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CORNERS OF GRAPH ALGEBRAS

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ABSTRACT. It is known that given a directed graph E and a subset X of vertices, the sum $\sum\limits_{v\in X} P_v$ of vertex projections in the C^*-algebra of E converges (strictly, in the multiplier algebra) to a projection P_X. Here we give a construction which, in certain cases, produces a directed graph F such that $C^*(F)$ is isomorphic to the corner $P_X C^*(E) P_X$. Corners of this type arise naturally as the fixed-point algebras of discrete coactions on graph algebras related to labellings. We prove this fact, and show that our construction is applicable to such a case whenever the labelling satisfies an analogue of Kirchhoff's voltage law.

KEYWORDS: *Graph algebra, labelling of a directed graph, fixed-point algebra of a discrete coaction.*

MSC (2000): 46L55.

1. INTRODUCTION

The C^*-algebra of a directed graph $E = (E^0, E^1, s, r)$ is a universal object generated by Hilbert space operators satisfying certain relations, where the relations reflect the path structure of the graph [4], [7], [13], [20], [21], [24], [25]. The vertices v of E correspond to projections P_v onto mutually orthogonal subspaces, and the edges e correspond to partial isometries S_e which map between these subspaces. Given any subset X of vertices, the sum $\sum\limits_{v\in X} P_v$ converges strictly to a projection, which we denote by P_X, in the multiplier algebra $M(C^*(E))$ [4]. Corners of the type $P_X C^*(E) P_X$ associated to certain sets X of vertices arise often in the study of graph algebras: see [3], [5], [9], [11], [16], [27], and Section 4. See also the work of Allen on corners of k-graphs, [1].

It is in general very useful to be able to identify an abstract C^*-algebra with a graph algebra. This is because the graphical presentation encodes a great deal of structural information about its associated C^*-algebra, and allows one to compute the algebra's invariants via straightforward calculations [2], [10], [17], [25]. So, given a directed graph E and a subset X of vertices, it might be useful to realize the corner $P_X C^*(E) P_X$ itself as the C^*-algebra of a directed graph. It is to this goal that the first part of this paper is devoted. That is, we shall give a construction which, in certain cases, produces a graph for this corner from the graph E.

Probably the best-known example of such a construction is the procedure described in the literature as "adding a tail to a sink", which is used to approximate the C^*-algebra of a graph containing sinks as a full corner of the algebra of a graph without sinks, as in Lemma 1.2 of [4]. In a similar vein, if the graph E contains infinite-emitters, one may realize $C^*(E)$ as a full corner of the C^*-algebra of a row-finite graph, via a construction due to Drinen and Tomforde [11]. This construction was generalized in [3], and further in [5].

The conditions of Theorem 3.1 in [5] are in practice quite limiting: for example, the theorem is not applicable if the hereditary complement of the set X contains loops, sinks or infinite-emitters. In order to overcome these restrictions, the approach taken here is substantially different to that of [3], [5], [11]. The prototype for our present construction comes from [27], where the graph E was assumed to be finite, and the set X to consist of a single vertex. We generalize and simplify this construction, and fix up a slight error. Unfortunately, this new approach is not applicable to our basic (nonunital) examples, adding a tail at a sink and the desingularization of [11]. However, our construction applies in particular to all unital graph algebras (Lemma 3.6), and it has been shown that any graph algebra can be approximated as a direct limit of unital graph algebras [25].

One context in which corners of graph algebras arise naturally is as fixed-point algebras of certain discrete coactions on graph algebras. Indeed, the motivating example for this research was the construction in [16] of quantum lens spaces as the fixed-point algebras of certain actions of finite cyclic groups on quantum spheres, by analogy with construction of the classical lens spaces. The actions in question arise from labellings, in the sense of Kumjian and Pask [19], who showed that the crossed product of a graph algebra $C^*(E)$ by such a group action is itself isomorphic to the C^*-algebra of a directed graph, called the skew product graph. The work of Kumjian and Pask was generalized in [8] and [18] to cover labellings of directed graphs by discrete (not necessarily abelian) groups. Labellings of this sort give rise to discrete group coactions, rather than the compact group actions of [19], but the realization of the crossed product as the graph algebra of a skew product still works. In Theorem 4.6 we show that, just as in the case of the quantum lens spaces, the fixed-point algebras of these discrete coactions may be recovered as corners of the skew product graph algebras, and then give a condition on labellings which ensures that we may use the construction of Section 3 to realize these corners as graph algebras.

2. PRELIMINARIES

We adopt the "historical" nomenclature of directed graphs and graph algebras, as found in [4], for example. Note in particular that our conventions differ from those used in Raeburn's recent book [24].

The following terminology may be nonstandard:

DIRECTED GRAPHS. Let $E = (E^0, E^1, s, r)$ be a directed graph. A (nonzero) *path* in E is a nonempty ordered set $\mu = \{e_i\}_{i=k}^{n}$ of edges, where $k = 1$ or $-\infty$, $n \in \{1, 2, \dots, \infty\}$ and $s(e_{i+1}) = r(e_i)$ for all i. If $k = 1$ and $n \leqslant \infty$, we say μ is *finite*. If this is not the case, we say μ is *infinite*, and further distinguish between the following three cases: if $k = 1$ and $n = \infty$, μ is *right-infinite*; if $k = -\infty$ and $n < \infty$, μ is *left-infinite*; if $k = -\infty$ and $n = \infty$ we say μ is *bi-infinite*. The *length* of a nonzero path is defined to be the number of elements in the corresponding ordered set. Vertices are considered to be paths of length zero, and for each nonnegative integer m, E^m denotes the collection of all paths in E of length m. E^* denotes the set of all finite paths, and E^∞ denotes the set of all infinite paths. In practice, we usually dispense with the set notation when referring to paths, and simply write $\mu = e_1 e_2 \cdots$ as a concatenation of edges.

Let E be a directed graph, and let $m, n \in \mathbb{N} \cup \{\infty\}$ be such that $n \geqslant m$. If $\mu \in E^m$ and $\nu \in E^n$ are paths of length m and n respectively, such that $\nu_i = \mu_i$ for all $i = 1, \dots, m$, then we say that μ is an *initial subpath* of ν, and write $\mu \prec \nu$.

Each finite path $\mu \in E^*$ gives a finite sequence $s(\mu_1), r(\mu_1), r(\mu_2), \dots, r(\mu)$ of vertices. The path μ is called *vertex-simple* if this sequence contains no repeated vertices (i.e. if μ contains no loops). Similarly, an infinite path $\nu \in E^\infty$ is called vertex-simple if its corresponding right-, left- or bi-infinite ordered set of vertices contains no repetition. Each path of length zero (i.e. each vertex) is also defined to be vertex-simple. A graph E for which $E^* \cup E^\infty$ contains only vertex-simple paths is called an *acyclic* graph. A graph E for which E^∞ contains no vertex-simple paths is called a *path-finite* graph.

If E is a directed graph and F a subgraph of E, then for vertices $u, v \in E^0$ we write $u \geqslant_F v$ to mean that there is a path $\mu \in F^*$ with $s(\mu) = u$ and $r(\mu) = v$. A subset $X \subseteq E^0$ is said to be *hereditary* if it has the property that for all $v \in X$ and $u \in E^0$, $v \geqslant_E u$ implies $u \in X$. For any subset $Y \subseteq E^0$ we shall denote by $H_E(Y)$ the smallest hereditary subset of E^0 containing Y. The set $H_E(Y) \setminus Y$ is referred to as the *hereditary complement* of Y in E.

