

INFINITE DIMENSIONAL GRASSMANNIANS

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ABSTRACT. We study the differentiable structure and the homotopy type of some spaces related to the Grassmannian of closed linear subspaces of an infinite dimensional Hilbert space, such as the space of Fredholm pairs, the Grassmannian of compact perturbations of a given space, and the essential Grassmannians. We define a determinant bundle over the space of Fredholm pairs. We lift the composition of Fredholm operators to the Quillen determinant bundle, and we show how this map can be used in several constructions involving the determinant bundle over the space of Fredholm pairs. We deduce some properties of suitable orientation bundles.

KEYWORDS: *Determinant bundle, compact perturbation, essential Grassmannian, Fredholm operator, Fredholm pair.*

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INTRODUCTION

Infinite dimensional Grassmannians play a relevant role in many fields of mathematics. For instance, they appear as classifying spaces in the homotopy theory of classical groups (see [7]), they provide a natural setting to study completely integrable systems and loop groups (see [27], [28], [25], [15], [3]). Furthermore, they turn out to be significant in the Morse theory of infinite dimensional manifolds (see [9], [2], [16]).

This paper is devoted to a systematic study of the analytic and homotopic properties of some infinite dimensional Grassmannians.

After having established some useful facts about the set of symmetric idempotent elements of a Banach $*$ -algebra, we recall the properties of the Grassmannian $\text{Gr}(H)$ of all closed subspaces of a real or complex Hilbert space H , and then we focus our analysis on the following topics.

We study the space of Fredholm pairs $\text{Fp}(H)$, i.e. the set of pairs of closed linear subspaces (V, W) of H having finite dimensional intersection and finite codimensional sum, and its open and closed subset $\text{Fp}^*(H)$ consisting of the Fredholm pairs (V, W) with $\dim V = \dim W = \infty$. These are analytic Banach

manifolds, and we show that each component of $\text{Fp}^*(H)$ has the homotopy type of BO , in the real case, or BU , in the complex case, the classifying spaces of the infinite orthogonal group, respectively of the infinite unitary group (see [4] for similar results about the homotopy type of the space of skew-adjoint Fredholm operators).

We study the Grassmannian $\text{Gr}_c(V, H)$ of all compact perturbations of the closed linear subspace $V \subset H$, i.e. of all the closed linear subspaces $W \subset H$ such that the orthogonal projection onto W differs from the orthogonal projection onto V by a compact operator. This space is called restricted Grassmannian by some authors (see [9], where this space is considered with a weaker topology, or [25], where the ideal of compact operators is replaced by the ideal of Hilbert-Schmidt operators). It has the structure of an analytic Banach manifold, but when $\dim V = \text{codim} V = \infty$, as a subset of $\text{Gr}(H)$ it is just a topological submanifold. In the latter case, each component of $\text{Gr}_c(V, H)$ has the homotopy type of BO , if H is real, or of BU , if H is complex. A similar result is proved in [25], the Hilbert-Schmidt case is treated in that reference, but the difference turns out to be irrelevant from the point of view of homotopy theory. The proof we give here is somehow more direct.

We study the essential Grassmannian $\text{Gr}_e(H)$, i.e. the quotient of $\text{Gr}(H)$ by the equivalence relation $V \sim W$ if and only if V is a compact perturbation of W . This is also an analytic Banach manifold, isometric to the space of symmetric idempotent elements in the Calkin algebra, and its homotopy type is easily determined. Quotient spaces by finer equivalence relations, taking the Fredholm index into account, are also introduced and studied.

We define the determinant bundle over $\text{Fp}(H)$ as an analytic line bundle whose fiber at $(V, W) \in \text{Fp}(H)$ is the top degree component in the exterior algebra of the finite dimensional space $(V \cap W) \oplus (H/(V+W))^*$. The space of Fredholm pairs is in some sense the larger space on which the determinant bundle can be defined, and from this it is possible to deduce the construction of other line bundles, such as the determinant bundle over the space of Fredholm operators introduced by Quillen [26], or the determinant bundle over the Grassmannian of compact perturbations of a given infinite dimensional and infinite codimensional space (see [28], [25]). We also show that the composition of Fredholm operators can be lifted analytically to the Quillen determinant bundles, and that such a lift has a useful associativity property. These facts are used to show how the sum of a finite dimensional space to one of the elements in a Fredholm pair can be lifted analytically to the determinant bundles, and to prove associativity for such a construction.

In the case of a real Hilbert space H , the determinant bundle over the space of Fredholm pairs $\text{Fp}^*(H)$ induces a double covering of $\text{Fp}^*(H)$, called its the orientation bundle. This is an explicit presentation of the universal covering of $\text{Fp}^*(H)$. The associativity property of determinant bundles implies that the orientation bundle can be used quite effectively to determine orientations of finite

dimensional linear spaces which are found as transverse intersections of infinite dimensional ones. These facts are used in [2] to associate a chain complex with integer coefficients to functionals defined on infinite dimensional manifolds, having critical points with infinite Morse index.

Two appendices conclude the paper. In the first one, we prove the existence of a continuous global section for a continuous linear surjective map between Banach spaces, a result which is used several times in the paper. In the second one, we prove the above mentioned associativity for the composition of Fredholm operators lifted to the determinant bundles.

1. A FEW FACTS ABOUT BANACH ALGEBRAS

Let \mathcal{A} be a real or complex Banach $*$ -algebra, that is a real or complex Banach algebra \mathcal{A} endowed with an involution map $*$. The involution $x \mapsto x^*$ satisfies the following properties:

$$x^{**} = x, \quad (xy)^* = y^*x^*, \quad \|x^*\| = \|x\|, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \bar{\lambda}x^*,$$

where λ is a real respectively a complex number. Let

$$\text{Sym}(\mathcal{A}) = \{x \in \mathcal{A} : x^* = x\}, \quad \text{and} \quad \text{Skew}(\mathcal{A}) = \{x \in \mathcal{A} : x^* = -x\},$$

be the subspaces of symmetric and skew-symmetric elements of \mathcal{A} , so that $\mathcal{A} = \text{Sym}(\mathcal{A}) \oplus \text{Skew}(\mathcal{A})$. Denote by $\text{U}(\mathcal{A})$ the group of unitary elements of \mathcal{A} ,

$$\text{U}(\mathcal{A}) := \{u \in \mathcal{A} : u^*u = uu^* = 1\},$$

and by $\text{P}^s(\mathcal{A})$ the set of symmetric idempotent elements of \mathcal{A} ,

$$\text{P}^s(\mathcal{A}) := \{p \in \text{Sym}(\mathcal{A}) : p^2 = p\}.$$

The aim of this section is to prove the following facts.

PROPOSITION 1.1. *The set $\text{P}^s(\mathcal{A})$ of symmetric idempotent elements of the Banach $*$ -algebra \mathcal{A} is an analytic Banach submanifold of \mathcal{A} .*

PROPOSITION 1.2. *Consider the analytic group action*

$$\text{U}(\mathcal{A}) \times \text{P}^s(\mathcal{A}) \rightarrow \text{P}^s(\mathcal{A}), \quad (u, p) \mapsto upu^{-1}.$$

The orbit of every $p \in \text{P}^s(\mathcal{A})$, $\text{U}(\mathcal{A}) \cdot p := \{upu^{-1} : u \in \text{U}(\mathcal{A})\}$, is open and closed in $\text{P}^s(\mathcal{A})$, and the map

$$\text{U}(\mathcal{A}) \rightarrow \text{U}(\mathcal{A}) \cdot p, \quad u \mapsto upu^{-1},$$

is an analytic principal $I(p)$ -bundle, where $I(p)$ is the isotropy subgroup of p ,

$$I(p) := \{u \in \text{U}(\mathcal{A}) : upu^{-1} = p\}.$$

PROPOSITION 1.3. *If $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is a surjective homomorphism of Banach $*$ -algebras, its restriction to the space of symmetric idempotent elements*

$$\Phi|_{\mathbb{P}^s(\mathcal{A})} : \mathbb{P}^s(\mathcal{A}) \rightarrow \mathbb{P}^s(\mathcal{B}),$$

is a C^0 fiber bundle (possibly with a non-constant fiber). It is an analytic fiber bundle if and only if the linear subspace $\text{Ker } \Phi \cap \text{Skew}(\mathcal{A})$ has a direct summand in \mathcal{A} .

REMARK 1.4. The above results hold also in a Banach algebra setting, by dropping the symmetry requirement, and by replacing the group of unitary elements by the group of invertible elements.

1.1. SQUARE ROOTS OF 1. An element $p \in \mathcal{A}$ is symmetric and idempotent if and only if $2p - 1$ is a symmetric square root of 1. This fact allows us to deduce the above propositions by analogous statements for the space

$$\mathbb{Q}^s(\mathcal{A}) := \{q \in \text{Sym}(\mathcal{A}) : q^2 = 1\}.$$

Denote by $\|\cdot\|$ the norm of \mathcal{A} , and by $B_r(a)$ the open ball of radius r centered in $a \in \mathcal{A}$. The set $\mathbb{Q}^s(\mathcal{A})$ is a closed subset of $\text{U}(\mathcal{A})$, in particular every $q \in \mathbb{Q}^s(\mathcal{A})$ has norm 1. The elements 1 and -1 are isolated in $\mathbb{Q}^s(\mathcal{A})$. More precisely

$$(1.1) \quad \mathbb{Q}^s(\mathcal{A}) \cap B_2(1) = \{1\}, \quad \mathbb{Q}^s(\mathcal{A}) \cap B_2(-1) = \{-1\}.$$

Indeed, $q^2 = 1$ implies $2(1 - q) = (1 - q)^2$, whence $2\|1 - q\| \leq \|1 - q\|^2$, and hence either $\|1 - q\| = 0$ or $\|1 - q\| \geq 2$, proving the first identity. The second identity follows, because $-\mathbb{Q}^s(\mathcal{A}) = \mathbb{Q}^s(\mathcal{A})$.

Notice that every $q \in \mathbb{Q}^s(\mathcal{A})$ determines a decomposition $\text{Sym}(\mathcal{A}) = \text{A}^s(q) \oplus \text{C}^s(q)$ into the symmetric anti-commutant and the symmetric commutant of q :

$$\text{A}^s(q) = \{a \in \text{Sym}(\mathcal{A}) : aq = -qa\}, \quad \text{C}^s(q) = \{a \in \text{Sym}(\mathcal{A}) : aq = qa\}.$$

Indeed, the corresponding projections are

$$\text{Sym}(\mathcal{A}) \rightarrow \text{A}^s(q), \quad a \mapsto \frac{1}{2}(a - qa); \quad \text{Sym}(\mathcal{A}) \rightarrow \text{C}^s(q), \quad a \mapsto \frac{1}{2}(a + qa).$$

LEMMA 1.5. *Let $q \in \mathbb{Q}^s(\mathcal{A})$. Then the map*

$$\phi_q : B_{\sqrt{3}/2}(0) \cap \text{A}^s(q) \rightarrow \text{C}^s(q), \quad x \mapsto (1 - x^2)^{1/2}q,$$

is analytic, and $\mathbb{Q}^s(\mathcal{A})$ is locally the graph of ϕ_q :

$$\mathbb{Q}^s(\mathcal{A}) \cap [(B_{\sqrt{3}/2}(0) \cap \text{A}^s(q)) \times (B_{1/2}(q) \cap \text{C}^s(q))] = \text{graph } \phi_q.$$

Proof. Let $x \in B_{\sqrt{3}/2}(0) \cap \text{A}^s(q)$. Since $\sqrt{3}/2 < 1$, a square root of $1 - x^2$ is well-defined:

$$z = (1 - x^2)^{1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{1/2}{k} x^{2k} = 1 - \sum_{k=1}^{\infty} \left| \binom{1/2}{k} \right| x^{2k}.$$

So ϕ_q is analytic on $B_{\sqrt{3}/2}(0)$. Moreover, z is invertible and

$$z^{-1} = (1 - x^2)^{-1/2} = \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k} = \sum_{k=0}^{\infty} \left| \binom{-1/2}{k} \right| x^{2k}.$$

Since the coefficients of the above series are positive, we obtain the inequalities:

$$(1.2) \quad \|1 - z\| \leq 1 - (1 - \|x\|^2)^{1/2} < \frac{1}{2},$$

$$(1.3) \quad \|z^{-1}\| \leq (1 - \|x\|^2)^{-1/2} < 2,$$

$$(1.4) \quad \|1 - z^{-1}\| \leq (1 - \|x\|^2)^{-1/2} - 1 < 1.$$

From (1.2) we have

$$\|q - \phi_q(x)\| = \|q - zq\| \leq \|1 - z\| \|q\| < \frac{1}{2}.$$

Since x anti-commutes with q , z commutes with q , and so does $\phi_q(x) = zq$. Therefore, ϕ_q maps $B_{\sqrt{3}/2}(0) \cap A^s(q)$ into $B_{1/2}(q) \cap C^s(q)$.

Let $p \in Q^s(\mathcal{A})$, $p = x + y$ with $x \in B_{\sqrt{3}/2}(0) \cap A^s(q)$, $y \in B_{1/2}(q) \cap C^s(q)$. We claim that $y = \phi_q(x)$, that is $z^{-1}yq = 1$, with z as before. By (1.1), it is enough to show that $(z^{-1}yq)^2 = 1$ and $\|z^{-1}yq - 1\| < 2$. Since

$$x^2 + y^2 \in C^s(q), \quad xy + yx \in A^s(q),$$

and $x^2 + y^2 + xy + yx = p^2 = 1 \in C^s(q)$, from $\text{Sym}(\mathcal{A}) = A^s(q) \oplus C^s(q)$ we deduce that $x^2 + y^2 = 1$ and $xy + yx = 0$. Therefore, $[y, x^2] = 0$, whence $[y, z^{-1}] = 0$. Using also $[q, z^{-1}] = 0$, we obtain that $(z^{-1}yq)^2 = 1$. Moreover,

$$\begin{aligned} \|z^{-1}yq - 1\| &\leq \|z^{-1}yq - z^{-1}\| + \|z^{-1} - 1\| \leq \|z^{-1}\| \|yq - q^2\| + \|z^{-1} - 1\| \\ &\leq \|z^{-1}\| \|y - q\| + \|z^{-1} - 1\|, \end{aligned}$$

so by (1.3) and (1.4),

$$\|z^{-1}yq - 1\| \leq 2\|y - q\| + 1 < 2,$$

concluding the proof of the claim, by (1.1).

On the other hand, let $p = x + \phi_q(x)$, with $x \in B_{\sqrt{3}/2}(0) \cap A^s(q)$. Then, since $xq = -qx$ and $zq = qz$, we conclude that $p^2 = 1$. ■

The above lemma has the following consequence, which implies Proposition 1.1.

PROPOSITION 1.6. *The set $Q^s(\mathcal{A})$ of symmetric square roots of 1 in the Banach $*$ -algebra \mathcal{A} is an analytic submanifold of \mathcal{A} , and its tangent space at $q \in Q^s(\mathcal{A})$ is the symmetric anti-commutant $A^s(q)$ of q .*

1.2. THE ACTION OF $U(\mathcal{A})$ ON $Q^s(\mathcal{A})$. The group $U(\mathcal{A})$ acts analytically on $Q^s(\mathcal{A})$, by

$$U(\mathcal{A}) \times Q^s(\mathcal{A}) \rightarrow Q^s(\mathcal{A}), \quad (u, q) \mapsto uqu^{-1},$$

and the isotropy subgroup of $q \in Q^s(\mathcal{A})$ is

$$I(q) := \{u \in U(\mathcal{A}) : uqu^{-1} = q\} = U(\mathcal{A}) \cap C(q),$$

where $C(q)$ denotes the commutant subalgebra of the element q . The bijection $P^s(\mathcal{A}) \rightarrow Q^s(\mathcal{A})$, $p \mapsto 2p - 1$, commutes with the respective actions of $U(\mathcal{A})$, so Proposition 1.2 is implied by the following:

PROPOSITION 1.7. *The orbit of every $q \in Q^s(\mathcal{A})$, $U(\mathcal{A}) \cdot q := \{uqu^{-1} : u \in U(\mathcal{A})\}$, is open and closed in $Q^s(\mathcal{A})$, and the following map is an analytic principal $(U(\mathcal{A}) \cap C(q))$ -bundle:*

$$\psi_q : U(\mathcal{A}) \rightarrow U(\mathcal{A}) \cdot q, \quad u \mapsto uqu^{-1}.$$

Proof. By a well known simple argument (see for instance Section 7.4 in [30]), it is enough to prove that the map ψ_q has an analytic local section near q . Let $q \in Q^s(\mathcal{A})$. Note that $B_1(q)$ consists of invertible elements: if $a \in B_1(q)$, then $\|aq - 1\| = \|aq - q^2\| \leq \|a - q\| < 1$, thus aq is invertible, and so is a . If $p \in B_2(q) \cap Q^s(\mathcal{A})$, then $(p + q)/2 \in B_1(q)$. So $p + q$ is invertible, and the map

$$s : B_2(q) \cap Q^s(\mathcal{A}) \rightarrow U(\mathcal{A}), \quad p \mapsto (p + q)|p + q|^{-1}q$$

is analytic. Since q commutes with $(p + q)^2$, it commutes also with $|p + q|^{-1}$, so

$$ps(p) = (p + q)q|p + q|^{-1}q = (p + q)|p + q|^{-1} = s(p)q,$$

implying that s is an analytic local section near q for the map ψ_q . ■

1.3. SURJECTIVE HOMOMORPHISMS. Since the bijection $P^s(\mathcal{A}) \rightarrow Q^s(\mathcal{A})$, $p \mapsto 2p - 1$, commutes with Banach $*$ -algebra homomorphisms, statement (ii) of the following proposition implies Proposition 1.3.

