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AN EXTENDED CLASS OF INTEGRABLE OPERATORS

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ABSTRACT. We consider an extended class of the integrable operators. This class is connected with the class of the Riemann–Hilbert problems. We show that the solution $W(z)$ of the Riemann–Hilbert problem coincides with the monodromy matrix of the corresponding differential system. It follows from this result that $W(z)$ can be represented in two forms:
1. In the form of the transfer matrix function.
2. In the form of the multiplicative integral.
The analogues of the Plemelj formulas are deduced in the paper for limiting values of the multiplicative integral $W(z)$.

KEYWORDS: *Riemann–Hilbert problem, differential system, monodromy matrix, multiplicative integral, triangular factorization.*

MSC (2000): Primary 15A52; Secondary 47A48, 47G20, 81R12.

1. INTEGRAL OPERATORS AND RIEMANN–HILBERT PROBLEMS

We consider the operator S which acts in the space $L^2_k(a,b)$ and has the form

$$Sf = L_1(x)f(x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_a^b \frac{F_1(x)F_2^{\star}(t)}{x-t}f(t)\mathrm{d}t, \tag{1.1}$$

where $L_1(x)$, $F_1(x)$, $F_2(x)$ and $f(x)$ are matrix functions of the orders $k\times k$, $k\times m$, $k\times m$ and $k\times 1$ respectively. The symbol P.V. indicates that the integral is understood as principal value. We suppose that all the entries of $L_1(x)$, $F_1(x)$ and $F_2(x)$ are measurable functions and that the following conditions are fulfilled:

(I) For some M_1 we have

$$\|L(x)\| + \|F_1(x)\| + \|F_2(x)\| \leqslant M_1, \quad a\leqslant x\leqslant b. \tag{1.2}$$

(II) The operator S is invertible and the matrix functions

$$G_1(x) = S^{-1}F_1, \quad G_2(x) = (S^*)^{-1}F_2, \tag{1.3}$$

are such that for some M_2 we have

$$\|G_1(x)\| + \|G_2(x)\| \leqslant M_2, \quad a \leqslant x \leqslant b. \tag{1.4}$$

In (1.3) and below, we understand that the operators on matrix valued functions act column-wise.

REMARK 1.1. The operator V defined as

$$Vf = \frac{1}{\pi}\text{P.V.}\int_{-\infty}^{+\infty} \frac{f(t)}{x-t}\text{d}t, \quad f(t) \in L^2(-\infty, +\infty) \tag{1.5}$$

is unitary in the space $L^2(-\infty, +\infty)$. This fact and condition (1.2) imply that the operator S is bounded.

Let us introduce the operators

$$\Pi_1 g = \frac{1}{\sqrt{2\pi}} F_1(x) g, \quad \Pi_2 g = -\text{i}\frac{1}{\sqrt{2\pi}} F_2(x) g, \tag{1.6}$$

where g are constant $m \times 1$ vectors. Now we can write the operator identity

$$QS - SQ = \Pi_1 \Pi_2^\star, \tag{1.7}$$

where $Qf = xf(x), f(x) \in L_k^2(a, b)$. From the operator identity (1.7) we obtain that

$$TQ - QT = \Gamma_1 \Gamma_2^\star, \tag{1.8}$$

where $T = S^{-1}$ and

$$\Gamma_1 g = T\Pi_1 g = \frac{1}{\sqrt{2\pi}} G_1(x) g, \quad \Gamma_2 g = T^* \Pi_2 g = -\text{i}\frac{1}{\sqrt{2\pi}} G_2(x) g. \tag{1.9}$$

Using relations (1.1), (1.6) and (1.9) we deduce that

$$Tf = L_2(x) f(x) + \frac{\text{i}}{2\pi}\text{P.V.}\int_a^b \frac{G_1(x) G_2^\star(t)}{t-x} f(t)\text{d}t, \tag{1.10}$$

where

$$\|L_2(x)\| \leqslant M_3, \quad a \leqslant x \leqslant b. \tag{1.11}$$

REMARK 1.2. (i) The operators S of the form (1.1) were investigated in the article [8] by the operator identity method. These operators are connected with the spectral theory of the non-self-adjoint operators (see [8], [12]). Later in the work [5] the important subclass of the operators S, when

$$k = 1, \quad L(x) = 1, \quad F_1(x) F_2^\star(x) = 0, \tag{1.12}$$

was studied in detail. The operators of the class (1.12) are called integrable operators [1]–[3], [5]. We use the term integrable operators for a wider class of operators S of form (1.1).