A subgraph T of a directed graph E is called a *directed forest* in E if it is acyclic and if $|T^1 \cap r^{-1}(v)| \leqslant 1$ for each vertex $v \in T^0$ (that is, if each vertex in T^0 receives at most one edge in T^1). If T is a directed forest in E, let T^{r} denote the subset of T^0 consisting of those vertices v with $|T^1 \cap r^{-1}(v)| = 0$ (these vertices are called the *roots* of T). Let T^{l} denote the subset of T^0 consisting of those vertices v with $|T^1 \cap s^{-1}(v)| = 0$ (these vertices are called the *leaves* of T).

REMARK 2.1. Considering the underlying undirected graph $\mathcal{T}$ (obtained by ignoring the orientations of the edges of T), the above definition implies that each connected component of $\mathcal{T}$ is a tree (that is, has trivial fundamental group). Kumjian and Pask [19] call a directed graph a "directed tree" if its underlying undirected graph is a tree. Our definition thus requires that the connected components of T be directed trees in the sense of Kumjian and Pask, and also that they satisfy the property that no two edges have the same range. Note that some authors include this second property in the definition of a "directed tree". The terminology "directed forest" was chosen with a view toward consistency with the existing nomenclature, and certainly not for its aesthetic appeal.

The concept of a directed forest (in particular, a row- and path-finite one) is central to our construction in Section 3, and the following lemma points out several basic and useful facts about such graphs.

LEMMA 2.2. *Let T be a row-finite, path-finite directed forest in a directed graph E. Then the following hold:*

(i) *For each $v \in T^0$ there exists a unique path $\tau(v)$ in T^* with source in T^{r} and range v. Moreover, for $u, v \in T^0$, $v \geqslant_T u$ if and only if $\tau(v) \prec \tau(u)$.*

(ii) *For each $v \in T^0$ there exist at most finitely many vertices $u \in T^0$ with $v \geqslant_T u$.*

(iii) *For each $v \in T^0$ there exists at least one $u \in T^1$ such that $v \geqslant_T u$.*

(iv) *Suppose $u, v \in T^0$ have $\tau(v) \prec \tau(u)$ and $u \neq v$. Then there exists a unique edge $e \in s^{-1}(v) \cap T^1$ such that $\tau(v)e \prec \tau(u)$. If $f \in s^{-1}(v) \cap T^1$ satisfies $\tau(u) \prec \tau(v)f$, then $f = e$ and $\tau(v)e = \tau(u)$.*

Proof. (i) Fix $v \in T^0$. If $v \in T^{\mathrm{r}}$ then $\tau(v) = v$. If not, then v receives exactly one edge $e_1 \in T^1$. If $s(e_1)$ is in T^{r} then $\tau(v) = e_1$. If not, then $s(e_1)$ receives exactly one edge $e_2 \in T^1$. As T is path-finite we get a path $\tau(v) = e_n e_{n-1} \cdots e_1$ with source in T^{r} after finitely many iterations of this construction. Uniqueness of $\tau(v)$ follows from the fact that each $u \in T^0$ receives at most one edge, and this same fact gives the equivalence $v \geqslant_T u \iff \tau(v) \prec \tau(u)$.

(ii) Suppose $v \in T^0$ is such that infinitely many such u exist. As T is row-finite, there is an edge $e_1 \in s^{-1}(v) \cap T^1$ such that e_1 is the first edge in infinitely many distinct paths in T^* (by the pigeonhole principle). We may apply this same argument to the vertex $r(e_1)$, giving an edge $e_2 \in s^{-1}(r(e_1)) \cap T^1$ such that $e_1 e_2$ is an initial subpath of infinitely many distinct paths in T^*. Continuing this construction gives a path $e_1 e_2 \cdots \in T^\infty$, which must be vertex-simple because T is acyclic. This contradicts the assumption that T is path-finite, proving the claim.

(iii) Fix $v \in T^0$. If $v \in T^1$ then we are done. Otherwise choose $e_1 \in s^{-1}(v) \cap T^1$. If $r(e) \in T^1$ then we are done; otherwise find $e_2 \in s^{-1}(r(e_1)) \cap T^1$. This construction must terminate after finitely many iterations, because T is path-finite.

(iv) Fix $u, v \in T^0$ with $\tau(v) \prec \tau(u)$ and $u \neq v$. Clearly there exists an edge $e \in s^{-1}(v) \cap T^1$ such that $\tau(v)e \prec \tau(u)$. Suppose e' is another such edge.

Then $r(e) = r(e') \in T^0$, contradicting that each vertex in T^0 receives at most one edge. Hence e is unique. Now suppose $f \in s^{-1}(v) \cap T^1$ has $\tau(u) \prec \tau(v)f$. Since $\tau(v) \prec \tau(u)$ we must then have $\tau(u) = \tau(v)f$, so $\tau(v)f \prec \tau(u)$ and $f = e$ by uniqueness of e. ∎

GRAPH C^*-ALGEBRAS. A Cuntz-Krieger E-family is a set $\{P_v, S_e : v \in E^0, e \in E^1\}$ of operators on a Hilbert space such that the elements P_v are mutually orthogonal projections and the elements S_e are partial isometries with mutually orthogonal ranges, satisfying the following relations:

(CK1) $S_e^* S_e = P_{r(e)}$ for all $e \in E^1$;

(CK2) $S_e S_e^* \leqslant P_{s(e)}$ for each $e \in E^1$;

(CK3) $P_v = \sum\limits_{e \in s^{-1}(v)} S_e S_e^*$ for each $v \in E^0$ with $0 < |s^{-1}(v)| < \infty$.

The C^*-algebra of E, denoted $C^*(E)$, is the universal C^*-algebra generated by a Cuntz-Krieger E-family.

For any subset $X \subseteq E^0$, the sum $\sum\limits_{v \in X} P_v$ converges strictly to a projection P_X in $M(C^*(E))$ (Lemma 1.1 in [4]).

REMARK 2.3. As noted earlier, our definitions differ from those of [24] (and several other authors in this area). The chief difference is that in our notation, the partial isometries "go against the edges", in that the support of the partial isometry S_e is the subspace corresponding to the range of e (and vice-versa). We prefer our present convention in this case, as it allows a more intuitive definition of the graphical construction that is central to this work.

3. CORNERS OF DIRECTED GRAPHS

In this section we describe our procedure for constructing a graph for the corner $P_X C^*(E) P_X$ of a graph algebra $C^*(E)$ associated to a vertex set X. This construction is given in the following definition, and its relation to $P_X C^*(E) P_X$ is shown in Theorem 3.5.

DEFINITION 3.1. Let E be a directed graph, $X \subseteq E^0$, and let T be a row-finite, path-finite directed forest in E with $T^{\mathrm{r}} = X$ and $T^0 = H_E(X)$. Define a new directed graph, denoted $E(T)$ and called the *T-corner of E*, as follows:

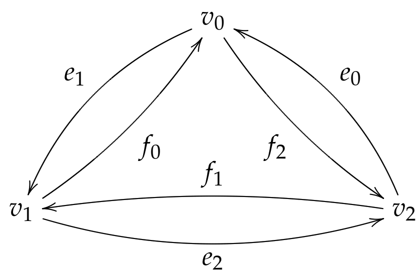
$$E(T)^0 := T^0 \setminus \{v \in T^0 : \emptyset \neq s^{-1}(v) \subseteq T^1\},$$
$$E(T)^1 := \{e_u : e \in s^{-1}(T^0) \setminus T^1, u \in E(T)^0, r(e) \geqslant_T u\},$$
$$s(e_u) = s(e),\ r(e_u) = u.$$

Informally, the vertices of $E(T)$ are those vertices of T which are sinks, or which emit at least one edge not belonging to T. If e is an edge whose source belongs to T but which does not itself form part of T, then there is one edge e_u in

$E(T)$ for each vertex u which can be joined to $r(e)$ by a path in T. We hope that the following examples will clarify this somewhat technical definition.

