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SIMPLE UNITAL C^*-ALGEBRAS WITH THE STABLE LOCALLY FINITE DIMENSIONAL PROPERTY

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ABSTRACT. The stable locally finite dimensional (SLF) property is the "opposite" of the tracial rank zero (TR0) property (defined by Lin). We conjecture that for unital separable simple C^*-algebras, SLF is equivalent to nuclearity and TR0. We prove the following:

THEOREM. *Let $\mathcal{A}$ be a unital simple separable C^*-algebra.*
(i) *If $\mathcal{A}$ is nuclear and has* TR0 *then $\mathcal{A}$ is* SLF.
(ii) *If $\mathcal{A}$ is* SLF *then $\mathcal{A}$ is nuclear, quasidiagonal, and has real rank zero, stable rank one and weakly unperforated K_0 group.*

We also show that if $\mathcal{A}$ is a unital simple separable C^*-algebra with the SLF property, then every embedding of a commutative C^*-algebra into $\mathcal{A}$ has a TR0 type property.

KEYWORDS: *Stable locally finite dimensional, quasidiagonal, nuclear, tracial rank zero, K-theory.*

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1. INTRODUCTION

In the K-theoretic classification program for simple nuclear stably finite C^*-algebras, greatest progress has been made in the case where the C^*-algebras have real rank zero (plus other properties!). Among other things, Elliott and Gong have classified all simple unital AH-algebras with bounded dimension growth and real rank zero [11]. This class of C^*-algebras is currently the class of model algebras for the real rank zero stably finite case (since it exhausts the current range of the invariant for the aforementioned case [11]) and much work has been done to show that various classes of stably finite nuclear real rank zero C^*-algebras are subclasses of this class. (For example, Elliott and Evans showed that all irrational rotation algebras are in the class of Elliott and Gong [10]. This answered a longstanding question of Effros [9].)

An interesting advance in this direction is Lin's tracial rank zero (TR0) property which is an abstract characterization of the class of Elliott and Gong.

DEFINITION 1.1. Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then $\mathcal{A}$ is said to have the *tracial rank zero (TR0)* property if for every $\varepsilon > 0$, for every finite subset $\mathcal{F} \subseteq \mathcal{A}$ and for every nonzero positive element $a \in \mathcal{A}$, there is a projection $p \in \mathcal{A}$ such that the following hold:

(i) p is Murray–von Neumann equivalent to a projection in $\mathrm{Her}(a)$ (the hereditary sub-C^*-algebra of $\mathcal{A}$ generated by a);

(ii) pf is within ε of fp for all $f \in \mathcal{F}$; and

(iii) there is a finite dimensional sub-C^*-algebra $\mathcal{D} \subseteq \mathcal{A}$ with $1_{\mathcal{D}} = 1_{\mathcal{A}} - p$ such that $(1_{\mathcal{A}} - p)f(1_{\mathcal{A}} - p)$ is within ε of an element of $\mathcal{D}$ for all $f \in \mathcal{F}$.

Lin showed that the class of Elliott and Gong (i.e., the class of all simple unital AH-algebras with bounded dimension growth and real rank zero) is exactly the class of all unital simple separable nuclear TR0 C^*-algebras that satisfy the universal coefficient theorem [16]. This abstract characterization of Elliott and Gong's class has turned out to be rather useful. (For instance, Phillips used Lin's characterization to show that every (higher dimensional) simple noncommutative torus is in the class of Elliott and Gong [20]. This solved a longstanding question. Furthermore, Lin's characterization has also been used to show that many simple C^*-algebras coming from minimal dynamical systems are in the class of Elliott and Gong [18].)

Another interesting point of view is that, under appropriate hypotheses, Lin's TR0 property is similar to Popa's characterization of quasidiagonality. A unital C^*-algebra $\mathcal{A}$ is said to be *quasidiagonal* if there exists a faithful $*$-representation $\pi : \mathcal{A} \to \mathbb{B}(\mathcal{H})$ such that $\pi(\mathcal{A})$ is a set of quasidiagonal operators in $\mathbb{B}(\mathcal{H})$ — i.e., there exists an increasing sequence $\{p_n\}_{n=1}^{\infty}$ of finite rank projections in $\mathbb{B}(\mathcal{H})$ such that:

(i) p_n converges to $1_{\mathbb{B}(\mathcal{H})}$ in the strong operator topology; and

(ii) for each $b \in \pi(\mathcal{A})$, $\|bp_n - p_n b\| \to 0$ as $n \to \infty$.

Note that statements (i) and (ii) in the previous paragraph are equivalent to saying that $\pi(\mathcal{A})$ is a set of (simultaneously) block diagonal operators in $\mathbb{B}(\mathcal{H})$, modulo the compacts. (The blocks are finite dimensional.)

Notions around quasidiagonality play an interesting role in many places (see, for example, the survey paper [5] and the references therein). Among other things, it is an open problem (in classification theory) whether for simple unital separable nuclear C^*-algebras, stable finiteness is equivalent to quasidiagonality. Note that quasidiagonal C^*-algebras are automatically stably finite (see Proposition 3.19 of [5]). A recent discussion concerning the (interesting) converse (under appropriate hypotheses) can be found in 6.6 of Brown's paper [6].

Moreover, we point out that all present examples of simple stably finite nuclear C^*-algebras that have been classified are quasidiagonal. (See, for example,

[6], [17] and [23]; we also point out that unital simple separable TR0 C^*-algebras are quasidiagonal ([5], Theorem 12.1).) Hence, it is interesting to better understand the notion of quasidiagonality (especially in the presence of nuclearity!).

Popa gave a characterization of a large class of quasidiagonal C^*-algebras.