(ii) The introduced operators S and T are connected with a certain Riemann–Hilbert problem (RHP). For different partial cases this connection was investigated in a number of papers (see [1]–[3], [5]).

Now we consider the general case (1.1). We use the $m \times m$ matrix functions:

$$W(z) = I_m - \Gamma_2^{\star}(Q - zI)^{-1}\Pi_1, \tag{1.13}$$

$$V(z) = I_m + \Pi_2^{\star}(Q - zI)^{-1}\Gamma_1. \tag{1.14}$$

Formulas (1.13) and (1.14) are realizations of $W(z)$ and $V(z)$. It means that $W(z)$ and $V(z)$ can be interpreted as the transfer matrix functions. Due to (1.6), (1.9) and (1.13), (1.14) we can rewrite $W(z)$ and $V(z)$ in the form

$$W(z) = I_m + \frac{1}{2\pi \mathrm{i}}\int_a^b \frac{G_2^{\star}(t)F_1(t)}{t - z}\mathrm{d}t, \tag{1.15}$$

$$V(z) = I_m - \frac{1}{2\pi \mathrm{i}}\int_a^b \frac{F_2^{\star}(t)G_1(t)}{t - z}\mathrm{d}t. \tag{1.16}$$

It is known (see Chapter 1 of [11]), that

$$W(z)V(z) = I_m, \quad z \notin [a, b]. \tag{1.17}$$

We define $V_{\pm}(x)$ by the equalities

$$V_{\pm}(x) = \lim V(z), \quad z = x + \mathrm{i}y, \quad y \to \pm 0. \tag{1.18}$$

It follows from relation (1.16) that (see Chapter 2 of [7]).

$$V_+(x) - V_-(x) = -F_2^{\star}(x)G_1(x), \tag{1.19}$$

$$V_+(x) = I_m - \frac{1}{2}F_2^{\star}(x)G_1(x) - \frac{1}{2\pi \mathrm{i}}\mathrm{P.V.}\int_a^b \frac{F_2^{\star}(t)G_1(t)}{t - x}\mathrm{d}t. \tag{1.20}$$

By comparing formulas (1.1), (1.20) and $SG_1 = F_1$ we obtain the equality

$$F_1(x)V_+(x) = \Big[L_1(x) - \frac{1}{2}F_1(x)F_2^{\star}(x)\Big]G_1(x). \tag{1.21}$$

LEMMA 1.3 (see [8]). *Let conditions* (I) *and* (II) *be fulfilled. Then the following equalities are true:*

$$\lim \mathrm{e}^{\mathrm{i}uQ}S\mathrm{e}^{-\mathrm{i}uQ}f = \Big[L_1(x) \mp \frac{1}{2}F_1(x)F_2^{\star}(x)\Big]f, \quad u \to \pm\infty, \tag{1.22}$$

$$\lim \mathrm{e}^{\mathrm{i}uQ}T\mathrm{e}^{-\mathrm{i}uQ}f = \Big[L_2(x) \pm \frac{1}{2}G_1(x)G_2^{\star}(x)\Big]f, \quad u \to \pm\infty. \tag{1.23}$$

Using relations $ST = I$ we deduce that

$$\Big[L_1(x) \mp \frac{1}{2}F_1(x)F_2^{\star}(x)\Big]\Big[L_2(x) \pm \frac{1}{2}G_1(x)G_2^{\star}(x)\Big] = I_k. \tag{1.24}$$

According to (1.24) the following relations

$$\det\Big[L_1(x) \mp \frac{1}{2}F_1(x)F_2^\star(x)\Big] \neq 0, \tag{1.25}$$

$$L_1(x)L_2(x) - \frac{1}{4}F_1(x)F_2^\star(x)G_1(x)G_2^\star(x) = I_k, \tag{1.26}$$

$$L_1(x)G_1(x)G_2^\star(x) = F_1(x)F_2^\star(x)L_2(x), \tag{1.27}$$

are valid. It follows from (1.21) and (1.25) that equality

$$G_1(x) = \Big[L_1(x) - \frac{1}{2}F_1(x)F_2^\star(x)\Big]^{-1}F_1(x)V_+(x) \tag{1.28}$$

is true. From relations (1.19) and (1.28) we deduce that

$$V_-(x) = u(x)V_+(x), \tag{1.29}$$

where

$$u(x) = I_m + F_2^\star(x)\Big[L_1(x) - \frac{1}{2}F_1(x)F_2^\star(x)\Big]^{-1}F_1(x). \tag{1.30}$$

So we have proved the following assertion.