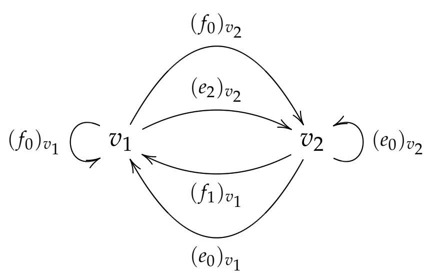
EXAMPLE 3.2. Suppose X is a hereditary subset of E^0. We then have $T^0 = X = T^{\mathrm{r}}$, and since no root may receive an edge in T we infer that T^1 is empty. Thus each $v \in T^0$ is either a sink, or emits an edge which does not belong to T^1; this implies that $E(T)^0 = T^0 = X$. Furthermore, for each edge e with source in T^0, and each vertex $u \in T^0$, $r(e) \geqslant_T u$ if and only if $r(e) = u$, because $T^* = T^0$. Hence $E(T)^1 = \{e_{r(e)} : s(e) \in X\}$, where each $e_{r(e)}$ has the same range and source as e. Thus $E(T)$ is nothing but the graph $(X, s^{-1}(X), s, r)$.

EXAMPLE 3.3. Let E be the graph



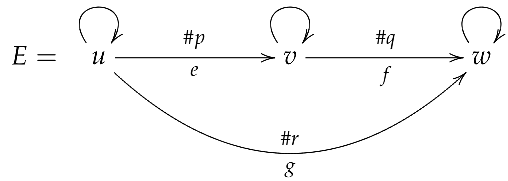
and let $T^0 = E^0$, $T^1 = \{e_1, f_2\}$. T is a row- and path-finite directed forest in E with root set $X = \{v_0\}$, such that $T^0 = H_E(X)$. The vertex v_0 is not a sink, and each edge with source v_0 belongs to T^1, so $v_0 \notin E(T)^0$. On the other hand, both v_1 and v_2 emit edges in E which are not part of T (for example, f_0 and e_0 respectively), so both belong to $E(T)^0$.

Now constructing the edge set $E(T)^1$, we consider in turn each edge in $s^{-1}(T^0) \setminus T^1 = \{e_0, e_2, f_0, f_1\}$; let us start with e_0. The range v_0 of e_0 satisfies $v_0 \geqslant_T v_1$, because e_1 is a path in T^* with source v_0 and range v_1. Since v_1 belongs to $E(T)^0$ there will be an edge $(e_0)_{v_1}$ in $E(T)^1$, with $s((e_0)_{v_1}) = s(e_0) = v_2$ and $r((e_0)_{v_1}) = v_1$. Similarly, there will be an edge $(e_0)_{v_2}$ with source v_2 and range v_2. Notice that, although $v_0 = r(e_0) \geqslant_T v_0$, there is no edge $(e_0)_{v_0}$ because $v_0 \notin E(T)^0$. Considering the remaining edges $e_2, f_0, f_1 \in s^{-1}(T^0) \setminus T^1$ in a similar way, we obtain that $E(T)^1 = \{(e_0)_{v_1}, (e_0)_{v_2}, (e_2)_{v_2}, (f_0)_{v_1}, (f_0)_{v_2}, (f_1)_{v_1}\}$, so $E(T)$ is the following graph:

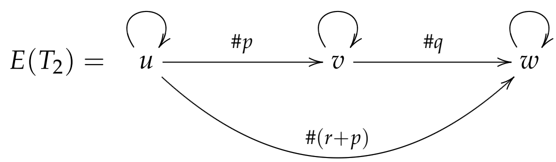


There will often be more than one choice of forest with the desired properties, giving nonisomorphic graphs $E(T)$:

EXAMPLE 3.4. Let p, q, r be positive integers, and let E be as shown:



Here the label "#n" above an arrow indicates that that arrow represents n edges, and a label of "x" below an arrow means that we will distinguish one of those edges and call it x. Then the subgraph T_1 with $T_1^0 = E^0$ and $T_1^1 = \{e, g\}$ is a finite directed forest in E with root $X_1 = \{u\}$, satisfying $T_1^0 = H_E(X_1)$. For this T_1, the construction of Definition 3.1 gives $E(T_1) \cong E$. On the other hand, let T_2 be the finite forest $T_2^0 = E^0$, $T_2^1 = \{e, f\}$. This forest also has root set $X_2 = X_1 = \{u\}$ and $T_2^0 = H_E(X_2)$, but now the construction gives the following graph:



We now state the main result of this section.

THEOREM 3.5. *Let E, X, T and $E(T)$ be as in Definition* 3.1. *Then* $C^*(E(T)) \cong P_X C^*(E) P_X$.

Before we prove this result, we prove the following lemma, which indicates the scope of the main theorem.

LEMMA 3.6. *If $H_E(X) \setminus X$ is finite, then there is a forest which satisfies the conditions of Theorem* 3.5. *If X is finite and $H_E(X)$ infinite, then there is no such forest.*

Proof. First suppose $H_E(X) \setminus X$ is finite. For each $v \in H_E(X)$ let $d(v)$ be the length of a shortest path in E^* with source in X and range v (such a path must exist: if there is no path with source in X and range v, then v does not belong to the hereditary set $H_E(X)$ generated by X). Let $T^0 = H_E(X)$ and construct the edge set T^1 recursively as follows. For each $n \in \mathbb{N}$ and each $v \in H_E(X) \setminus X$ with $d(v) = n$ choose one edge $e_v \in E^1$ such that $r(e_v) = v$ and $d(s(e_v)) = n - 1$. Let $T^1 = \{e_v : v \in H_E(X) \setminus X\}$. Then the subgraph T of E is row- and path-finite by finiteness of $H_E(X) \setminus X$. On the other hand, suppose X is finite and $H_E(X)$ infinite, and suppose T is a directed forest in E with roots X and vertex set $H_E(X)$. By Lemma 2.2(i) and the pigeonhole principle, there must be a vertex

$v \in X$ such that $v \geqslant_T u$ for infinitely many vertices $u \in H_E(X)$. Hence, by part (ii) of Lemma 2.2, T cannot be row- and path-finite. ∎

The proof of Theorem 3.5 will proceed in three main steps: first we find a Cuntz-Krieger family for $E(T)$ inside $C^*(E)$, so that the universal property of $C^*(E(T))$ gives a homomorphism $\phi : C^*(E(T)) \to C^*(E)$. Next we show that this ϕ is injective, using the gauge-invariant uniqueness theorem (Corollary 1.4 in [26]). Finally we show that the range of ϕ is equal to $P_X C^*(E) P_X$ using an inductive argument.