PROPOSITION 1.8. *Let $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a surjective homomorphism of Banach $*$ -algebras.*

(i) *The following group homomorphism is a C^0 principal bundle (possibly with a non constant fiber):*

$$\Phi_U := \Phi|_{U(\mathcal{A})} : U(\mathcal{A}) \rightarrow U(\mathcal{B}).$$

(ii) *The following restriction is a C^0 fiber bundle (possibly with a non-constant fiber):*

$$\Phi_{Q^s} := \Phi|_{Q^s(\mathcal{A})} : Q^s(\mathcal{A}) \rightarrow Q^s(\mathcal{B}).$$

(iii) The maps Φ_U and Φ_{Q^s} are compatible with the group actions, meaning that the following diagram commutes:

$$\begin{array}{ccc} U(\mathcal{A}) \times Q^s(\mathcal{A}) & \xrightarrow{\Phi_U \times \Phi_{Q^s}} & U(\mathcal{B}) \times Q^s(\mathcal{B}) \\ \downarrow & & \downarrow \\ Q^s(\mathcal{A}) & \xrightarrow{\Phi_{Q^s}} & Q^s(\mathcal{B}) \end{array}$$

The bundles in (i) and (ii) are analytic if and only if the linear subspace $\text{Ker } \Phi \cap \text{Skew}(\mathcal{A})$ has a direct summand in \mathcal{A} .

Proof. The restriction of Φ to the subspaces of skew-symmetric elements,

$$\Phi|_{\text{Skew}(\mathcal{A})} : \text{Skew}(\mathcal{A}) \rightarrow \text{Skew}(\mathcal{B}),$$

is surjective, because \mathcal{A} and \mathcal{B} are the direct sum of the subspaces of their symmetric and skew-symmetric elements, and because Φ is surjective. By Proposition A.1 of Appendix A, the above map has a continuous global section, which can be chosen to be linear if and only if the linear subspace $\text{Ker } \Phi \cap \text{Skew}(\mathcal{A})$ has a direct summand in $\text{Skew}(\mathcal{A})$, or equivalently in \mathcal{A} . The map $\Phi|_{\text{Skew}(\mathcal{A})}$ is locally conjugated at 0 to Φ_U by the exponential map. Hence the map Φ_U has a continuous local section mapping $1_{\mathcal{B}}$ to $1_{\mathcal{A}}$. As already mentioned, this is equivalent to (i).

Let $q \in Q^s(\mathcal{B})$. By Proposition 1.7, the map

$$\psi_q : U(\mathcal{B}) \rightarrow U(\mathcal{B}) \cdot q, \quad u \mapsto uqu^{-1},$$

has an analytic local section mapping q to $1_{\mathcal{B}}$, so by composition we get a continuous local section for the map $\psi_q \circ \Phi_U$: there exists a neighborhood $\mathcal{N} \subset U(\mathcal{B}) \cdot q$ of q , and a map

$$s : \mathcal{N} \rightarrow U(\mathcal{A}), \quad s(q) = 1_{\mathcal{A}},$$

such that $\Phi(s(p))q\Phi(s(p))^{-1} = p$ for every $p \in \mathcal{N}$. Therefore, the continuous map

$$\mathcal{N} \times \Phi_{Q^s}^{-1}(\{q\}) \rightarrow \Phi_{Q^s}^{-1}(\mathcal{N}), \quad (p, x) \mapsto s(p)xs(p)^{-1}.$$

is a local trivialization for Φ_{Q^s} near q , proving (ii).

Claim (iii) is readily checked. The last statement follows from the fact that Φ_U has an analytic local section if and only if $\text{ker } \Phi \cap \text{Skew}(\mathcal{A})$ has a direct summand in \mathcal{A} . ■

REMARK 1.9. The maps Φ_U and Φ_{Q^s} need not be surjective. Indeed, if $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ is the quotient map from the Banach algebra of bounded linear operators on a Hilbert space onto its Calkin algebra, then Φ_U is not onto: indeed, the image of Φ_U is the set of unitary elements in the Calkin algebra having index zero. As for the map Φ_{Q^s} , notice that if X is a compact Hausdorff space and Y is a closed subspace, the map $\Phi : C(X) \rightarrow C(Y)$, $f \mapsto f|_Y$, is surjective, but Φ_{Q^s} is surjective if and only if each connected component of X contains at most one connected

component of Y (by the Gel'fand duality, see Theorem 4.29 in [12], this situation covers the general case of commutative complex C^* -algebras).

2. THE HILBERT GRASSMANNIAN AND THE SPACE OF FREDHOLM PAIRS

By $L(E, F)$, respectively $L_c(E, F)$, we denote the space of continuous linear, respectively compact linear, maps from the Banach space E to the Banach space F . If $F = E$ we also use the short forms $L(E)$ and $L_c(E)$. The group of invertible linear maps on E is denoted by $GL(E)$, and $GL_c(E)$ denotes the subgroup of the compact perturbations of the identity. The norm of the operator $T \in L(E, F)$ is denoted by $\|T\|$.

Let now H be a real or complex infinite dimensional separable Hilbert space. The orthogonal projection onto a closed subspace $V \subset H$ is denoted by P_V , while V^\perp denotes the orthogonal complement of V in H . The unitary group of H is denoted by

$$U(H) = \{U \in GL(H) : U^*U = UU^* = I\},$$

(in the real case it would be more appropriate to call $U(H)$ the orthogonal group, and to denote it by $O(H)$, but since we wish to deal with the complex and the real case simultaneously, we do not make such a distinction).

Let $\text{Gr}(H)$ be the *Grassmannian of H* , i.e. the set of closed linear subspaces of H . The assignment $V \mapsto P_V$ is an inclusion of $\text{Gr}(H)$ into $L(H)$, onto the closed subset of the orthogonal projectors of H . We can therefore define, for any $V, W \in \text{Gr}(H)$ the distance

$$\text{dist}(V, W) := \|P_V - P_W\|,$$

which makes $\text{Gr}(H)$ a complete metric space. We always consider the topology on $\text{Gr}(H)$ induced by this distance. The weak topologies of $L(H)$ induce other interesting topologies on the Grassmannian, whose properties are examined in [29].

2.1. ANALYTIC STRUCTURE OF $\text{Gr}(H)$. For any $V \in \text{Gr}(H)$, the unit open ball $B_1(V) \subset \text{Gr}(H)$ is contractible, as shown by the homeomorphism

$$\Psi_V : B_1(V) \rightarrow L(V, V^\perp), \quad W \mapsto P_{V^\perp}(P_V|_W)^{-1},$$

the inverse mapping being

$$\Psi_V^{-1} : L(V, V^\perp) \rightarrow B_1(V), \quad A \mapsto \text{graph } A.$$

Indeed, Ψ_V is well-defined because $P_V|_W$ is the restriction of the isomorphism

$$I - (P_W - P_V)(P_W - P_{W^\perp}) = P_V P_W + P_{V^\perp} P_{W^\perp} : W \oplus W^\perp \rightarrow V \oplus V^\perp.$$

The collection $\{\Psi_V\}_{V \in \text{Gr}(H)}$ is an analytic atlas, with transition maps

$$\Psi_W \circ \Psi_V^{-1}(A) = P_{W^\perp}(I + A)[P_W(I + A)]^{-1}.$$

Therefore, $\text{Gr}(H)$ is an analytic Banach manifold. Actually, the identification $V \mapsto P_V$ allows to see $\text{Gr}(H)$ as an analytic submanifold of $L(H)$, by Proposition 1.1.

REMARK 2.1. It easily seen that the analytic structure on $\text{Gr}(H)$ does not depend on the choice of the Hilbert inner product on H . In other words, if $H = V \oplus V'$ then the following map is an analytic coordinate system:

$$L(V, V') \rightarrow \text{Gr}(H), \quad A \mapsto \text{graph } A.$$

2.2. TRANSVERSE INTERSECTIONS. We often use the fact that the map

$$\{(V, W) \in \text{Gr}(H) \times \text{Gr}(H) : V + W = H\}, \quad (V, W) \mapsto V \cap W,$$

is analytic. To prove this fact, let (V, W) be a pair of closed linear subspaces such that $V + W = H$ and set $H_0 := V \cap W$, $H_1 := H_0^\perp \cap V$, $H_2 := H_0^\perp \cap W$, so that

$$H = H_0 \oplus H_1 \oplus H_2, \quad V = H_0 \oplus H_1, \quad W = H_0 \oplus H_2.$$

Consider the analytic coordinate system

$L(H_0 \oplus H_1, H_2) \times L(H_0 \oplus H_2, H_1) \rightarrow \text{Gr}(H) \times \text{Gr}(H)$, $(A, B) \mapsto (\text{graph } A, \text{graph } B)$, mapping $(0, 0)$ into (V, W) . If $\|A\| < 1$ and $\|B\| < 1$, then

$$\text{graph } A \cap \text{graph } B = \text{graph } T(A, B),$$

where $T(A, B) \in L(H_0, H_1 \oplus H_2)$ is the linear mapping

$$\begin{aligned} T(A, B) &= (I_{H_0 \oplus H_1} - BA)^{-1}(B + BA) + (I_{H_0 \oplus H_2} - AB)^{-1}(A + AB) \\ &= A + B + AB + BA + ABA + BAB + ABAB + BABA + \dots, \end{aligned}$$

which depends analytically on (A, B) , proving the claim.

We may also express $P_{V \cap W}$ in terms of P_V and P_W . First notice that if $V + W$ is closed then

$$(2.1) \quad \|P_V P_W - P_{V \cap W}\| < 1.$$

Indeed, in the case $H = V \oplus W$, W is the graph of some $L \in L(V^\perp, V)$, and (2.1) reduces to

$$\|P_V P_W\| = \|P_V|_W\| = \frac{\|L\|}{\sqrt{1 + \|L\|^2}} < 1.$$

The general case of a closed sum follows because

$$P_V P_W - P_{V \cap W} = P_{V'} P_{W'},$$

where $V' := (V \cap W)^\perp \cap V$, $W' := (V \cap W)^\perp \cap W$ are in direct sum in the closed linear subspace $(V \cap W)^\perp \cap (V + W)$.

From the fact that $X \subset Y$ implies $P_X P_Y = P_Y P_X = P_X$ it follows that

$$(P_V P_W)^n - P_{V \cap W} = (P_V P_W - P_{V \cap W})^n.$$

Together with (2.1), this implies that when $V + W$ is closed,

$$P_{V \cap W} = \lim_{n \rightarrow \infty} (P_V P_W)^n,$$

in the operator norm. Notice also that the above limit is uniform on the set of those pairs (V, W) with $\|P_V P_W - P_{V \cap W}\| < \theta < 1$. Since the function $(V, W) \mapsto \|P_V P_W - P_{V \cap W}\|$ is continuous on the open set of pairs (V, W) such that $V + W = H$, we conclude that on this space the analytic function $P_{V \cap W}$ is the locally uniform limit of the sequence of monomials $(P_V P_W)^n$.

2.3. HOMOTOPY OF $\text{Gr}(H)$. The connected components of $\text{Gr}(H)$ are the subsets

$$\text{Gr}_{n,k}(H) := \{V \in \text{Gr}(H) : \dim V = n, \text{codim} V = k\}, \quad n, k \in \mathbb{N} \cup \{\infty\}, \quad n+k = \infty.$$

The unitary group $U(H)$ acts analytically and transitively on each component $\text{Gr}_{n,k}(H)$, and if $V \in \text{Gr}_{n,k}(H)$, the map

$$U(H) \rightarrow \text{Gr}_{n,k}(H), \quad U \mapsto UV,$$

defines an analytic principal $U(V) \times U(V^\perp)$ -bundle, by Proposition 1.2 (see also Corollary 8.1 in [20]). In particular, the above map is a fibration. It is a well known result of Kuiper's that $U(H)$ is contractible when H is an infinite dimensional Hilbert space (see [18]), so the exact homotopy sequence associated to the above fibration yields the following isomorphisms:

$$\pi_i(\text{Gr}_{n,\infty}(H)) \cong \pi_i(\text{Gr}_{\infty,n}(H)) = \begin{cases} 0 & \text{if } i=0, \\ \pi_{i-1}(\text{GL}(n)) & \text{if } i \geq 1, \end{cases} \quad \pi_i(\text{Gr}_{\infty,\infty}(H)) = 0, \quad \forall i \in \mathbb{N}.$$

Therefore, a theorem by Whitehead (see the Corollary after Theorem 15 in [23]) implies that $\text{Gr}_{\infty,\infty}(H)$ is contractible, while $\text{Gr}_{n,\infty}(H)$ and $\text{Gr}_{\infty,n}(H)$ have the homotopy type of the classifying space of $\text{GL}(n)$.

2.4. STIEFEL SPACES. Let $n, k \in \mathbb{N} \cup \infty$, $n+k = \infty$. The *Stiefel spaces* $\text{St}_{n,k}(H)$ are the sets

$$\text{St}_{n,\infty}(H) := \{T \in L(\mathbb{R}^n, H) : T \text{ is injective}\}, \quad n \in \mathbb{N};$$

$$\text{St}_{\infty,n}(H) := \{T \in L(H) : T \text{ is semi-Fredholm of index } -n \text{ and injective}\}, \quad n \in \mathbb{N} \cup \{\infty\};$$

endowed with the operator norm topology. The map

$$\text{St}_{n,k}(H) \rightarrow \text{Gr}_{n,k}(H), \quad T \mapsto \text{ran } T,$$

defines a $\text{GL}(n)$ -principal bundle, in the case $n < \infty$, and a $\text{GL}(H)$ -principal bundle, in the case $n = \infty$ (see Theorem 8.6 in [20]). The Stiefel spaces $\text{St}_{n,\infty}(H)$ are contractible, for every $n \in \mathbb{N} \cup \{\infty\}$, while, for $n \in \mathbb{N}$ (see Corollary 8.4 in [20]),

$$\pi_i(\text{St}_{\infty,n}(H)) = \begin{cases} 0 & \text{if } i = 0, \\ \pi_{i-1}(\text{GL}(n)) & \text{if } i \geq 1. \end{cases}$$

2.5. FREDHOLM PAIRS. A pair (V, W) of closed subspaces of H is said to be a *Fredholm pair* if $V \cap W$ is finite dimensional, and $V + W$ is finite codimensional, hence closed (see Section IV paragraph 4 in [17]). In this situation, the *Fredholm index* of (V, W) is the integer number

$$\text{ind}(V, W) = \dim V \cap W - \text{codim}(V + W).$$

The pair (V, W) is Fredholm if and only if the operator $P_{W^\perp}|_V \in L(V, W^\perp)$ is Fredholm, in which case $\text{ind}(V, W) = \text{ind}(P_{W^\perp}|_V)$.

The set of Fredholm pairs in H is denoted by $\text{Fp}(H)$. If $X, Y \in \text{Gr}(H)$, with $\dim X < \infty$ and $\text{codim} Y < \infty$, then (X, Y) is a Fredholm pair, of index

$$\text{ind}(X, Y) = \dim X - \text{codim} Y.$$

If (V, W) is a Fredholm pair, and $V \in \text{Gr}_{\infty, \infty}(H)$, then also $W \in \text{Gr}_{\infty, \infty}(H)$. Therefore we set

$$\text{Fp}^*(H) := \{(V, W) \in \text{Fp}(H) : V, W \in \text{Gr}_{\infty, \infty}(H)\}.$$

2.6. HOMOTOPY OF $\text{Fp}(H)$. The space $\text{Fp}(H)$ is open in $\text{Gr}(H) \times \text{Gr}(H)$, and the index is a continuous function on $\text{Fp}(H)$. It is easily checked that the connected components of $\text{Fp}(H)$ are the subsets

$$\begin{aligned} & \text{Gr}_{n, \infty}(H) \times \text{Gr}_{\infty, m}(H), \quad \text{Gr}_{\infty, n}(H) \times \text{Gr}_{m, \infty}(H), \quad n, m \in \mathbb{N}; \\ & \text{Fp}_k^*(H) := \{(V, W) \in \text{Fp}^*(H) : \text{ind}(V, W) = k\}, \quad k \in \mathbb{Z}. \end{aligned}$$

The homotopy type of the first two families of spaces can be deduced from the results seen before. As for the homotopy type of $\text{Fp}_k^*(H)$, we recall some definitions and basic facts.

Let GL be the infinite linear group, that is the inductive limit $GL = \varinjlim GL(n)$. In the complex case, GL has the homotopy type of $U = \varinjlim U(n)$, the infinite unitary group, while in the real case it has the homotopy type of $O = \varinjlim O(n)$, the infinite orthogonal group. By the Bott periodicity theorem [7], for $i \geq 0$,

$$(2.2) \quad \pi_i(U) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is odd,} \\ 0 & \text{if } i \text{ is even;} \end{cases} \quad \pi_i(O) = \begin{cases} \mathbb{Z}_2 & \text{if } i \equiv 0, 1 \pmod{8}, \\ 0 & \text{if } i \equiv 2, 4, 5, 6 \pmod{8}, \\ \mathbb{Z} & \text{if } i \equiv 3, 7 \pmod{8}. \end{cases}$$

By BGL we denote the classifying space (see Section 7 in [11]) of the topological group GL : in the complex case, BGL has the homotopy type of BU , the classifying space of U , while in the real case it has the homotopy type of BO , the classifying space of O . The spaces BU and BO are path connected, and (2.2) implies that for $i \geq 1$,

$$\pi_i(BU) = \begin{cases} \mathbb{Z} & \text{if } i \text{ is even,} \\ 0 & \text{if } i \text{ is odd;} \end{cases} \quad \pi_i(BO) = \begin{cases} \mathbb{Z} & \text{if } i \equiv 0, 4 \pmod{8}, \\ \mathbb{Z}_2 & \text{if } i \equiv 1, 2 \pmod{8}, \\ 0 & \text{if } i \equiv 3, 5, 6, 7 \pmod{8}. \end{cases}$$

THEOREM 2.2. *The components $\text{Fp}_k^*(H)$ are pairwise homeomorphic, and they have the homotopy type of BU, if H is a complex space, or BO, if H is a real space.*

The proof of this result is given at the end of Section 4.

3. THE GRASSMANNIAN OF COMPACT PERTURBATIONS

We say that the subspace W is a *compact perturbation* of V if its orthogonal projector P_W is a compact perturbation of P_V . It is an equivalence relation.