DEFINITION 1.2. Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then $\mathcal{A}$ is said to have the *Popa* property if for every $\varepsilon > 0$ and for every finite subset $\mathcal{F} \subseteq \mathcal{A}$, there is a projection $p \in \mathcal{A}$ such that p is not equal to $1_{\mathcal{A}}$ and the following hold:

(i) pf is within ε of fp for all $f \in \mathcal{F}$; and

(ii) there is a (nonzero) finite dimensional sub-C^*-algebra $\mathcal{D} \subseteq \mathcal{A}$ with $1_{\mathcal{D}} = 1_{\mathcal{A}} - p$ such that $(1_{\mathcal{A}} - p)f(1_{\mathcal{A}} - p)$ is within ε of an element of $\mathcal{D}$ for all $f \in \mathcal{F}$.

Popa showed that a unital separable simple real rank zero C^*-algebra is quasidiagonal if-and-only-if it has the Popa property [21]. One of the reasons why Lin's TR0 property is so fascinating is because of its similarity to the Popa property. (Basically, the TR0 property is the Popa property together with the additional constraint that the projection p can be chosen to be "arbitrarily small".)

Finally, in view of the interesting difficulties with proving TR0 for nonnuclear simple quasidiagonal real rank zero C^*-algebras satisfying all the (remaining) desirable properties (see [6] and [7]), it seems timely to comment on the combination of nuclearity and quasidiagonality. In this paper, we make a modest contribution to the discussion by looking at simple concepts which contain both of these properties.

We first define a property (which we call the *weak stable locally finite dimensional* (weak SLF) property) which is, in a sense, the "dual" or "opposite" of the Popa property (Definition 2.1). It turns out that for simple unital separable C^*-algebras, the weak SLF property is equivalent to nuclearity and quasidiagonality.

Next, motivated by the weak SLF property, we define a property (which we call the *stable locally finite dimensional* (SLF) property) which is, in a sense (similar to the previous paragraph) the "dual" or "opposite" of the TR0 property of Lin. Moreover, the relationship between the SLF property and the weak SLF property is similar to the relationship between the TR0 property and the Popa property (a certain projection is allowed to be "arbitrarily small"; Definition 3.1). We conjecture that for unital simple separable C^*-algebras, the SLF property is equivalent to nuclearity and TR0. We prove the following:

THEOREM 1.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra.*

(i) *If $\mathcal{A}$ is nuclear and has* TR0, *then $\mathcal{A}$ has the* SLF *property.*

(ii) *If $\mathcal{A}$ has the* SLF *property, then $\mathcal{A}$ is nuclear, quasidiagonal, and has real rank zero, stable rank one and weakly unperforated K_0 group.*

Finally, for simple unital separable C^*-algebras $\mathcal{A}$ with the SLF property, we show that every embedding of a commutative C^*-algebra into $\mathcal{A}$ has a TR0 type property. This implies that for self-adjoint finite subsets of $\mathcal{A}$ consisting of commuting elements, the TR0 property holds.

2. THE WEAK STABLE LOCALLY FINITE DIMENSIONAL PROPERTY

Roughly speaking the *Popa* property (see Definition 1.2) says that given a finite subset, one can "remove" a possibly "big piece" (though not as big as the unit) such that the remainder is close to a (nonzero) finite dimensional C^*-algebra. The *weak stable locally finite dimensional* (*weak SLF*) property is (roughly) the opposite of the Popa property. Roughly speaking, the weak SLF property says that given a finite subset, one can "add" a possibly "big piece" to get close to a finite dimensional C^*-algebra.

To get to the precise definition of weak SLF, we first introduce the following terminology: Let $\mathcal{A}$, $\mathcal{B}$ be C^*-algebras. Let $\varepsilon > 0$ and let a finite subset $\mathcal{F} \subseteq \mathcal{A}$ be given. A linear map $\psi : \mathcal{A} \to \mathcal{B}$ is said to be *$\mathcal{F}$-ε-multiplicative* if $\|\psi(ab) - \psi(a)\psi(b)\| < \varepsilon$ for all $a, b \in \mathcal{F}$.

Here is the precise definition of weak SLF:

DEFINITION 2.1. Let $\mathcal{A}$ be a unital separable C^*-algebra. Then $\mathcal{A}$ is said to have the *weak stable locally finite dimensional property* (weak SLF) if for every $\varepsilon > 0$ and for every finite subset $\mathcal{F} \subseteq \mathcal{A}$, there are an integer n and completely positive contractive maps $\phi : \mathcal{A} \to \mathrm{M}_n(\mathcal{A})$ and $\Phi : \mathcal{A} \to \mathrm{M}_{n+1}(\mathcal{A})$ with the range of Φ contained in a finite dimensional sub-C^*-algebra $\mathcal{D}$ of $\mathrm{M}_{n+1}(\mathcal{A})$ such that:
(i) ϕ and Φ are both $\mathcal{F}$-ε-multiplicative; and
(ii) $a \oplus \phi(a)$ is within ε of $\Phi(a)$ for all $a \in \mathcal{F}$.

The next lemma allows us to assume that all the maps in Definition 2.1 bring the unit to a projection (projection depending on the map) in the codomain algebra.