For the first step, let $\{P_v, S_e\}$ be the canonical Cuntz-Krieger generators of $C^*(E)$. For each $v \in T^0$ let $\tau(v) \in T^*$ be the path given by part (i) of Lemma 2.2 (in particular, for $v \in X$, $\tau(v) = v$). Now for each $v \in T^0$, define

$$Q_v := S_{\tau(v)} S^*_{\tau(v)} - \sum_{e \in T^1 \cap s^{-1}(v)} S_{\tau(v)e} S^*_{\tau(v)e}.$$

Since T is row-finite, this sum is finite and each Q_v is an element of $C^*(E)$. The relations (CK1)–(CK3) in $C^*(E)$ imply that each Q_v is a projection. These projections will correspond to the vertex projections of $E(T)$, and we shall need to know that they are nonzero:

LEMMA 3.7. *For each $v \in T^0$, $Q_v = 0$ if and only if $\varnothing \neq s^{-1}(v) \subseteq T^1$. Also,*

$$S_{\tau(v)} S^*_{\tau(v)} = \sum_{u \in T^0,\, v \geqslant_T u} Q_u. \tag{3.1}$$

Proof. For the first claim, first suppose $\varnothing \neq s^{-1}(v) \subseteq T^1$. The subgraph T is row-finite, and each edge with source v belongs to T^1, so $0 < |s^{-1}(v)| < \infty$. The Cuntz-Krieger relation (CK3) in $C^*(E)$ then gives $P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$, so we have

$$\begin{aligned} Q_v &= S_{\tau(v)} S^*_{\tau(v)} - \sum_{e \in s^{-1}(v)} S_{\tau(v)e} S^*_{\tau(v)e} \quad (\text{since } T^1 \cap s^{-1}(v) = s^{-1}(v)) \\ &= S_{\tau(v)} P_v S^*_{\tau(v)} - S_{\tau(v)} \Big(\sum_{e \in s^{-1}(v)} S_e S_e^* \Big) S^*_{\tau(v)} = 0. \end{aligned}$$

Conversely, if v is a sink in E then $Q_v = S_{\tau(v)} S^*_{\tau(v)} \neq 0$. If v emits an edge $f \in E^1 \setminus T^1$ then the relations (CK1)–(CK3) in $C^*(E)$ imply that $S_{\tau(v)f} S^*_{\tau(v)f}$ is a subprojection of $S_{\tau(v)} S^*_{\tau(v)}$ orthogonal to $\sum_{e \in T^1 \cap s^{-1}(v)} S_{\tau(v)e} S^*_{\tau(v)e}$, so $Q_v \geqslant S_{\tau(v)f} S^*_{\tau(v)f} \neq 0$.

For the second claim, first notice that the sum is finite by part (ii) of Lemma 2.2. For each vertex $v \in T^0$, let $c(v)$ be the number of elements in the set $\{u \in T^0 : v \geqslant_T u\}$. The formula (3.1) will be derived by induction on $c(v)$. For the basis step, note that if $c(v) = 1$ then $T^1 \cap s^{-1}(v) = \varnothing$, so $Q_v = S_{\tau(u)} S^*_{\tau(u)}$ as desired. For $n \in \mathbb{N}$, suppose the formula (3.1) holds for all $w \in T^0$ with

$c(w) \leqslant n-1$, and let $v \in T^0$ have $c(v) = n$. Now

$$S_{\tau(v)}S^*_{\tau(v)} = Q_v + \sum_{e \in T^1 \cap s^{-1}(v)} S_{\tau(v)e}S^*_{\tau(v)e}, \tag{3.2}$$

and for each $e \in T^1 \cap s^{-1}(v)$ we have $\tau(v)e = \tau(r(e))$ and $c(r(e)) < c(v)$, so $S_{\tau(v)e}S^*_{\tau(v)e} = \sum_{u \in T^0,\, r(e) \geqslant_T u} Q_u$ by the inductive hypothesis. Substituting this into (3.2) gives the formula (3.1) for the vertex v. ∎

Now for $e_u \in E(T)^1$ define $T_{e_u} := S_{\tau(s(e))e}S^*_{\tau(r(e))}Q_u$.

PROPOSITION 3.8. *The family* $\{Q_v, T_{e_u} : v \in E(T)^0,\, e_u \in E(T)^1\}$ *is a Cuntz-Krieger family for the graph* $E(T)$.

Proof. The proof of this proposition requires some technical manipulations of the relations (CK1)–(CK3), but is theoretically straightforward. Lemma 3.7 implies that for each $v \in E(T)^0$, Q_v is a nonzero projection. To see that they are mutually orthogonal, first notice that for each v, Q_v is a subprojection of $S_{\tau(v)}S^*_{\tau(v)}$. Suppose v and w are distinct elements of $E(T)^0$ such that $Q_vQ_w \neq 0$. We must have $S^*_{\tau(v)}S_{\tau(w)} \neq 0$, and hence one of $\tau(v)$ and $\tau(w)$ is an initial subpath of the other (this implication is a consequence of the fact that the S_e have mutually orthogonal ranges). Assume, without loss of generality, that $\tau(w) \prec \tau(v)$, and let $f \in T^1 \cap s^{-1}(w)$ be the edge given by Lemma 2.2(iv). Then

$$\sum_{e \in T^1 \cap s^{-1}(w)} S^*_{\tau(v)}S_{\tau(w)e}S^*_{\tau(w)e} = S^*_{\tau(v)}S_{\tau(w)f}S^*_{\tau(w)f},$$

because f is the unique edge in $T^1 \cap s^{-1}(w)$ with the property that $\tau(w)f \prec \tau(v)$. Now $S^*_{\tau(v)}S_{\tau(w)f}S^*_{\tau(w)f} = S^*_{\tau(v)}$, and thus

$$\begin{aligned} Q_vQ_w &= Q_vS_{\tau(v)}S^*_{\tau(v)}\Big(S_{\tau(w)}S^*_{\tau(w)} - \sum_{e \in T^1 \cap s^{-1}(w)} S_{\tau(w)e}S^*_{\tau(w)e}\Big) \\ &= Q_v(S_{\tau(v)}S^*_{\tau(v)} - S_{\tau(v)}S^*_{\tau(v)}) = 0. \end{aligned}$$

Hence $Q_vQ_w \neq 0$ if and only if $v = w$.

Turning our attention to the T_{e_u}, fix $e_u \in E(T)^1$. By definition of $E(T)^1$ we must have $\tau(r(e)) \prec \tau(u)$, so $Q_u \leqslant S_{\tau(u)}S^*_{\tau(u)} \leqslant S_{\tau(r(e))}S^*_{\tau(r(e))}$. Therefore

$$\begin{aligned} T^*_{e_u}T_{e_u} &= Q_uS_{\tau(r(e))}(S^*_{\tau(s(e))e}S_{\tau(s(e))e})S^*_{\tau(r(e))}Q_u = Q_uS_{\tau(r(e))}P_{r(e)}S^*_{\tau(r(e))}Q_u \\ &= Q_u(S_{\tau(r(e))}S^*_{\tau(r(e))})Q_u = Q_u. \end{aligned}$$

Thus the T_{e_u} are nonzero partial isometries with $T^*_{e_u}T_{e_u} = Q_{r(e_u)}$. To see that they have mutually orthogonal ranges, take e_u and f_v in $E(T)^1$ and suppose $T^*_{e_u}T_{f_v} \neq 0$. Now

$$T^*_{e_u}T_{f_v} = Q_uS_{\tau(r(e))}S^*_{\tau(s(e))e}S_{\tau(s(f))f}S^*_{\tau(r(f))}Q_v, \tag{3.3}$$

and in order for this product to be nonzero we must have either $\tau(s(f))f \prec \tau(s(e))e$ or $\tau(s(e))e \prec \tau(s(f))f$. Since neither e nor f belongs to T^1 (so that neither may be part of any $\tau(w)$), this implies that $\tau(s(e))e = \tau(s(f))f$, and so $e = f$. Putting $e = f$ in (3.3) gives

$$T_{e_u}^* T_{f_v} = Q_u S_{\tau(r(e))} S_{\tau(r(e))}^* Q_v = Q_u Q_v,$$

and in order for this product to be nonzero we must have $u = v$. Thus $e_u = f_v$.