If V is a compact perturbation of W , then (V, W^\perp) is a Fredholm pair, and the operator $P_W|_V : V \rightarrow W$ is Fredholm. The *relative dimension of V with respect to W* is the integer

$$(3.1) \quad \dim(V, W) := \text{ind}(V, W^\perp) = \dim V \cap W^\perp - \dim V^\perp \cap W = \text{ind}(P_W|_V : V \rightarrow W).$$

See [1] for a proof of the identities above. See also Remark 4.9 in [8] for an early appearance of the concepts of compact perturbation and of relative dimension for linear subspaces. When V and W are finite dimensional (respectively finite codimensional), we have

$$\dim(V, W) = \dim V - \dim W \quad (\text{respectively } \dim(V, W) = \text{codim} W - \text{codim} V).$$

It is easy to see that the image by a linear isomorphism T of a compact perturbation of some $V \in \text{Gr}(H)$ is a compact perturbation of TV , and that the relative dimension is preserved (see [1], Proposition 2.4). Hence nor the notion of compact perturbation, neither the relative dimension depend on the choice of an inner product in H .

PROPOSITION 3.1. *If (W, Z) is a Fredholm pair of subspaces and V is a compact perturbation of W , then (V, Z) is a Fredholm pair, with*

$$\text{ind}(V, Z) = \text{ind}(W, Z) + \dim(V, W).$$

Proof. From the identity

$$(3.2) \quad P_{Z^\perp}|_V = P_{Z^\perp}|_W \circ P_W|_V + P_{Z^\perp}(P_V - P_W)|_V,$$

we see that $P_{Z^\perp}|_V \in \text{L}(V, Z^\perp)$ is a compact perturbation of the Fredholm operator

$$V \xrightarrow{P_W|_V} W \xrightarrow{P_{Z^\perp}|_W} Z^\perp.$$

Therefore $P_{Z^\perp}|_V$ is Fredholm, and the conclusion follows from (3.1) and from the additivity of the index by composition. ■

In particular, if each of the subspaces V, W, Y is a compact perturbation of the other subspaces, there holds

$$\dim(Y, V) = \dim(Y, W) + \dim(W, V).$$

The following fact is proved in [1], Proposition 2.3.

PROPOSITION 3.2. *Let H_1, H_2 be Hilbert spaces, let $T, T' \in L(H_1, H_2)$ be such that T' is a compact perturbation of T , $\text{ran } T$ and $\text{ran } T'$ are closed. Then $\ker T'$ is a compact perturbation of $\ker T$, $\text{ran } T'$ is a compact perturbation of $\text{ran } T$, and*

$$\dim(\text{ran } T', \text{ran } T) = -\dim(\ker T', \ker T).$$

If $V \in \text{Gr}(H)$, the Grassmannian of compact perturbations of V

$$\text{Gr}_c(V, H) := \{W \in \text{Gr}(H) : W \text{ is a compact perturbation of } V\}$$

is a closed subspace of $\text{Gr}(H)$. If V has finite dimension (respectively finite codimension), then

$$\text{Gr}_c(V, H) = \bigcup_{n \in \mathbb{N}} \text{Gr}_{n, \infty}(H), \quad (\text{respectively } = \bigcup_{n \in \mathbb{N}} \text{Gr}_{\infty, n}(H)).$$

The interesting case arises when V has both infinite dimension and infinite codimension. In such a situation, $\text{Gr}_c(V, H)$ is a closed proper subset of $\text{Gr}_{\infty, \infty}(H)$. Moreover, the continuity of the Fredholm index implies that the function

$$\dim(\cdot, \cdot) : \text{Gr}_c(V, H) \times \text{Gr}_c(V, H) \rightarrow \mathbb{Z}$$

is continuous. So the subsets

$$\text{Gr}_{c, n}(V, H) := \{W \in \text{Gr}_c(V, H) : \dim(W, V) = n\}, \quad n \in \mathbb{Z}.$$

are open and closed in $\text{Gr}_c(V, H)$. It is easily checked that the above sets are connected, so they are the connected components of $\text{Gr}_c(V, H)$.

3.1. ANALYTIC STRUCTURE OF $\text{Gr}_c(V, H)$. The restriction of the map Ψ_V defined in Section 2 to the unit open ball $B_1(V) \cap \text{Gr}_c(V, H)$ is a homeomorphism onto $L_c(V, V^\perp)$. The collection of these homeomorphisms defines an analytic atlas on $\text{Gr}_c(V, H)$, which is therefore an analytic Banach manifold modeled on $L_c(V, V^\perp)$. Actually, Proposition 1.1 implies that $\text{Gr}(V, H)$ can be seen as an analytic Banach submanifold of the Banach affine space $P_V + L_c(H)$. However, if both V and V^\perp are infinite-dimensional, $\text{Gr}_c(V, H)$ is not a C^1 -submanifold of $\text{Gr}(H)$, because in this case the subspace $L_c(V, V^\perp)$ does not admit a direct summand in $L(V, V^\perp)$. Indeed, if H is an infinite dimensional separable Hilbert space, $L_c(H)$ is not complemented in $L(H)$, a fact which can be deduced from the analogous and more known fact that c_0 is not complemented (actually, in ℓ^∞ every infinite dimensional complemented subspace of ℓ^∞ is isomorphic to ℓ^∞ , see Theorem 2.a.7 in [19]). In fact, let us fix an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$ in H . Consider the canonical inclusion j of ℓ^∞ into the space of diagonal operators with respect to this basis, and its left inverse $r : L(H) \rightarrow \ell^\infty$ mapping $T \in L(H)$ into the sequence $(\langle Te_n, e_n \rangle)_{n \in \mathbb{N}} \in \ell^\infty$. Then $j(c_0) \subset L_c(H)$, and $r(L_c(H)) = c_0$. If by contradiction $P : L(H) \rightarrow L(H)$ is a linear projection onto $L_c(H)$, the above properties of j and r imply that $PjrP = jrP$. It follows that $rPj : \ell^\infty \rightarrow \ell^\infty$ is a projection onto c_0 . The fact that every surjective continuous linear map between Banach spaces admits a (in general non differentiable) global section (see Appendix A), implies that $\text{Gr}_c(V, H)$ is a C^0 -submanifold of $\text{Gr}(H)$.

3.2. THE ACTION OF $\mathrm{GL}_c(H)$ ON $\mathrm{Gr}_c(V, H)$. The group $\mathrm{GL}_c(H)$, consisting of the linear automorphisms of H which are compact perturbations of the identity, is an open subset of the affine Banach space $I + \mathrm{L}_c(H)$. It inherits therefore the structure of an analytic group, although it is just a C^0 -subgroup of $\mathrm{GL}(H)$ (again, because $\mathrm{L}_c(H)$ does not have a direct summand in $\mathrm{L}(H)$). By Proposition 3.2, there is an analytic group action

$$\mathrm{GL}_c(H) \times \mathrm{Gr}_c(V, H) \rightarrow \mathrm{Gr}_c(V, H), \quad (L, W) \mapsto LW,$$

which preserves the connected components of $\mathrm{Gr}_c(V, H)$, and such an action is easily seen to be transitive on each component. Fix some $V \in \mathrm{Gr}(H)$. We claim that the map

$$(3.3) \quad \mathrm{GL}_c(H) \rightarrow \mathrm{Gr}_{c,0}(V, H), \quad L \mapsto LV,$$

is an analytic principal G -bundle, with $G = \{L \in \mathrm{GL}_c(H) : LV = V\}$. Indeed, an analytic local section of (3.3) near V is

$$s : B_1(V) \cap \mathrm{Gr}_{c,0}(V, H) \rightarrow \mathrm{GL}_c(H), \quad W \mapsto P_W P_V + P_{W^\perp} P_{V^\perp} = I - (P_V - P_W)(P_V - P_{V^\perp}),$$

and such a local section defines a local trivialization near any $W_0 \in \mathrm{Gr}_{c,0}(V, H)$ in the standard way: fixing $L_0 \in \mathrm{GL}_c(H)$ such that $L_0 V = W_0$, the map

$$L_0[B_1(V) \cap \mathrm{Gr}_{c,0}(V, H)] \times G \rightarrow \mathrm{GL}_c(H), \quad (W, L) \mapsto L_0 s(L_0^{-1} W) L,$$

is an analytic trivialization of (3.3) near W_0 .

3.3. HOMOTOPY OF $\mathrm{Gr}_c(V, H)$. Let $V \in \mathrm{Gr}_{\infty, \infty}(H)$. The components $\mathrm{Gr}_{c,n}(V, H)$ are pairwise analytically diffeomorphic. More generally, $\mathrm{Gr}_{c,0}(V, H)$ is analytically diffeomorphic to $\mathrm{Gr}_{c,0}(W, H)$, for $V, W \in \mathrm{Gr}_{\infty, \infty}(H)$, a diffeomorphism being induced by an $L \in \mathrm{GL}(H)$ mapping V onto W .

In order to study the homotopy type of $\mathrm{Gr}_{c,0}(V, H)$, we introduce the *Stiefel space of compact perturbations of $V \hookrightarrow H$* ,

$$\mathrm{St}_c(V, H) := \{T \in \mathrm{L}(V, H) : T \text{ is injective, } Tx = x + Kx \text{ with } K \in \mathrm{L}_c(V, H)\},$$

endowed with the norm topology. The set $\mathrm{St}_c(V, H)$ is open in the affine Banach space $I + \mathrm{L}_c(V, H)$, so it is an analytic manifold. The map

$$(3.4) \quad \mathrm{St}_c(V, H) \rightarrow \mathrm{Gr}_{c,0}(V, H), \quad T \mapsto \mathrm{ran} T,$$

is an analytic principal $\mathrm{GL}_c(V)$ -bundle. Indeed, if $W_0 \in \mathrm{Gr}_{c,0}(V, H)$ and $s_0 : \mathcal{U} \rightarrow \mathrm{GL}_c(H)$ is an analytic local section of the fiber bundle (3.3), with \mathcal{U} a neighborhood of W_0 in $\mathrm{Gr}_{c,0}(V, H)$, the map

$$\mathcal{U} \times \mathrm{GL}_c(V) \rightarrow \mathrm{St}_c(V, H), \quad (W, A) \mapsto s_0(W)A,$$

is an analytic local trivialization of (3.4) near W_0 .

Let (V_n) be an increasing sequence of subspaces of V , with $\dim V_n = n$ and such that their union is dense in V . Then the union of the closed subspaces

$$\mathcal{K}_n = \{K \in \mathrm{L}_c(V, H) : KP_{V_n^\perp} = 0\}, \quad n \in \mathbb{N},$$

is dense in $L_c(V, H)$. A well known result by Palais (see [23], Corollary after Theorem 17) implies that $\text{St}_c(V, H)$ is homotopically equivalent to the inductive limit of the spaces $\text{St}_c(V, H) \cap (I + \mathcal{K}_n)$, equipped with the limit topology. Moreover, the set $\text{St}_c(V, H) \cap (I + \mathcal{K}_n)$ is homeomorphic to the Stiefel space $\text{St}_{n, \infty}(H)$, introduced in Section 2. Since such a space is contractible, this proves that $\text{St}_c(V, H)$ is also contractible. In particular, the principal bundle (3.4) is the universal principal bundle for the topological group $\text{GL}_c(V)$, and $\text{Gr}_{c,0}(V, H)$ is the classifying space for such a group. The exact homotopy sequence now implies that

$$\pi_i(\text{Gr}_{c,0}(V, H)) \cong \pi_{i-1}(\text{GL}_c(V)), \quad \text{for } i \geq 1.$$

Palais [22] has shown that $\text{GL}_c(V)$ has the homotopy type of the infinite dimensional general linear group GL , hence of U in the complex case, and of O in the real case. Therefore, we have proved the following result.

THEOREM 3.3. *Let $V \in \text{Gr}_{\infty, \infty}(H)$. Then the components $\text{Gr}_{c,n}(V, H)$ are pairwise analytically diffeomorphic, and $\pi_i(\text{Gr}_{c,n}(V, H)) \cong \pi_{i-1}(GL)$ for $i \geq 1$, so that $\text{Gr}_{c,n}(V, H)$ has the homotopy type of BU , if H is complex, or BO , if H is real.*

4. ESSENTIAL GRASSMANNIANS

We have seen that the notion of compact perturbation produces an equivalence relation on $\text{Gr}(H)$. The *essential Grassmannian* of H is the space of the equivalence classes for such a relation, endowed with the quotient topology, and it is denoted by $\text{Gr}_e(H)$. Denote by

$$\Pi : \text{Gr}(H) \rightarrow \text{Gr}_e(H)$$

the quotient projection. This space can be described in terms of the Calkin algebra of H , i.e. the quotient C^* -algebra $\mathcal{C}(H) := L(H)/L_c(H)$. Denote by $\pi : L(H) \rightarrow \mathcal{C}(H)$ the quotient projection. The closed subspace V is a compact perturbation of W if and only if $\pi(P_V) = \pi(P_W)$, so π maps the set of orthogonal projectors into the set of symmetric idempotent elements of the Calkin algebra. The next result shows that every symmetric idempotent element of the Calkin algebra is the image of some orthogonal projector (compare this result with Remark 1.9).

PROPOSITION 4.1. *The restriction of the map π to the set of the (orthogonal) projectors of H is onto the set of (symmetric) idempotent elements of $\mathcal{C}(H)$.*

Proof. In any Banach $*$ -algebra, an element x is (symmetric) idempotent if and only if $2x - 1$ is a (symmetric) square root of the identity. It is therefore equivalent to show that the quotient projector π maps the (symmetric) square roots of I in $L(H)$ onto the set of (symmetric) square roots of 1 in $\mathcal{C}(H)$.

Let $Q \in L(H)$ and $K \in L_c(H)$ be such that $Q^2 = I - K$. We have to find $J \in L_c(H)$ such that $(Q - J)^2 = I$. The operator J will be self-adjoint if Q and K are.

The spectral mapping theorem implies that

$$\sigma(Q)^2 = \sigma(Q^2) = \sigma(I - K),$$

so the set $\sigma(Q) \setminus \{-1, 1\}$ consists of isolated eigenvalues of finite multiplicity.

Let $U = \{z \in \mathbb{C} : |z^2 - 1| < 1\}$, and let $H = H_0 \oplus H_1$, $Q = Q_0 \oplus Q_1$ be the spectral decomposition of the operator Q corresponding to the decomposition of $\sigma(Q)$ into the closed sets $\sigma(Q) \setminus U$ and $\sigma(Q) \cap U$. Denote by P_0, P_1 the corresponding projectors. Then H_0 is finite dimensional, and Q_1 is invertible, with $\sigma(Q_1) \subset U$. Therefore, the spectrum of the operator $K_1 := K|_{H_1} = I - Q_1^2 \in L_c(H_1)$ is contained in the unit ball $\{z \in \mathbb{C} : |z| < 1\}$. The analytic function

$$f(z) := 1 - z - (1 - z)^{1/2} = -\frac{z}{2} + \sum_{n=2}^{\infty} \left| \binom{1/2}{n} \right| z^n, \quad |z| < 1,$$

verifies the identity

$$(4.1) \quad \frac{f(z)^2}{1 - z} - 2f(z) - z = 0.$$

Set $J_0 := Q_0 - P_0 \in L(H_0)$, $J_1 := Q_1^{-1}f(K_1) \in L_c(H_1)$. Then $J := J_0 \oplus J_1$ is compact, and by (4.1) it satisfies $(Q - J)^2 = I$. ■

The above result implies that π induces a homeomorphism between the essential Grassmannian $\text{Gr}_e(H)$ and the space of idempotent symmetric elements of $\mathcal{C}(H)$. Therefore $\text{Gr}_e(H)$ inherits the structure of a complete metric space. Furthermore, Proposition 1.1 shows that $\text{Gr}_e(H)$ can be given the structure of an analytic Banach submanifold of $\mathcal{C}(H)$.

4.1. HOMOTOPY OF $\text{Gr}_e(H)$. The finite dimensional and the finite codimensional spaces represent two isolated points in the essential Grassmannian. What remains is a connected component, which is the image of $\text{Gr}_{\infty, \infty}(H)$ by Π , and which is denoted by $\text{Gr}_e^*(H)$. By Proposition 1.3, the map

$$\Pi : \text{Gr}_{\infty, \infty}(H) \rightarrow \text{Gr}_e^*(H)$$

is a C^0 fiber bundle, with typical fiber $\text{Gr}_c(V, H)$, for some $V \in \text{Gr}_{\infty, \infty}(H)$. In particular, the above map is a fibration, and since its total space is contractible, the exact homotopy sequence implies that

$$\pi_i(\text{Gr}_e^*(H)) \cong \pi_{i-1}(\text{Gr}_c(V, H)).$$

Therefore, Theorem 3.3 has the following consequence.

COROLLARY 4.2. *The space $\text{Gr}_e^*(H)$ is path connected, its fundamental group is infinite cyclic, and $\pi_i(\text{Gr}_e^*(H)) \cong \pi_{i-2}(GL)$ for $i \geq 2$.*

4.2. THE (m) -ESSENTIAL GRASSMANNIAN. In order to represent the coverings of $\text{Gr}_e^*(H)$, we consider the quotient of $\text{Gr}(H)$ by stronger equivalence relations, which take the relative dimension into account: if $m \in \mathbb{N}$, we define the (m) -essential Grassmannian $\text{Gr}_{(m)}(H)$ to be the quotient of $\text{Gr}(H)$ by the equivalence relation

$$\{(V, W) \in \text{Gr}(H) \times \text{Gr}(H) : V \text{ is a compact perturbation of } W, \text{ and } \dim(V, W) \in m\mathbb{Z}\}.$$

Denote by

$$\Pi_m : \text{Gr}_{\infty, \infty}(H) \rightarrow \text{Gr}_{(m)}(H)$$

the quotient map. With this terminology, the (1) -essential Grassmannian is just the essential Grassmannian, and the (0) -essential Grassmannian distinguishes commensurable spaces according to their relative dimension. We can endow $\text{Gr}_{(m)}(H)$ with the quotient topology induced by $\text{Gr}(H)$.

Every set $\text{Gr}_{n, \infty}(H)$ or $\text{Gr}_{\infty, n}(H)$, $n \in \mathbb{N}$, represents an isolated point in $\text{Gr}_{(0)}(H)$, which has thus infinitely many isolated points. If $m \geq 1$, the sets

$$\bigcup_{n \in m\mathbb{Z} + k} \text{Gr}_{n, \infty}(H) \quad \text{and} \quad \bigcup_{n \in m\mathbb{Z} + k} \text{Gr}_{\infty, n}(H), \quad k = 0, 1, \dots, m-1,$$

represent distinct isolated points in $\text{Gr}_{(m)}(H)$, which has thus $2m$ isolated points.

