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AMENABILITY OF THE UNITARY GROUP OF THE MULTIPLIER ALGEBRA OF A STABLE AH-ALGEBRA

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ABSTRACT. Let A be a unital AH-algebra, and let $G := \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $A \otimes \mathcal{K}$, given the strict topology. Then G is amenable in the following sense: Every affine continuous action of G on a compact convex set has a fixed point.

KEYWORDS: *Amenability, multiplier algebras, C^*-algebras, KK-theory.*

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1. INTRODUCTION

We say that a topological group G is *amenable* if every affine continuous action of G on a compact convex set has a fixed point. One would perhaps expect this property to hold primarily in the case of locally compact groups, but there exist non-locally-compact groups which are amenable in this sense. For instance, if $\mathcal{U}(\mathcal{H})$ is the group of unitaries over a separable infinite-dimensional Hilbert space, then $\mathcal{U}(\mathcal{H})$, with the strong topology, is an amenable topological group which is not locally compact. De la Harpe proved that a von Neumann algebra M is injective if and only if the unitary group $\mathcal{U}(M)$ of M, given the weak*-topology, is an amenable topological group [5]. Paterson, building on previous work, proved that a unital C^*-algebra A is nuclear if and only if the unitary group $\mathcal{U}(A)$ of A, given the relative weak topology, is an amenable topological group [14]. In general, it is interesting to find connections between properties of an operator algebra and amenability of an associated unitary group. In particular, there has been very little attention paid to amenability with respect to the strict topology of a multiplier algebra (we define these terms shortly) and this paper is a contribution towards filling the gap.

The multiplier algebra $\mathcal{M}(B)$ of a C^*-algebra B is the largest C^*-algebra inside which the C^*-algebra B is an essential ideal. Naturally, the multiplier algebra $\mathcal{M}(B)$ plays an important role in the extension theory (and hence, KK-theory) of

B. In particular, there is a one-to-one correspondence between extensions of B and $*$-homomorphisms into $\mathcal{M}(B)/B$ (the latter is the Busby invariant of the corresponding extension). Moreover, for the purposes of extension theory, we are often interested in the case where the canonical ideal B (of $\mathcal{M}(B)$) is a stable C^*-algebra. Among other things, stability of B implies that $\mathcal{M}(B)$ contains two isometries generating a unital copy of O_2. A construction based on these isometries then allows us to add two extensions of B (the so-called BDF-sum of extensions — this is, of course, only well-defined up to appropriate unitary equivalence). This paper (and future ones) will explore the connections between extension theory (i.e., KK-theory), the properties of operator algebras, and the properties of associated unitary groups.

The multiplier algebra $\mathcal{M}(B)$ has a natural norm topology, extensively used in C^*-algebra theory, the weak topology inherited from B^{**} (used by Paterson) and another natural topology, the strict topology. In fact, $\mathcal{M}(B)$ could, with due care, be defined to be a closure of B with respect to the strict topology; see Proposition 2.3.5 of [16]. In the case of function algebras on a topological space, the weak topology coincides with the strong topology and is the topology of pointwise convergence, the norm topology is the topology of uniform convergence, and the strict topology is the topology of uniform convergence on compact subsets.

If B is stable, the unitary group of $\mathcal{M}(B)$ cannot be amenable in the norm topology (since this would imply the existence of a tracial state — which is ruled out by the existence of the copy of O_2 mentioned in the previous paragraph).

The strict topology on $\mathcal{M}(B)$ is the initial topology with respect to the family of maps $L_b, R_b : \mathcal{M}(B) \longrightarrow B$, given by elements $b \in B$. (Here, L_b and R_b are left and right multiplications by b respectively.) In other words, the strict topology is the weakest topology on $\mathcal{M}(B)$ with respect to which the left and right product maps from $\mathcal{M}(B)$ to B with the norm topology are continuous.

There are natural inclusions $B \subseteq \mathcal{M}(B) \subseteq B^{**}$; and the strict topology on the unit ball of $\mathcal{M}(B)$ is in general stronger than the usual strong and weak topologies, but weaker than the norm topology. The strict topology coincides with the norm topology if and only if the C^*-algebra B is unital. For $B = \mathcal{K}(\mathcal{H})$, the C^*-algebra of compact operators on a Hilbert space H, the multiplier algebra is the C^*-algebra of bounded operators $B(\mathcal{H})$ and the strict topology coincides with the strong topology. The strict topology has been studied primarily on bounded subsets of a C^*-algebra, for in this case (if the C^*-algebra B has a countable approximate unit $\{b_n\}_{n=1}^{\infty}$) then the strict topology, on bounded subsets of the multipliers, is determined by the norm $m \mapsto \|bm\| + \|mb\|$, where b is a strictly positive element of B. In other words, the strict topology is in this case metrizable on bounded subsets — and in fact, it seems fair to say that the strict topology on unbounded subsets of a C^*-algebra is not well understood (however, we only shall need to consider the unit ball).