For the inequality $T_{e_u} T_{e_u}^* \leqslant Q_{s(e_u)}$, we calculate

$$\begin{aligned} S_{\tau(s(e))e}^* Q_{s(e)} &= S_{\tau(s(e))e}^* \Big(S_{\tau(s(e))} S_{\tau(s(e))}^* - \sum_{f \in T^1 \cap s^{-1}(s(e))} S_{\tau(s(e))f} S_{\tau(s(e))f}^* \Big) \\ &= S_{\tau(s(e))e}^* S_{\tau(s(e))e} S_{\tau(s(e))e}^* - 0 = S_{\tau(s(e))e}^*, \end{aligned}$$

since $e \notin T^1$ implies that $\tau(s(e))e$ is not an initial subpath of any $\tau(s(e))f$ for $f \in T^1$. Thus

$$T_{e_u} T_{e_u}^* Q_{s(e_u)} = T_{e_u} Q_v S_{\tau(r(e))} (S_{\tau(s(e))e}^* Q_{s(e)}) = T_{e_u} Q_v S_{\tau(r(e))} S_{\tau(s(e))e}^* = T_{e_u} T_{e_u}^*.$$

To prove the remaining identity (CK3), we need to know the following fact about singular vertices in $E(T)$:

LEMMA 3.9. *Each edge e in $E^1 \setminus T^1$ with $s(e) \in T^0$ gives at least one edge in $E(T)$ with source $s(e)$. In particular, if $v \in T^0$ is a singular vertex of E then v is a singular vertex of $E(T)$.*

Proof. Let e be an edge in $E^1 \setminus T^1$ with $s(e) \in T^0$, and let $s(e) = v$, $r(e) = u$. Since T^0 is a hereditary subset of E^0 we must have $u \in T^0$. By part (iii) of Lemma 2.2, there exists at least one vertex $u' \in T^0$ with $u \geqslant_T u'$ and $s^{-1}(u') \cap T^1 = \emptyset$. Then by definition of $E(T)$ we have $u' \in E(T)^0$, and there is an edge $e_{u'}$ in $E(T)^1$ with source v.

For the second claim, suppose $v \in T^0$ is a sink in E. Then $\emptyset = s^{-1}(v)$ gives $v \in E(T)$, and since there is no edge in E^1 with source v there is no edge e_u in $E(T)^1$ with source v. Hence v is a sink in $E(T)$. On the other hand, suppose $v \in T^0$ emits infinitely many edges. Since T is row-finite, infinitely many of these edges must belong to $E^1 \setminus T^1$. Each of these edges gives at least one edge in $E(T)$ with source v by the preceding paragraph, so v is an infinite-emitter in $E(T)$. ∎

Now suppose $v \in E(T)^0$ is nonsingular in $E(T)$. Then v is nonsingular in E by the preceding lemma, so the Cuntz-Krieger relation (CK3) in $C^*(E)$ gives

$P_v = \sum_{e \in s^{-1}(v)} S_e S_e^*$. Now

$$\begin{aligned} Q_v &= S_{\tau(v)} S_{\tau(v)}^* - \sum_{e \in T^1 \cap s^{-1}(v)} S_{\tau(v)e} S_{\tau(v)e}^* \\ &= S_{\tau(v)} P_v S_{\tau(v)}^* - S_{\tau(v)} \Big(\sum_{e \in T^1 \cap s^{-1}(v)} S_e S_e^* \Big) S_{\tau(v)}^* \qquad (3.4) \\ &= S_{\tau(v)} \Big(P_v - \sum_{e \in T^1 \cap s^{-1}(v)} S_e S_e^* \Big) S_{\tau(v)}^* = \sum_{e \in s^{-1}(v) \setminus T^1} S_{\tau(v)e} S_{\tau(v)e}^*. \end{aligned}$$

Fix an edge $e \in s^{-1}(v) \setminus T^1$. This edge gives one edge e_u in $E(T)$ with source v for each vertex $u \in E(T)^0$ with $r(e) \geqslant_T u$. The formula (3.1) of Lemma 3.7 gives

$$\begin{aligned} S_{\tau(v)e} S_{\tau(v)e}^* &= S_{\tau(v)e} P_{r(e)}^3 S_{\tau(v)e}^* = S_{\tau(v)e} S_{\tau(r(e))}^* (S_{\tau(r(e))} S_{\tau(r(e))}^*)^2 S_{\tau(r(e))} S_{\tau(v)e}^* \\ &= S_{\tau(v)e} S_{\tau(r(e))}^* (S_{\tau(r(e))} S_{\tau(r(e))}^*) (S_{\tau(v)e} S_{\tau(r(e))}^* (S_{\tau(r(e))} S_{\tau(r(e))}^*))^* \\ &= \Big(S_{\tau(v)e} S_{\tau(r(e))}^* \Big(\sum_{u \in T^0, r(e) \geqslant_T u} Q_u \Big) \Big) \Big(S_{\tau(v)e} S_{\tau(r(e))}^* \Big(\sum_{u \in T^0, r(e) \geqslant_T u} Q_u \Big) \Big)^* \\ &= \Big(\sum_{u \in E(T)^0, r(e) \geqslant_T u} T_{e_u} \Big) \Big(\sum_{u \in E(T)^0, r(e) \geqslant_T u} T_{e_u}^* \Big). \end{aligned}$$

Since for $u \neq u'$ we have $T_{e_u} T_{e_{u'}}^* = 0$, this product expands as

$$S_{\tau(v)e} S_{\tau(v)e}^* = \sum_{u \in E(T)^0, r(e) \geqslant_T u} T_{e_u} T_{e_u}^*.$$

Substituting into (3.4) now gives the Cuntz-Krieger identity $Q_v = \sum_{s(e_u)=v} T_{e_u} T_{e_u}^*$, and this final identity completes the proof of the proposition. ∎

Now the universal property of $C^*(E(T))$ gives a $*$-homomorphism

$$\phi : C^*(E(T)) \to C^*(E)$$

which maps each canonical generator of $C^*(E(T))$ to its corresponding element of the family $\{Q_v, T_{e_u}\}$. The following two propositions show that ϕ is injective and has range $P_X C^*(E) P_X$.

PROPOSITION 3.10. *The map ϕ defined above is injective.*

Proof. Arguing as in Section 1 of [4], the universal property of $C^*(E)$ implies that there exists an action $\sigma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ given on generators by $\sigma_t(P_v) = P_v$ for all $v \in E^0$, and

$$\sigma_t(S_e) = \begin{cases} S_e & \text{for } e \in T^1, \\ tS_e & \text{otherwise.} \end{cases}$$

This action does not move any $S_{\tau(v)}$ for $v \in E(T)^0$, and hence does not move any Q_v either. For $e_u \in E(T)^1$ we have $e \in E^1 \setminus T^1$, and so for $t \in \mathbb{T}$,

$$\sigma_t(T_{e_u}) = \sigma_t(S_{\tau(s(e))} S_e S^*_{\tau(r(e))} Q_v) = S_{\tau(s(e))} t S_e S^*_{\tau(r(e))} Q_v = t T_{e_u}.$$

Thus if γ denotes the gauge action on $C^*(E(T))$ we have $\phi \circ \gamma = \sigma \circ \phi$, and all Q_v are nonzero, so the gauge-invariant uniqueness theorem (Corollary 1.4 of [26]) implies that ϕ is injective. ∎

PROPOSITION 3.11. $\phi(C^*(E(T))) = P_X C^*(E) P_X$.

