \in H$ by the properties of E. Furthermore, for every $f \in C(K)$ and every $x \in H$ we have $\int_K f \mathrm{d}E'_{x,x} = \langle x, \varphi'(f)x \rangle = \langle Ux, \varphi(f)Ux \rangle = \int_K f \mathrm{d}F_{x,x}$ but

$E'_{x,x}$ is also a regular positive Borel measure on K by definition, hence by Riesz's representation theorem and the polarization identity $E' = F$.

Now, for the situation in Theorem 3.2 and with $\widetilde{\varphi}' := \widetilde{\varphi}'_{\mathfrak{T}}$, consider any Borel $V \subset K_T$, then $E'_T(V) = \widetilde{\varphi}'_T(\chi_V) = \widetilde{\varphi}' \circ \widetilde{\psi}_T(\chi_V) = \widetilde{\varphi}'(\chi_{\iota_T^{-1}(V)}) = E'(\iota_T^{-1}(V)) = U^* E(\iota_T^{-1}(V))U = U^* E_T(V)U$ by Corollary 3.3 and Theorem 3.2.

For any $f \in B_\infty(K_T)$ and $x \in H$ it now follows (using the notation of Theorem 3.2's proof) that $\langle x, \widetilde{\varphi}'_T(f)x\rangle = \int_{K_T} f \mathrm{d}E'_{T,x} = \int_K f \mathrm{d}E_{T,Ux} = \langle Ux, \widetilde{\varphi}_T(f)Ux\rangle = \langle x, U^*\widetilde{\varphi}_T(f)Ux\rangle$. ∎

These results will be used in the next section.

The remainder of this section shows how the imbedding $\psi_T : C(K_T) \to C(K)$ that appears in Theorem 3.2, or to be more precise, the extended imbedding $\widetilde{\psi}_T$ in Corollary 3.3, can be carried over to the pure representations of $C(K_T)$ and $C(K)$. This rounds off the discussion in this section, but is less important for the rest of the paper. Note that by the term *imbedding* we mean an injective $*$-homomorphism from one C^*-algebra to another. By standard theory of C^*-algebras such an imbedding is automatically norm preserving, and its image a C^*-algebra.

Let $\mathfrak{T}$, K_t, K_T, K and ψ_T be as in Theorem 3.2, let (H_T, θ_T) be a pure representation of $C(K_T)$ for every non-empty $T \subset \mathfrak{T}$ and set $(H, \pi) := (H_{\mathfrak{T}}, \theta_{\mathfrak{T}})$. Keep in mind that $\widetilde{\theta}_T(B_\infty(K_T)) \subset L(H_T)$ is a C^*-algebra, since $B_\infty(K_T)$ is a C^*-algebra and $\widetilde{\theta}_T$ is a $*$-homomorphism. We simply want to show that there is an imbedding $\eta_T : \widetilde{\theta}_T(B_\infty(K_T)) \to L(H)$ such that $\eta_T \circ \widetilde{\theta}_T = \widetilde{\pi} \circ \widetilde{\psi}_T$, i.e. we have a commutative diagram, which means that the imbedding $\widetilde{\psi}_T : B_\infty(K_T) \to B_\infty(K)$ has been carried over to $\widetilde{\theta}_T(B_\infty(K_T)) \to L(H)$ in a consistent way.

To do this it will be notationally convenient to write pure representations in a slightly more concrete form. For any compact Hausdorff space K it is easily seen that the GNS representation of the pure state ω_x previously used in constructing a pure representation (H, π) of $C(K)$ can be taken to be $(H_x, \pi_x, \Omega_x) = (\mathbb{C}, \omega_x, 1)$ where here we view $\omega_x(f)$ as an element of $L(\mathbb{C})$ by $\mathbb{C} \to \mathbb{C} : z \mapsto \omega_x(f)z$ for all $f \in C(K)$. This pure representation is then given by