The remaining part of $\text{Gr}_{(m)}(H)$ consists of the quotient of $\text{Gr}_{\infty, \infty}(H)$, denoted by $\text{Gr}_{(m)}^*(H)$. Since $\text{Gr}_{\infty, \infty}(H)$ is connected, so is $\text{Gr}_{(m)}^*(H)$, for any $m \in \mathbb{N}$.

4.3. THE ACTION OF $U(H)$ ON $\text{Gr}_{(m)}^*(H)$. The action of $U(H)$ on $\text{Gr}_{\infty, \infty}(H)$ induces a transitive action of $U(H)$ on the quotient $\text{Gr}_{(m)}^*(H)$. This action presents $\text{Gr}_{(m)}^*(H)$ as a quotient of $U(H)$: fixing $V \in \text{Gr}_{\infty, \infty}(H)$, the map

$$h_m : U(H) \rightarrow \text{Gr}_{(m)}^*(H), \quad U \mapsto \Pi_m(UV),$$

is indeed open and onto. Let us determine the isotropy subgroup $h_m^{-1}([V])$. By the identity

$$P_{UV} - P_V = UP_VU^* - P_V = [U, P_V]U^*, \quad \forall U \in U(H),$$

the subspace UV is a compact perturbation of V if and only if U belongs to the closed subgroup

$$\mathcal{H} := \{U \in U(H) : [U, P_V] \in \text{L}_c(H)\}.$$

If $U \in \mathcal{H}$, since

$$P_VUP_V + P_{V^\perp}UP_{V^\perp} = U - [[U, P_V], P_V] \in U + \text{L}_c(H),$$

the operator $P_VUP_V + P_{V^\perp}UP_{V^\perp}$ is Fredholm of index 0. So P_VUP_V and $P_{V^\perp}UP_{V^\perp}$ are Fredholm operators on V and V^\perp , respectively, with the relation $\text{ind}(P_VUP_V)$

$= -\text{ind}(P_{V^\perp} U P_{V^\perp})$. By (3.1), and by the additivity of the Fredholm index with respect to composition,

$$\begin{aligned} \dim(V, UV) &= \text{ind}(P_{UV}|_V : V \rightarrow UV) = \text{ind}(U P_V U^*|_V : V \rightarrow UV) \\ &= \text{ind}(P_V U^*|_V : V \rightarrow V) + \text{ind}(U|_V : V \xrightarrow{\sim} UV) = \text{ind}(P_V U^* P_V) \\ &= -\text{ind}(P_V U P_V). \end{aligned}$$

Therefore,

$$\dim(UV, V) = \text{ind}(P_V U P_V), \quad \forall U \in \mathcal{H},$$

which shows that $h_m^{-1}([V])$ is the open and closed subgroup of \mathcal{H} ,

$$\mathcal{H}_m := \{U \in \mathcal{H} : \text{ind}(P_V U P_V) \in m\mathbb{Z}\}.$$

Therefore the space $\text{Gr}_{(m)}^*(H)$ is homeomorphic to the space of left cosets $U(H)/\mathcal{H}_m$. Now assume that $n \in \mathbb{N}^*$ divides $m \in \mathbb{N}$. Since \mathcal{H}_m is an open subgroup of \mathcal{H}_n , the quotient map

$$\Pi_n^m : \text{Gr}_{(m)}^*(H) \rightarrow \text{Gr}_{(n)}^*(H)$$

is a covering, with fiber $n\mathbb{Z}/m\mathbb{Z}$. We claim that the map h_m is a C^0 principal \mathcal{H}_m -bundle. Since $h_1 = \Pi_1^m \circ h_m$, it is enough to prove that h_1 is a C^0 principal \mathcal{H}_1 -bundle. In this case

$$\text{Gr}_{(1)}^*(H) = \text{Gr}_e^*(H) = \text{Ps}(\mathcal{C}) \setminus \{1, 0\} = \{u \pi(P_V) u^{-1} : u \in U(\mathcal{C}(H))\},$$

so by Proposition 1.2 the map

$$U(\mathcal{C}(H)) \rightarrow \text{Gr}_{(1)}^*(H), \quad u \mapsto u \pi(P_V) u^{-1},$$

has a continuous local section near $\Pi_1(V)$, mapping $\Pi_1(V)$ to 1. By Proposition 1.8 (i), the quotient map $U(H) \rightarrow U(\mathcal{C}(H))$ has a continuous local section near 1. By composition we get a continuous local section for h_1 , which is therefore a C^0 principal \mathcal{H}_1 -bundle by the standard trivialization.

4.4. THE HOMOTOPY OF $\text{Gr}_{(m)}^*(H)$. We claim that the quotient map

$$\Pi_m|_{\text{Gr}_{\infty, \infty}(H)} : \text{Gr}_{\infty, \infty}(H) \rightarrow \text{Gr}_{(m)}^*(H)$$

is a C^0 fiber bundle, with typical fiber

$$\{W \in \text{Gr}_c(V, H) : \dim(W, V) \in m\mathbb{Z}\}.$$

Indeed, if $s : \mathcal{U} \subset \text{Gr}_{(m)}^*(H) \rightarrow U(H)$ is a continuous local section for the map h_m , the map

$$\mathcal{U} \times \{W \in \text{Gr}_c(V, H) : \dim(W, V) \in m\mathbb{Z}\} \rightarrow \text{Gr}_{\infty, \infty}(H), \quad (\alpha, W) \mapsto s(\alpha)W,$$

is a local trivialization for the map $\Pi_m|_{\text{Gr}_{\infty, \infty}(H)}$.

Since $\text{Gr}_{\infty, \infty}(H)$ is contractible, by the exact homotopy sequence of a fibration we immediately get the following result.

COROLLARY 4.3. *Let $m \in \mathbb{N}$. The space $\text{Gr}_{(m)}^*(H)$ is connected, its fundamental group is trivial for $m = 0$, and infinite cyclic for $m \geq 1$. Moreover, for every $m \in \mathbb{N}$,*

$$\pi_i(\text{Gr}_{(m)}^*(H)) \cong \pi_i(\text{Gr}_e(H)) \cong \pi_{i-2}(GL), \quad \text{for } i \geq 2.$$

In particular, the quotient map

$$\Pi_1^0 : \text{Gr}_{(0)}^*(H) \rightarrow \text{Gr}_{(1)}^*(H) = \text{Gr}_e^*(H)$$

is the universal covering of $\text{Gr}_e^*(H)$, and the covering corresponding to the subgroup $m\mathbb{Z} \subset \mathbb{Z} = \pi_1(\text{Gr}_e^*(H))$, $m \in \mathbb{N}$, is the quotient map

$$\Pi_1^m \text{Gr}_{(m)}^*(H) \rightarrow \text{Gr}_{(1)}^*(H) = \text{Gr}_e^*(H).$$

These are all the coverings of the essential Grassmannian.

We end this section by determining the homotopy groups of the space of Fredholm pairs, thus proving Theorem 2.2.

4.5. PROOF OF THEOREM 2.2. For $k \in \mathbb{Z}$, the map

$$Q_k : \text{Fp}_k^*(H) \rightarrow \text{Gr}_{\infty, \infty}(H), \quad (V, W) \mapsto W,$$

defines a fiber bundle. Indeed, if $W_0 \in \text{Gr}_{\infty, \infty}(H)$, the map

$$B_1(W_0) \times Q_k^{-1}(\{W_0\}) \rightarrow \text{Fp}_k^*(H), \quad (W, V) \mapsto ((P_W P_{W_0} + P_{W^\perp} P_{W_0^\perp})V, W),$$

defines a local trivialization near W_0 . Since $\text{Gr}_{\infty, \infty}(H)$ is contractible, $\text{Fp}_k^*(H)$ is homeomorphic to the product $\text{Gr}_{\infty, \infty}(H) \times Q_k^{-1}(\{W_0\})$, for any $W_0 \in \text{Gr}_{\infty, \infty}(H)$. Let W_1 be a compact perturbation of W_0 with $\dim(W_0, W_1) = k$. Then by Proposition 3.1, $Q_k^{-1}(\{W_0\}) = Q_0^{-1}(\{W_1\})$, so $\text{Fp}_k^*(H)$ is homeomorphic to $\text{Fp}_0^*(H)$, and it has the homotopy type of the fiber of Q_0 .

There remains to determine the homotopy type of $Q_0^{-1}(\{W_0\})$, for some fixed W_0 in the component $\text{Gr}_{\infty, \infty}(H)$. Consider the fiber bundle

$$\Pi : \text{Gr}_{\infty, \infty}(H) \rightarrow \text{Gr}_e^*(H),$$

and the open subset of $\text{Gr}_e^*(H)$

$$\Omega := \{\Pi(\text{graph } T) : T \in L(W_0^\perp, W_0)\}.$$

The space Ω is contractible: indeed the homotopy

$$\Omega \times [0, 1] \rightarrow \Omega, \quad (\Pi(\text{graph } T), \lambda) \mapsto \Pi(\text{graph } \lambda T)$$

is well defined, because the graphs of $T, S \in L(W_0^\perp, W_0)$ are commensurable if and only if $T - S$ is compact, by Proposition 3.2. Moreover, using Proposition 3.1 it is easy to see that

$$\Pi^{-1}(\Omega) = \{V \in \text{Gr}_{\infty, \infty}(H) : (V, W_0) \in \text{Fp}(H)\},$$

and that $Q^{-1}(\{W_0\})$ is a connected component of $\Pi^{-1}(\Omega)$, that is

$$Q^{-1}(\{W_0\}) = \{V \in \text{Gr}_{c,0}(\text{graph } T, H) : T \in L(W_0^\perp, W_0)\}.$$

Therefore, the restriction of the map Π ,

$$\Pi_0 : Q^{-1}(\{W_0\}) \rightarrow \Omega, \quad V \mapsto \Pi(V),$$

defines a fiber bundle, with typical fiber

$$\Pi_0^{-1}(\Pi(W_0^\perp)) = \text{Gr}_{c,0}(W_0^\perp, H).$$

Since the base space of Π_0 is contractible, $Q_0^{-1}(\{W_0\})$ has the homotopy type of $\text{Gr}_{c,0}(W_0^\perp, H)$, and the conclusion follows from Theorem 3.3. ■

5. THE FUNCTOR Det

We denote by Det the functor which associates to any finite dimensional real vector space the one-dimensional real vector space $\Lambda_{\max}(X)$, the component of top degree of the exterior algebra of X , and to any linear map $T : X \rightarrow Y$ the linear map

$$\text{Det}(T) = \begin{cases} \Lambda_{\max}(T) & \text{if } \dim X = \dim Y, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\text{Det}(T) \neq 0$ if and only if T is an isomorphism.

As it is well known, an exact sequence T of finite dimensional vector spaces

$$0 \xrightarrow{T_0} X_1 \xrightarrow{T_1} X_2 \xrightarrow{T_2} \dots \xrightarrow{T_{n-1}} X_n \xrightarrow{T_n} 0$$

induces an isomorphism

$$\phi_T : \bigotimes_{i \text{ odd}} \text{Det}(X_i) \cong \bigotimes_{i \text{ even}} \text{Det}(X_i).$$

Actually, there are different conventions in the literature concerning the choice of the sign of such an isomorphism. As shown in the following sections, the choice of proper signs is important, so let us exhibit an explicit construction for the standard isomorphism ϕ_T , and let us fix notations to denote the opposite isomorphism.

Consider an exact sequence S

$$0 \xleftarrow{S_1} X_1 \xleftarrow{S_2} X_2 \xleftarrow{S_3} \dots \xleftarrow{S_n} X_n \xleftarrow{S_{n+1}} 0$$

such that for every i

$$(5.1) \quad T_{i-1}S_i + S_{i+1}T_i = I_{X_i}.$$

Such an exact sequence is uniquely determined by the choice of an algebraic linear complement V_i for each subspace $\ker T_i$ in X_i . In fact, T_i restricts to an isomorphism from V_i to $\ker T_{i+1}$, and we can define $S_{i+1} : X_{i+1} \rightarrow X_i$ to coincide with the inverse of such an isomorphism on $\ker T_{i+1}$ and to be zero on V_{i+1} .

The linear maps $P_i := T_{i-1}S_i$ and $Q_i := S_{i+1}T_i$ are the projectors associated to the splitting $X_i = \ker T_i \oplus \ker S_i$.

The linear map

$$\Phi_T^S : \bigoplus_{i \text{ odd}} X_i \longrightarrow \bigoplus_{i \text{ even}} X_i, \quad \Phi_T^S = \bigoplus_{i \text{ odd}} (T_i + S_i)$$

is an isomorphism, its inverse being

$$\Phi_T^{S^{-1}} = \bigoplus_{i \text{ even}} (T_i + S_i).$$

If S' is another exact sequence satisfying (5.1), then

$$\Phi_T^{S'^{-1}} \circ \Phi_T^S = \bigoplus_{i \text{ odd}} (S'_{i+1}T_i + T_{i-1}S_i) = I + \bigoplus_{i \text{ odd}} T_{i-1}(S_i - S'_i),$$

and the last term is nilpotent,

$$\left(\bigoplus_{i \text{ odd}} T_{i-1}(S_i - S'_i) \right)^2 = 0.$$

It follows that

$$\det \Phi_T^{S'^{-1}} \circ \Phi_T^S = 1,$$

which implies

$$\text{Det}(\Phi_T^S) = \text{Det}(\Phi_T^{S'}).$$

Recalling that the exterior algebra of a direct sum is naturally identified with the tensor product of the corresponding exterior algebras, we can associate to the exact sequence T the isomorphism

$$\phi_T : \bigotimes_{i \text{ odd}} \text{Det}(X_i) \cong \bigotimes_{i \text{ even}} \text{Det}(X_i), \quad \phi_T := \text{Det}(\Phi_T^S),$$

where S is any sequence satisfying (5.1). Such an isomorphism is natural, in the sense that if we have an isomorphism of exact sequences

$$(5.2) \quad \begin{array}{ccccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{T_1} & X_2 & \xrightarrow{T_2} & X_3 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & 0 \\ & & R_1 \downarrow & & \downarrow R_2 & & \downarrow R_3 & & & & \downarrow & & \\ 0 & \longrightarrow & X'_1 & \xrightarrow{T'_1} & X'_2 & \xrightarrow{T'_2} & X'_3 & \longrightarrow & \cdots & \longrightarrow & X'_n & \longrightarrow & 0 \end{array}$$

there holds

$$(5.3) \quad \left(\bigotimes_{i \text{ even}} \text{Det}(R_i) \right) \circ \phi_T = \phi_{T'} \circ \left(\bigotimes_{i \text{ odd}} \text{Det}(R_i) \right).$$

This immediately follows from the fact that if S satisfies (5.1) for T then the sequence S' defined by $S'_i := R_{i-1}S_iR_i^{-1}$ satisfies (5.1) for T' .

The exact sequence A

$$0 \rightarrow A_1 \rightarrow A_1 \oplus A_2 \rightarrow A_2 \oplus A_3 \rightarrow \cdots \rightarrow A_{n-1} \oplus A_n \rightarrow A_n \rightarrow 0,$$

consisting of inclusions and projections produces the isomorphism ϕ_A given by

$$\phi_A(\alpha_1 \otimes (\alpha_2 \wedge \alpha_3) \otimes \cdots) = (\alpha_1 \wedge \alpha_2) \otimes (\alpha_3 \wedge \alpha_4) \otimes \cdots,$$

where α_i generates $\text{Det}(A_i)$. In the case of a general exact sequence T , if $\text{Det}(X_i)$ is generated by $T_{i-1*}\alpha_{i-1} \wedge \alpha_i$, the isomorphism ϕ_T is given by

$$(5.4) \quad \phi_T(\alpha_1 \otimes (T_{2*}\alpha_2 \wedge \alpha_3) \otimes \cdots) = (T_{1*}\alpha_1 \wedge \alpha_2) \otimes (T_{3*}\alpha_3 \wedge \alpha_4) \otimes \cdots .$$

For every subset $J \subset \{1, \dots, n\}$, the isomorphism ϕ_T induces an isomorphism

$$\phi_T^J : \bigotimes_{j \in J} \text{Det}(X_j)^{*(j-1)} \cong \bigotimes_{j \notin J} \text{Det}(X_j)^{*(j)},$$

where $V^{*(j)}$ denotes V when j is even, V^* when j is odd. The convention here is that a tensor product over an empty set of indices produces the field of scalars, \mathbb{R} or \mathbb{C} . The isomorphism ϕ_T^J is defined by tensorizing ϕ_T by the identity on $\text{Det}(X_j)^*$, for every even $j \in J$ and every odd $j \notin J$, and identifying each tensor product $V \otimes V^*$ with the field of scalars, \mathbb{R} or \mathbb{C} , by the duality pairing. The original isomorphism ϕ_T corresponds to choosing J to be the subset of $\{1, \dots, n\}$ consisting of odd numbers.

Naturality now means that if we are given the isomorphism (5.2) between the exact sequences T and T' , there holds

$$(5.5) \quad \left(\bigotimes_{j \notin J} \text{Det}(R_j^{(-1)^j})^{*(j)} \right) \circ \phi_T^J = \phi_{T'}^J \circ \left(\bigotimes_{j \in J} \text{Det}(R_j^{(-1)^{j-1}})^{*(j-1)} \right).$$

Since the naturality property only involves pairs of exact sequences where the corresponding vector spaces have identical dimension, we still achieve naturality if we multiply each ϕ_T^J by a non-zero number which depends only on the dimensions of X_1, \dots, X_n . In particular, we are interested in changing ϕ_T^J just by a factor ± 1 . We can summarize the above discussion into the following:

PROPOSITION 5.1. *For any choice of the subset $J \subset \{1, \dots, n\}$ and of the function $\sigma : \mathbb{N}^n \rightarrow \{-1, 1\}$, the transformation $T \mapsto \phi_T^{J,\sigma}$ which associates the isomorphism*

$$\phi_T^{J,\sigma} = \sigma(\dim X_1, \dots, \dim X_n) \phi_T^J : \bigotimes_{j \in J} \text{Det}(X_j)^{*(j-1)} \cong \bigotimes_{j \notin J} \text{Det}(X_j)^{*(j)},$$

to the exact sequence T of finite dimensional vector spaces

$$0 \xrightarrow{T_0} X_1 \xrightarrow{T_1} X_2 \xrightarrow{T_2} \cdots \xrightarrow{T_{n-1}} X_n \xrightarrow{T_n} 0,$$

is natural with respect to isomorphisms of exact sequences.