LEMMA 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ containing the identity of $\mathcal{A}$. For every positive real number $\varepsilon > 0$, there exists a positive real number $\delta > 0$ such that the following holds: If $\mathcal{B}$ is a C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ is a $\mathcal{F}$-δ-multiplicative, completely positive contractive map, then there exists a $\mathcal{F}$-ε-multiplicative, completely positive contractive map $\psi : \mathcal{A} \to \mathcal{B}$ such that $\psi(1_{\mathcal{A}})$ is a projection in $\mathcal{B}$ and*

$$\|\phi(a) - \psi(a)\| < \varepsilon, \quad \forall a \in \mathcal{F}.$$

Sketch of proof. Since ϕ is almost multiplicative, $\phi(1_{\mathcal{A}})$ is close to $\phi(1_{\mathcal{A}})^2$. Hence, the spectrum of (the positive element) $\phi(1_{\mathcal{A}})$ is concentrated near 0 and 1. Hence, choose a continuous function $f \geqslant 0$ on $[0,1]$ such that $f(\phi(1_{\mathcal{A}}))\phi(1_{\mathcal{A}})$ is a projection and such that $f(\phi(1_{\mathcal{A}}))$, $\phi(1_{\mathcal{A}})$ and $f(\phi(1_{\mathcal{A}}))\phi(1_{\mathcal{A}})$ are all close to each other. Take $h := f(\phi(1_{\mathcal{A}}))^{1/2}$ and let $\psi : \mathcal{A} \to \mathcal{B}$ be given by $\psi(a) := h\phi(a)h$ for all $a \in \mathcal{A}$. ∎

For the proof of the next result, we will need the following terminology: Say that $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras. Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subseteq \mathcal{A}$ be given. We say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is *$\mathcal{F}$-ε-isometric* if $|\|\phi(a)\| - \|a\|| < \varepsilon$ for all $a \in \mathcal{F}$.

THEOREM 2.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then the following conditions are equivalent:*

(i) *$\mathcal{A}$ is nuclear and quasidiagonal.*

(ii) *$\mathcal{A}$ has the weak* SLF *property.*

Proof. We first show that (ii) implies (i). Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subseteq \mathcal{A}$ be given.

Since $\mathcal{A}$ has the weak SLF property, choose maps Φ and ϕ, and a finite dimensional sub-$C*$-algebra $\mathcal{D}$ that satisfy the statements in Definition 2.1. Now let $\psi : \mathrm{M}_{n+1}(\mathcal{A}) \to \mathcal{A}$ be the completely positive contractive map given by taking the cut down to the 1 by 1 entry. Since $\mathcal{D}$ is a sub-C^*-algebra of $\mathrm{M}_{n+1}(\mathcal{A})$, we get the restricted map $\psi : \mathcal{D} \to \mathcal{A}$ (which we also denote by "ψ"). Moreover, $\psi \circ \Phi(f)$ is within ε of f for all $f \in \mathcal{F}$.

Also, note that the map Φ is a completely positive contractive $\mathcal{F}$-ε-multiplicative map that is also $\mathcal{F}$-ε-isometric.

Since $\mathcal{F}$ and ε are arbitrary, it follows from the equivalence of nuclearity and the completely positive approximation property, and from Voiculescu's abstract characterization of quasidiagonality (see Theorem 4.2 of [22]) that $\mathcal{A}$ is nuclear and quasidiagonal.

The proof that (i) implies (ii) follows from Proposition 2 of [8] and Proposition 6.1.6 of [3]. ∎

3. NUCLEAR TR0 C^*-ALGEBRAS HAVE THE SLF PROPERTY

Just as the weak SLF property is the opposite of the Popa property, the SLF property is the opposite of the tracial rank zero (TR0) property of Lin. Recall that the TR0 property roughly says the following (see Definition 1.1): Given a finite subset, one can "remove" an arbitrarily "small piece" such that the remainder is close to a finite dimensional C^*-algebra. The "dual" or opposite property, which is the SLF property, roughly says the following: Given a finite subset, one can "add" an arbitrarily "small piece" such that the result is close to a finite dimensional C^*-algebra.

From still another point of view: The TR0 property is the Popa property with the additional constraint that the projection p (the "piece" that is "removed") is arbitrarily small. Similarly, the SLF property is the weak SLF property with the additional constraint that the projection p (the "piece" that is "added") is arbitrarily small.

DEFINITION 3.1. Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then $\mathcal{A}$ is said to have the *stable locally finite dimensional property* (SLF) if for every $\varepsilon > 0$, for any finite subset $\mathcal{F}$ of $\mathcal{A}$ and for every nonzero positive element $a \in \mathcal{A}$, there exists a projection $p \in \mathcal{A}$ and completely positive contractive maps $\phi : \mathcal{A} \to p\mathcal{A}p$ and $\Phi : \mathcal{A} \to (1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ with the range of Φ contained in a finite dimensional sub-C^*-algebra of $(1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ such that:

(i) ϕ and Φ are $\mathcal{F}$-ε-multiplicative;

(ii) $a \oplus \phi(a)$ is within ε of $\Phi(a)$ for all $a \in \mathcal{F}$; and

(iii) p is Murray–von Neumann equivalent to a projection in $\mathrm{Her}(a)$.

Once more using Lemma 2.2, all the maps in Definition 3.1 can be chosen to bring the unit to a projection (projection depending on the map) in the codomain algebra.

We also need the following perturbation result for projections, which is Lemma 2.5.1 in [15]:

LEMMA 3.2. *Let $\mathcal{D}$ be a unital C^*-algebra. Suppose that p, q are projections in $\mathcal{D}$ such that $\|p - q\| < 1$. Then there exists a unitary $u \in \mathcal{D}$ such that $upu^* = q$ and $\|u - 1_{\mathcal{D}}\| \leqslant \sqrt{2}\|p - q\|$.*

THEOREM 3.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. If $\mathcal{A}$ is nuclear and has* TR0 *then $\mathcal{A}$ has the* SLF *property.*

Proof. Since $\mathcal{A}$ has the TR0 property, it is quasidiagonal ([5], Theorem 12.1). Hence, $\mathcal{A}$ is nuclear and quasidiagonal. Hence, by Theorem 2.3, $\mathcal{A}$ has the weak SLF property.