PROPOSITION 1.4. *Let the conditions* (I), (II) *be fulfilled. Then $V(z)$ defined by relation* (1.16) *solves the following $m \times m$ matrix Riemann–Hilbert problem:*

(i) *matrix function $V(z)$ is holomorphic for $z \neq \overline{z}$.*

(ii) $\lim V(z) = I_m, z \to \infty$.

(iii) $V_-(x) = u(x)V_+(x), x \in (a,b)$.

We can add to Proposition 1.4 the following assertion.

PROPOSITION 1.5. *Let the conditions* (I), (II) *be fulfilled. Then the inequality*

$$\det u(x) \neq 0 \tag{1.31}$$

is true and the solution $V(z)$ of RHP (i)–(iii) *is unique.*

Proof. From formula (1.15) we deduce that the limit matrices $W_+(x)$ and $W_-(x)$ exist. Hence the matrices $V_+(x)$ and $V_-(x)$ are invertible (see (1.17)). From this fact and relation (1.29) we obtain inequality (1.31). Suppose that RHP (i)–(iii) has another solution $\widetilde{V}(z)$. Then the matrix function $\widetilde{V}(z)V^{-1}(z)$ has no jumps across the segment $[a,b]$ and

$$\widetilde{V}(z)V^{-1}(z) \to I_m, \quad z \to \infty. \tag{1.32}$$

This means that $\widetilde{V}(z) = V(z)$. The proposition is proved. ∎

2. EXAMPLES AND APPLICATIONS

In this section we show that a number of concrete examples satisfy conditions (I) and (II) of the previous section. We remark that the examples of this section are encountered in many applied and theoretical problems.

EXAMPLE 2.1. Let us consider the case when

$$L_1(x) = L_2(x). \tag{2.1}$$

In view of (1.27) we have

$$F_1(x)F_2^\star(x) = G_1(x)G_2^\star(x). \tag{2.2}$$

We introduce the notations

$$L_1(x) = L(x), \quad F_1(x)F_2^\star(x) = B(x) \tag{2.3}$$

and suppose that

$$L(x) > 0, \quad B(x) = B^\star(x). \tag{2.4}$$

It follows from (1.26) and (2.3) that

$$L(x) = \left[I_k + \frac{1}{4}B^2(x)\right]^{\frac{1}{2}}. \tag{2.5}$$

Case (2.1)–(2.4) plays an essential role in the spectral theory of non-selfadjoint operators (see [8], [9]).

EXAMPLE 2.2. Let us consider the case when

$$L_1(x) = I_k, \quad F_1(x)F_2^\star(x) = 0. \tag{2.6}$$

This case is used in the random matrix theory (see [5], [13]). According to (1.30) and (2.6) the matrix function $u(x)$ takes the form

$$u(x) = I_m + F_2^\star(x)F_1(x). \tag{2.7}$$

From (2.6) and (2.7) we deduce the next assertion.

PROPOSITION 2.3. *Let conditions* (2.6) *be fulfilled. Then the following equality is valid:*

$$[u(x) - I_m]^2 = 0. \tag{2.8}$$

Relation (2.8) implies that all the eigenvalues of $u(x)$ are equal to 1 and so

$$\det u(x) = 1. \tag{2.9}$$

It follows from (2.7) that

$$u^{-1}(x) = I_m - F_2^\star(x)F_1(x). \tag{2.10}$$

Hence the matrix function $W(z) = V^{-1}(z)$ gives the solution of the following RHP

$$W_+(x)u^{-1}(x) = W_-(x). \tag{2.11}$$

EXAMPLE 2.4 (Ising chain). The problems of Ising chain are connected with the operator (see [1], [2])

$$Sf = f(x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_{-1}^{1} \frac{F_1(x)JF_1^\star(t)}{x-t} f(t)\mathrm{d}t, \tag{2.12}$$

where

$$F_1(x) = \sqrt{a(x)}[\mathrm{e}^{-xu}, -\mathrm{e}^{xu}], \quad a(x) = \tanh\big(\beta\sqrt{1-x^2}\big). \tag{2.13}$$