Proof. For $v \in E(T)^0$ we have $P_X Q_v P_X = P_{s(\tau(v))} Q_v P_{s(\tau(v))} = Q_v$, and for $e_u \in E(T)^1$ we have $P_X T_{e_u} P_X = P_{s(\tau(s(e)))} T_{e_u} P_{s(\tau(u))} = T_{e_u}$. Hence $\phi(C^*(E(T))) \subseteq P_X C^*(E) P_X$, and it remains to show the opposite inclusion. To do this, we must show that the range of ϕ contains all products $S_\mu S_\nu^*$ such that $\mu, \nu \in E^*$; $s(\mu)$, $s(\nu) \in X$; and $r(\mu) = r(\nu)$. Since for such μ and ν we have

$$S_\mu S_\nu^* = S_\mu S^*_{\tau(r(\mu))} S_{\tau(r(\mu))} S_\nu^* = (S_\mu S^*_{\tau(r(\mu))})(S_\nu S^*_{\tau(r(\nu))})^*,$$

we may assume that $\nu = \tau(r(\mu))$. The proof is by induction on the length of μ.

If $|\mu| = 0$ then $\mu = s(\mu) \in X$ and so $\mu = \tau(r(\mu))$. Then $S_\mu S^*_{\tau(r(\mu))} = S_{\tau(r(\mu))} S^*_{\tau(r(\mu))}$, which is in the range of ϕ by Lemma 3.7. Now for $n \in \mathbb{N}$, suppose $|\mu| = n$ and suppose that $S_\nu S^*_{\tau(r(\nu))}$ belongs to the range of ϕ for all paths ν of length $n-1$. Let e be the final edge of μ, and write $\mu = \mu' e$. Then

$$\begin{aligned} S_\mu S^*_{\tau(r(\mu))} &= S_{\mu'} S_e S^*_{\tau(r(e))} = S_{\mu'} P_{r(\mu')} S_e S^*_{\tau(r(e))} = S_{\mu'} (S^*_{\tau(r(\mu'))} S_{\tau(r(\mu'))}) S_e S^*_{\tau(r(\mu))} \\ &= (S_{\mu'} S^*_{\tau(r(\mu'))})(S_{\tau(r(\mu'))e} S^*_{\tau(r(e))}), \end{aligned}$$

where $S_{\mu'} S^*_{\tau(r(\mu'))}$ belongs to the range of ϕ by the inductive hypothesis. If $e \in T^1$ then $\tau(r(\mu'))e = \tau(r(e))$, and so $S_{\tau(r(\mu'))e} S^*_{\tau(r(e))}$ belongs to the range of ϕ by Lemma 3.7. If e does not belong to T^1, then once again we use Lemma 3.7 to give

$$\begin{aligned} S_{\tau(r(\mu'))e} S^*_{\tau(r(e))} &= S_{\tau(r(\mu'))e} S^*_{\tau(r(e))} (S_{\tau(r(e))} S^*_{\tau(r(e))}) \\ &= S_{\tau(s(e))e} S^*_{\tau(r(e))} \Big(\sum_{u \in E(T)^0,\, r(e) \geqslant_T u} Q_u \Big) = \sum_{u \in E(T)^0,\, r(e) \geqslant_T u} T_{e_u} \end{aligned}$$

which belongs to the range of ϕ. This completes the proof by induction. ∎

Propositions 3.8, 3.10 and 3.11 prove Theorem 3.5.

EXAMPLE 3.12. Theorem 3.5 is already known in the case where X is hereditary: in this case we have seen that $E(T)$ is just the subgraph of E consisting of the vertices X and each edge whose source belongs to this set. Now the isomorphism of $C^*(E(T))$ with the corner $P_X C^*(E) P_X$ is implied by Theorem 4.1(c) in [4] and its proof.

EXAMPLE 3.13. Let p, q, r be positive integers, and for each nonnegative integer n let

$$E_n = \begin{tikzcd} u \arrow[loop above] \arrow[r, "\#p"] \arrow[rr, bend right, "\#(r+np)"'] & v \arrow[loop above] \arrow[r, "\#q"] & w \arrow[loop above] \end{tikzcd}$$

Example 3.4 and Theorem 3.5 show that $C^*(E_n) \cong P_u C^*(E_0) P_u$ for each n. Graphs of this type arise, for example, in the study of quantum lens spaces [16].

4. LABELLINGS OF DIRECTED GRAPHS AND DISCRETE COACTIONS

Corners of graph algebras arise as the fixed-point algebras of certain discrete coactions on graph algebras. In this section we make this precise, and then link up with Theorem 3.5.

Throughout this section, G will be a group with identity element 1_G, or 0_G if G is abelian (we will sometimes omit the subscript G to avoid clutter). We will assume throughout that G is discrete, with the exception of Corollary 4.9 and the paragraph immediately following the statement of Theorem 4.6. We denote by $C^*(G)$ the full group C^*-algebra, and for $s \in G$ we simply write s to denote the image of the group element under the canonical mapping $G \to C^*(G)$. We denote by λ the left-regular representation of G (and $C^*(G)$) on $l^2(G)$, and by M the representation of $C_0(G)$ on $l^2(G)$ by multiplication. For $s \in G$, χ_s denotes the characteristic function of $\{s\}$. All C^*-algebra tensor products considered here will involve at least one nuclear C^*-algebra, and we write $A \otimes B$ to mean the completion of the algebraic tensor product in its unique C^*-norm.

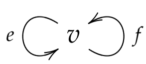
COACTIONS OF DISCRETE GROUPS ON C^*-ALGEBRAS. In line with [8] we adopt the following notations and conventions. Let $\delta_G : C^*(G) \to C^*(G) \otimes C^*(G)$ be the *comultiplication* $s \mapsto s \otimes s$ for $s \in G$. A *coaction* of G on a C^*-algebra A is an injective, nondegenerate homomorphism $\delta : A \to A \otimes C^*(G)$ such that $(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta$. (Here "nondegenerate" means that $\operatorname{span} \delta(A) A \otimes C^*(G)$ is dense in $A \otimes C^*(G)$.) For each $s \in G$ let $A_s = \{a \in A : \delta(a) = a \otimes s\}$, and write a_s to denote a general element of A_s. The span of the subalgebras $A_s : s \in G$ is dense in A (here it is important that G be discrete). The *fixed-point algebra* A^δ of δ is defined as $A^\delta = A_{1_G}$.