Let $\varepsilon > 0$ and a finite subset $\mathcal{F} \subseteq \mathcal{A}$ be given. For simplicity, we may assume that all the elements of $\mathcal{F}$ have norm less than or equal to one. We may also assume that $1_{\mathcal{A}} \in \mathcal{F}$. We start with a positive number $\delta_0 > 0$. Plug $\mathcal{F}$ and δ_0 into Lemma 2.2 to get a number $\delta_0' > 0$ with $\delta_0' < \delta_0$. Let $\delta_1 > 0$ be a number such that $\delta_1 < \frac{\delta_0'}{10}$. Since $\mathcal{A}$ has the weak SLF property, plug δ_1 and $\mathcal{F}$ into Definition 2.1 to get an integer n, maps ϕ, Φ, and a finite dimensional C^*-algebra $\mathcal{D}$ (all the notation as in Definition 2.1 except that ε is replaced by δ_1).

Note that $\mathcal{D}$ is both a semiprojective C^*-algebra and an injective von Neumann algebra. Also, if $\mathcal{C}$ is a C^*-algebra and if $\rho : \mathcal{A} \to \mathcal{C}$ is a linear map that is almost multiplicative then the induced maps $\rho^{(n)} : \mathrm{M}_n(\mathcal{A}) \to \mathrm{M}_n(\mathcal{C})$ and $\rho^{(n+1)} : \mathrm{M}_{n+1}(\mathcal{A}) \to \mathrm{M}_{n+1}(\mathcal{C})$ are also almost multiplicative (though with different finite sets and estimates). Moreover, since $\phi : \mathcal{A} \to \mathrm{M}_n(\mathcal{A})$ is already $\mathcal{F}$-δ_1-multiplicative, if ρ is sufficiently multiplicative (large enough finite set and small enough positive constant for the estimate), then $\rho^{(n)} \circ \phi$ is also $\mathcal{F}$-δ_1-multiplicative. Hence, let $\mathcal{G} \subseteq \mathcal{A}$ be a finite subset with $\mathcal{F} \subseteq \mathcal{G}$ and let $\delta_2 > 0$ with $\delta_2 < \delta_1$ be such that if $\mathcal{C}$ is a unital C^*-algebra and if $\rho : \mathcal{A} \to \mathcal{C}$ is a completely positive contractive $\mathcal{G}$-δ_2-multiplicative map then:

(i.) $\rho^{(n)} \circ \phi : \mathcal{A} \to \mathrm{M}_n(\mathcal{C})$ is $\mathcal{F}$-δ_1-multiplicative;

(ii.) there exists a finite dimensional sub-C^*-algebra $\mathcal{D}' \subseteq \mathrm{M}_{n+1}(\mathcal{C})$ and there exists a completely positive contractive $\mathcal{F}$-δ_1-multiplicative map $\Phi' : \mathcal{A} \to \mathcal{D}'$

such that $\Phi'(a)$ is within δ_1 of $\rho^{(n+1)} \circ \Phi(a)$ for all $a \in \mathcal{F}$; hence, $\Phi'(a)$ is within $2\delta_1$ of $\rho(a) \oplus \rho^{(n)} \circ \phi(a)$ for all $a \in \mathcal{F}$. (Here $\mathcal{D}'$ is obtained by considering the image of $\mathcal{D}$ under $\rho^{(n+1)}$ and using semiprojectivity. Φ' is obtained by using, among other things, injectivity of $\mathcal{D}'$.)

Let us collectively denote the above conditions by "($*$)". In the above, for each m, $\rho^{(m)} : \mathrm{M}_m(\mathcal{A}) \to \mathrm{M}_m(\mathcal{C})$ is the natural completely positive contractive map induced by $\rho : \mathcal{A} \to \mathcal{C}$.

Now since $\mathcal{A}$ has the TR0 property, let q be a projection in $\mathcal{A}$ such that the following hold:

(a) $\tau(q) < \frac{\varepsilon}{10n}$ for all $\tau \in T(\mathcal{A})$;
(b) the map $\mathcal{A} \to q\mathcal{A}q$ given by $a \mapsto qaq$ is $\mathcal{G}$-δ_2-multiplicative;
(c) there is a finite dimensional sub-C^*-algebra $\mathcal{E}$ of $\mathcal{A}$ with unit $1_{\mathcal{E}} = 1_{\mathcal{A}} - q$ and there is a completely positive contractive $\mathcal{G}$-δ_2-multiplicative map $\xi : \mathcal{A} \to \mathcal{E}$ such that $(1_{\mathcal{A}} - q)a(1_{\mathcal{A}} - q)$ and a are within δ_2 of $\xi(a)$ and $qaq + \xi(a)$ respectively for all $a \in \mathcal{G}$.

We collectively denote the above conditions by "($**$)".

By statement (b) of ($**$), we can apply ($*$) to the completely positive contractive map $\rho : \mathcal{A} \to q\mathcal{A}q : a \mapsto qaq$ (take $\mathcal{C}$ to be $q\mathcal{A}q$). We thus get the following:

(1) $\rho^{(n)} \circ \phi : \mathcal{A} \to \mathrm{M}_n(q\mathcal{A}q)$ is an $\mathcal{F}$-δ_1-multiplicative, completely positive contractive map.
(2) There is a finite dimensional sub-C^*-algebra $\mathcal{D}' \subseteq \mathrm{M}_{n+1}(q\mathcal{A}q)$.
(3) There is an $\mathcal{F}$-δ_1-multiplicative, completely positive contractive map $\Phi' : \mathcal{A} \to \mathcal{D}'$ such that $\Phi'(a)$ is within $2\delta_1$ of $qaq \oplus \rho^{(n)} \circ \phi(a)$ for all $a \in \mathcal{F}$.

We collectively denote the above conditions by "($***$)".