In this paper, we investigate amenability with respect to the strict topology for the unitary group of the multiplier algebra of a stable C^*-algebra. Our main result is the following, where an AH-algebra is defined to be the unital direct limit of certain unital building blocks — each building block being the finite direct sum of C^*-algebras of the form $P(C(X) \otimes M_n)P$ where X is a compact, second countable, metric space, and where P is a projection in $C(X) \otimes M_n$.

THEOREM 1.1. *Let A be a unital AH-algebra, and let $G := \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $A \otimes \mathcal{K}$, given the strict topology. Then G is an amenable topological group.*

We note that amenability in this sense implies amenability in the usual C^*-algebraic sense, as is proven in Proposition 3.1.

A key step in the proof of Theorem 1.1 is a stabilization result (Lemma 1.2) which more or less follows from the results of Dadarlat, Eilers, Elliott and Gong.

Our stabilization result is basically the following construction from KK-theory as applied to the classification program: Given a $*$-homomorphism $\phi : C(X) \otimes M_n \longrightarrow A$, where X is a finite CW complex, and A is a unital separable C^*-algebra, it is well-known that we can add on a $*$-homomorphism ξ into some abstract matrix algebra $M_{m-1}(A)$ such that $\phi \oplus \xi : C(X) \otimes M_n \longrightarrow M_m(A)$ is trivial in KK-theory. In fact, this idea can be pushed further. The map ξ can be chosen so that $\phi \oplus \xi$ will be homotopy trivial in the sense of Elliott and Gong ([6], Lemma 2.9). This means that there is a $*$-homomorphism ψ homotopic to $\phi \oplus \xi$ that has finite-dimensional range: $\psi(C(X) \otimes M_n) \subseteq M_m(\mathbb{C}1_A) \subseteq M_m(A)$. Since KK is a homotopy invariant (bi)functor, this finite-dimensional map ψ will have the same KK-class as $\phi \oplus \xi$. Now, by the main result of Dadarlat and Eilers ([3], Theorem 4.5), we have that given an arbitrary finite set in $C(X) \otimes M_n$ and $\varepsilon > 0$, there exists an integer $l \geqslant 0$ and a unitary u in $M_{m+\ell}(A)$ such that

$$\left\| u \begin{pmatrix} \phi(f) \oplus \xi(f) & \\ & \kappa_\ell(f) \end{pmatrix} u^* - \begin{pmatrix} \psi(f) & \\ & \kappa_\ell(f) \end{pmatrix} \right\| < \varepsilon, \tag{1.1}$$

for all f in the given finite set. The homomorphism κ_ℓ is a cutdown of an arbitrary absorbing quasidiagonal representation $\kappa : C(X) \otimes M_n \longrightarrow 1_A \otimes \mathcal{M}(\mathcal{K}) \subseteq \mathcal{M}(A \otimes \mathcal{K})$, which we can take to be the GNS representation found by Kasparov [12] and proven by him to be absorbing (the so-called Kasparov extension). Note that Kasparov's construction is just to take an appropriate full $*$-representation into $\mathcal{M}(\mathcal{K}) = B(\mathcal{H})$ and tensor it by 1_A. Hence, since $C(X) \otimes M_n$ is residually finite dimensional, we can take κ to be an appropriate l_∞-direct sum of finite-dimensional $*$-representations of $C(X) \otimes M_n$ (the image of each summand should give finite rank operators over the Hilbert space of the representation κ). Hence, we can take κ_l be a finite-dimensional $*$-homorphism of $C(X) \otimes M_n$ into some matrix algebra of the form $\mathbb{M}_{n_1}(\mathbb{C}1_A)$. We thus immediately obtain the following result:

LEMMA 1.2. *Let X be a finite* CW *complex, and let A be a unital C^*-algebra. Suppose that $\phi : M_n(C(X)) \to A$ is a unital $*$-homomorphism. We have that given an arbitrary finite set in $M_n(C(X))$ and $\varepsilon > 0$, there exists an integer N, a unitary $u \in M_N(A)$, and $*$-homomorphisms ϑ, Ψ into appropriate matrix algebras over A, with Ψ having finite-dimensional range $\mathbb{M}_N(\mathbb{C}1_A)$, and, for all f in the finite set,*

$$\left\| u(\ \phi(f) \oplus \vartheta(f))\ \ u^* - \Psi(f) \right\| < \varepsilon.$$

The main result of our paper is easily shown when the canonical ideal is an AF-algebra (indeed, we can even show extreme amenability for this special case), and the above result gives us approximating maps Ψ that are maps into AF-algebras. This type of "stabilized norm approximation" is exactly what we need, since we will be dealing with the strict topology on multiplier unitaries.

Indeed, the strategy of the proof of our main result (see Lemma 3.8) is to exploit the fact that the group of constant unitaries — the group of unitaries in $1 \otimes \mathcal{M}(\mathcal{K})$ — is most definitely *not* a normal subgroup of the unitary group of $\mathcal{M}(C(X) \otimes \mathcal{K})$! In fact, Lemma 1.2 will be used to approximate, in the strict topology, finite sets of multiplier unitaries by conjugates of elements of $\mathcal{U}(1 \otimes \mathcal{M}(\mathcal{K}))$. Since the unitary group of $1 \otimes \mathcal{M}(K)$ is known to be amenable (notice that the strict and strong topologies coincide), this will do the job. The main steps in the proof are Lemmas 1.2 and 3.6.