We conclude this section by considering the particular case of a sequence with four terms, that is the exact sequence T

$$0 \rightarrow \ker T \xrightarrow{i} X \xrightarrow{T} Y \xrightarrow{\pi} \text{coker } T \rightarrow 0.$$

induced by some homomorphism $T : X \rightarrow Y$ between finite dimensional vector spaces. We shall associate to such an exact sequence the isomorphism

$$(5.6) \quad \begin{aligned} \psi_T &: \text{Det}(\ker T) \otimes \text{Det}(\text{coker } T)^* \cong \text{Det}(X) \otimes \text{Det}(Y)^*, \\ \alpha \otimes (\pi_* \gamma)^* &\mapsto (i_*(\alpha) \wedge \beta) \otimes (\gamma \wedge T_* \beta)^*, \end{aligned}$$

where α generates $\text{Det}(\ker T)$, $i_*(\alpha) \wedge \beta$ generates $\text{Det}(X)$, $\gamma \wedge T_* \beta$ generates $\text{Det}(Y)$, and the superscript $*$ indicates the dual generator. Comparing this definition with formula (5.4), we notice an exchange in the exterior product between the generators γ and $T_* \beta$ ψ_T , which produces a sign change equal to

$$(-1)^{(\dim \text{ran } T)(\dim \text{coker } T)}.$$

Therefore, ψ_T coincides with the isomorphism $\phi_T^{J,\sigma}$, where

$$J = \{1, 4\}, \quad \sigma(d_1, d_2, d_3, d_4) = (-1)^{d_4(d_3 - d_4)}.$$

Let $X = X_0 \oplus Z$, $Y = Y_0 \oplus Z$ be finite dimensional vector spaces, and let $T \in L(X, Y)$ be such that $TX_0 \subset Y_0$ and $QT|_Z$ is an automorphism of Z , where $Q : Y_0 \oplus Z \rightarrow Z$ denotes the projection. Denote by $T_0 \in L(X_0, Y_0)$ the restriction of T . The fact that $QT|_Z$ is an isomorphism implies that there are natural identifications

$$(5.7) \quad \ker T_0 \cong \ker T, \quad \text{coker } T_0 \cong \text{coker } T.$$

Let

$$\begin{aligned} \psi_T &: \text{Det}(\ker T) \otimes \text{Det}(\text{coker } T)^* \cong \text{Det}(X) \otimes \text{Det}(Y)^*, \\ \psi_{T_0} &: \text{Det}(\ker T_0) \otimes \text{Det}(\text{coker } T_0)^* \cong \text{Det}(X_0) \otimes \text{Det}(Y_0)^*, \end{aligned}$$

be the isomorphisms defined above. By the identifications (5.7) we can consider the composition

$$\psi_T \circ \psi_{T_0}^{-1} : \text{Det}(X_0) \otimes \text{Det}(Y_0)^* \cong \text{Det}(X) \otimes \text{Det}(Y)^*.$$

The next lemma justifies the choice of the sign in the definition of ψ_T :

LEMMA 5.2. *Let x_0, y_0, z be generators of $\text{Det}(X_0), \text{Det}(Y_0), \text{Det}(Z)$, respectively. Then*

$$\psi_T \circ \psi_{T_0}^{-1}(x_0 \otimes y_0^*) = \det(QT|_Z)(x_0 \wedge z) \otimes (y_0 \wedge z)^*.$$

In particular, the function mapping a homomorphism $T \in L(X, Y)$ with the above properties into $\psi_T \circ \psi_{T_0}^{-1}$ depends analytically on T .

Proof. We can write $x_0 = \alpha \wedge x_1$, where α is a generator of $\text{Det}(\ker T_0)$, and $y_0 = y_1 \wedge T_{0*} x_1$, where y_1 is a generator of $\text{coker } T_0 \cong \text{coker } T$, identified with a subspace of Y_0 . Then

$$\begin{aligned} \psi_{T_0}(\alpha \otimes y_1^*) &= (\alpha \wedge x_1) \otimes (y_1 \wedge T_{0*} x_1)^* = x_0 \otimes y_0^*, \\ \psi_T(\alpha \otimes y_1^*) &= (\alpha \wedge x_1 \wedge z) \otimes (y_1 \wedge T_*(x_1 \wedge z))^* = (x_0 \wedge z) \otimes (y_1 \wedge T_*(x_1 \wedge z))^*. \end{aligned}$$

Notice that $y_1 \wedge T_*(x_1 \wedge z) = y_1 \wedge T_{0*}x_1 \wedge T_*z = y_0 \wedge QT_*z + y_0 \wedge (I - Q)T_*z = \det(QT|_Z)y_0 \wedge z$. It follows that $\psi_T(\alpha \otimes y_1^*) = \det(QT|_Z)(x_0 \wedge z) \otimes (y_0 \wedge z)^*$, so, as claimed:

$$\psi_T \circ \psi_{T_0}^{-1}(x_0 \otimes y_0^*) = \det(QT|_Z)(x_0 \wedge z) \otimes (y_0 \wedge z)^*. \quad \blacksquare$$

Another useful property of the isomorphism ψ_T defined in (5.6) is that

$$(5.8) \quad \psi_{T^*}^{-1} = \psi_T^*,$$

whereas the corresponding formula relating a general isomorphism $\phi_T^{J,\sigma}$ to $\phi_{T^*}^{J,\sigma}$ is more complicated, and may involve a change of sign.

6. THE DETERMINANT BUNDLE OVER THE SPACE OF FREDHOLM PAIRS

The determinant bundle over $\text{Fp}(H)$ is an analytic line bundle with base space $\text{Fp}(H)$, and fiber at (V, W)

$$\text{Det}(V, W) := \text{Det}(V \cap W) \otimes \text{Det}\left(\frac{H}{V+W}\right)^*.$$

It is denoted by

$$p : \text{Det}(\text{Fp}(H)) \rightarrow \text{Fp}(H), \quad \text{Det}(\text{Fp}(H)) := \bigsqcup_{(V,W) \in \text{Fp}(H)} \text{Det}(V, W).$$

The construction of an analytic structure on the total space $\text{Det}(\text{Fp}(H))$, making p an analytic bundle map, is not immediate because the subspaces $V \cap W$ and $V + W$ do not depend even continuously on $(V, W) \in \text{Fp}(H)$. We will describe such an analytic structure first on the counter image by p of those components of $\text{Fp}(H)$ where one of the two spaces is finite dimensional, and then on the other components.

6.1. ANALYTIC STRUCTURE ON $p^{-1}(\text{Fp}(H) \setminus \text{Fp}^*(H))$. If $(V, W) \in \text{Gr}_{n,\infty}(H) \times \text{Gr}_{\infty,m}(H)$, $n, m \in \mathbb{N}$, the exact sequence of inclusions and quotient projections

$$0 \rightarrow V \cap W \rightarrow V \rightarrow \frac{H}{W} \rightarrow \frac{H}{V+W} \rightarrow 0$$

yields to the isomorphism

$$(6.1) \quad \psi : \text{Det}(V, W) \cong \text{Det}(V) \otimes \text{Det}\left(\frac{H}{W}\right)^*,$$

defined in (5.6). The lines $\text{Det}(V) \otimes \text{Det}(H/W)^*$ are clearly the fibers of an analytic bundle over $\text{Gr}_{n,\infty}(H) \times \text{Gr}_{\infty,m}(H)$, and we can endow $p^{-1}(\text{Gr}_{n,\infty}(H) \times \text{Gr}_{\infty,m}(H))$ with the analytic structure which makes (6.1) an isomorphism of analytic bundles. The same construction works for $p^{-1}(\text{Gr}_{\infty,n}(H) \times \text{Gr}_{m,\infty}(H))$.

6.2. ANALYTIC STRUCTURE ON $p^{-1}(\text{Fp}^*(H))$. We proceed to define a structure of analytic bundle on the restriction of p to $p^{-1}(\text{Fp}^*(H))$. Consider the covering of $\text{Fp}^*(H)$ consisting of the open sets

$$\mathcal{U}_X := \{(V, W) \in \text{Fp}^*(H) : X + V + W = H, X \cap V = (0)\},$$

for X varying among all finite dimensional linear subspaces of H . If X is finite dimensional, the map $V \mapsto X + V$ is analytic on the space of $V \in \text{Gr}(H)$ such that $X \cap V = (0)$: indeed in this case $X + V$ is closed, so $X + V = (X^\perp \cap V^\perp)^\perp$ and $X^\perp + V^\perp = (X \cap V)^\perp = H$, hence the analyticity of the sum follows from the analyticity of the transverse intersection. As a consequence, the map $\mathcal{U}_X \rightarrow \text{Gr}(H)$, $(V, W) \mapsto (X + V) \cap W$, is analytic. Therefore, we can consider the analytic line bundle $p_X : \mathcal{E}_X \rightarrow \mathcal{U}_X$, whose fiber at (V, W) is the line

$$p_X^{-1}(\{(V, W)\}) := \text{Det}((X + V) \cap W) \otimes \text{Det}(X)^*.$$

Let $(V, W) \in \mathcal{U}_X$. Since $X \cap V = (0)$, we can define a linear map $(X + V) \cap W \rightarrow X$ to be the following composition of inclusions, quotient projections, and their inverses:

$$(X + V) \cap W \rightarrow X + V \rightarrow \frac{X + V}{V} \cong X.$$

The kernel of this linear map is $V \cap W$, while its range is $X \cap (V + W)$, so using also the identity $X + V + W = H$, we obtain the exact sequence

$$0 \rightarrow V \cap W \rightarrow (X + V) \cap W \rightarrow X \rightarrow \frac{H}{V + W} \rightarrow 0.$$

Therefore formula (5.6) produces the isomorphism

$$\psi_{(V,W)}^X : \text{Det}(V, W) \rightarrow \text{Det}((X + V) \cap W) \otimes \text{Det}(X)^*,$$

and we can endow $p^{-1}(\mathcal{U}_X)$ with the analytic structure which makes the map $p^{-1}(\mathcal{U}_X) \rightarrow \mathcal{E}_X$, $\xi \mapsto \psi_{(V,W)}^X \xi$ for $p(\xi) = (V, W)$, an isomorphism of analytic bundles.

In order to have an analytic structure on the whole $p^{-1}(\text{Fp}^*(H))$, there remains to show that if X_1, X_2 are finite dimensional linear subspaces of H , the transition map

$$\psi_{X_1}^{X_2} : p_{X_1}^{-1}(\mathcal{U}_{X_1} \cap \mathcal{U}_{X_2}) \rightarrow p_{X_2}^{-1}(\mathcal{U}_{X_1} \cap \mathcal{U}_{X_2}), \quad \xi \mapsto \psi_{(V,W)}^{X_2} (\psi_{(V,W)}^{X_1})^{-1} \xi, \quad \text{for } p_{X_1}(\xi) = (V, W),$$

is analytic. When $X_1 \subset X_2$ or $X_2 \subset X_1$, this fact follows immediately from Lemma 5.2. In the general case, let $(V, W) \in \mathcal{U}_{X_1} \cap \mathcal{U}_{X_2}$. Then there exists a finite dimensional linear subspace $X \subset H$ such that

$$(6.2) \quad X + V + W = H, \quad (X_1 + X) \cap V = (0), \quad (X_2 + X) \cap V = (0).$$

Indeed, setting $k = \text{codim}(V + W)$ and recalling that V has infinite codimension, we have that the set of $X \in \text{Gr}_{k,\infty}(H)$ satisfying each of the identities (6.2) is open and dense. Therefore,

$$(V, W) \in \mathcal{U}_{X_1} \cap \mathcal{U}_{X_1+X} \cap \mathcal{U}_X \cap \mathcal{U}_{X_2+X} \cap \mathcal{U}_{X_2},$$

and on the inverse image by p of such a neighborhood $\psi_{X_1}^{X_2}$ is the composition of analytic maps, by the case seen above. Hence $\psi_{X_1}^{X_2}$ is analytic on $p^{-1}(\mathcal{U}_{X_1} \cap \mathcal{U}_{X_2})$.

6.3. TRANSPOSITION. Note that, although the fiber $\text{Det}(V, W)$ depends symmetrically on (V, W) , in our construction of the analytic structure on $p^{-1}(\text{Fp}^*(H))$ such a symmetry is lost. However, if we exchange the role of V and W in the definition above, we obtain the same analytic structure. In other words, the map $\text{Det}(\text{Fp}(H)) \rightarrow \text{Det}(\text{Fp}(H))$ which lifts the transposition involution

$$\tau : \text{Fp}(H) \rightarrow \text{Fp}(H), \quad (V, W) \mapsto (W, V),$$

and is the identity on the fibers, is an analytic bundle isomorphism. This fact is obvious for the components where one of the spaces is finite dimensional, so we just have to check it locally at some $(V_0, W_0) \in \text{Fp}^*(H)$. We can choose a finite dimensional linear subspace $X \subset H$ such that $X \cap (V_0 + W_0) = (0)$ and $X + V_0 + W_0 = H$. Then both (V_0, W_0) and (W_0, V_0) belong to the τ -invariant open set

$$\mathcal{U}_X \cap \tau(\mathcal{U}_X) = \{(V, W) \in \text{Fp}^*(H) : X \cap V = X \cap W = (0), X + V + W = H\},$$

and we must check that the isomorphism

$$\psi_{(W,V)}^X \circ (\psi_{(V,W)}^X)^{-1} : \text{Det}((X+V) \cap W) \otimes \text{Det}(X)^* \rightarrow \text{Det}((X+W) \cap V) \otimes \text{Det}(X)^*$$

depends analytically on (V, W) in such a set. If (V, W) belong to this set, we can find an isomorphism

$$T_{(V,W)} : (X+W) \cap V \xrightarrow{\cong} (X+V) \cap W,$$

that makes the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V \cap W & \longrightarrow & (X+V) \cap W & \longrightarrow & X & \longrightarrow & \frac{H}{V+W} & \longrightarrow & 0 \\ & & \text{id} \downarrow & & T_{(V,W)} \downarrow & & \text{id} \downarrow & & \text{id} \downarrow & & \\ 0 & \longrightarrow & W \cap V & \longrightarrow & (X+W) \cap V & \longrightarrow & X & \longrightarrow & \frac{H}{W+V} & \longrightarrow & 0 \end{array}$$

commutative. Such an isomorphism is uniquely determined by the choice of two linear complements of $V \cap W$ in $(X+V) \cap W$ and $(X+W) \cap V$, and if we choose these to be the orthogonal complements, $T_{(V,W)}$ depends analytically on (V, W) . By the naturality property stated in Proposition 5.1, we have

$$\psi_{(W,V)}^X \circ (\psi_{(V,W)}^X)^{-1} = \text{Det}(T_{(V,W)}) \otimes \text{id},$$

which depends analytically on $(V, W) \in \mathcal{U}_X \cap \tau(\mathcal{U}_X)$, proving the claim.

This fact shows that it is possible to define a determinant bundle also on the space of unordered Fredholm pairs.

6.4. **NONTRIVIALITY.** The determinant bundle $p : \text{Det}(\text{Fp}(H)) \rightarrow \text{Fp}(H)$ is nontrivial on every connected component of $\text{Fp}(H)$, with the exception of the trivial components $\text{Gr}_{0,\infty}(H) \times \text{Gr}_{\infty,0}(H)$ and $\text{Gr}_{\infty,0}(H) \times \text{Gr}_{0,\infty}(H)$. We shall check this fact for the connected components $\text{Fp}_k^*(H)$, $k \in \mathbb{Z}$, the case of the other components being analogous. First notice that if H_0 is 2-dimensional, then the determinant bundle on $\text{Gr}_{1,1}(H_0)$ is nontrivial, being isomorphic to the tautological line bundle on the projective line. Fix a splitting $H = H_0 \oplus H_1$, $\dim H_0 = 2$, a Fredholm pair $(V, W) \in \text{Fp}_{k-1}^*(H_1)$ and a generator ξ of $\text{Det}(V \cap W) \otimes \text{Det}(H_1/(V + W))^*$, so that for every $L \in \text{Gr}_{1,1}(H_0)$,

$$(L \oplus V) \cap (H_0 \oplus W) = L \oplus (V \cap W), \quad \frac{H}{(L \oplus V) + (H_0 \oplus W)} = \frac{H_1}{V + W}.$$

Then the embedding

$$f : \text{Gr}_{1,1}(H_0) \rightarrow \text{Fp}_k(H), \quad L \mapsto (L \oplus V, H_0 \oplus W)$$

can be lifted to an analytic bundle isomorphism

$$F : \text{Det}(\text{Gr}_{1,1}(H_0)) \rightarrow p^{-1}(f(\text{Gr}_{1,1}(H_0))) \subset \text{Det}(\text{Fp}_k^*(H)), \quad \eta \mapsto \eta \otimes \xi,$$

and the determinant bundle is nontrivial on $\text{Fp}_k^*(H)$.

6.5. **THE DETERMINANT BUNDLE OVER $\text{Gr}_c(V, H)$.** Let $V \in \text{Gr}_{\infty,\infty}(H)$. Then the Grassmannian $\text{Gr}_c(V, H)$ of compact perturbations of V has a natural inclusion into $\text{Fp}(H)$ given by

$$(6.3) \quad \text{Gr}_c(V, H) \hookrightarrow \text{Fp}(H), \quad W \mapsto (W, V^\perp).$$

The above map allows us to consider the pull-back of the determinant bundle $\text{Det}(\text{Fp}(H)) \rightarrow \text{Fp}(H)$ defined on $\text{Gr}_c(V, H)$, obtaining an analytic line bundle $\text{Det}(\text{Gr}_c(V, H)) \rightarrow \text{Gr}_c(V, H)$ with fiber at $W \in \text{Gr}_c(V, H)$

$$\text{Det}(W \cap V^\perp) \otimes \text{Det}\left(\frac{H}{W + V^\perp}\right)^* \cong \text{Det}((W \cap V^\perp) \oplus (W^\perp \cap V)).$$

The argument for the nontriviality used above shows that the above line bundle is nontrivial on every connected component $\text{Gr}_{c,n}(V, H)$, for $n \in \mathbb{Z}$.