$$H := \bigoplus_{x\in K} \mathbb{C} \tag{3.3}$$

$$\pi := \bigoplus_{x\in K} \omega_x \tag{3.4}$$

so $\pi(f) = \bigoplus_{x\in K} f(x)$ for all $f \in C(K)$ where $\bigoplus_{x\in K} a_x$, with any bounded $K \ni x \mapsto a_x \in \mathbb{C}$, denotes an element of $L(H)$ defined by $\left(\bigoplus_{x\in K} a_x\right)(v_x)_{x\in K} := (a_x v_x)_{x\in K}$ for $(v_x)_{x\in K} \in H$. When we use the form (3.3) and (3.4) we will say that π is in *diagonal form*. Since the GNS representation of ω_x can always be written in the form $(\mathbb{C}, \omega_x, 1)$, we know that a pure representation can always be written in diagonal form.

PROPOSITION 3.5. *Let K be a compact Hausdorff space and (H, π) a pure representation of $C(K)$ in diagonal form. Then*

$$\widetilde{\pi}(f) = \bigoplus_{x \in K} f(x)$$

for all $f \in B_\infty(K)$.

Proof. Let E be the spectral resolution of π and define $\Omega_x \in H$ by $\Omega_x := (\Omega_{x,y})_{y \in K}$ where $\Omega_{x,y} = 1$ for $y = x$ and $\Omega_{x,y} = 0$ otherwise. Hence Ω_x, $x \in K$, is the total orthonormal set in H that we used previously (but now in π 's diagonal form). Setting $E_{x,y} := \langle \Omega_x, E(\cdot)\Omega_y \rangle$ for all $x, y \in K$ it follows from Theorem 3.1 that

$$E_{x,x}(V) = \begin{cases} 1 & \text{if } x \in V, \\ 0 & \text{if } x \notin V, \end{cases}$$

for all Borel $V \subset K$, while $E_{x,y} = 0$ for $x \neq y$. Hence

$$\langle \Omega_x, \widetilde{\pi}(f)\Omega_y \rangle = \int_K f \mathrm{d}E_{x,y} = f(x)\langle \Omega_x, \Omega_y \rangle = \left\langle \Omega_x, \Big[\bigoplus_{z \in K} f(z)\Big]\Omega_y \right\rangle$$

and since Ω_x, $x \in K$, is a total orthonormal set in H, the result follows. ∎

COROLLARY 3.6. *Let K be a compact Hausdorff space and (H, π) a pure representation of $C(K)$, then $\widetilde{\pi} : B_\infty(K) \to L(H)$ is injective.*

Proof. Without loss we can put π in diagonal form, hence $\widetilde{\pi}$ is given by Proposition 3.5. Now for any bounded $K \ni x \mapsto a_x \in \mathbb{C}$ and $K \ni x \mapsto b_x \in \mathbb{C}$ with $\bigoplus_{x \in K} a_x = \bigoplus_{x \in K} b_x$ as elements of $L(H)$, we have

$$a_x = \left\langle \Omega_x, \Big(\bigoplus_{y \in K} a_y\Big)\Omega_x \right\rangle = \left\langle \Omega_x, \Big(\bigoplus_{y \in K} b_y\Big)\Omega_x \right\rangle = b_x$$

for every $x \in K$, with Ω_x as in Proposition 3.5's proof. In particular, if $\widetilde{\pi}(f) = \widetilde{\pi}(g)$ for $f, g \in B_\infty(K)$, then $f = g$. ∎

COROLLARY 3.7. *For the situation in Theorem 3.2, but with $\varphi = \pi$ a pure representation of $C(K)$, it follows that $\widetilde{\pi}_T$ is injective, and with π in diagonal form it is given by*

$$\widetilde{\pi}_T(f) = \bigoplus_{x \in K} f(\iota_T(x)) \tag{3.5}$$

for all $f \in B_\infty(K_T)$.