Perhaps there might eventually be a characterization of stably finite nuclear C^*-algebras in terms of some sort of generalized stabilization that is similar to the above lemma.

2. THE STRICT AND STRONG TOPOLOGIES ON $\mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$

Given that the multiplier algebra of $\mathcal{K}$ is $B(\mathcal{H})$, and that the unitary group of $B(\mathcal{H})$ has the property that the strict, strong, and weak topologies all coincide, it is reasonable to wonder if unitary groups of multiplier algebras $\mathcal{M}(A \otimes \mathcal{K})$ might not always have the property that the strict and strong topologies coincide.

In this section, we show that for a primitive, separable C^*-algebra A that is not of type I, the strict topology on $\mathcal{U}(\mathcal{M}((A \otimes \mathcal{K})))$ is not the same as the restricted strong topology. Since on $\mathcal{U}((A \otimes \mathcal{K})^{**})$, the strong topology is the same as the w*-topology, this shows that amenability of $\mathcal{U}((A \otimes \mathcal{K})^{**})$ in the w*-topology does not immediately imply amenability of $\mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ in the strict topology. Furthermore, note that since simple C^*-algebras are automatically primitive, we immediately have this result for the case where A is simple and nontype I.

PROPOSITION 2.1. *Let A be a primitive separable C^*-algebra that is not of type* I. *Let $G := \mathcal{U}(\mathcal{M}(A))$ be the unitary group of the multiplier algebra of A. Then the strict and strong topologies on G are distinct.*

Proof. Since A is primitive, we take it to be faithfully and irreducibly represented on a separable Hilbert space H. This representation extends to a representation of the multipliers of A with the same properties.

Since A is not of type I, there exists a positive element s whose spectrum contains an interval. We can as well assume that the element has norm 1 and that 1 is in the continuous spectrum. Considering approximate eigenvectors with respect to points of the continuous spectrum, which we can do since s is positive ([11], Section 31), we see that the image $s(\mathcal{H}_1)$ of the unit ball of $\mathcal{H}$ contains infinitely many orthogonal vectors ξ_i of norm greater than $1-\varepsilon$. The Kadison transitivity theorem ([15], 2.7.5) lets us find unitaries $U_m \in \mathcal{M}(A)$ that act as the unit on the first m vectors of a basis for $\mathcal{H}$ but map some ξ_i to approximately $-\xi_i$. These unitaries converge strongly to $1_{\mathcal{M}(A)}$ but for each m there is a ξ_i such that $(U_m s - s)\xi_i$ has norm greater than 1. Since this in particular shows that $(U_m - 1)s$ does not go to zero, it follows that U_m does not go to zero strictly. It follows that the strong and strict topologies are distinct, as claimed.

We want to show that the strict topology on G is distinct from the strong topology on G.

Let p be a nonzero projection in $A \otimes \mathcal{K}$. Let 1 be the unit of the multiplier algebra $\mathcal{M}(A \otimes \mathcal{K})$. Consider the strictly open neighbourhood N of 1 in G, given by $N := \{V \in G : \|Vp - p\| < \varepsilon\}$, where ε is a real number such that $0 < \varepsilon < 1$.

We will show that N is not an open set in the strong topology restricted to G. Suppose, to the contrary, that N is an open set in the strong topology restricted to G. Let $h_1, h_2, \ldots, h_n$ be vectors in the Hilbert space H, and let $\delta > 0$ be a positive real number such that for every unitary $V \in G$, if $\|Vh_i - h_i\| < \delta$ for every i, then V is an element of N. Let O be the (strongly) open set consisting of V as in the hypothesis of the previous conditional (i.e., $\|Vh_i - h_i\| < \delta$ for all i). Hence, $O \subseteq N$.

Since A is simple and not type I, if q is a nonzero projection in $A \otimes \mathcal{K}$ then q sits as an infinite-dimensional projection on H. Hence, there is a nonzero vector h in pH such that h is orthogonal to h_i for $1 \leqslant i \leqslant n$. Hence, since A is simple, nontype I and real rank zero, and since $A \otimes \mathcal{K}$ sits faithfully and irreducibly on H, let r be a nonzero projection in $p(A \otimes \mathcal{K})p$ such that for each i, rh_i has (Hilbert space) norm strictly less than γ.

Let W be the unitary in $\mathcal{M}(A \otimes \mathcal{K})$, given by $W = (1-r) - r$. (In particular, $W \in G$.)

Now for every i, $\|Wh_i - h_i\| = \|-rh_i - rh_i\| = 2\|rh_i\| < 2\gamma$. Also, $\|Wp - p\| = \|-r-r\| = 2\|r\| = 2$. Hence, if γ was chosen so that 2γ is strictly less than δ, then W is an element of O and W is not an element of N. This is a contradiction. ∎

3. AMENABILITY

We first show that the type of amenability under discussion implies ordinary C^*-algebraic amenability, at least for unital separable C^*-algebras.