7. THE DETERMINANT BUNDLE OVER THE SPACE OF FREDHOLM OPERATORS

Let $\text{Fr}(H_1, H_2)$ be the open subset of $L(H_1, H_2)$ consisting of Fredholm operators. The inclusion

$$i : \text{Fr}(H_1, H_2) \rightarrow \text{Fp}(H_1 \times H_2), \quad T \mapsto (\text{graph } T, H_1 \times (0)),$$

allows to define the determinant bundle over $\text{Fr}(H_1, H_2)$ as the pull-back of the determinant bundle over $\text{Fp}(H_1 \times H_2)$ by the map i . We obtain a line bundle

$$q : \text{Det}(\text{Fr}(H_1, H_2)) \rightarrow \text{Fr}(H_1, H_2),$$

whose fiber at T is

$$\text{Det}(T) := \text{Det}(\ker T) \otimes \text{Det}(\text{coker } T)^*.$$

This is the determinant bundle over the space of Fredholm operators defined by Quillen in [26] (see also [6]).

A direct construction of the analytic bundle structure on $\text{Det}(\text{Fr}(H_1, H_2))$ goes as follows. Let X be a finite dimensional linear subspace of H_2 , and consider the open set

$$(7.1) \quad \mathcal{U}_X(H_1, H_2) := \{T \in \text{Fr}(H_1, H_2) : T \text{ is transverse to } X\}.$$

If $T \in \mathcal{U}_X(H_1, H_2)$, the exact sequence

$$0 \rightarrow \ker T \longrightarrow T^{-1}X \xrightarrow{T} X \longrightarrow \text{coker } T \rightarrow 0$$

induces an isomorphism

$$(7.2) \quad \psi_T : \text{Det}(T) \rightarrow \text{Det}(T^{-1}X) \otimes \text{Det}(X)^*$$

by formula (5.6). Since the lines $\text{Det}(T^{-1}X) \otimes \text{Det}(X)$ are the fibers of an analytic line bundle over $\mathcal{U}_X(H_1, H_2)$, we obtain an analytic line bundle structure for $\text{Det}(\text{Fr}(H_1, H_2))$ over $\mathcal{U}_X(H_1, H_2)$. Lemma 5.2 can again be used to show that the transition maps are analytic.

An alternative construction for the analytic structure of q is the following. If T is an element of $\text{Fr}(H_1, H_2)$, the operator $T^*T \in L(H_1)$ is self-adjoint, positive, and Fredholm. In particular, if $\varepsilon > 0$ is small enough, the spectrum of T^*T has finite multiplicity in $[0, \varepsilon]$, meaning that $\sigma(T^*T) \cap [0, \varepsilon]$ is a finite set of eigenvalues with finite multiplicity. Denote by $V_\varepsilon(T^*T)$ the corresponding finite dimensional eigenspace. Therefore the sets

$$\mathcal{V}_\varepsilon(H_1, H_2) := \{T \in \text{Fr}(H_1, H_2) : \varepsilon \notin \sigma(T^*T) \text{ and } \sigma(T^*T) \cap [0, \varepsilon] \text{ has finite multiplicity}\},$$

for $\varepsilon > 0$, constitute an open covering of $\text{Fr}(H_1, H_2)$. Clearly, the operators

$$\begin{aligned} T^*T|_{(\ker T)^\perp} &: (\ker T)^\perp \rightarrow (\ker T)^\perp = \text{ran } T^*, \\ TT^*|_{(\ker T^*)^\perp} &: (\ker T^*)^\perp \rightarrow (\ker T^*)^\perp = \text{ran } T, \end{aligned}$$

are conjugated by $T|_{(\ker T)^\perp} : (\ker T)^\perp \rightarrow (\ker T^*)^\perp$. Since $\ker T^*T = \ker T$ and $\ker TT^* = \ker T^*$, it follows that any $\lambda > 0$ belongs to $\sigma(T^*T)$ if and only if it belongs to $\sigma(TT^*)$, and that it is an eigenvalue of T^*T corresponding to the eigenvector x if and only if it is an eigenvalue of TT^* corresponding to the eigenvector Tx , with the same multiplicity. Hence, if $T \in \mathcal{V}_\varepsilon(H_1, H_2)$, then $T^* \in \mathcal{V}_\varepsilon(H_2, H_1)$, and the sequence

$$(7.3) \quad 0 \rightarrow \ker T \rightarrow V_\varepsilon(T^*T) \xrightarrow{T} V_\varepsilon(TT^*) \rightarrow \ker T^* \rightarrow 0$$

is exact. From the isomorphism

$$\psi_T^\varepsilon : \text{Det}(T) \cong \text{Det}(V_\varepsilon(T^*T))^* \otimes \text{Det}(V_\varepsilon(TT^*))$$

defined by (5.6) and from the analyticity of the maps $T \mapsto V_\varepsilon(T^*T)$ and $T \mapsto V_\varepsilon(TT^*)$ on $\mathcal{V}_\varepsilon(H_1, H_2)$, we obtain the analytic structure of q over $\mathcal{V}_\varepsilon(H_1, H_2)$. The analyticity of the transition maps is an immediate consequence of Lemma 5.2, because if $0 \leq \varepsilon < \varepsilon'$ and $T \in \mathcal{V}_\varepsilon(H_1, H_2) \cap \mathcal{V}_{\varepsilon'}(H_1, H_2)$, there holds $V_\varepsilon(T^*T) \subset V_{\varepsilon'}(T^*T)$, and $V_\varepsilon(TT^*) \subset V_{\varepsilon'}(TT^*)$.

We just mention that an analogous spectral approach provides an alternative construction of the determinant bundle on $\text{Fp}(H)$.

7.1. LEFT AND RIGHT ACTION OF $\text{GL}(H)$. The group $\text{GL}(H_2)$ acts on $\text{Fr}(H_1, H_2)$ by left multiplication $(G, T) \mapsto GT$, while the group $\text{GL}(H_1)$ acts on $\text{Fr}(H_1, H_2)$ by right multiplication $(G, T) \mapsto TG$. Since

$$\ker GT = \ker T, \quad \text{coker } GT = \tilde{G} \text{ coker } T,$$

where $\tilde{G} : H_2/\text{ran } T \rightarrow H_2/G \text{ ran } T$ is induced by G , the left action lifts to a bundle action on $\text{Det}(\text{Fr}(H_1, H_2))$ defined fiberwise by

$$\text{Det}(T) \rightarrow \text{Det}(GT), \quad \zeta \otimes \eta^* \mapsto \zeta \otimes (\text{Det}(\tilde{G}^{-1})^* \eta^*).$$

This action is analytic because of the naturality property of Proposition 5.1. Similarly, since

$$\ker TG = G^{-1} \ker T, \quad \text{coker } TG = \text{coker } T,$$

the right action lifts to an analytic bundle action on $\text{Det}(\text{Fr}(H_1, H_2))$ defined fiberwise by

$$\text{Det}(T) \rightarrow \text{Det}(TG), \quad \zeta \otimes \eta^* \mapsto (\text{Det}(G^{-1})\zeta) \otimes \eta^*.$$

7.2. ADJOINT. The adjoint map

$$\text{Fr}(H_1, H_2) \rightarrow \text{Fr}(H_2, H_1), \quad T \mapsto T^*,$$

has analytic lift to the determinant bundles. Indeed, by using the identifications

$$\ker T^* = (\text{ran } T)^\perp \cong (\text{coker } T)^*, \quad \text{coker } T^* = \frac{H_1^*}{(\ker T)^\perp} \cong (\ker T)^*,$$

the lift is defined fiberwise by

$$\text{Det}(T) \rightarrow \text{Det}(T^*), \quad \zeta \otimes \eta^* \mapsto \eta^* \otimes \zeta.$$

Let us check that this lift is analytic. If T belongs to the open set $\mathcal{V}_\varepsilon(H_1, H_2)$, then T^* belongs to the open set $\mathcal{V}_\varepsilon(H_2, H_1)$. The adjoint of the exact sequence (7.3) is

$$0 \rightarrow \ker T^* \rightarrow V_\varepsilon(TT^*) \xrightarrow{T} V_\varepsilon(T^*T) \rightarrow \ker T \rightarrow 0,$$

which is precisely the sequence producing the isomorphism

$$\psi_{T^*}^\varepsilon : \text{Det}(T^*) \cong \text{Det}(V_\varepsilon(TT^*)) \otimes \text{Det}(V_\varepsilon(T^*T))^*.$$

Then property (5.8) allows to conclude.

7.3. TWO ANALYTIC SECTIONS. The determinant bundle $\text{Det}(\text{Fr}(H_1, H_2))$ has the global analytic section

$$s(T) = \begin{cases} 0 & \text{if } T \text{ is not invertible,} \\ 1 \in \text{Det}(T) = \mathbb{C} \otimes \mathbb{C}^* & \text{if } T \text{ is invertible.} \end{cases}$$

This section vanishes precisely at the non-invertible elements of $\text{Fr}(H_1, H_2)$.

The restriction of the determinant bundle $\text{Det}(\text{Fr}(H, H))$ to the space of self-adjoint Fredholm operators on H has a nowhere vanishing analytic section σ which associates to each self-adjoint Fredholm operator T the element

$$1 \in \text{Det}(\ker T) \otimes \text{Det}(\ker T)^* = \text{Det}(\ker T) \otimes \text{Det}(\text{coker } T)^* = \text{Det}(T).$$

In particular, the restriction of the determinant bundle over the space of self-adjoint Fredholm operators on H is trivial (as observed by Furutani in [14], Theorem 4.1).

7.4. COMPOSITION. Let us show how the composition of Fredholm operators

$$\text{Fr}(H_1, H_2) \times \text{Fr}(H_2, H_3) \rightarrow \text{Fr}(H_1, H_3), \quad (S, T) \mapsto TS,$$

lifts to the determinant bundles, producing the bundle morphism

$$\text{Det}(\text{Fr}(H_1, H_2)) \otimes \text{Det}(\text{Fr}(H_2, H_3)) \rightarrow \text{Det}(\text{Fr}(H_1, H_3)),$$

where the domain is seen as a line bundle over the product $\text{Fr}(H_1, H_2) \times \text{Fr}(H_2, H_3)$.

If $S \in \text{Fr}(H_1, H_2)$ and $T \in \text{Fr}(H_2, H_3)$, we have the exact sequence

$$(7.4) \quad 0 \rightarrow \ker S \rightarrow \ker TS \xrightarrow{S} \ker T \xrightarrow{\pi} \text{coker } S \xrightarrow{T} \text{coker } TS \rightarrow \text{coker } T \rightarrow 0,$$

where π denotes the restriction of the quotient projection. This exact sequence induces the isomorphism

$$(7.5) \quad \begin{aligned} \phi_{S,T} : \text{Det}(S) \otimes \text{Det}(T) &\rightarrow \text{Det}(TS), \\ \alpha_1 \otimes (\pi_* \alpha_3 \wedge \alpha_4)^* \otimes (\alpha_3 \wedge S_* \alpha_2) \otimes \alpha_5^* &\mapsto (-1)^{\rho(S,T)} (\alpha_1 \wedge \alpha_2) \otimes (\alpha_5 \otimes T_* \alpha_4)^*, \end{aligned}$$

with

$$\rho(S, T) := (\dim \ker S + \dim \ker TS)(\dim \text{coker } T + \dim \text{coker } TS),$$

where α_1 generates $\text{Det}(\ker S)$, $\alpha_1 \wedge \alpha_2$ generates $\text{Det}(\ker TS)$, $\alpha_3 \wedge S_* \alpha_2$ generates $\text{Det}(\ker T)$, $\pi_* \alpha_3 \wedge \alpha_4$ generates $\text{Det}(\text{coker } S)$, $\alpha_5 \wedge T_* \alpha_4$ generates $\text{Det}(\text{coker } TS)$, and α_5 generates the space $\text{Det}(\text{coker } T)$. If we denote by (S, T) the exact sequence (7.4), the isomorphism $\phi_{S,T}$ coincides with the natural isomorphism $\phi_{(S,T)}^{J,\sigma}$ of Section 5, with

$$(7.6) \quad J = \{1, 3, 4, 6\}, \quad \sigma(d_1, d_2, d_3, d_4, d_5, d_6) = (-1)^{d_4(d_2-d_1)+d_6(d_5-d_6)}.$$

Let us check that the bundle map defined by the isomorphisms $\phi_{S,T}$ is analytic. It is convenient to see the analytic structure on the determinant bundle of $\text{Fr}(H_1, H_2)$ in terms of the open sets $\mathcal{U}_X(H_1, H_2)$ and of the isomorphisms ψ_T introduced in (7.1) and (7.2).

Let Y be a finite dimensional linear subspace of H_3 . Notice that the composition TS is transverse to Y if and only if T is transverse to Y and S is transverse to $X := T^{-1}Y$. In particular, the composition maps $\mathcal{U}_X(H_1, H_2) \times \mathcal{U}_Y(H_2, H_3)$ into $\mathcal{U}_Y(H_1, H_3)$. Let

$$\begin{aligned}\psi_S &: \text{Det}(S) \rightarrow \text{Det}(S^{-1}X) \otimes \text{Det}(X)^*, \\ \psi_T &: \text{Det}(T) \rightarrow \text{Det}(T^{-1}Y) \otimes \text{Det}(Y)^* = \text{Det}(X) \otimes \text{Det}(Y)^*, \\ \psi_{TS} &: \text{Det}(TS) \rightarrow \text{Det}((TS)^{-1}Y) \otimes \text{Det}(Y)^* = \text{Det}(S^{-1}X) \otimes \text{Det}(Y)^*,\end{aligned}$$

be the isomorphisms defined in (7.2). We must show that the upper horizontal isomorphism which makes the following diagram commutative,

$$\begin{array}{ccc}\text{Det}(S^{-1}X) \otimes \text{Det}(X)^* \otimes \text{Det}(X) \otimes \text{Det}(Y)^* & \longrightarrow & \text{Det}(S^{-1}X) \otimes \text{Det}(Y)^* \\ \psi_S \otimes \psi_T \uparrow & & \uparrow \psi_{TS} \\ \text{Det}(\ker S) \otimes \text{Det}(\text{coker } S)^* \otimes \text{Det}(\ker T) \otimes \text{Det}(\text{coker } T)^* & \xrightarrow{\phi_{S,T}} & \text{Det}(\ker TS) \otimes \text{Det}(\text{coker } TS)^*\end{array}$$

depends analytically on $(S, T) \in \mathcal{U}_X(H_1, H_2) \times \mathcal{U}_Y(H_2, H_3)$.

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ be as above. Let ζ be a generator of $\text{Det}(Y)$. Since TS is transverse to Y , we can find an element $\alpha_0 \in \Lambda_*((TS)^{-1}Y) = \Lambda_*(S^{-1}X)$ such that $\zeta = \alpha_5 \wedge T_*\alpha_4 \wedge T_*S_*\alpha_0$. Then $\xi := \alpha_1 \wedge \alpha_2 \wedge \alpha_0$ generates $\text{Det}(S^{-1}X)$. Moreover, $\eta := \alpha_3 \wedge \alpha_4 \wedge S_*\alpha_2 \wedge S_*\alpha_0$ generates $\text{Det}(X)$. By (5.6), we have

$$\begin{aligned}\psi_S(\alpha_1 \otimes (\pi_*\alpha_3 \wedge \alpha_4)^*) &= (\alpha_1 \wedge \alpha_2 \wedge \alpha_0) \otimes (\alpha_3 \wedge \alpha_4 \wedge S_*\alpha_2 \wedge S_*\alpha_0)^* = \xi \otimes \eta^*, \\ \psi_T((\alpha_3 \wedge S_*\alpha_2) \otimes \alpha_5^*) &= (\alpha_3 \wedge S_*\alpha_2 \wedge \alpha_4 \wedge S_*\alpha_0) \otimes (\alpha_5 \wedge T_*\alpha_4 \wedge T_*S_*\alpha_0)^* \\ &= (-1)^{|\alpha_2| |\alpha_4|} \eta \otimes \zeta^*, \\ \psi_{TS}((\alpha_1 \wedge \alpha_2) \otimes (\alpha_5 \wedge T_*\alpha_4)^*) &= (\alpha_1 \wedge \alpha_2 \wedge \alpha_0) \otimes (\alpha_5 \wedge T_*\alpha_4 \wedge T_*S_*\alpha_0)^* = \xi \otimes \zeta^*.\end{aligned}$$

Since the parity of $|\alpha_2| |\alpha_4|$ equals the parity of $\rho(S, T)$, the above formulas imply that

$$\psi_{TS} \circ \phi_{S,T} \circ (\psi_S \otimes \psi_T)^{-1}(\xi \otimes \eta^* \otimes \eta \otimes \zeta^*) = \xi \otimes \zeta^*,$$

so this isomorphism depends analytically on (S, T) , as claimed.

7.5. ASSOCIATIVITY. The commutative diagram

$$\begin{array}{ccc}\text{Fr}(H_1, H_2) \times \text{Fr}(H_2, H_3) \times \text{Fr}(H_3, H_4) & \longrightarrow & \text{Fr}(H_1, H_2) \times \text{Fr}(H_2, H_4) \\ \downarrow & & \downarrow \\ \text{Fr}(H_1, H_3) \times \text{Fr}(H_3, H_4) & \longrightarrow & \text{Fr}(H_1, H_4)\end{array}$$

lifts to a commutative diagram between the corresponding determinant bundles. In other words, if $T_i \in \text{Fr}(H_i, H_{i+1})$ then the diagram

$$(7.7) \quad \begin{array}{ccc} \text{Det}(T_1) \otimes \text{Det}(T_2) \otimes \text{Det}(T_3) & \xrightarrow{\text{id} \otimes \phi_{T_2, T_3}} & \text{Det}(T_1) \otimes \text{Det}(T_3 T_2) \\ \downarrow \phi_{T_1, T_2} \otimes \text{id} & & \downarrow \phi_{T_1, T_3 T_2} \\ \text{Det}(T_2 T_1) \otimes \text{Det}(T_3) & \xrightarrow{\phi_{T_2 T_1, T_3}} & \text{Det}(T_3 T_2 T_1) \end{array}$$

commutes. Although completely elementary, the proof of the commutativity of the above diagram is quite long. It is contained in Appendix B.