Proof. By Corollary 3.3 we have $\widetilde{\pi}_T(f) = \widetilde{\pi}(f \circ \iota_T)$, and since $\widetilde{\pi}$ is injective by Corollary 3.6 while ι_T is surjective, it follows that $\widetilde{\pi}_T$ is injective. With π in diagonal form, (3.5) follows immediately from Corollary 3.3 and Proposition 3.5. ∎

THEOREM 3.8. *For the situation in Theorem* 3.2*, let* (H_T, θ_T) *be a pure representation of* $C(K_T)$*, and set* $\pi := \theta_{\mathfrak{T}}$ *and* $\pi_T := \pi \circ \psi_T$ *for every non-empty* $T \subset \mathfrak{T}$*. Then for every such* T *there is a unique function*

$$\eta_T : \widetilde{\theta}_T(B_\infty(K_T)) \to L(H)$$

such that $\eta_T \circ \widetilde{\theta}_T = \widetilde{\pi}_T$*. Furthermore,* η_T *is an injective norm preserving unital* $*$*-homomorphism. With* π *and* θ_T *in diagonal form, it is given by*

$$\eta_T\Big(\bigoplus_{x \in K_T} f(x)\Big) = \bigoplus_{x \in K} f(\iota_T(x)) \tag{3.6}$$

for all $f \in B_\infty(K_T)$*.*

Proof. The existence and uniqueness of η_T follow from the injectivity of $\widetilde{\theta}_T$ given by Corollary 3.6. It is a $*$-homomorphism, since $\widetilde{\theta}_T$ and $\widetilde{\pi}_T$ are, and it is injective, since $\widetilde{\pi}_T$ is injective according to Corollary 3.7. Hence η_T is norm preserving. It is unital since $\widetilde{\theta}_T$ and $\widetilde{\pi}_T$ are. In diagonal form $\widetilde{\theta}_T$ is given by Proposition 3.5, hence (3.6) follows directly from (3.5). ∎

COROLLARY 3.9. *For the situation in Theorem* 3.8*, and with* E_T *and* F_T *the spectral resolutions of* π_T *and* θ_T *respectively, we have* $\eta_T \circ F_T = E_T$*.*

Proof. For any Borel $V \subset K_T$ we have $\eta_T(F_T(V)) = \eta_T(\widetilde{\theta}_T\,(\chi_V)) = \widetilde{\pi}_T(\chi_V) = E_T(V)$. ∎

4. EVOLUTION INTEGRALS

Now we apply the ideas of the previous section to find an analogue of path integrals for abstract dynamical systems. Fix an arbitrary set $\mathfrak{T}$. We will allow T to be any subset of $\mathfrak{T}$. We will view $\mathfrak{T}$ as the set of all points in "time" (corresponding to $\mathbb{R}$ in usual quantum mechanics), and the T 's as "time intervals". Let A be a W^*-algebra as in Section 2, and set

$$X_T := \overline{\mathrm{Aut}(A)}^T$$

which is the evolution space over T. Write $X := X_{\mathfrak{T}}$. Our goal is to do integrals over X_T to represent dynamics on a "higher level" as discussed in the introduction, the higher level being a Hilbert space obtained from a pure representation. However, we would like to use the same Hilbert space for different T, since then we can interpret the integrals for different T 's to represent the dynamics of the same system but over different "time intervals". Therefore we will imbed $C(X_T)$ canonically into $C(X)$ and then consider a pure representation of $C(X)$. To do this let ι_T and ψ_T be defined as in Theorem 3.2 in terms of $K = X$ and $K_T = X_T$. Then ψ_T is a well defined injective norm preserving unital $*$-homomorphism, which can be viewed as a canonical imbedding of $C(X_T)$ into $C(X)$. Let (H, π) be a pure representation of $C(X)$, which makes H independent of T, and then consider the spectral resolution $E_T := E_{\pi_T}$ of the injective unital $*$-homomorphism

$$\pi_T := \pi \circ \psi_T : C(X_T) \to L(H).$$

This spectral measure E_T can be viewed as being natural, since π and ψ_T are both canonical (also see Section 3), and allows us to do integrals over the evolution space X_T, namely

$$\widetilde{\pi}_T(f) = \int\limits_{X_T} f \mathrm{d}E_T \tag{4.1}$$

is defined for all $f \in B_\infty(X_T)$. We will call integrals of the form (4.1) *evolution integrals*.