PROPOSITION 3.1. *Let A be a unital, separable C^*-algebra. Let $G := \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $A \otimes \mathcal{K}$, given the strict topology. If this topological group is amenable then A is a nuclear C^*-algebra.*

Proof. Let $\mathcal{U}((A \otimes \mathcal{K})^{**})$ be the unitary group of the second dual of $A \otimes \mathcal{K}$, given the weak*-topology.

Suppose that α is an affine weak*-continuous action of $\mathcal{U}((A \otimes \mathcal{K})^{**})$ on a compact convex set Y. Then, restricting α to $G = \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ still gives us a continuous action (with respect to the weak*-topology restricted to G). Since the strict topology on G is stronger than the restricted w*-topology, $\alpha|G$ must also be continuous with respect to the restricted strict topology. Hence, since G is amenable with respect to the strict topology, let $y_0 \in Y$ be a fixed point of $\alpha|G$.

Since $G = \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ is w*-dense in $\mathcal{U}((A \otimes \mathcal{K})^{**})$, this fixed point is also a fixed point for the unitary group of the double dual in the w*-topology. Therefore $\mathcal{U}((A \otimes \mathcal{K})^{**})$ is an amenable group with respect to the weak*-topology — which is well known to imply [5], [2] that $(A \otimes \mathcal{K})^{**}$ is an injective von Neumann algebra and that $A \otimes \mathcal{K}$ is a nuclear C^*-algebra. ∎

We now proceed towards our main result. We first state two lemmas, whose proofs are short exercises in topology.

LEMMA 3.2. *Let G be a topological group with metric d. Let X be a compact convex set, and let $\mathcal{O}$ be an open neighbourhood of zero in the topological vector space containing X. Suppose that $\alpha : G \to \mathrm{Aut}(X)$ is an affine continuous action of G on X. Then for every $g_0 \in G$, there exists $\delta > 0$ such that if $d(g_0, g) < \delta$ then $\alpha(g_0)x - \alpha(g)x \in \mathcal{O}$ for every $x \in X$.*

LEMMA 3.3. *Let V be a topological vector space, and let O_1 be an open neighbourhood of zero in V. There is an open neighbourhood O_2 of zero in V such that the closure $\overline{O}_2 \subseteq O_1$.*

LEMMA 3.4. *Let G be a topological group with metric d, and let X be a compact convex subset of a locally convex seminormed space. Suppose that $\alpha : G \to \mathrm{Aut}(X)$ is an affine continuous action of G on X. Then for any open neighbourhood $N(\alpha(g_0)x_0)$ of $\alpha(g_0)x_0$ in X, there exists a neighbourhood $N_\delta(g_0)$ of $g_0 \in G$ such that for every $x_0 \in X$, the natural map $\beta : G \times X \to X : (g, x) \mapsto \alpha(g)x$ maps $N_\delta(g_0) \times \{x_0\}$ into $N(g_0 x_0)$.*

Proof. Given a neighbourhood N as above, choose a basis element in N, containing $\alpha(g_0)x_0$. The basis element is defined by some finite set $\mathcal{F}$ of seminorms. The compactness of X implies that the group action is continuous with respect to the semimetric

$$d(g,g') := \sup_{f\in\mathcal{F},x\in X} \|\alpha_g(x) - \alpha_{g'}(x)\|_f$$

on G and the usual topology on X. It follows that given $x_0 \in X$, there is a δ such that β maps $N_\delta(g_0) \times \{x_0\}$ into $N(g_0x_0)$. Using the compactness of X one more time, we can find one δ that will work for all $x_0 \in X$. ∎

The next lemma is related to Lemma 2.20 in [8]. Let G be a topological group, with topology given by a metric. We say that a net of subgroups $\{H_\lambda\}_{\lambda\in\Lambda}$ of G is *pointwise dense* in G if for every finite set $\{g_i\}_{i=1}^n$ in G, there is a $\lambda_0 \in \Lambda$ such that for $\lambda \geqslant \lambda_0$ and for $1 \leqslant i \leqslant n$, $\mathrm{dist}(g_i, H_\lambda) < \varepsilon$.

LEMMA 3.5. *Let G be a topological group with metric d. If G has a pointwise dense net $\{H_\lambda\}_{\lambda\in\Lambda}$ consisting of amenable subgroups, then G is amenable.*

Proof. Suppose that $\alpha : G \to \mathrm{Aut}(Y)$ is an affine continuous action of G on a compact convex set Y. For each λ, α restricts to an affine continuous action of H_λ on Y. Since the H_λs are amenable topological groups, the corresponding restrictions of α all have fixed points. From this and Lemma 3.2, it follows that for every nonempty finite set $\mathcal{F}$ in G, and for every open neighbourhood $\mathcal{O}$ of zero in the topological vector space containing Y, the set $Y(\mathcal{F},\mathcal{O}) := \{y \in Y : \alpha(g)y - y \in \overline{\mathcal{O}}, \forall g \in \mathcal{F}\}$ is nonempty. Note that $Y(\mathcal{F},\mathcal{O})$ is a closed subset of Y (in the definition, we took the closure of $\mathcal{O}$).