The matrix J has the form

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{2.14}$$

We note that

$$F_1(x)JF_1^\star(t) = \sqrt{a(x)}\sqrt{a(t)}[\mathrm{e}^{(x-t)u} - \mathrm{e}^{-(x-t)u}], \quad J^\star = -J. \tag{2.15}$$

Comparing formulas (2.6) and (2.15) we have

$$F_2(x) = -F_1(x)J. \tag{2.16}$$

It is easy to see that in Example 2.4 conditions (I) and (II) of Section 1 are fulfilled. We remark that in the considered case

$$G_2(x) = -(S^*)^{-1}F_1(x)J. \tag{2.17}$$

Proposition 1.4 implies the assertion.

PROPOSITION 2.5. *Let conditions* (2.13) *and* (2.14) *be fulfilled. Then the matrix function*

$$W(z) = I_2 + \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1} \frac{G_2^\star(t)F_1(t)}{t-z}\mathrm{d}t, \tag{2.18}$$

is the solution of RHP (2.11), *where*

$$v(x) = u^{-1}(x) = \begin{bmatrix} 1+a(x) & -a(x)\mathrm{e}^{2xu} \\ a(x)\mathrm{e}^{-2xu} & 1-a(x) \end{bmatrix} \tag{2.19}$$

and $a(x) = \tanh\big(\beta\sqrt{1-x^2}\big)$.

3. DIFFERENTIAL SYSTEMS

3.1. We consider the operator S_ξ which acts in the space $L_k^2(a,\xi)$ and has the form

$$S_\xi f = f(x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_a^\xi \frac{F_1(x)JF_1^\star(t)}{x-t}f(t)\mathrm{d}t, \tag{3.1}$$

where $a \leqslant x \leqslant \xi \leqslant b$, J is a constant $m \times m$ matrix. We suppose that S_ξ satisfies conditions (I) and (II) of Section 1 and

$$F_1(x)JF_1^\star(x) = 0, \quad a \leqslant x \leqslant b. \tag{3.2}$$

The operators S_ξ of form (3.1), (3.2) play an important role in the random matrix theory [1], [2], [13].

PROPOSITION 3.1. *The operator S_ξ^{-1} has the form*

$$S_\xi^{-1} f = f(x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_a^\xi \frac{G_1(x,\xi)G_2^\star(t,\xi)}{t-x} f(t)\mathrm{d}t, \tag{3.3}$$

where

$$G_1(x,\xi) = S_\xi^{-1}F_1(x), \quad G_2(x) = -(S^*)^{-1}F_1(x)J. \tag{3.4}$$

Proof. Using equalities (1.27), and (3.2) we have

$$L_2(x) = I_k, \quad G_1(x,\xi)JG_1^\star(x,\xi) = 0, \quad a \leqslant x \leqslant \xi. \tag{3.5}$$

In view of (1.10) and (3.2) relation (3.3) is valid. The proposition is proved. ∎

According to (2.7) and (2.16) we have

$$u(x) = I_m + A(x), \tag{3.6}$$

where

$$A(x) = JF_1^\star(x)F_1(x). \tag{3.7}$$

We introduce the matrix function

$$r(x) = I_m + \frac{1}{2}A(x). \tag{3.8}$$

It is easy to see that

$$u(x) = r^2(x). \tag{3.9}$$

3.2. Let us consider the $m \times m$ matrix function

$$B(\xi) = \frac{1}{2\pi}\int_a^\xi F_1^\star(x)G_1(x,\xi)\mathrm{d}x. \tag{3.10}$$

We suppose that the operator S_b admits the triangular factorization (see Chapter 4 of [4], and [10]), i.e. S_b can be represented in the form

$$S_b = S_- S_+, \tag{3.11}$$

where $S_-^{\pm 1}$, $S_+^{\pm 1}$ are bounded operators and

$$S_+^{\pm 1}P_\xi = P_\xi S_+^{\pm 1}P_\xi, \quad Q_\xi S_-^{\pm 1} = Q_\xi S_-^{\pm 1}Q_\xi,$$

where $Q_\xi = I - P_\xi$, $P_\xi f = f(x)$, $a \leqslant x < \xi$ and $P_\xi f = 0$, $\xi \leqslant x \leqslant b$, $f(x) \in L_k^2(a,b)$. From relations (3.10) and (3.11) we deduce the next assertion.