Theorem 3.8 tells us that it would in fact be mathematically equivalent to work on the various Hilbert spaces H_T, $T \subset \mathfrak{T}$, however we will express everything in terms of H, since this is a simpler point of view as far as the dynamical system on the higher level is concerned.

The arguments in Section 3 that $E := E_{\mathfrak{T}}$ attaches the same weight to all the points of $K = X$ can be interpreted as each evolution over $\mathfrak{T}$ having the same weight. Via Theorem 3.2 we can then also say that E_T attaches the same weight to each of the evolutions over T, namely to each point of X_T. This is analogous to path integrals, as explained at the beginning of Section 3, and hence is exactly the type of structure that we intuitively want.

Our interpretation of H in a pure representation of $C(K)$ in Section 3 gives us a nice picture in the case where $K = X$, namely the vectors $\{\Omega'_\alpha\}_{\alpha \in X}$ defined as in the case of K represent the evolutions of A over the entire $\mathfrak{T}$ as a total orthonormal set in H. In a similar way an evolution $\beta \in X_T$ corresponds to the set of evolutions in X projected onto β by ι_T, and hence to the set of Ω'_α 's with $\iota_T(\alpha) = \beta$.

The basic idea for getting dynamics on H is to consider a unitary $u_T \in B_\infty(X_T)$, and then set $U_T = \widetilde{\pi}_T(u_T)$ which is a unitary operator on H, since $\widetilde{\pi}_T$ is a unital $*$-homomorphism. We will interpret U_T as representing dynamics on H over the set T, and will discuss this in more detail below, and in the next section.

Note that since $B_\infty(X_T)$ is abelian and $\widetilde{\pi}_T$ is a homomorphism, all U_T 's obtained in this way will commute with each other. This is where the fact that a pure representation of $C(K)$ is not unique, as mentioned in Section 3, comes into play. To obtain unitaries on H which do not commute with these U_T 's, we can use a unitary transformation of π to get another pure representation of $C(K)$, namely $\pi' := U^* \pi(\cdot) U$ with U a unitary operator on H, and then replace E_T by the spectral resolution E'_T of $\pi'_T := \pi' \circ \psi_T$. By Proposition 3.4 we get $U'_T := \int\limits_{X_T} u_T \mathrm{d}E'_T = U^* U_T U$ instead of U_T. In this sense evolution integrals differ from path integrals. Instead of having one $\mathbb{R}^+ \cup \{\infty\}$-valued measure, we have many projection valued measures.

Although we will not discuss detailed examples and applications in this paper, in this paragraph we give a brief description of how simple examples can be obtained, before we resume with the theory. As mentioned above we are interested in unitary operators on H given by $U_T = \widetilde{\pi}_T(u_T)$ with $u_T \in B_\infty(X_T)$ unitary. A simple case of this would be $u_T = \mathrm{e}^{\mathrm{i}S_T}$ where $S_T : X_T \to \mathbb{R}$ is continuous. Hence we look at a simple class of examples of such an S_T. Let T be a finite subset of $\mathfrak{T}$ and let $f_t \in L(A)_*$ for every $t \in T$. Let $g_t : \mathbb{C} \to \mathbb{R}$ be continuous for every $t \in T$, for example $g_t = |\cdot|$ or $g_t = |\cdot|^2$. Fix any $(\tau_t)_{t\in T} \in X_T$. For $f \in L(A)_*$ and $\alpha \in \overline{\mathrm{Aut}(A)}$ we view α as a linear functional on $L(A)_*$ and denote its value at f by $f(\alpha)$. Now set $S_T(\alpha) := \sum_{t\in T} g_t(f_t(\alpha_t - \tau_t))$ for all $\alpha = (\alpha_t)_{t\in T} \in X_T$. Then S_T is continuous, since T is finite and we are using the product topology on X_T. This example gives some indication that we would have to be careful when attempting to construct examples for the case where T is not finite.