Applying Lemma 2.2 to the maps in ($***$), we obtain the following:

(i.) There is an $\mathcal{F}$-δ_0-multiplicative, completely positive contractive map $\phi' : \mathcal{A} \to \mathrm{M}_n(q\mathcal{A}q)$ such that $\phi'(a)$ is within δ_0 of $\rho^{(n)} \circ \phi(a)$ for all $a \in \mathcal{F}$.
(ii.) There exists an $\mathcal{F}$-δ_0-multiplicative, completely positive contractive map $\Phi'' : \mathcal{A} \to \mathcal{D}'$ such that $\Phi''(a)$ is within δ_0 of $\Phi'(a)$ for all $a \in \mathcal{F}$.
(iii.) Both $\phi'(1_{\mathcal{A}})$ and $\Phi''(1_{\mathcal{A}})$ are projections.

We collectively denote the above conditions by "($****$)".

From ($****$) (i.), (ii.) and ($***$) (3), we get that $\Phi''(a)$ is within $2\delta_0 + 2\delta_1$ of $qaq \oplus \phi'(a)$ for all $a \in \mathcal{F}$. In particular, $\Phi''(1_{\mathcal{A}})$ is within $2\delta_0 + 2\delta_1$ of $q \oplus \phi'(1_{\mathcal{A}})$ (and both are, by ($****$) (iii.), projections). Hence, by Lemma 3.2, there exists a unitary $u \in \mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ (where $\mathcal{M}(\mathcal{A} \otimes \mathcal{K})$ is the multiplier algebra of the stabilization of $\mathcal{A}$) such that $\|u - 1_{\mathcal{M}(\mathcal{A}\otimes\mathcal{K})}\| \leqslant 2\sqrt{2}(\delta_0 + \delta_1)$ and $u\Phi''(1_{\mathcal{A}})u^* = q \oplus \phi'(1_{\mathcal{A}})$. Note that (since the elements of $\mathcal{F}$ all have norm less than or equal to one) $u\Phi''(a)u^*$ is within $4\sqrt{2}(\delta_0 + \delta_1)$ of $\Phi''(a)$ for all $a \in \mathcal{F}$. Hence, $u\Phi''(a)u^*$ is within $(2 + 4\sqrt{2})(\delta_0 + \delta_1)$ of $qaq \oplus \phi'(a)$ for all $a \in \mathcal{F}$. Let us denote this last statement by "(+)".

Let r_1 be the projection in $\mathcal{A} \otimes \mathcal{K}$ given by $r_1 := \phi'(1_{\mathcal{A}})$. Let $\mathcal{D}_1$ be the finite dimensional unital sub-C^*-algebra of $(1_{\mathcal{A}} \oplus r_1)(\mathcal{A} \otimes \mathcal{K})(1_{\mathcal{A}} \oplus r_1)$ that is given by

$$\mathcal{D}_1 := \mathcal{E} \oplus (q \oplus r_1)(u\mathcal{D}'u^*)(q \oplus r_1).$$

Let $\phi_1 := \phi' : \mathcal{A} \to r_1(\mathcal{A} \otimes \mathcal{K})r_1$ which is a completely positive unital $\mathcal{F}$-δ_1-multiplicative map. Finally, let $\Phi_1 : \mathcal{A} \to \mathcal{D}_1$ be given by

$$\Phi_1(a) := \xi(a) \oplus u\Phi''(a)u^*$$

for all $a \in \mathcal{A}$. Then Φ_1 is a completely positive unital $\mathcal{F}$-δ_0-multiplicative map.

By $(**)$ statement (c), for all $a \in \mathcal{F}$, $a \oplus \phi_1(a) = a \oplus \phi'(a)$ is within δ_2 of $\xi(a) \oplus qaq \oplus \phi'(a)$. By (+), the latter is within $(2 + 4\sqrt{2})(\delta_0 + \delta_1)$ of $\xi(a) \oplus u\Phi''(a)u^* = \Phi_1(a)$. Hence, $a \oplus \phi_1(a)$ is within $\delta_2 + (2 + 4\sqrt{2})(\delta_0 + \delta_1) < 21\delta_0$ of $\Phi_1(a)$ for all $a \in \mathcal{F}$. We denote this last statement by "(++)".

For a unital trace τ on $\mathcal{A}$, let "τ" also denote the extension to a semifinite trace on $\mathcal{A} \otimes \mathcal{K}$. Recall that $r_1 = \phi'(1_{\mathcal{A}}) \in \mathrm{M}_n(q\mathcal{A}q)$. Hence, by $(**)$ statement (a), we must have that $\tau(r_1) < \frac{\varepsilon}{10}$ for all $\tau \in T(\mathcal{A})$. From this and (++), if we choose $\delta_0 > 0$ to be such that $21\delta_0 < \varepsilon$, then we would almost have the required result except that statement (iii) in Definition 3.1 is replaced by the statement that the projection is small in trace. But unital simple separable TR0 C^*-algebras have both real rank zero and comparison of projections. Hence, we have that $\mathcal{A}$ has the SLF property as required. ∎

4. C^*-ALGEBRAS WITH THE SLF PROPERTY

In this section, we show that unital simple separable SLF C^*-algebras have stable rank one, real rank zero and weakly unperforated K_0 group. It is clear from the definition that every unital separable simple C^*-algebra $\mathcal{A}$ with the stable local finite dimensional property has the (SP) property, i.e. every nonzero hereditary sub-C^*-algebra of $\mathcal{A}$ has a nonzero projection.

We first show that (under appropriate hypotheses) SLF is preserved by hereditary subalgebras.