Also, if $\mathcal{F}_1, \mathcal{F}_2$ are nonempty finite subsets of G, and if $\mathcal{O}_1, \mathcal{O}_2$ are open neighbourhoods of zero in the topological vector space containing Y, it follows that $Y(\mathcal{F}_1,\mathcal{O}_1) \cap Y(\mathcal{F}_2,\mathcal{O}_2) \supseteq Y(\mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{O}_1 \cap \mathcal{O}_2)$ must be nonempty. Hence, the family of all sets of the form $Y(\mathcal{F},\mathcal{O})$ (where $\mathcal{F}$ is a nonempty finite subset of G and $\mathcal{O}$ is an open neighbourhood of zero in the topological vector space containing Y) is a family of closed subsets of Y, with the finite intersection property. Hence, since Y is compact, this family must have a nonempty intersection. Let y_0 be a point in this intersection. Then, by Lemma 3.3, y must be a fixed point of the action α. ∎

The next lemma lets us approximate, in the strict topology, a multiplier unitary by a sequence of "appropriate" multiplier unitaries by controlling what happens inside an ascending sequence of unital hereditary subalgebras of the canonical ideal.

LEMMA 3.6. *Let A_0 be a unital separable C^*-algebra. Let U be a unitary in $\mathcal{M}(A_0 \otimes \mathcal{K})$, and let $\{\widetilde{p}_i\}_{i=1}^\infty$ be an approximate unit for $A_0 \otimes \mathcal{K}$, consisting of a sequence of projections. Given an integer $M > 0$, there exists an integer $M_2 > M$ and a unitary u in $\widetilde{p}_{M_2}(A_0 \otimes \mathcal{K})\widetilde{p}_{M_2}$ with $u\widetilde{p}_M$ approximating $U\widetilde{p}_M$ in norm within any specified error δ.*

Proof. Let v be a partial isometry with initial projection $\widetilde{p}_M$ that approximates $U\widetilde{p}_M$ within δ in norm. We may suppose, since we are only looking for an approximation, that the range projection is majorized by $\widetilde{p}_{M_1}$ for some $M_1 > M$.

Hence, $\widetilde{p}_{M_1} - v^*v$ and $\widetilde{p}_{M_1} - vv^*$ must give the same element in the Grothendieck K-theory group $K_0(A_0)$. By the definition of equivalence in a Grothendieck group, there exists a projection $q \in A_0 \otimes \mathcal{K}$ such that $(\widetilde{p}_{M_1} - v^*v) + q$ is Murray-von Neumann equivalent to $(\widetilde{p}_{M_1} - vv^*) + q$ in $A_0 \otimes \mathcal{K}$. We can moreover suppose that q is majorized by $\widetilde{p}_{M_2} - \widetilde{p}_{M_1}$ for some $M_2 > M_1$.

Let w be a partial isometry in $A_0 \otimes \mathcal{K}$ that implements this equivalence. In other words,

$$w^*w = (\widetilde{p}_{M_1} - v^*v) + q, \quad ww^* = (\widetilde{p}_{M_1} - vv^*) + q.$$

Let

$$u := v + w + (\widetilde{p}_{M_2} - \widetilde{p}_{M_1} - q).$$

The cross-terms in uu^* and u^*u are zero, and u is thus a unitary in $\widetilde{p}_{M_2}(A_0 \otimes \mathcal{K})\widetilde{p}_{M_2}$ such that $u\widetilde{p}_M = v$ is within δ in norm of $U\widetilde{p}_M$. ∎

The next lemma is a key step. It is a proof that the family $\{gHg^{-1} : g \in G\}$, where H is an appropriate amenable subgroup of G, gives a pointwise dense net in G.

In more detail: Let X be a finite CW complex. Let $G = \mathcal{U}(\mathcal{M}(C(X) \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $C(X) \otimes \mathcal{K}$, given the strict topology. Let H be the subgroup of G given by $H := \mathcal{U}(1_{C(X)} \otimes \mathcal{M}(\mathcal{K}))$. Let b be a strictly positive element of $C(X) \otimes \mathcal{K}$ such that $\|b\| \leqslant 1$, and let d be the strict topology metric on G given by $d(U, V) := \|Ub - Vb\| + \|bU - bV\|$ for $U, V \in G$. We will show that with respect to d, the conjugates of H give a pointwise dense net for G.

LEMMA 3.7. *Let G, H and d be as in the previous paragraph. Then for any finite set $(U_i)_1^N$ in G, there exist $g \in G$ and $h_i \in H$ such that for $1 \leqslant i \leqslant N$, U_i can be approximated by gh_ig^{-1} in d within any given specified error.*

Proof. We first define some notation.

Let $\{p_m\}_{m=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $C(X) \otimes \mathcal{K}$ such that $1_{\mathcal{M}(C(X)\otimes\mathcal{K})} = \sum_{m=1}^{\infty} p_m$, where the sum converges in the strict topology in $\mathcal{M}(C(X) \otimes \mathcal{K})$. Since $C(X)$ is unital, we can certainly assume that for each m, p_m has the form $1_{C(X)} \otimes r$, where r is a rank one projection in $\mathcal{K}$. (Hence, each p_m also has rank one.) Finally, the partial sums naturally form an approximate unit for $C(X) \otimes \mathcal{K}$, which we denote by $\{\widetilde{p}_m\}_{m=1}^{\infty}$.