Next we refine our idea for obtaining dynamics on H, by taking $\mathfrak{T}$ to be a measure space $(\mathfrak{T}, \Sigma, \mu)$ with Σ a σ-algebra in $\mathfrak{T}$ and μ a usual positive measure on Σ. In the following section this will allow us to clarify the analogy with path integrals. Let $\mathcal{U}(\mathfrak{A})$ denote the set of all unitary elements of any unital C^*-algebra $\mathfrak{A}$.

DEFINITION 4.1. Let $(\mathfrak{T}, \Sigma, \mu)$ be a measure space, and $\Sigma_0 \subset \Sigma$ a set such that $T_1 \cup T_2 \in \Sigma_0$ when $T_1, T_2 \in \Sigma_0$. Now consider a function

$$u : \Sigma_0 \to \bigcup_{T\in\Sigma_0} B_\infty(X_T) : T \mapsto u_T$$

such that

$$u_T \in \mathcal{U}(B_\infty(X_T))$$

for all $T \in \Sigma_0$,

$$u_{T_1\cup T_2}(\iota_{T_1\cup T_2}(\alpha)) = u_{T_1}(\iota_{T_1}(\alpha))u_{T_2}(\iota_{T_2}(\alpha)) \tag{4.2}$$

for all $\alpha \in X$ and all $T_1, T_2 \in \Sigma_0$ with $\mu(T_1 \cap T_2) = 0$, and

$$u_T = 1 \tag{4.3}$$

for all $T \in \Sigma_0$ with $\mu(T) = 0$. Such a u, or (u, Σ_0, μ) to be more complete, will be called an *action weight* for $(A, \mathfrak{T})$. If $u_T \in \mathcal{U}(C(X_T))$ for all $T \in \Sigma_0$, we will call u a *continuous action weight*.

The word "action" in the term action weight is borrowed from the classical action which appears in usual path integrals in quantum mechanics. Note that when using an action weight, we no longer allow all subsets $T \subset \mathfrak{T}$, but only $T \in \Sigma_0$. A typical situation might be where $\mathfrak{T}$ is a locally compact Hausdorff group, Σ its Borel σ-algebra, μ its Haar-measure, and Σ_0 the Borel sets with finite measure.

So let u be an action weight as in Definition 4.1, and define

$$U_T = \int_{X_T} u_T \mathrm{d}E_T \tag{4.4}$$

for all $T \in \Sigma_0$ with the convention that $U_\varnothing = 1$ if $\varnothing \in \Sigma_0$. Note that U_T is defined in terms of a fixed pure representation π of $C(X)$, so all the U_T 's commute with one another as discussed earlier. Although u corresponds to the classical level in path integrals, and U to the quantum level, one should keep in mind that A need not be abelian, hence we in general do not view u as coming from "classical" dynamics, unless A is abelian. Next we show that the dynamics on H given by the U_T 's have familiar grouplike properties:

PROPOSITION 4.2. *Let U_T be defined as in* (4.4)*. Then we have*

$$U_{T_1}U_{T_2} = U_{T_1\cup T_2}$$

for all $T_1, T_2 \in \Sigma_0$ with $\mu(T_1 \cap T_2) = 0$, and

$$U_T = 1$$

for all $T \in \Sigma_0$ with $\mu(T) = 0$.