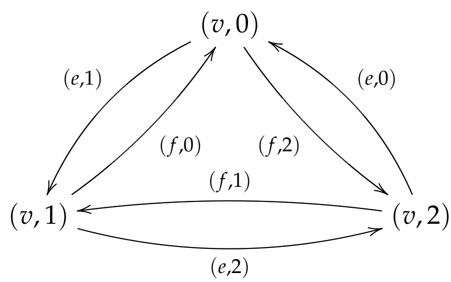
A *covariant representation* of the triple (A, G, δ) is a pair (π, μ), where $\pi : A \to B(\mathcal{H})$ and $\mu : C_0(G) \to B(\mathcal{H})$ are nondegenerate representations on a Hilbert space $\mathcal{H}$ satisfying $\pi(a_s)\mu(\chi_t) = \mu(\chi_{st})\pi(a_s)$ for all $s, t \in G$ and $a_s \in A_s$. Given a nondegenerate representation π of A on $\mathcal{H}$, and letting λ denote the left-regular representation of $C^*(G)$ on $l^2(G)$ and M the representation of $C_0(G)$ on $l^2(G)$ by multiplication, there is a covariant representation $((\pi \otimes \lambda) \circ \delta, 1 \otimes M)$

of (A, G, δ) on $\mathcal{H} \otimes l^2(G)$, called the *regular covariant representation induced by* π (Proposition 2.6 in [23]). The coaction δ is *normal* if there is a covariant representation (π, μ) with π faithful. The *crossed product* $A \times_\delta G$ is the universal C^*-algebra generated by a covariant representation (j_A, j_G), and is densely spanned by elements of the form $j_A(a_s)j_G(\chi_t)$. The nondegenerate representations of $A \times_\delta G$ are in one-to-one correspondence with the covariant representations of (A, G, δ) (cf. Definition 2.8 in [23]). In particular, any nondegenerate representation π of A on a Hilbert space $\mathcal{H}$ induces a nondegenerate representation of the crossed product on $\mathcal{H} \otimes l^2(G)$, corresponding to the regular covariant representation induced by π. We denote this representation of $A \times_\delta G$ by $\operatorname{Ind} \pi$.

LABELLINGS OF DIRECTED GRAPHS. Once again, conventions are adopted from [8]; see also [14], [19]. A *labelling* of a directed graph E by a discrete group G is a function $c : E^1 \to G$. Given a labelling c, the *skew product* $E \times_c G$ (called the *voltage graph* in [14]) is the graph with vertex set $E^0 \times G$, edge set $E^1 \times G$ and whose source and range maps are given by $s(e, s) = (s(e), c(e)s)$ and $r(e, s) = (r(e), s)$ (there are several slightly different definitions of the skew product, all yielding isomorphic graphs [8], [14], [18], [19]). As further notation, if c is a labelling of a directed graph E by a group G and if μ is an element of E^*, we shall denote by $c(\mu)$ the element $c(\mu_1)c(\mu_2)\cdots c(\mu_{|\mu|})$ of G.

EXAMPLE 4.1. Let E be the graph with one vertex v and two edges e and f. Define a labelling c of E by $\mathbb{Z}_3$ as $c(e) = 2$, $c(f) = 1$. Then the skew product graph $E \times_c \mathbb{Z}_3$ is as follows:

$E =$ 

$E \times_c \mathbb{Z}_3 =$ 

Any labelling of a graph E by a discrete group G induces a normal coaction of G on $C^*(E)$, and the graph C^*-algebra of the skew product is naturally isomorphic to the crossed product of $C^*(E)$ by this coaction; this is the content of the following results from [18] and [8], which we recall here for convenience:

LEMMA 4.2 (cf. Lemma 2.3 in [18] and Lemma 3.3 in [8]). *Let c be a labelling of a directed graph E by a discrete group G. Then there is a normal coaction δ of G on $C^*(E)$ such that*

$$\delta(S_e) = S_e \otimes c(e) \quad \textit{and} \quad \delta(P_v) = P_v \otimes 1_G \quad \textit{for } e \in E^1, v \in E^0.$$

THEOREM 4.3 (cf. Theorem 3.4 in [8]). *Let c be a labelling of a directed graph E by a discrete group G, with corresponding coaction δ. Then*

$$C^*(E \times_c G) \cong C^*(E) \times_\delta G$$

under the isomorphism ϕ given on generators by

$$\phi(P_{(v,s)}) = j_{C^*(E)}(P_v)j_G(\chi_s) \quad \text{and} \quad \phi(S_{(e,s)}) = j_{C^*(E)}(S_e)j_G(\chi_s)$$

for $v \in E^0$, $e \in E^1$ and $s \in G$.

EXAMPLE 4.4. Let G be a discrete group and $S \subseteq G$ a generating subset of cardinality $n \in \mathbb{N} \cup \{\infty\}$. Theorem 2.2.3 in [14] implies that any Cayley graph Γ for the pair (G, S) is isomorphic to a skew product, by G, of the graph B_n with one vertex and n edges. Hence by Theorem 3.4 in [8], there is a normal coaction δ of G on $C^*(B_n)$ ($= \mathcal{O}_n$, the Cuntz algebra generated by n nonunitary isometries with mutually orthogonal ranges [6]), such that

$$C^*(\Gamma) \cong \mathcal{O}_n \times_\delta G.$$

REMARK 4.5. Theorem 3.4 in [8] is more general than the version used here. [8] considered "coactions of a homogeneous space" G/H, and defined an analogue of the crossed product $A \times_\delta (G/H)$. Similarly, one may define skew products of graphs by discrete homogeneous spaces rather than discrete groups. Theorem 3.4 in [8] then says that $C^*(E \times_c (G/H)) \cong C^*(E) \times_\delta (G/H)$. At this stage it is not clear how to extend our Theorem 4.6 to this more general setting, as we do not have an obvious analogue of the fixed-point algebra.

FIXED-POINT ALGEBRAS ASSOCIATED TO LABELLINGS. We shall use the isomorphism of Theorem 3.4 in [8] to prove the following:

THEOREM 4.6. *Let c be a labelling of a directed graph E by a discrete group G, with corresponding coaction δ. Then*

$$C^*(E)^\delta \cong P_{E^0 \times \{1_G\}} C^*(E \times_c G) P_{E^0 \times \{1_G\}}.$$

This will follow immediately from the next lemma, which is a weaker version of a result proved by Quigg (Corollary 2.5 in [22]); we give a proof here for the sake of completeness.

LEMMA 4.7. *Let δ be a coaction of a discrete group G on a C^*-algebra A, and let (j_A, j_G) be the universal covariant representation of (A, G, δ). Then the fixed-point algebra A^δ is isomorphic to the corner $j_G(\chi_1)(A \times_\delta G)j_G(\chi_1)$.*

Proof. Consider the linear map $\psi : A^\delta \to A \times_\delta G$, $a_1 \mapsto j_A(a_1)j_G(\chi_1)$. For $a_1, b_1 \in A^\delta$ the covariance property gives

$$j_A(a_1 b_1)j_G(\chi_1) = j_A(a_1)j_A(b_1)j_G(\chi_1)^2 = j_A(a_1)j_G(\chi_1)j_A(b_1)j_G(\chi_1)$$

and

$$j_A(a_1^*)j_G(\chi_1) = (j_G(\chi_1)j_A(a_1))^* = (j_A(a_1)j_G(\chi_1))^*,$$

so ψ is a homomorphism. Furthermore, it is injective: let π be a faithful representation of A on a Hilbert space $\mathcal{H}$, and consider the induced representation $\operatorname{Ind}\pi$ of $A \times_\delta G$ on $\mathcal{H} \otimes l^2(G)$. Then for each nonzero $a_1 \in A^\delta$ we have

$$\operatorname{Ind}\pi(\psi(a_1)) = (\pi(a_1) \otimes \lambda_1)(1 \otimes M(\chi_1)) = \pi(a_1) \otimes M(\chi_1) \neq 0.$$

Hence we have shown that $A^\delta \cong j_A(A^\delta) j_G(\chi_1)$.