We can now begin the proof:

Let $\varepsilon > 0$ and an element U of G be given. Fix an integer $M \geqslant 1$ such that:

(i) The element $\widetilde{p}_M b$ is within δ of b and $U\widetilde{p}_M b$ is within δ of Ub.

By Lemma 3.6 there is a unitary u in $\widetilde{p}_{M_2}(C(X) \otimes \mathcal{K})\widetilde{p}_{M_2}$ such that:

(ii) $u\widetilde{p}_M =: v$ is within δ of $U\widetilde{p}_M$.

Applying Lemma 1.2 to the identity map $\widetilde{p}_{M_2}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_2} \longrightarrow \widetilde{p}_{M_2}(C(X) \otimes\mathcal{K})\widetilde{p}_{M_2}$, and the finite set $\{u\}$, we obtain a map $\vartheta : \widetilde{p}_{M_2}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_2} \longrightarrow \widetilde{p}_{M_3}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_3}$ and a unitary y such that $y(u\oplus\vartheta(u))y^*$ is close to $\Psi(u)$, where $\Psi : \widetilde{p}_{M_2}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_2} \longrightarrow \widetilde{p}_{M_4}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_4}$ has finite-dimensional range.

Actually, the map Ψ has image contained inside $1\otimes\mathcal{K}$. (This is true by Lemma 1.2 and the special form of the projections p_m.) Thus we have shown that:

(iii) $y(u\oplus\vartheta(u))y^*$ is a unitary (in $\widetilde{p}_{M_4}(C(X)\otimes\mathcal{K})\widetilde{p}_{M_4}$) within δ in norm of a unitary $\Psi(u)$ in $\widetilde{p}_{M_4}(1_{C(X)}\otimes\mathcal{K})\widetilde{p}_{M_4}\subset 1_{C(X)}\otimes\mathcal{K}$.

Let $\widetilde{u}$ be the unitary in $1\otimes\mathcal{M}(K)$ given by $\widetilde{u}=\Psi(u)+(1_H-\widetilde{p}_{M_4})$. This is thus an element of the subgroup H in G. Let $\widetilde{y}\in G$ be the unitary $y+1-p_{M_4}$.

Using first (i), then the fact that $\widetilde{y}^*\widetilde{u}\widetilde{y}\widetilde{p}_M = y^*\Psi(u)y\widetilde{p}_M$, and then (iii) and (ii), we have that:

$$\begin{aligned} Ub-\widetilde{y}^*\widetilde{u}\widetilde{y}b &\approx_{2\delta} U\widetilde{p}_Mb-\widetilde{y}^*\widetilde{u}\widetilde{y}\widetilde{p}_Mb = U\widetilde{p}_Mb-\widetilde{y}^*\Psi(u)\widetilde{y}\widetilde{p}_Mb \\ &\approx_\delta U\widetilde{p}_Mb-(u\oplus\vartheta(u))\widetilde{p}_Mb = U\widetilde{p}_Mb-u\widetilde{p}_Mb\approx_\delta 0. \end{aligned}$$

We have here introduced the notation $\approx_\varepsilon$ to indicate that two expressions differ by at most ε in norm. Since all the approximations can equally be made over finite sets of unitaries — with the conjugating unitary $\widetilde{y}$ chosen to be dependent only on the finite set (independent of specific element in the finite set) — we see that we have the same approximation if $\{U\}$ is replaced by $\{U,U^*\}$ or for that matter by $\{U_i,U_i^* : i=1,\dots,n\}$ where $\{U_1,\dots,U_n\}$ is a given finite set. Thus, $d(U_i,\widetilde{y}^*\widetilde{u}_i\widetilde{y})<12\delta$ for finitely many i, where d is the strict metric on G as defined before the statement of this theorem. ∎

LEMMA 3.8. *Let X be a compact, second countable, metric space and let $G=\mathcal{U}(\mathcal{M}(C(X)\otimes\mathcal{K}))$ be the unitary group of the multiplier algebra of $C(X)\otimes\mathcal{K}$, given the strict topology. Then G is an amenable topological group.*

Proof. First, we look at the case where X is a finite CW complex.

Let H be the subgroup of G given by $H:=1_{C(X)}\otimes\mathcal{U}(\mathcal{M}(\mathcal{K}))$, with the strict topology. Since all choices of the strictly positive element b in the definition of the metric are equivalent, we may as well pick $b:=1\otimes k$, where k is strictly positive in $\mathcal{K}$ — implying that H is isomorphic to the unitary group of $B(\mathcal{H})$ with the strict topology. In this case, the strict and strong topologies coincide, and the subgroup H is therefore an amenable topological group (c.f. [5], [10], [8]). By Lemma 3.7, the subgroups $\{gHg^{-1} : g\in G\}$ are a pointwise dense net, in G, with respect to an appropriate directed set (defined implicitly in Lemma 3.7). Therefore, by Lemma 3.5, the group G is amenable.