Proof. If any of T_1, T_2 or T are empty, the result is trivial, so assume they are not empty. Using Corollary 3.3 and its notation, we have

$$\begin{aligned} U_{T_1}U_{T_2} &= \widetilde{\pi}_{T_1}(u_{T_1})\widetilde{\pi}_{T_2}(u_{T_2}) = \widetilde{\pi}(\widetilde{\psi}_{T_1}(u_{T_1}))\widetilde{\pi}(\widetilde{\psi}_{T_2}(u_{T_2})) \\ &= \widetilde{\pi}(\widetilde{\psi}_{T_1}(u_{T_1})\widetilde{\psi}_{T_2}(u_{T_2})) = \widetilde{\pi}(\widetilde{\psi}_{T_1\cup T_2}(u_{T_1\cup T_2})) = U_{T_1\cup T_2} \end{aligned}$$

by (4.2). From (4.3) on the other hand, we immediately have $U_T = \widetilde{\pi}_T(u_T) = \widetilde{\pi}_T(1) = 1$. ∎

These are the properties that one would have in quantum mechanics coming from path integrals over closed intervals, say $T_1 = [t_0, t_1]$ and $T_2 = [t_1, t_2]$, with $\mathfrak{T} = \mathbb{R}$, μ the Lebesgue measure on $\mathbb{R}$, and Σ_0 for example being the sets with finite Lebesgue measure. There one would interpret these properties as the group structure of the quantum dynamics, with inverses of the unitaries providing the group inverse. Since all the U_T 's commute, this grouplike structure is inherently abelian.

REMARK 4.3. Any unital $*$-homomorphism $\varphi : C(X) \to L(H)$, with H any Hilbert space, in principle gives us another definition of evolution integrals if we replace π by φ, but π seems most canonical, and also gives a very simple interpretation of H as discussed above. Note that Proposition 4.2 still holds if we replace π by such a φ. One other choice besides π that would be canonical, is the universal representation of $C(X)$.

A different approach we could have followed in setting up our framework is to begin with a Hilbert space H_0 instead of A, and then consider the set $\mathcal{U}(H_0)$ of unitary operators $H_0 \to H_0$ instead of $\mathrm{Aut}(A)$. Since $\mathcal{U}(H_0) \subset L(H_0)$ with

$L(H_0)$ a von Neumann algebra which therefore has a unique predual, we could look at the closure $\overline{\mathcal{U}(H_0)}$ in the resulting weak* topology on $L(H_0)$. Then replace $\overline{\mathrm{Aut}(A)}^T$ in the framework we set up above with the compact Hausdorff space $\overline{\mathcal{U}(H_0)}^T$, the elements of which we would interpret as evolutions in H_0. However, to start with A instead of H_0 seems more natural simply because the algebraic formulation is a more natural way to formulate a dynamical system, with a Hilbert space being a specific way of representing such a system. This is especially clear if we start with a classical system on the lower level. This raises the question if the dynamical system on the higher level, namely the Hilbert space H and the dynamics on it, is in some natural way a representation of an algebraic formulation of the same dynamical system.

5. THE ANALOGY WITH PATH INTEGRALS

We now make the analogy between evolution integrals and path integrals more explicit. We will continue using the notation from the previous section. In particular, Σ is still a σ-algebra in $\mathfrak{T}$. For $T \in \Sigma$, set $\Sigma|_T := \{V \cap T : V \in \Sigma\}$ which is a σ-algebra in T. Denote the vector space of bounded $\Sigma|_T$-measurable functions $T \to \mathbb{C}$ with the sup-norm by $B_\infty(\Sigma|_T)$. Although this space is a C^*-algebra, we will only use its normed space structure.

DEFINITION 5.1. Let $\Sigma_0 \subset \Sigma$ be the sets with finite μ-measure. Consider a mapping $\mathcal{L} : T \mapsto \mathcal{L}_T$ on Σ_0 such that $\mathcal{L}_T : X_T \to B_\infty(\Sigma|_T) : \alpha \mapsto \mathcal{L}_{T,\alpha}$ is continuous, $\mathcal{L}_{T,\alpha}$ is real-valued and $\mathcal{L}_{T,\alpha}|_{T'} = \mathcal{L}_{T',\alpha|_{T'}}$ for all $T, T' \in \Sigma_0$ with $T' \subset T$. We will call an $\mathcal{L}$ with these properties a *Lagrangian*. Define $S_T : X_T \to \mathbb{R}$ by

$$S_T(\alpha) := \int_T \mathcal{L}_{T,\alpha} \mathrm{d}\mu$$

for all $\alpha \in X_T$ and all $T \in \Sigma_0$. The mapping $S : T \mapsto S_T$ defined on Σ_0 will be called the *action* of $\mathcal{L}$.