LEMMA 3.2. *If the operator S_b admits the triangular factorization* (3.11) *then the matrix $B(x)$ is absolutely continuous and*

$$H(x) = \frac{\mathrm{d}}{\mathrm{d}x}B(x) = \frac{1}{2\pi}g^\star(x)h(x), \tag{3.12}$$

where

$$g(x) = (S_+^\star)^{-1}F_1, \quad h(x) = S_-^{-1}F_1. \tag{3.13}$$

Now let us consider the system of the equations

$$W(x,z) = I + \mathrm{i}J\int_a^x \frac{\mathrm{d}B(\xi)}{z-\xi}W(\xi,z). \tag{3.14}$$

PROPOSITION 3.3. *The monodromy matrix $W(b,z)$ of system* (3.14) *coincides with the solution $W(z)$ of* RHP (2.11), (3.6) *and* (3.7).

COROLLARY 3.4. *The integral system* (3.14) *is equivalent to the differential system*

$$\frac{\mathrm{d}W(x,z)}{\mathrm{d}x} = \frac{\mathrm{i}JH(x)}{z-x}W(x,z) \tag{3.15}$$

with the boundary condition $W(a,z) = I_m$. Here the matrix function H(x) is defined by relation (3.12).

Due to (3.14) the following relation

$$W(x,z) = I + \frac{M_1(x)}{z} + \frac{M_2(x)}{z^2} + \cdots \tag{3.16}$$

is fulfilled in the neighborhood of $z = \infty$. It follows from (3.15) and (3.16) that

$$M_1(x) = \mathrm{i}JB(x). \tag{3.17}$$

THEOREM 3.5. *If the operators S_ξ, $(a < \xi \leqslant b)$ are invertible in the corresponding spaces $L^2_k(a,\xi)$ and the kernel*

$$k(x,t) = \frac{F_1(x)JF_1^\star(t)}{x-t} \tag{3.18}$$

satisfies the condition

$$\int_a^b\int_a^b \|k(x,t)\|^2 \mathrm{d}x\mathrm{d}t < \infty, \tag{3.19}$$

then the operator S_b defined by relation (3.3) *admits the triangular factorization* (3.11).

This theorem follows directly from the M.G. Krein theorem (see Chapter 4 of [4]).

REMARK 3.6. The operators S_ξ of Example 2.4 satisfy the conditions of Theorem 3.5.

4. LIMITING VALUES OF THE MULTIPLICATIVE INTEGRAL

Let $\beta_1(x)$ and $\beta_2(x)$ be $k \times m$ matrix functions ($k \leqslant m$). We consider the canonical system of the form

$$\frac{\mathrm{d}}{\mathrm{d}x}W(x,z) = \frac{H(x)}{z-x}W(x,z), \quad W(a,z) = I_m, \tag{4.1}$$

where $H(x) = \beta_1^*(x)\beta_2(x), a \leqslant x \leqslant b$. Systems (4.1) play an important role in the theory of non-selfadjoint operators [9], in the Riemann–Hilbert problem [1]–[3], [5], [8], in the theory of random matrices [1]–[3], [12], [13]. The solution of systems (4.1) can be represented in the form of the multiplicative integral

$$W(x,z) = \int_a^{\curvearrowleft x} \mathrm{e}^{\frac{1}{z-t}\mathrm{d}E(t)}, \tag{4.2}$$

where $E(x) = \int_a^x H(t)\mathrm{d}t$. The multiplicative integral is defined by the relation

$$\int_a^{\curvearrowleft b} \mathrm{e}^{f(t)\mathrm{d}E(t)} = \lim_{\max \Delta t_j \to 0} \mathrm{e}^{f(t_{n-1})\Delta E(t_{n-1})}\mathrm{e}^{f(t_{n-2})\Delta E(t_{n-2})} \cdots \mathrm{e}^{f(t_0)\Delta E(t_0)}, \tag{4.3}$$

where $a = t_0 < t_1 < \cdots < t_n = b$. The analogues of the Plemelj formulas were deduced for the limiting values

$$W_{\pm}(b,\sigma) = \lim_{y \to \pm 0} W(b,z), \quad z = \sigma + \mathrm{i}y \tag{4.4}$$

of the multiplicative integral (see Chapter 1 of [9]). It was in particular supposed that the matrix function $H(x)$ for each x is linearly similar to a certain selfadjoint matrix. Now we shall consider the case when

$$\beta_2(x)\beta_1^*(x) = 0. \tag{4.5}$$

It follows from (4.5) that

$$[H(x)]^2 = 0. \tag{4.6}$$

Thus the matrix function $H(x)$ is a nilpotent one and hence it is not similar to a selfadjoint matrix function. In this case as well the analogue of the Plemelj formula is true [12].