LEMMA 4.1. *Suppose that $\mathcal{A}$ is a unital simple separable C^*-algebra with the* SLF *property. Then every unital hereditary sub-C^*-algebra of $\mathcal{A}$ has the* SLF *property.*

Proof. We will only give a sketch of the proof. Suppose e is a nonzero projection of $\mathcal{A}$. Let $\mathcal{F}$ be any finite subset of $e\mathcal{A}e$ and let a be any nonzero positive element of $e\mathcal{A}e$. Let $\mathcal{G}$ be the union of $\mathcal{F}$ and $\{e\}$. Then, there exist a projection p such that p is Murray–von Neumann equivalent to a projection in $\mathrm{Her}(a)$ and almost multiplicative (on $\mathcal{G}$) completely positive contractive maps $\varphi : \mathcal{A} \to p\mathcal{A}p$ and $\Phi : \mathcal{A} \to (1_A \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ such that $\mathrm{id} \oplus \varphi$ is close to Φ on $\mathcal{G}$ and the range of Φ is contained in a finite dimensional sub-C^*-algebra of $(1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$.

Restricting φ and Φ to $e\mathcal{A}e$ and using Lemma 2.2, we get a projection p_0 in $e\mathcal{A}e$ with p_0 Murray–von Neumann equivalent to a projection in $\mathrm{Her}(a)$ and completely positive contractive maps $\varphi_0 : e\mathcal{A}e \to p_0\mathcal{A}p_0$ and $\Phi_0 : e\mathcal{A}e \to (e \oplus p_0)\mathrm{M}_2(e\mathcal{A}e)(e \oplus p_0)$ which are close to φ and Φ. Hence, $y \oplus \varphi_0(y)$ is close to $\Phi_0(y)$ for all y in $\mathcal{F}$ and φ_0 and Φ_0 are almost multiplicative on $\mathcal{F}$. ∎

If p is a projection in a C^*-algebra $\mathcal{A}$ and n is a positive integer, then np will denote the projection $\bigoplus_{k=1}^{n} p$ in $\mathrm{M}_n(\mathcal{A})$. The next lemma shows that SLF is a "stable property" for C^*-algebras.

LEMMA 4.2. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra. Then $\mathcal{A}$ has the* SLF *property if and only if for all positive integers n, the C^*-algebra of n by n matrices with entries in $\mathcal{A}$ has the* SLF *property.*

Proof. The only if direction is trivial: Just take $n = 1$. We now prove the if direction. Suppose that $\mathcal{A}$ has the SLF property. Let $\mathcal{F}$ be a finite subset of $\mathrm{M}_n(\mathcal{A})$, let a be a positive element of $\mathrm{M}_n(\mathcal{A})$, and let ε be a positive number. Since $\mathcal{A}$ is a simple C^*-algebra with the (SP) property, there exists a nonzero projection e in $\mathcal{A}$ such that ne is Murray–von Neumann equivalent to a projection in $\mathrm{Her}(a)$. (See, for example, Definition 3.5.2, Proposition 3.5.3 and Lemma 3.5.7 of [15].)

Let $\mathcal{G}$ be a finite subset of $\mathcal{A}$ such that every entry of elements in $\mathcal{F}$ is in $\mathcal{G}$. Since $\mathcal{A}$ has the SLF property, there exist a projection p in $\mathcal{A}$ and $\mathcal{G}$-$\frac{\varepsilon}{n^2}$-multiplicative completely positive contractive maps $\varphi : \mathcal{A} \to p\mathcal{A}p$ and $\Phi : \mathcal{A} \to (1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ such that p is Murray–von Neumann equivalent to a subprojection of e, the range of Φ is contained in a finite dimensional sub-C^*-algebra of $(1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{A} \oplus p)$, and

$$\|x \oplus \varphi(x) - \Phi(x)\| < \frac{\varepsilon}{n^2}$$

for all x in $\mathcal{G}$.

Let v be a partial isometry in $\mathrm{M}_{2n}(\mathcal{A})$ such that $v^*v = n(1_{\mathcal{A}} \oplus p)$ and $vv^* = (n1_{\mathcal{A}}) \oplus (np)$. Define $\psi : \mathrm{M}_n(\mathcal{A}) \to (np)\mathrm{M}_n(\mathcal{A})(np)$ by $\psi = \varphi^{(n)}$ and define $\Psi : M_n(\mathcal{A}) \to [(n1_{\mathcal{A}}) \oplus (np)]\mathrm{M}_{2n}(\mathcal{A})[(n1_{\mathcal{A}}) \oplus (np)]$ by $\mathrm{Ad}(v) \circ \Phi^{(n)}$. It is easy to check that ψ and Ψ are $\mathcal{F}$-ε-multiplicative completely positive contractive maps with the range of Ψ contained in a finite dimensional sub-C^*-algebra of $[(n1_{\mathcal{A}}) \oplus (np)]\mathrm{M}_{2n}(\mathcal{A})[(n1_{\mathcal{A}}) \oplus (np)]$ and

$$\|x \oplus \psi(x) - \Psi(x)\| < \varepsilon$$

for all x in $\mathcal{F}$. Since p is Murray–von Neumann equivalent to a subprojection of e and ne is Murray–von Neumann equivalent to a subprojection of a, we have that np is Murray–von Neumann equivalent to a subprojection of a. ∎

THEOREM 4.3. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra with the* SLF *property. Then $\mathcal{A}$ has real rank zero, i.e. the self-adjoint invertible elements in $\mathcal{A}$ are dense in the self-adjoint elements in $\mathcal{A}$.*

Proof. By Theorem 2.6 in [4], to show that $\mathcal{A}$ has real rank zero, it suffices to prove the following (equivalent statement): for every positive number $\varepsilon > 0$ and for all positive orthogonal elements a and b of $\mathcal{A}$, there exists a projection e in $\mathcal{A}$ such that

$$\|(1-e)b\| < \varepsilon \quad \text{and} \quad \|ea\| < \varepsilon.$$

Let us assume that $\varepsilon < 1$. Suppose that a and b are positive orthogonal elements of $\mathcal{A}$. The case a or b is zero is trivial. So we may assume a and b are nonzero elements of $\mathcal{A}$ with norm less than or equal to one. In particular, zero is in the spectrum of b (else if b is invertible then $a = 0$).