Now suppose that X is an arbitrary compact second countable metric space. By [7], let $\{X_n,\rho_n\}_{n=1}^\infty$ be an inverse system consisting of finite CW-complexes, such that X is the inverse limit $X=\lim_{\leftarrow}X_n$. This gives direct limits $C(X)=$

$\lim_{\to} C(X_n)$ and $C(X) \otimes \mathcal{K} = \lim_{\to}(C(X_n) \otimes \mathcal{K})$. Now, for each n, the induced map $C(X_n) \otimes \mathcal{K} \to C(X) \otimes \mathcal{K}$ brings approximate units to approximate units. Hence, for each n, we naturally have a map $\Delta_n : \mathcal{M}(C(X_n) \otimes \mathcal{K}) \to \mathcal{M}(C(X) \otimes \mathcal{K})$. Note that for each n, Δ_n is continuous with respect to the strict topology on bounded subsets. Hence, by the first part of this proof, we have that for each n, $\Delta_n(\mathcal{U}(\mathcal{M}(C(X_n) \otimes \mathcal{K})))$ is an amenable subgroup of $G = \mathcal{U}(\mathcal{M}(C(X) \otimes \mathcal{K}))$ (where the subgroup is given the strict topology from G). Moreover, the (increasing) union $\bigcup_{n=1}^{\infty} \Phi(\mathcal{U}(\mathcal{M}(C(X_n) \otimes \mathcal{K})))$ is dense in G with the strict topology. Hence, G, with the strict topology, is an amenable topological group. ∎

LEMMA 3.9. *Let B be a separable, stable C^*-algebra. Suppose that P is a projection in B. Then the projection $1_{\mathcal{M}(B)} - P$, in $\mathcal{M}(B)$, is Murray–von Neumann equivalent to $1_{\mathcal{M}(B)}$.*

Proof. Since P is contained in B, and since B is stable, let S be an isometry in $\mathcal{M}(B)$, such that $S^* pS$ has norm strictly less than ε. Hence, $S^*(1_{\mathcal{M}(B)} - P)S$ is within ε of $1_{\mathcal{M}(B)}$, implying that $r^*(1 - P)r = 1$ for some r, and hence $1_{\mathcal{M}(B)}$ is Murray–von Neumann equivalent to a subprojection of $1_{\mathcal{M}(B)} - p$. Hence, by a well-known result originally due to Mingo, $1_{\mathcal{M}(B)} - p$ is Murray–von Neumann equivalent to the unit of $\mathcal{M}(B)$. ∎

THEOREM 3.10. *Let G be the unitary group of the multiplier algebra of a separable stable* AH*-algebra, with the strict topology. Then G is amenable.*

Proof. A general AH-algebra can be taken to be the unital direct limit of certain unital building blocks — each building block being the finite direct sum of C^*-algebras of the form $P(C(X) \otimes M_n)P$ where X is a compact, second countable, metric space, and where P is a projection in $C(X) \otimes M_n$.

Let the building blocks be denoted by A_n, and let the limit algebra be A. The connecting maps $\phi_n : A_n \longrightarrow A$ are assumed to be both unital and injective (since a quotient of a building block is still a building block). Stabilizing, we have an inductive limit of the form $A \otimes \mathcal{K} = \lim_{n\to\infty}(A_n \otimes \mathcal{K})$, where the connecting maps $\phi_n \otimes 1 : A_n \otimes \mathcal{K} \to A \otimes \mathcal{K}$ map an approximate unit to an approximate unit. Hence, these connecting maps extend naturally to (unital) $*$-homomorphisms $\Phi_n : \mathcal{M}(A_n \otimes \mathcal{K}) \to \mathcal{M}(A \otimes \mathcal{K})$ of the multipliers. Note that these maps are strictly continuous on bounded subsets. We thus get strictly continuous topological group homomorphisms (where the groups are given the strict topology) $\mathcal{U}(\mathcal{M}(A_n \otimes \mathcal{K})) \to \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$.

Now for each n, $A_n \otimes \mathcal{K}$ has the form $C(Z_n) \otimes \mathcal{K}$, where Z_n is a compact, second countable, metric space. Hence, by Lemma 3.8, for each n the unitaries $\mathcal{U}(\mathcal{M}(A_n \otimes \mathcal{K}))$ are an amenable topological group. Happily, a topological group quotient (by a continuous homomorphism) of an amenable group is an amenable group. Hence, it follows that for each n, the group $G_n := \Phi_n(\mathcal{U}(\mathcal{M}(A_n \otimes \mathcal{K})))$ is an amenable topological subgroup of G.

Since the union $\bigcup \Phi_n(A_n \otimes \mathcal{K})$ is norm-dense in $A \otimes \mathcal{K}$, the unitary group of $\bigcup \Phi_n(\mathcal{M}(A_n \otimes \mathcal{K}))$ is strictly dense in the unitary group of $\mathcal{M}(A \otimes \mathcal{K})$. By a suitable form of the Kaplansky density theorem, it follows that the unitaries $\bigcup G_n$ are dense in G. By Lemma 3.5 (or by Lemma 2.20 in [8]) the amenability of the G_n implies the amenability of G.

Claim. $G = \overline{\bigcup_{n=1}^{\infty} G_n}$, where the closure is in the topology of G (the strict topology of $\mathcal{M}(C(X) \otimes \mathcal{K})$, restricted to G).