LEMMA 4.1. *Let the $k \times m$ matrix functions $\beta_1(x)$ and $\beta_2(x)$ be continuous on the segment $[a,b]$ and satisfy the estimates*

$$\|\beta_k(x)\| \leqslant M, \quad (k=1,2); \quad \left\| \frac{\beta_2(x)\beta_1^*(t)}{x-t} \right\| \leqslant M, \quad a \leqslant x, \quad t \leqslant b. \tag{4.7}$$

Then there exist the limits

$$V_1(x,\sigma) = \lim_{y \to +0} [W(x,\sigma+\mathrm{i}y) - W(x,\sigma-\mathrm{i}y)], \tag{4.8}$$

$$V_2(x,\sigma) = \lim_{y\to+0}[W^{-1}(x,\sigma+\mathrm{i}y) - W^{-1}(x,\sigma-\mathrm{i}y)], \tag{4.9}$$

and for some M_1 *the following inequality is true:*

$$\|V_k(x,\sigma)\| \leqslant M_1, \quad (k=1,2). \tag{4.10}$$

THEOREM 4.2. *Let the conditions of Lemma* 4.1 *be fulfilled. Then the equalities are valid:*

$$V_1(x,\sigma) = \lim_{\varepsilon\to+0}\Big(\int_{\sigma+\varepsilon}^{\overset{\curvearrowleft}{x}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}(-2\mathrm{i}\pi H(\sigma))\int_{a}^{\overset{\curvearrowleft}{\sigma-\varepsilon}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}\Big), \tag{4.11}$$

$$V_2(x,\sigma) = \lim_{\varepsilon\to+0}\Big(\int_{a}^{\overset{\curvearrowleft}{\sigma-\varepsilon}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}\Big)^{-1}(2\mathrm{i}\pi H(\sigma))\Big(\int_{\sigma+\varepsilon}^{\overset{\curvearrowleft}{x}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}\Big)^{-1}. \tag{4.12}$$

Here $a < \sigma < x, E(x) = \int_a^x H(t)\mathrm{d}t$. Lemma 4.1 and Theorem 4.2 follow directly from results of our paper [12].

REMARK 4.3. Equality (4.11) can be written in the form

$$\lim_{y\to+0}(W(x,\sigma+\mathrm{i}y) - W(x,\sigma-\mathrm{i}y))$$

$$= \lim_{\varepsilon\to+0}\Big(\int_{\sigma+\varepsilon}^{\overset{\curvearrowleft}{x}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}(\mathrm{e}^{-\mathrm{i}\pi H(\sigma)} - \mathrm{e}^{\mathrm{i}\pi H(\sigma)})\int_{a}^{\overset{\curvearrowleft}{\sigma-\varepsilon}} \mathrm{e}^{\frac{1}{\sigma-t}\mathrm{d}E(t)}\Big). \tag{4.13}$$

Here we use relation (4.6).

PROPOSITION 4.4. *Let the conditions of Lemma* 4.1 *be fulfilled. The corresponding matrix* $u(x)$ *satisfies the relation*

$$(u(x) - I)^2 = 0. \tag{4.14}$$

Proof. Let us consider $W(z) = W(b,z)$. From (4.8),(4.9) and (4.11), (4.12) we deduce that

$$[W_+^{-1}(x) - W_-^{-1}(x)]\cdot[W_+(x) - W_-(x)] = 0. \tag{4.15}$$

Due to (2.11) and (4.15) the following relation holds:

$$2I_m - u(x) - u^{-1}(x) = 0. \tag{4.16}$$

The last equality can be written in form (4.14). The proposition is proved. ∎

Thus under some additional conditions we deduce the formula (4.14) from (4.6).