First assume that zero is an isolated point of the spectrum of b. In this case, $e = \chi_{(0,\|b\|]}(b)$ is a projection such that $eb = b$ and $ea = 0$, where $\chi_{(0,\|b\|]}$ is the characteristic function of $(0, \|b\|]$. Now assume that zero is not an isolated point of the spectrum of b. Therefore, there exists a positive number δ such that $\delta < \frac{3\varepsilon}{20}$ and any interval in $[0, \delta)$ with left endpoint zero has nonempty intersection with the spectrum of b.

Define g_1, g_2, g_3 in $C([0,1])$ (all of which take the value zero at zero) as follows:

$$g_1(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \frac{\delta}{2}, \\ \text{linear} & \text{if } \frac{\delta}{2} \leqslant t \leqslant \delta, \\ t & \text{otherwise;} \end{cases} \qquad g_2(t) = \begin{cases} \text{linear} & \text{if } 0 \leqslant t \leqslant \frac{\delta}{8}, \\ 1 & \text{if } \frac{\delta}{8} \leqslant t \leqslant \frac{\delta}{4}, \\ \text{linear} & \text{if } \frac{\delta}{4} \leqslant t \leqslant \frac{\delta}{2}, \\ 0 & \text{otherwise;} \end{cases}$$

$$g_3(t) = \begin{cases} 0 & \text{if } 0 \leqslant t \leqslant \frac{\delta}{8}, \\ \text{linear} & \text{if } \frac{\delta}{8} \leqslant t \leqslant \frac{\delta}{6}, \\ 1 & \text{if } \frac{\delta}{6} \leqslant t \leqslant \frac{5\delta}{24}, \\ \text{linear} & \text{if } \frac{5\delta}{24} \leqslant t \leqslant \frac{\delta}{4}, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $g_2 g_3 = g_3$, $g_2 g_1 = 0$, and g_1, g_2, g_3 are nonzero continuous functions on the spectrum of b. Set $b_i = g_i(b)$. Then b_1, b_2, b_3 are nonzero positive elements of A, $b_2 b_3 = b_3$, and $b_2 b_1 = 0$. Also, $\|b - b_1\| < \delta < \frac{3\varepsilon}{20}$.

Let $\mathcal{F} = \{b_1, a^{1/2}, b_2^{1/2}\}$. Since $\mathcal{A}$ has the SLF property, there exist a projection p in $\mathcal{A}$, a completely positive unital map $\varphi : \mathcal{A} \to p\mathcal{A}p$, and a completely positive unital map $\Phi : \mathcal{A} \to (1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ with range contained in a finite dimensional sub-C*-algebra $\mathcal{D}$ of $(1_{\mathcal{A}} \oplus p)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus p)$ such that:

(1) φ and Φ are $\mathcal{F}$-ε-multiplicative;

(2) $\|x \oplus \varphi(x) - \Phi(x)\| < \left(\frac{\varepsilon}{20\sqrt{2}}\right)^2$ for all x in $\mathcal{F}$; and

(3) p is Murray–von Neumann equivalent to a projection q in $\mathrm{Her}(b_3)$.

Since b_1 is orthogonal to $a^{1/2}$ and $b_2^{1/2}$, it is easy to check that

$$\|\Phi(b_1)\Phi(a^{1/2} + b_2^{1/2})\| < \left(\frac{\varepsilon}{20}\right)^2.$$

Since $\mathcal{D}$ is a finite dimensional C^*-algebra, $\mathcal{D}$ has real rank zero. Hence, by Theorem 2.6 in [4], there exists a projection e_0 in $\mathcal{D}$ such that

$$\|e_0\Phi(b_1) - \Phi(b_1)\| < \frac{\varepsilon}{20} \quad \text{and} \quad \|e_0\Phi(a^{1/2} + b_2^{1/2})\| < \frac{\varepsilon}{20}.$$

An easy computation shows that

$$\begin{aligned}\|e_0(b_1 \oplus \varphi(b_1)) - (b_1 \oplus \varphi(b_1))\| &< \frac{\varepsilon}{20} + 2\Big(\frac{\varepsilon}{20\sqrt{2}}\Big)^2 < \frac{\varepsilon}{10},\\ \|e_0[(a^{1/2} + b_2^{1/2}) \oplus \varphi(a^{1/2} + b_2^{1/2})]\| &< \frac{\varepsilon}{20} + 2\Big(\frac{\varepsilon}{20\sqrt{2}}\Big)^2 < \frac{\varepsilon}{10}.\end{aligned}$$

Therefore, we have that

$$\|e_0(b_1 \oplus 0) - b_1 \oplus 0\| < \frac{\varepsilon}{10}, \quad \|e_0(a \oplus 0)\| < \frac{\varepsilon}{10}, \quad \text{and} \quad \|e_0(b_2 \oplus 0)\| < \frac{\varepsilon}{10}.$$

Since p is Murray–von Neumann equivalent to a projection q in $\mathrm{Her}(b_3)$, there exists a partial isometry v in $\mathcal{A}$ such that $v^*v = p$ and $vv^* = q$. Set $w = \begin{pmatrix} 1_{\mathcal{A}} - q & v \\ 0 & 0 \end{pmatrix}$. Then

$$w^*w = (1_{\mathcal{A}} - q) \oplus p \quad \text{and} \quad ww^* = 1_{\mathcal{A}} \oplus 0.$$