8. OPERATIONS ON THE DETERMINANT BUNDLE OVER FREDHOLM PAIRS

Besides the transposition involution already considered in Section 6, other operations on the space of Fredholm pairs can be lifted to the determinant bundle.

8.1. THE ACTION OF $\text{GL}(H)$. The analytic action of $\text{GL}(H)$ on $\text{Fp}(H)$, $(T, (V, W)) \mapsto (TV, TW)$, lifts to an analytic bundle action of $\text{GL}(H)$ on $\text{Det}(\text{Fp}(H))$, defined fiberwise by

$$\text{Det}(V, W) \rightarrow \text{Det}(TV, TW), \quad \xi \otimes \eta^* \mapsto \text{Det}(T)\xi \otimes \text{Det}(\tilde{T}^{-1})^*\eta^*,$$

for $\xi \in \text{Det}(V \cap W)$, $\eta^* \in \text{Det}(H/(V+W))^*$, where $\tilde{T} \in \text{L}(H/(V+W), H/(TV+TW))$ is induced by the linear operator $T \in \text{GL}(H)$. The analyticity of such an action follows immediately from the naturality property stated in Proposition 5.1.

8.2. SUM. If X, Y, Z are finite dimensional linear spaces, the isomorphism

$$S(X, Y) : \text{Det}(X) \otimes \text{Det}(Y) \rightarrow \text{Det}(X \oplus Y), \quad \xi \otimes \eta \mapsto \xi \wedge \eta,$$

is induced by the exact sequence

$$0 \rightarrow X \rightarrow X \oplus Y \rightarrow Y \rightarrow 0.$$

It is readily seen that the diagram

$$(8.1) \quad \begin{array}{ccc} \text{Det}(X) \otimes \text{Det}(Y) \otimes \text{Det}(Z) & \xrightarrow{S(X, Y) \otimes \text{id}_{\text{Det}(Z)}} & \text{Det}(X \oplus Y) \otimes \text{Det}(Z) \\ \text{id}_{\text{Det}(X)} \otimes S(Y, Z) \downarrow & & \downarrow S(X \oplus Y, Z) \\ \text{Det}(X) \otimes \text{Det}(Y \oplus Z) & \xrightarrow{S(X, Y \oplus Z)} & \text{Det}(X \oplus Y \oplus Z) \end{array}$$

commutes. We would like to extend this construction to Fredholm pairs.

Let $X \in \text{Gr}(H)$ be finite dimensional, and let $(V, W) \in \text{Fp}(H)$ be such that $X \cap V = (0)$. The isomorphism

$$S(X, (V, W)) : \text{Det}(X) \otimes \text{Det}(V, W) \rightarrow \text{Det}(X + V, W)$$

is induced by the exact sequence

$$(8.2) \quad 0 \rightarrow V \cap W \rightarrow (X+V) \cap W \rightarrow \frac{X+V}{V} \cong X \rightarrow \frac{H}{V+W} \rightarrow \frac{H}{X+V+W} \rightarrow 0.$$

More precisely, if (X, V, W) denotes the above exact sequence, $S(X, (V, W))$ is defined to be the isomorphism $\phi_{(X, V, W)}^{J, \sigma}$ from Section 5, where

$$(8.3) \quad J = \{1, 3, 4\}, \quad \sigma(d_1, d_2, d_3, d_4, d_5) = (-1)^{d_3 d_2 + d_5(d_4 - d_5)}.$$

The reason for this choice of the sign is that the above exact sequence can be seen as the exact sequence associated to the composition of two Fredholm operators. Indeed, let

$$R = R_{X, (V, W)} : V \oplus W \hookrightarrow (X+V) \oplus W$$

be the inclusion, and let

$$T = T_{X, (V, W)} : (X+V) \oplus W \rightarrow H, \quad (v, w) \mapsto v - w,$$

be the difference mapping. Then R and T are Fredholm operators, and the exact sequence (7.4) associated to their composition is precisely (8.2):

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & V \cap W & \longrightarrow & (X+V) \cap W & \longrightarrow & \frac{X+V}{V} \cong X & \longrightarrow & \frac{H}{V+W} & \longrightarrow & \frac{H}{X+V+W} & \longrightarrow & 0 \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\ 0 & \longrightarrow & \ker R & \longrightarrow & \ker TR & \longrightarrow & \ker T & \longrightarrow & \operatorname{coker} R & \longrightarrow & \operatorname{coker} TR & \longrightarrow & \operatorname{coker} T & \longrightarrow & 0 \end{array}$$

The choice of the sign in (8.3) is the same as the one in (7.6).

8.3. ANALYTICITY OF THE SUM. Consider the set

$$\mathcal{S}(H) := \left\{ (X, (V, W)) \in \left(\bigcup_{n \in \mathbb{N}} \operatorname{Gr}_{n, \infty}(H) \right) \times \operatorname{Fp}(H) : X \cap V = (0) \right\},$$

and the analytic map

$$s : \mathcal{S}(H) \rightarrow \operatorname{Fp}(H), \quad (X, (V, W)) \mapsto (X+V, W).$$

The space $\mathcal{S}(H)$ is the base space of the analytic line bundle $\operatorname{Det}(\mathcal{S}) \rightarrow \mathcal{S}(H)$, whose fiber at $(X, (V, W))$ is $\operatorname{Det}(X) \otimes \operatorname{Det}(V, W)$. The collection of the isomorphisms $S(X, (V, W))$ defines a bundle morphism

$$S : \operatorname{Det}(\mathcal{S}(H)) \rightarrow \operatorname{Det}(\operatorname{Fp}(H)),$$

which lifts the map s . We can use the fact that the composition of Fredholm operators has an analytic lift to the determinant bundles to show that the above bundle morphism is analytic. Indeed, the diagram

$$\begin{array}{ccc} \operatorname{Det}(X) \otimes \operatorname{Det}(V, W) & \xrightarrow{S(X, (V, W))} & \operatorname{Det}(X+V, W) \\ \parallel & & \parallel \\ \operatorname{Det}(R)^* \otimes \operatorname{Det}(TR) & \longrightarrow & \operatorname{Det}(T) \end{array}$$

commutes, and the lower isomorphism is induced by the inverse of the composition lift tensorized by the identity on $\text{Det}(R)^*$.

8.4. ASSOCIATIVITY OF THE SUM. The operation S is associative, meaning that if $X, Y \in \text{Gr}(H)$ are finite dimensional, $(V, W) \in \text{Fp}(H)$, and $X \cap Y = (X + Y) \cap V = (0)$, then the diagram

$$\begin{array}{ccc} \text{Det}(X) \otimes \text{Det}(Y) \otimes \text{Det}(V, W) & \xrightarrow{\text{id} \otimes S(Y, (V, W))} & \text{Det}(X) \otimes \text{Det}(Y + V, W) \\ \downarrow s(X, Y) \otimes \text{id} & & \downarrow S(X, (Y + V, W)) \\ \text{Det}(X + Y) \otimes \text{Det}(V, W) & \xrightarrow{S(X + Y, (V, W))} & \text{Det}(X + Y + V, W) \end{array}$$

commutes. This follows from the associativity property for the composition lift of Fredholm operator and from the commutativity of the diagram

$$\begin{array}{ccc} V \oplus W & \xrightarrow{R_{Y, (V, W)}} & (Y + V) \oplus W \\ \downarrow R_{X + Y, (V, W)} & \nearrow R_{X, (Y + V, W)} & \downarrow T_{Y, (V, W)} \\ (X + Y + V) \oplus W & \xrightarrow{T_{X + Y, (V, W)} = T_{X, (Y + V, W)}} & H. \end{array}$$

9. ORIENTATION BUNDLES

The determinant bundle over the space of real Fredholm operators is often used in global analysis to orient a finite dimensional manifold, which is obtained as the zero set, or more generally as the inverse image of some other finite dimensional manifold, of some nonlinear real Fredholm map. See for instance [13]. Similarly, the determinant bundle over the space of Fredholm pairs can be used to orient finite dimensional submanifolds which are obtained as transverse intersections of infinite dimensional manifolds. See [2] for applications of the objects introduced in this section to infinite dimensional Morse theory.

The advantage of using determinant bundles for these kind of problems lies in the associativity property. See also [24] for other approaches to the orientation question.

9.1. THE ORIENTATION BUNDLE OVER THE SPACE OF FREDHOLM PAIRS. We assume throughout this section that the Hilbert space H is real, so that the determinant bundle over $\text{Fp}(H)$ is a real line bundle. Its \mathbb{Z}_2 -reduction defines a double covering

$$\text{Or}(\text{Fp}(H)) \rightarrow \text{Fp}(H),$$

whose fiber at $(V, W) \in \text{Fp}(H)$ is the quotient

$$\text{Or}(V, W) := \text{Det}(V, W) \setminus \{0\} / \sim, \quad \text{where } \xi \sim \eta \text{ if and only if } \xi = \lambda \eta \text{ with } \lambda > 0.$$

Such a \mathbb{Z}_2 -bundle will be said the *orientation bundle over* $\text{Fp}(H)$. Since the determinant bundle is nontrivial over each connected component of $\text{Fp}(H)$, except the two trivial ones, so is the orientation bundle. Since in the real case the fundamental group of each connected component of $\text{Fp}^*(H)$ is \mathbb{Z}_2 (see Theorem 2.2), $\text{Or}(\text{Fp}^*(H)) \rightarrow \text{Fp}^*(H)$ is the universal covering of $\text{Fp}^*(H)$. If $(V, W) \in \text{Fp}(H)$, $X \in \text{Gr}(H)$ is finite dimensional, and $X \cap V = (0)$, then any orientation of two of

$$X, \quad (V, W), \quad (X + V, W),$$

determines, by the isomorphism $S(X, (V, W))$, an orientation of the third one. By the properties of S , this way of summing orientations is continuous and associative.

9.2. CO-ORIENTATIONS. Let $V, W \in \text{Gr}(H)$, W a compact perturbation of V . If V' is a linear complement of V in H , then (W, V') is a Fredholm pair (by Proposition 3.1). Moreover, the set

$$\mathcal{C}(W, V) := \{(W, V') : V' \text{ is a linear complement of } V \text{ in } H\}$$

is contractible, being homeomorphic to the Banach space $L(V^\perp, V)$. In particular, the restriction of the orientation bundle of Fredholm pairs to $\mathcal{C}(W, V)$ is trivial, and we can give the following:

DEFINITION 9.1. Let W be a compact perturbation of V . A *co-orientation* of (W, V) is the choice of one of the two continuous sections of the trivial double covering

$$\text{Or}(\text{Fp}(H))|_{\mathcal{C}(W, V)}.$$

If H is endowed with a preferred inner product, we can identify a co-orientation of (W, V) with an orientation of (W, V^\perp) , hence with an orientation of the finite dimensional space $(W \cap V^\perp) \oplus (W^\perp \cap V)$. It follows that the concept of co-orientation is symmetric, meaning that a co-orientation of (W, V) canonically induces a co-orientation of (V, W) .

The set of the two co-orientations of (W, V) is denoted by $\text{coOr}(V, W)$. These sets are the fibers of the co-orientation bundle $\text{coOr}(H)$, a double covering of the space

$$\{(W, V) \in \text{Gr}(H) \times \text{Gr}(H) : W \text{ is a compact perturbation of } V\}.$$

If we restrict this double covering to the Grassmannian $\text{Gr}_c(V, H)$ of compact perturbations of a fixed $V \in \text{Gr}_{\infty, \infty}(H)$, we obtain a nontrivial \mathbb{Z}_2 -bundle

$$\text{coOr}(\text{Gr}_c(V, H)) \rightarrow \text{Gr}_c(V, H),$$

called the *co-orientation bundle over* $\text{Gr}_c(V, H)$. By Theorem 3.3, this is the universal covering of $\text{Gr}_c(V, H)$.

9.3. INDUCED ORIENTATIONS AND CO-ORIENTATIONS. Now let (V, Z) be a Fredholm pair, and let W be a compact perturbation of V . We know from Proposition 3.1 that (W, Z) is also a Fredholm pair. Let us show how the choice of two among the following three objects

$$(9.1) \quad o_{(V,Z)} \in \text{Or}(V, Z), \quad o_{(W,Z)} \in \text{Or}(W, Z), \quad co_{(W,V)} \in \text{coOr}(W, V),$$

determines the third one. The argument mimics the proof of Proposition 3.1.

The pair (W, Z) is Fredholm if and only if the operator $P_{Z^\perp}|_W \in \text{L}(W, Z^\perp)$ is Fredholm, and

$$\begin{aligned} \text{Det}(P_{Z^\perp}|_W) &= \text{Det}(W \cap Z) \otimes \text{Det}(Z^\perp / P_{Z^\perp} W) \cong \text{Det}(W \cap Z) \otimes \text{Det}(H / (W + Z)) \\ &= \text{Det}(W, Z), \end{aligned}$$

where we have used the fact that $P_{Z^\perp} W + Z = W + Z$. Since W is a compact perturbation of V , we can apply the above facts to the Fredholm pair (V, W^\perp) , obtaining that $P_W|_V \in \text{L}(V, W)$ is Fredholm and

$$\text{Det}(P_W|_V) = \text{Det}(V, W^\perp).$$

By (3.2) we have

$$P_{Z^\perp}|_V = P_{Z^\perp}|_W \circ P_W|_V + P_{Z^\perp} \circ (P_V - P_W)|_V.$$

The last term is a compact operator, so $P_{Z^\perp}|_V$ is a compact perturbation of the composition $T := P_{Z^\perp}|_W \circ P_W|_V$. Using the composition lift isomorphism given by (7.5), we obtain an isomorphism

$$\text{Det}(T) \cong \text{Det}(P_W|_V) \otimes \text{Det}(P_{Z^\perp}|_W) \cong \text{Det}(V, W^\perp) \otimes \text{Det}(W, Z).$$

Since $\text{L}_c(V, Z^\perp)$ is simply connected (it is actually contractible), an orientation of $\text{Det}(T)$ determines an orientation of $\text{Det}(T')$ for each compact perturbation T' of T . In particular, an orientation of $\text{Det}(T)$ determines an orientation of

$$\text{Det}(P_{Z^\perp}|_V) \cong \text{Det}(V, Z).$$

Therefore an orientation of

$$\text{Det}(V, W^\perp) \otimes \text{Det}(W, Z)$$

determines an orientation of $\text{Det}(V, Z)$. We conclude that the choice of two among the three objects in (9.1) determines the third one.

If we exchange the role of V and W in the above construction, we still get the same way of inducing orientations. This follows from the fact that the diagram

$$\begin{array}{ccc} & & P_V|_W \\ & \curvearrowright & \\ V & & W \\ & \curvearrowleft & \\ & & P_W|_V \\ P_{Z^\perp}|_V & \searrow & \swarrow P_{Z^\perp}|_W \\ & Z^\perp & \end{array}$$

commutes up to compact perturbations. Moreover, the way of inducing orientations does not depend on the choice of the Hilbert product.

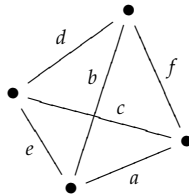
Here is a typical application of this construction: We are given a Hilbert manifold \mathcal{M} with a preferred linear subbundle \mathcal{V} of the tangent bundle $T\mathcal{M}$. Then we have two submanifolds \mathcal{W} and \mathcal{Z} of \mathcal{M} , such that for every $p \in \mathcal{W}$ the tangent space of \mathcal{W} at p is a compact perturbation of \mathcal{V}_p , whereas for every $p \in \mathcal{Z}$ the pair $(T_p\mathcal{Z}, \mathcal{V}_p)$ is Fredholm. Then a co-orientation of $T\mathcal{Z}$ with respect to \mathcal{V} and an orientation of $(T\mathcal{Z}, \mathcal{V})$ determine an orientation of the Fredholm pair $(T\mathcal{W}, T\mathcal{Z})$, at every point in the intersection $\mathcal{W} \cap \mathcal{Z}$. If moreover this intersection is transverse, such an orientation is an orientation of the finite dimensional manifold $\mathcal{W} \cap \mathcal{Z}$. See [2] for an application of these concepts to infinite dimensional Morse theory.

9.4. ASSOCIATIVITY. Let V, W, Z be as before, and consider a compact perturbation Y of Z . Then also (V, Y) and (W, Y) are Fredholm pairs, and we can consider six elements

$$a \in \text{Or}(V, Z), \quad b \in \text{Or}(V, Y), \quad c \in \text{Or}(W, Z), \quad d \in \text{Or}(W, Y),$$

$$e \in \text{coOr}(V, W), \quad f \in \text{coOr}(Y, Z),$$

to be associated with the edges of the tetrahedron with faces $\{a, c, e\}$, $\{a, b, f\}$, $\{b, d, e\}$, $\{c, d, f\}$:



Then associativity can be stated in this way: If we are given five of these elements in such a way that the two triplets corresponding to the complete faces in the above tetrahedron are compatible, then there exists a unique element to be put on the remaining edge which makes also the triplets corresponding to the remaining faces compatible.