Suppose that U is an element of G (i.e., U is a unitary in $\mathcal{U}(\mathcal{M}(C(X) \otimes \mathcal{K}))$). Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of pairwise orthogonal projections in $C(X) \otimes \mathcal{K}$, such that $1_{\mathcal{M}(C(X)\otimes\mathcal{K})} = \sum_{m=1}^{\infty} p_m$, where the sum converges in the strict topology in $\mathcal{M}(C(X) \otimes \mathcal{K})$. We may assume that each p_m has the form $1_{C(X)} \otimes q$, where q is a projection in $\mathcal{K}$. Hence, $U = \sum_{m=1}^{\infty} Up_m$, where the sum converges in the strict topology in $\mathcal{M}(C(X) \otimes \mathcal{K})$. Also, for each m', we let $\widetilde{p}_{m'} := \sum_{m=1}^{m'} p_m$.

Now let $\delta > 0$ be given and fix an integer $M \geqslant 1$. We consider the partial isometry $U\widetilde{p}_M$ in $C(X) \otimes \mathcal{K}$. Since, $\mathcal{C}(X) \otimes \mathcal{K} = \overline{\bigcup_{n=1}^{\infty} \phi_n(C(X_n) \otimes \mathcal{K})}$ (where the closure is in the norm topology), let $N \geqslant M$ be a positive integer and let $v_{M,\delta}$ be a partial isometry in $\widetilde{p}_N(\phi_N(C(X_N) \otimes \mathcal{K}))\widetilde{p}_N$ such that $v_{M,\delta}$ is within δ of $U\widetilde{p}_M$. Let q_1 and q_2 be the initial and range projections, respectively, of $v_{M,\delta}$. Since q_1 and q_2 are elements of $\phi_N(C(X_N) \otimes \mathcal{K})$, it follows, by Lemma 3.9, that $1_{\mathcal{M}(C(X)\otimes\mathcal{K})} - q_1$ and $1_{\mathcal{M}(C(X)\otimes\mathcal{K})} - q_2$ are both Murray–von Neumann equivalent to $1_{\mathcal{M}(C(X)\otimes\mathcal{K})}$ in $\mathcal{M}(C(X_N) \otimes \mathcal{K})$ (note that $1_{\Phi_N(\mathcal{M}(C(X)\otimes\mathcal{K}))} = 1_{\mathcal{M}(C(X)\otimes\mathcal{K})}$). Hence, let $W_{M,\delta}$ be a partial isometry in $\Phi_N(\mathcal{M}(C(X_N) \otimes \mathcal{K}))$, such that $W_{M,\delta}$ has initial projection $1_{\mathcal{M}(C(X_N))} - q_1$ and range projection $1_{\mathcal{M}(C(X_N))} - q_2$. We thus get a unitary $U_{M,\delta} := v_{M,\delta} + W_{M,\delta}$ in $\Phi_N(\mathcal{M}(C(X_N) \otimes \mathcal{K}))$. And it is clear that as $M \to \infty$ and $\delta \to 0$, $U_{M,\delta}$ converges to U in the strict topology in $\mathcal{M}(C(X) \otimes \mathcal{K})$. This proves the claim.

Now suppose that $\alpha : G \to \mathrm{Aut}(Y)$ is an affine continuous action of G on a compact convex set Y. For each n, α restricts to an affine continuous action of $G_n = \Phi_n(\mathcal{U}(\mathcal{M}(C(X_n) \otimes \mathcal{K})))$ on Y. Now for each n, let $Y_n := \{y \in Y : \alpha(g)y = y, \forall g \in G_n\}$. Since each G_n is amenable, each Y_n is nonempty. Also, since $G_n \subseteq G_m$ for $n \leqslant m$, the family $\{Y_n\}_{n=1}^{\infty}$ satisfies the finite intersection property. Hence, by compactness of Y, let $y_0 \in Y$ be such that $y_0 \in \bigcap_{n=1}^{\infty} Y_n$. Hence, since the union of the G_ns is dense in G, we must have that y_0 is a fixed point of α. ∎

Inspired by the existing results on extreme amenability [8], and noting that $\mathcal{M}(\mathcal{K})$ is extremely amenable, by modifying the argument of Theorem 3.10, we get the following result (which is itself close to extreme amenability):

THEOREM 3.11. *Suppose that A is a unital* AH-*algebra, and let $G := \mathcal{U}(\mathcal{M}(A \otimes \mathcal{K}))$ be the unitary group of the multiplier algebra of $A \otimes \mathcal{K}$, given the strict topology. Then every continuous action of G, on a compact metric space, has a fixed point.*

Proof. The proof is the same as that of Theorem 3.10. The main difference is in Lemmas 3.2 and 3.5. There, we need to replace expressions like $\alpha(g_0)x - \alpha(g)x \in \mathcal{O}$ and $\alpha(g)y - y \in \mathcal{O}$ by expressions of the form $\rho(\alpha(g_0)x, \alpha(g)x) < \delta$ and $\rho(\alpha(g)y, y) < \delta$ respectively, where ρ is the metric on the compact metric space. ∎

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