The terminology in this definition is of course borrowed from classical mechanics. The simple example given in Section 4 is such an action.

PROPOSITION 5.2. *The function S_T in Definition 5.1 is continuous, hence S is a function $\Sigma_0 \to \bigcup_{T \in \Sigma_0} C(X_T)$ with $S_T \in C(X_T)$ for every $T \in \Sigma_0$. Setting $u_T := \mathrm{e}^{\mathrm{i}S_T}$ for all $T \in \Sigma_0$ makes the function u given by $\Sigma_0 \ni T \mapsto u_T$ a continuous action weight.*

Proof. For any $\alpha, \beta \in X_T$ we have

$$|S_T(\alpha) - S_T(\beta)| = \Big| \int_T (\mathcal{L}_{T,\alpha} - \mathcal{L}_{T,\beta}) \mathrm{d}\mu \Big| \leqslant \|\mathcal{L}_{T,\alpha} - \mathcal{L}_{T,\beta}\| \mu(T)$$

and since $\mu(T) < \infty$ while $\mathcal{L}_T$ is continuous by assumption, we know that for every $\varepsilon > 0$ there is a neighbourhood N of α in X such that

$$|S_T(\alpha) - S_T(\beta)| < \varepsilon$$

for all $\beta \in N$. Hence S_T is continuous. Since S_T is continuous and real-valued, we have $u_T \in \mathcal{U}(C(X_T))$. For $T_1, T_2 \in \Sigma_0$ with $\mu(T_1 \cap T_2) = 0$ we have for all $\alpha \in X$ that

$$\begin{aligned} S_{T_1\cup T_2}(\iota_{T_1\cup T_2}(\alpha)) &= \int\limits_{T_1\cup T_2} \mathcal{L}_{T_1\cup T_2, \iota_{T_1\cup T_2}(\alpha)} \mathrm{d}\mu = \int\limits_{T_1} \mathcal{L}_{T_1, \iota_{T_1}(\alpha)} \mathrm{d}\mu + \int\limits_{T_2} \mathcal{L}_{T_2, \iota_{T_2}(\alpha)} \mathrm{d}\mu \\ &= S_{T_1}(\iota_{T_1}(\alpha)) + S_{T_2}(\iota_{T_2}(\alpha)) \end{aligned}$$

from which (4.2) follows. For $T \in \Sigma_0$ with $\mu(T) = 0$ we immediately have $S_T = 0$, and (4.3) follows. ∎

The evolution integral (4.4) takes the form

$$U_T = \int\limits_{X_T} \mathrm{e}^{\mathrm{i}S_T} \mathrm{d}E_T$$

for all $T \in \Sigma_0$ in the case of the action weight given by Proposition 5.2, which makes the analogy with path integrals in quantum mechanics particularly clear, although a path integral gives an amplitude and therefore more properly corresponds to $\langle x, U_T y \rangle$ with $x, y \in H$.

6. CONCLUDING REMARKS

There are a number of aspects of evolution integrals that could merit further investigation. For example, is it possible to use only continuous evolutions $T \to \overline{\mathrm{Aut}(A)}$ where we assume T is a topological space? Or to use $\mathrm{Aut}(A)^T$ instead of $\overline{\mathrm{Aut}(A)}^T$? Also, if given a Hilbert space representation of an abstract dynamical system, is there a way to decide whether or not its dynamics is given by evolution integrals over some other system with certain properties, for example having an abelian W^*-algebra? And can evolution integrals be linked more closely with quantum physics, for example by trying to find the connection between the Hilbert space H used above and the usual quantum state space in the case where A is abelian and represents a classical physical system?

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