For $s, t \in G$ and $a_s \in A_s$ the covariance of (j_A, j_G) implies

$$j_G(\chi_1)(j_A(a_s)j_G(\chi_t))j_G(\chi_1) = \begin{cases} j_A(a_s)j_G(\chi_1) & \text{if } s = t = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The span of the elements $j_A(a_s)j_G(\chi_t)$ is dense in $A \times_\delta G$, and everything in sight is continuous, so this shows that

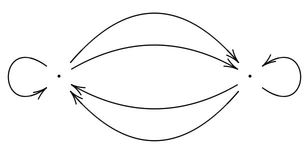
$$j_G(\chi_1)(A \times_\delta G)j_G(\chi_1) = \overline{j_A(A^\delta)j_G(\chi_1)} \cong A^\delta. \quad \blacksquare$$

Proof of Theorem 4.6. To simplify the notation, we shall write P for the element $P_{E^0 \times \{1_G\}}$ of $M(C^*(E \times_c G))$. The isomorphism ϕ of Theorem 4.3 extends to a strictly continuous isomorphism $\overline{\phi}$ of $M(C^*(E \times_c G))$ onto $M(C^*(E) \times_\delta G)$, such that

$$\overline{\phi}(P) = \sum_{v \in E^0} \phi(P_{(v,1)}) = \sum_{v \in E^0} j_{C^*(E)}(P_v) j_G(\chi_1) = j_G(\chi_1)$$

(because $\sum_{v \in E^0} P_v = 1$ in $M(C^*(E))$). Thus the corner in which we are interested is isomorphic to $j_G(\chi_1)(C^*(E) \times_\delta G)j_G(\chi_1)$, and so Lemma 4.7 proves the theorem. $\blacksquare$

EXAMPLE 4.8. Continuing on from Examples 3.3 and 4.1, the coaction $\delta : C^*(E) \to C^*(E) \otimes C^*(\mathbb{Z}_3)$ induced by the labelling c is defined on generators by $\delta(P_v) = P_v \otimes 0_{\mathbb{Z}_3}$, $\delta(S_e) = S_e \otimes 2$ and $\delta(S_f) = S_f \otimes 1$. Proposition 4.6 says that the fixed-point algebra $C^*(E)^\delta$ is isomorphic to the corner $P_{(v,0)}C^*(E \times_c \mathbb{Z}_3)P_{(v,0)}$. As in Example 3.3, let T be the row- and path-finite directed forest in $E \times_c \mathbb{Z}_3$ with vertices $E^0 \times \mathbb{Z}_3$ and edges $(e, 1)$ and $(f, 2)$. Then Theorem 3.5 implies that $C^*(E)^\delta$ is isomorphic to $C^*(E(T))$, where $E(T)$ is the graph



When the group G is abelian, coactions of G correspond (via the Gelfand transform) to actions of the dual group $\widehat{G}$ of group homomorphisms $\chi : G \to \mathbb{T}$ (cf. Remark 2.7 in [12]). The following corollary applies this fact to Theorem 4.6.

COROLLARY 4.9. *Let α be an action of a compact abelian group G on a graph algebra $C^*(E)$, such that for each $t \in G$, each $v \in E^0$ and each $e \in E^1$, $\alpha_t(P_v) = P_v$ and*

$\alpha_t(S_e) = \chi(e,t)S_e$ for some $\chi(e,t) \in \mathbb{C}$. Then there is a labelling c of E by $\widehat{G}$ such that

$$C^*(E)^\alpha \cong P_{E^0\times\{1_G\}}C^*(E \times_c \widehat{G})P_{E^0\times\{1_G\}}.$$

Proof. Each α_t is an automorphism of $C^*(E)$, so each $\chi(e,t)S_e$ is a partial isometry. This implies that for all $e \in E^1$ and $t \in G$, $\chi(e,t) \in \mathbb{T}$. Now as $\alpha : G \to \mathrm{Aut}(C^*(E))$ is a group homomorphism, we must have each $\chi(e,\cdot) : G \to \mathbb{T}$ a group homomorphism, so $\chi(e,\cdot) \in \widehat{G}$ for each $e \in E^1$. Let c be the labelling $e \mapsto \chi(e,\cdot)$ of E by the discrete group $\widehat{G}$, and let δ be the induced coaction of $\widehat{G}$ on $C^*(E)$. The Gelfand transform then gives $C^*(E)^\alpha = C^*(E)^\delta$, so the corollary follows from Theorem 4.6. ∎

The following key example was first brought to our attention by David Pask:

EXAMPLE 4.10. Let E be a directed graph, and let $c : E^1 \to \mathbb{Z}$ be the labelling $c(e) = 1$ for all $e \in E^1$. This labelling corresponds to the canonical gauge action γ of $\mathbb{T}$ on $C^*(E)$, so Corollary 4.9 implies that the AF-core $C^*(E)^\gamma$ is isomorphic to the corner $P_{E^0\times\{0_\mathbb{Z}\}}C^*(E \times_c \mathbb{Z})P_{E^0\times\{0_\mathbb{Z}\}}$.

Theorem 4.6 and its corollary lead us to seek conditions on the labelling c which allow us to apply Theorem 3.5 to find a graph for $C^*(E)^\delta$. That is, we seek conditions on c which ensure that $E \times_c G$ has a row- and path-finite directed forest with roots $E^0 \times \{1_G\}$ and vertex set $H_{E\times_c G}(E^0 \times \{1_G\})$. If E^0 and G are both finite, the graph $E \times_c G$ has finitely many vertices and so Lemma 3.6 tells us that we can always find such a forest. More generally, when E is row-finite we have the following:

PROPOSITION 4.11. *Let c be a labelling of a row-finite directed graph E by a discrete group G such that for each $\mu \in E^\infty$, there exists $i \in \mathbb{N}$ such that $c(\mu_1 \cdots \mu_i) = 1_G$. Then $E \times_c G$ has a row- and path-finite directed forest with roots $E^0 \times \{1_G\}$ and vertex set $H_{E\times_c G}(E^0 \times \{1_G\})$.*

(Note that when E is finite and $G = \mathbb{Z}$, the above condition is equivalent to the condition that $c(\lambda) = 0$ for each loop $\lambda \in E^*$, in analogy with Kirchhoff's voltage law.)

Proof. We may apply the iterative method used in the proof of Lemma 3.6 to find a directed forest T in $E \times_c G$ with the desired roots and vertex set. This forest is row-finite because E is, so it remains to show that it is path-finite. Suppose it is not, and let $\mu = (\mu_1, t_1)(\mu_2, t_2)\cdots$ be an infinite path in T. Lemma 2.2(i) implies that we may assume that the source of μ is in $E^0 \times \{1_G\}$, so that $t_1 = c(\mu_1)^{-1}$, and then by definition of the source and range maps in $E \times_c G$ we must have $r(\mu_i) = s(\mu_{i+1})$ and $t_i = c(\mu_1 \cdots \mu_i)^{-1}$ for each $i \in \mathbb{N}$. Now $\mu_1\mu_2\cdots$ is an infinite path in E^∞, so by assumption there exists an index $i \in \mathbb{N}$ such that $c(\mu_1 \cdots \mu_i) = 1_G$. For this i we have $r((\mu_i, t_i)) = (r(\mu_i), c(\mu_1 \cdots \mu_i)^{-1}) = (r(\mu_i), 1_G)$, so the vertex

$(r(\mu_i), 1_G)$ receives an edge in the forest T. This is a contradiction, since each vertex in $E^0 \times \{1_G\}$ is a root of T, and so we conclude that T is path-finite. ∎

REMARK 4.12. We can also prove the following analogue of Proposition 4.11 for coactions of homogeneous spaces: when E is row-finite and H is a subgroup of G such that for each $\mu \in E^\infty$ there exists $i \in \mathbb{N}$ with $c(\mu_1 \cdots \mu_i) \in H$, then there is a directed forest in $E \times_c (G/H)$ with the desired properties. The proof is virtually identical to the proof of the preceding proposition, and is therefore omitted.

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