In order to prove this fact, we may choose the Hilbert product in such a way that Y , hence also Z , is a compact perturbation of the orthogonal complement of V , hence also of W . The choice of the orientations for the triplets associated to the four faces is determined by the following four compositions of Fredholm operators

$$V \xrightarrow{P_W} W \xrightarrow{P_{Y^\perp}} Y^\perp, \quad V \xrightarrow{P_W} W \xrightarrow{P_{Z^\perp}} Z^\perp, \quad Z \xrightarrow{P_Y} Y \xrightarrow{P_{V^\perp}} V^\perp, \quad Z \xrightarrow{P_Y} Y \xrightarrow{P_{W^\perp}} W^\perp.$$

By the canonical isomorphism between the determinant line of a Fredholm pair and the determinant line of the pair consisting of the orthogonal complements,

we may replace the last two compositions by the compositions

$$Z^\perp \xrightarrow{P_{Y^\perp}} Y^\perp \xrightarrow{P_V} V, \quad Z^\perp \xrightarrow{P_{Y^\perp}} Y^\perp \xrightarrow{P_W} W.$$

Then the claim follows from the fact that the diagram

$$\begin{array}{ccc} V & \xrightarrow{P_W} & W \\ P_V \uparrow & \nearrow^{P_{Y^\perp}} & \downarrow P_{Z^\perp} \\ Y^\perp & \xleftarrow{P_{Y^\perp}} & Z^\perp \end{array}$$

commutes up to compact perturbations, together with the associativity property for the composition of Fredholm operators lifted to the determinant bundles.

9.5. FINAL REMARKS. Let us consider again the case of a Fredholm pair (V, Z) and of a compact perturbation W of V , and let us make some comments on the argument following (9.1).

Notice that the compact operator $P_{Z^\perp} \circ (P_V - P_W)|_V$ vanishes if and only if $P_{W^\perp}V \subset Z$. In this situation, $P_{Z^\perp}|_V$ coincides with the composition T , and there is a natural isomorphism already at the level of determinants,

$$\text{Det}(V, Z) \cong \text{Det}(V, W^\perp) \otimes \text{Det}(W, Z).$$

This isomorphism is induced by the exact sequence (7.4), which in this case is

$$0 \rightarrow V \cap W^\perp \rightarrow V \cap Z \xrightarrow{P_W} W \cap Z \rightarrow \frac{W}{P_W V} \xrightarrow{P_{Z^\perp}} \frac{Z^\perp}{P_{Z^\perp} P_W V} \rightarrow \frac{Z^\perp}{P_{Z^\perp} V} \rightarrow 0.$$

In particular, if X is a finite dimensional subspace such that $X \cap V = (0)$ and $W = X + V$, the above exact sequence reduces to

$$0 \rightarrow 0 \rightarrow V \cap Z \rightarrow (X + V) \cap Z \rightarrow \frac{X + V}{V} \cong X \rightarrow \frac{Z^\perp}{P_{Z^\perp} V} \rightarrow \frac{Z^\perp}{P_{Z^\perp}(X + V)} \rightarrow 0.$$

Since $P_{Z^\perp}V + Z = V + Z$, we have natural isomorphisms

$$\frac{Z^\perp}{P_{Z^\perp}V} \cong \frac{H}{V + Z'}, \quad \frac{Z^\perp}{P_{Z^\perp}(X + V)} \cong \frac{H}{X + V + Z'}.$$

Therefore, the above exact sequence is the one inducing the isomorphism

$$S(X, (V, W)) : \text{Det}(X) \otimes \text{Det}(V, Z) \cong \text{Det}(X + V, Z).$$

Hence, the way of inducing orientations presented in this section agrees with the sum construction of Section 8.

A. APPENDIX — SECTIONS

A continuous linear surjective map $T : E \rightarrow F$ between Banach spaces has a continuous linear global section, that is a continuous linear map $S : F \rightarrow E$ such that $TS = I_F$, if and only if the kernel of T has a direct summand in E . For instance, the quotient projection $\ell^\infty \rightarrow \ell^\infty/c_0$, where ℓ^∞ denotes the Banach space of bounded sequences, and c_0 denotes the closed subspace of infinitesimal sequences, has no continuous linear right inverse. Indeed, c_0 does not have a direct summand in ℓ^∞ (see Section 3.1).

However, Bartle and Graves [5] have shown that it is always possible to find a nonlinear continuous right inverse of T . In general, such a global section will not be Gateaux differentiable at any point (otherwise its differential would be a linear global section). The existence of a continuous global section could be seen as a consequence of more general selection theorems by Michael (see [21] and references therein, or [23], Theorems 10,11). The aim of this appendix is to show a direct proof of this fact.

PROPOSITION A.1. *Let $T : E \rightarrow F$ be a continuous linear surjective map between Banach spaces. Then there exists a continuous map $s : F \rightarrow E$ which is a global section of T .*

Proof. Since T is onto, by the open mapping theorem the quotient norm

$$\|x\| := \inf\{\|y\|_E : Ty = x\}$$

is equivalent to the norm of F . We will use such a quotient norm to define the uniform norm $\|\cdot\|_\infty$ of a F -valued map. Set $S := \{x \in F : \|x\| = 1\}$.

CLAIM. If $\Psi : S \rightarrow F$ is continuous and bounded, there exists $\Phi : S \rightarrow E$ continuous such that $\|\Phi\|_\infty \leq \|\Psi\|_\infty$ and $\|T\Phi - \Psi\|_\infty \leq (1/2)\|\Psi\|_\infty$.

For every $x \in S$, let $U(x)$ be an open neighborhood of x such that $\|\Psi(x') - \Psi(x)\| \leq \|\Psi\|_\infty/4$ for every $x' \in U(x)$. Let $\{V_i\}_{i \in I}$ be a locally finite open refinement of $\{U(x)\}_{x \in S}$, and let $\{\varphi_i\}_{i \in I}$ be a partition of unity subordinated to it. For every $i \in I$, let $x_i \in V_i$, and choose $y_i \in E$ such that $Ty_i = (2/3)\Psi(x_i)$, $\|y_i\|_E \leq \|\Psi\|_\infty$. Define $\Phi : S \rightarrow E$ as

$$\Phi(x) = \sum_{i \in I} \varphi_i(x)y_i.$$

Then $\|\Phi\|_\infty \leq \sup_{i \in I} \|y_i\|_E \leq \|\Psi\|_\infty$, and if $x \in S$,

$$T\Phi(x) - \Psi(x) = \sum_{i \in I} \varphi_i(x)[Ty_i - \Psi(x)] = \frac{2}{3} \sum_{i \in I} \varphi_i(x)[\Psi(x_i) - \Psi(x)] - \frac{1}{3}\Psi(x),$$

hence we have the following proving the claim:

$$\|T\Phi(x) - \Psi(x)\| \leq \frac{2}{3} \sup_{\substack{i \in I \\ x \in V_i}} \|\Psi(x_i) - \Psi(x)\| + \frac{1}{3}\|\Psi\|_\infty \leq \frac{1}{6}\|\Psi\|_\infty + \frac{1}{3}\|\Psi\|_\infty = \frac{1}{2}\|\Psi\|_\infty.$$

Applying the above claim iteratively, starting from the identity map, we obtain a sequence of continuous maps $\Phi_n : S \rightarrow E$, $n \geq 0$, such that

$$\|\Phi_n\|_\infty \leq 2^{-n}, \quad \left\| I - T \sum_{k=0}^{n-1} \Phi_k \right\|_\infty \leq 2^{-n}.$$

Therefore, the series $\sum_{n=0}^{\infty} \Phi_n$ converges uniformly on S to a map which lifts the identity, and the following map is a global section of T :

$$s(x) := \|x\| \sum_{n=0}^{\infty} \Phi_n \left(\frac{x}{\|x\|} \right), \quad s(0) = 0. \quad \blacksquare$$

REMARK A.2. From the existence of the global section s , it follows easily that every continuous linear surjective map $T : E \rightarrow F$ admits a natural structure of a trivial C^0 vector bundle over F , and the map $x \mapsto \|x - s(Tx)\|_E$ is a Finsler structure on such a vector bundle.

B. APPENDIX — ASSOCIATIVITY

Consider the Fredholm operators $T_i \in \text{Fr}(H_i, H_{i+1})$, $1 \leq i \leq 3$. We wish to prove that the diagram

$$(B.1) \quad \begin{array}{ccc} \text{Det}(T_1) \otimes \text{Det}(T_2) \otimes \text{Det}(T_3) & \xrightarrow{\text{id} \otimes \phi_{T_2, T_3}} & \text{Det}(T_1) \otimes \text{Det}(T_3 T_2) \\ \downarrow \phi_{T_1, T_2} \otimes \text{id} & & \downarrow \phi_{T_1, T_3 T_2} \\ \text{Det}(T_2 T_1) \otimes \text{Det}(T_3) & \xrightarrow{\phi_{T_2 T_1, T_3}} & \text{Det}(T_3 T_2 T_1) \end{array}$$

commutes.

Given $1 \leq i < j \leq 4$, set $T_{ij} = T_{j-1} \circ \cdots \circ T_i \in \text{Fr}(H_i, H_j)$. Using the left and right action of the general linear group, we are reduced to consider the following situation:

$$H_i = \bigoplus_{h \leq i \leq k} H_{hk}, \quad T_{ij} = \bigoplus_{\substack{h \leq i \\ k \geq j}} I_{H_{hk}},$$

where the space H_{hk} , $1 \leq h \leq k \leq 4$, has finite dimension d_{hk} , except for H_{14} which is infinite dimensional. Then

$$\ker T_{ij} = \bigoplus_{\substack{h \in [1, i] \\ k \in [i, j-1]}} H_{hk}, \quad \text{coker } T_{ij} = \bigoplus_{\substack{h \in [i+1, j] \\ k \in [j, 4]}} H_{hk}.$$

Fix $1 \leq i < j < \ell \leq 4$. The exact sequence associated to the composition $T_{i\ell} = T_{j\ell} \circ T_{ij}$ is

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \ker T_{ij} & \longrightarrow & \ker T_{i\ell} & \longrightarrow & \ker T_{j\ell} & \longrightarrow & \operatorname{coker} T_{ij} & \longrightarrow & \operatorname{coker} T_{i\ell} & \longrightarrow & \operatorname{coker} T_{j\ell} & \longrightarrow & 0 \\
& & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\
0 > & \bigoplus_{\substack{h \in [1,i] \\ k \in [i,j-1]}} H_{hk} & > & \bigoplus_{\substack{h \in [1,i] \\ k \in [i,\ell-1]}} H_{hk} & > & \bigoplus_{\substack{h \in [1,j] \\ k \in [j,\ell-1]}} H_{hk} & > & \bigoplus_{\substack{h \in [i+1,j] \\ k \in [j,\mathcal{A}]}} H_{hk} & > & \bigoplus_{\substack{h \in [i+1,\ell] \\ k \in [\ell,\mathcal{A}]}} H_{hk} & > & \bigoplus_{\substack{h \in [j+1,\ell] \\ k \in [\ell,\mathcal{A}]}} H_{hk} & > & 0.
\end{array}$$

Denote by θ_{hk} a generator of $\operatorname{Det}(H_{hk})$, for $(h, k) \neq (1, 4)$, and given two sets of consecutive integers $I, J \subset \{1, 2, 3, 4\}$, set

$$\Theta_I^J = \bigwedge_{h \in I} \bigwedge_{k \in J} \theta_{hk},$$

where we are considering the standard order in each wedge product. Let us show that the composition morphism

$$\phi_{T_{ij}, T_{j\ell}} : \operatorname{Det}(T_{ij}) \otimes \operatorname{Det}(T_{j\ell}) \rightarrow \operatorname{Det}(T_{i\ell})$$

equals

$$(B.2) \quad \phi_{T_{ij}, T_{j\ell}} : \Theta_{[1,i]}^{[i,j-1]} \otimes \Theta_{[i+1,j]}^{[j,\mathcal{A}]} \otimes \Theta_{[1,j]}^{[j,\ell-1]} \otimes \Theta_{[j+1,\ell]}^{[\ell,\mathcal{A}]} \mapsto (-1)^{\sigma(i,j,\ell)} \Theta_{[1,i]}^{[i,\ell-1]} \otimes \Theta_{[i+1,\ell]}^{[\ell,\mathcal{A}]},$$

where the coefficient $\sigma(i, j, \ell)$ is to be determined. Notice that:

(i) If we want to change the generator $\Theta_{[1,i]}^{[i,\ell-1]}$ of $\operatorname{Det}(\ker T_{i\ell})$ into $\Theta_{[1,i]}^{[i,j-1]} \wedge \Theta_{[1,i]}^{[j,\ell-1]}$, we have to exchange the position of θ_{hk} and $\theta_{h'k'}$ if and only if $h < h'$, $k \in [j, \ell - 1]$, and $k' \in [i, j - 1]$. Therefore,

$$\Theta_{[1,i]}^{[i,\ell-1]} = (-1)^{\sigma_1(i,j,\ell)} \Theta_{[1,i]}^{[i,j-1]} \wedge \Theta_{[1,i]}^{[j,\ell-1]}, \quad \text{where } \sigma_1(i, j, \ell) = \sum_{\substack{1 \leq h < h' \leq i \\ k \in [j, \ell - 1] \\ k' \in [i, j - 1]}} d_{hk} d_{h'k'}.$$

(ii) The generator $\Theta_{[1,j]}^{[j,\ell-1]}$ of $\operatorname{Det}(\ker T_{j\ell})$ can be rewritten as $\Theta_{[1,i]}^{[j,\ell-1]} \wedge \Theta_{[i+1,j]}^{[j,\ell-1]}$. Therefore,

$$\Theta_{[1,j]}^{[j,\ell-1]} = (-1)^{\sigma_2(i,j,\ell)} \Theta_{[i+1,j]}^{[j,\ell-1]} \wedge \Theta_{[1,i]}^{[j,\ell-1]}, \quad \text{where } \sigma_2(i, j, \ell) = \sum_{\substack{h \in [1,i] \\ h' \in [i+1,j] \\ k, k' \in [j, \ell - 1]}} d_{hk} d_{h'k'}.$$

(iii) Arguing as in (i), the generator $\Theta_{[i+1,j]}^{[j,\mathcal{A}]}$ of $\operatorname{Det}(\operatorname{coker} T_{ij})$ can be rewritten as

$$\Theta_{[i+1,j]}^{[j,\mathcal{A}]} = (-1)^{\sigma_3(i,j,\ell)} \Theta_{[i+1,j]}^{[j,\ell-1]} \wedge \Theta_{[i+1,j]}^{[\ell,\mathcal{A}]}, \quad \text{where } \sigma_3(i, j, \ell) = \sum_{\substack{i+1 \leq h < h' \leq j \\ k \in [\ell, \mathcal{A}] \\ k' \in [j, \ell - 1]}} d_{hk} d_{h'k'}.$$

(iv) As in (ii), the generator $\Theta_{[i+1,\ell]}^{[\ell,4]}$ of $\text{Det}(\text{coker } T_{i\ell})$ can be rewritten as

$$\Theta_{[i+1,\ell]}^{[\ell,4]} = (-1)^{\sigma_4(i,j,\ell)} \Theta_{[j+1,\ell]}^{[\ell,4]} \wedge \Theta_{[i+1,j]}^{[\ell,4]}, \quad \text{where } \sigma_4(i,j,\ell) = \sum_{\substack{h \in [i+1,j] \\ h' \in [j+1,\ell] \\ k,k' \in [\ell,4]}} d_{hk} d_{h'k'}.$$

Then formula (7.5) implies formula (B.2) with

$$\sigma(i,j,\ell) = \sigma_0(i,j,\ell) + \sigma_1(i,j,\ell) + \sigma_2(i,j,\ell) + \sigma_3(i,j,\ell) + \sigma_4(i,j,\ell),$$

where

$$\begin{aligned} \sigma_0(i,j,\ell) &:= (\dim \ker T_{ij} + \dim \ker T_{i\ell})(\dim \text{coker } T_{j\ell} + \dim \text{coker } T_{i\ell}) \\ &= \sum_{\substack{h \in [1,i] \\ k \in [j,\ell-1] \\ h' \in [i+1,j] \\ k' \in [\ell,4]}} d_{hk} d_{h'k'} \pmod{2}. \end{aligned}$$

Formula (B.2) implies that the diagram (B.1) commutes up to the sign

$$(-1)^{\sigma(1,2,3) + \sigma(1,2,4) + \sigma(1,3,4) + \sigma(2,3,4)},$$

so we have to show that the above exponent vanishes modulo 2. The following computations are modulo 2. We have

$$\sigma_0(i,j,\ell) + \sigma_2(i,j,\ell) = \sum_{\substack{h \in [1,i] \\ k \in [j,\ell-1] \\ h' \in [i+1,j] \\ k' \in [j,4]}} d_{hk} d_{h'k'},$$

and summing over the four possibilities for (i,j,ℓ) we obtain

$$\begin{aligned} &(\sigma_0 + \sigma_2)(1,2,3) + (\sigma_0 + \sigma_2)(1,2,4) + (\sigma_0 + \sigma_2)(1,3,4) + (\sigma_0 + \sigma_2)(2,3,4) \\ \text{(B.3)} \quad &= d_{13}d_{22} + d_{23}d_{33} + d_{23}d_{34}. \end{aligned}$$

The range of the sum defining $\sigma_1(i,j,\ell)$ is always empty, except for the case $i = 2, j = 3, \ell = 4$, so we have

$$\text{(B.4)} \quad \sigma_1(1,2,3) + \sigma_1(1,2,4) + \sigma_1(1,3,4) + \sigma_1(2,3,4) = \sigma_1(2,3,4) = d_{13}d_{22}.$$

The range of the sum defining $\sigma_3(i,j,\ell)$ is always empty, except for the case $i = 1, j = 3, \ell = 4$, so we have

$$\text{(B.5)} \quad \sigma_3(1,2,3) + \sigma_3(1,2,4) + \sigma_3(1,3,4) + \sigma_3(2,3,4) = \sigma_3(1,3,4) = d_{24}d_{33}.$$

Finally, taking the sum of $\sigma_4(i,j,\ell)$ over the four possibilities for (i,j,ℓ) we obtain

$$\text{(B.6)} \quad \sigma_4(1,2,3) + \sigma_4(1,2,4) + \sigma_4(1,3,4) + \sigma_4(2,3,4) = d_{23}d_{33} + d_{23}d_{34} + d_{24}d_{33}.$$

Taking the sum of (B.3), (B.4), (B.5), and (B.6), we find, as wished:

$$\sigma(1,2,3) + \sigma(1,2,4) + \sigma(1,3,4) + \sigma(2,3,4) = 0,$$

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