4.1. OPEN PROBLEM I. Find the conditions under which formula (4.6) follows from (4.14).

5. PARTIAL SOLUTION OF OPEN PROBLEM I

EXAMPLE 5.1. Let us consider the case when

$$J = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.1}$$

$$u(x) = \begin{bmatrix} 0 & \phi(x) \\ -\overline{\phi(x)} & 2 \end{bmatrix}, \quad 0 \leqslant x \leqslant r, \tag{5.2}$$

where $|\phi(x)|^2 = 1$. We remark that $u(x)$ has form (2.7), where

$$F_1(x) = [1, -\phi(x)], \quad F_2(x) = F_1(x)J. \tag{5.3}$$

Using (5.3) we obtain the relations:

$$F_1(x)JF_1^\star(x) = 0, \tag{5.4}$$

$$F_1(x)JF_1^\star(t) = \phi(x)\phi^\star(t) - 1. \tag{5.5}$$

Thus in case (5.3) we deduce from (3.1) and (5.5) that the operator S_ξ has the form

$$S_\xi f = f(x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_0^\xi \frac{\phi(x)\phi^\star(t) - 1}{x - t} f(t)\mathrm{d}t. \tag{5.6}$$

The fact that the operator V defined by (1.5) is unitary implies that in the space $L^2(0, \xi)$ we have

$$S_\xi \geqslant 0. \tag{5.7}$$

Further we suppose that the operator S_r is invertible in $L^2(0, r)$. So the operators S_ξ, $\xi \leqslant r$ are invertible in $L^2(0, \xi)$ as well.

REMARK 5.2. If $\phi(x)$ satisfies Hölder condition $|\phi(x) - \phi(t)| \leqslant |x - t|^\alpha, 0 < \alpha \leqslant 1$, then there exists such $r > 0$ that S_r is invertible in $L^2(0, r)$.

Using relation (3.4) we have

$$\Phi(\xi, x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_0^\xi \frac{\phi(x)\phi^\star(t) - 1}{x - t}\Phi(\xi, t)\mathrm{d}t = F_1(x), \tag{5.8}$$

where

$$\Phi(\xi, x) = [\Phi_1(\xi, x) \quad \Phi_2(\xi, x)]. \tag{5.9}$$

It follows directly from (5.3) and (5.8) that

$$\Phi_1(\xi, x) + \frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_0^\xi \frac{\phi(x)\phi^\star(t) - 1}{x - t}\Phi_1(\xi, t)\mathrm{d}t = 1, \tag{5.10}$$

$$\Phi_2(\xi,x)+\frac{\mathrm{i}}{2\pi}\mathrm{P.V.}\int_0^{\xi}\frac{\phi(x)\phi^{\star}(t)-1}{x-t}\Phi_2(\xi,t)\mathrm{d}t=-\phi(x). \tag{5.11}$$

Let the function $\Phi_1(\xi,x)$ be the solution of equation (5.10). It is easy to see that the function $[-\phi(x)\overline{\Phi_1(\xi,x)}]$ satisfies equation (5.11), i.e.

$$\Phi_2(\xi,x)=-\phi(x)\overline{\Phi_1(\xi,x)}. \tag{5.12}$$

Hence formula (3.10) takes the form:

$$B(\xi)=\frac{1}{2\pi}\int_0^{\xi}\begin{bmatrix}\Phi_1(\xi,x) & -\overline{\Phi_1(\xi,x)}\phi(x)\\ -\overline{\phi(x)}\Phi_1(\xi,x) & \overline{\Phi_1(\xi,x)}\end{bmatrix}\mathrm{d}x. \tag{5.13}$$

We suppose that $\phi(x)$ satisfies Hölder condition and $1/2<\alpha$. Hence the operator S_r admits the triangular factorization (see Chapter 4 of [4]). Comparing formulas (3.12) and (5.13) we deduce the representation

$$H(x)=B'(x)=a(x)\begin{bmatrix}1 & \mathrm{e}^{\mathrm{i}\alpha(x)}\\ \mathrm{e}^{-\mathrm{i}\alpha(x)} & 1\end{bmatrix}, \tag{5.14}$$

where $a(x)=|h_1(x)|^2\geqslant 0$, $\alpha(x)=\overline{\alpha(x)}$. Due to (5.14) we have the following assertion.

PROPOSITION 5.3. *Let the matrix function $u(x)$ have form* (5.2). *Then the matrix $JH(x)$ is nilpotent and*

$$[JH(x)]^2=0. \tag{5.15}$$

Proposition 5.3 gives a partial solution of Open Problem I.

REMARK 5.4. The special case of Example 5.1, when $\phi(x)=\mathrm{e}^{2\mathrm{i}ux}, u=\overline{u}$, plays an important role in the theory of random matrices (see [1], [2], [12], [13]).

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