Choose a positive element b_3xb_3 in $\mathrm{Her}(b_3)$ such that $\|b_3xb_3\| \leqslant 2$ and $\|q - b_3xb_3\| < \frac{\varepsilon}{10}$. Then

$$\begin{aligned}\|w^*we_0 - e_0\| = \|(q \oplus 0)e_0\| &\leqslant \|q \oplus 0 - (b_3xb_3) \oplus 0\| + \|b_3xb_3\|\|(b_2 \oplus 0)e_0\|\\ &< \frac{\varepsilon}{10} + \frac{2\varepsilon}{10} = \frac{3\varepsilon}{10}.\end{aligned}$$

Also, $\|e_0w^*w - e_0\| < \frac{3\varepsilon}{10}$. Therefore,

$$\|(we_0w^*)^2 - we_0w^*\| < \frac{3\varepsilon}{10}.$$

By Lemma 2.5.5 in [15], there exists a projection e in the sub-C^*-algebra generated by we_0w^* such that $\|e - we_0w^*\| < 2\|e_0w^*w - e_0\| < \frac{3\varepsilon}{5}$. Note that e is in $(1_{\mathcal{A}} \oplus 0)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus 0) = \mathcal{A}$.

Since $v^*(1_{\mathcal{A}} - q) = 0$, $qb_1 = 0$, and $qa = 0$, it is easy to check that $w^*[(1_{\mathcal{A}} - q) \oplus 0] = (1_{\mathcal{A}} - q) \oplus 0$, $w[(1_{\mathcal{A}} - q) \oplus 0] = (1_{\mathcal{A}} - q) \oplus 0$, $(1_{\mathcal{A}} - q)b_1 = b_1$, and $(1_{\mathcal{A}} - q)a = a$. Therefore,

$$\begin{aligned}\|e(a \oplus 0)\| &\leqslant \|e - we_0w^*\| + \|we_0w^*[(1_{\mathcal{A}} - q) \oplus 0](a \oplus 0)\| < \frac{3\varepsilon}{5} + \|we_0(a \oplus 0)\|\\ &< \frac{7\varepsilon}{10} < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\|e(b\oplus 0)-b\oplus 0\| &\leqslant 2\|b-b_1\|+\|e(b_1\oplus 0)-(b_1\oplus 0)\|\\ &< \frac{3\varepsilon}{10}+\|(e-we_0w^*)\|+\|we_0w^*(b_1\oplus 0)-(b_1\oplus 0)\|\\ &< \frac{9\varepsilon}{10}+\|we_0(b_1\oplus 0)-w(b_1\oplus 0)\|<\varepsilon.\end{aligned}$$

Hence, $\mathcal{A}$ has real rank zero. ∎

PROPOSITION 4.4. *If $\mathcal{A}$ is a unital separable simple C^*-algebra with the* SLF *property, then $\mathcal{A}$ is a stably finite C^*-algebra and the tracial state space of $\mathcal{A}$ is nonempty.*

Proof. Since $\mathcal{A}$ has the SLF property, it has the weak SLF property. Hence, by Theorem 2.3, $\mathcal{A}$ is nuclear and quasidiagonal. Hence, $\mathcal{A}$ is stably finite and has a tracial state. ∎

LEMMA 4.5. *Let $\mathcal{A}$ be a unital C^*-algebra. Let p and q be projections in $\mathrm{M}_k(\mathcal{A})$ and let e be a projection in $\mathrm{M}_\ell(\mathcal{A})$ such that $np\oplus e$ is Murray-von Neumann equivalent to a sub-projection of $nq\oplus e$. Then for any positive number ε, there exist a finite subset $\mathcal{F}$ of $\mathcal{A}$ and a positive number δ such that the following holds: If $\psi:\mathcal{A}\to\mathcal{B}$ is a $\mathcal{F}$-δ-multiplicative completely positive contractive map, then there exist projections p_0 and q_0 in $\mathrm{M}_k(\mathcal{B})$ and a projection e_0 in $\mathrm{M}_\ell(\mathcal{B})$ such that:*
(i) $\|p_0-\psi^{(k)}(p)\|<\varepsilon$;
(ii) $\|q_0-\psi^{(k)}(q)\|<\varepsilon$;
(ii) $\|e_0-\psi^{(\ell)}(e)\|<\varepsilon$; *and*
(iii) *$np_0\oplus e_0$ is Murray–von Neumann equivalent to a sub-projection of $nq_0\oplus e_0$,*
where $\psi^{(k)}$ is the map from $\mathrm{M}_k(\mathcal{A})$ to $\mathrm{M}_k(\mathcal{B})$ defined by $\psi^{(k)}(\{a_{ij}\})=\{\psi(a_{ij})\}$.

Proof. Let ε be a positive number less than $\frac{1}{12}$ and let δ be a positive number less than $\min\left\{\frac{\varepsilon}{2(nk\ell)^2},\frac{1}{12(nk\ell)^2}\right\}$. By assumption, there exists a partial isometry v in $\mathrm{M}_{nk+\ell}(\mathcal{A})$ such that $v^*v=np\oplus e$ and $vv^*\leqslant nq\oplus e$. Let $\mathcal{F}$ be a finite subset of $\mathcal{A}$ such that every entry of p, q, e, and v is in $\mathcal{F}$. Therefore,

(1) $\|\psi^{(k)}(p)^2-\psi^{(k)}(p)\|<\frac{\varepsilon}{2}$;
(2) $\|\psi^{(k)}(q)^2-\psi^{(k)}(q)\|<\frac{\varepsilon}{2}$;
(3) $\|\psi^{(\ell)}(e)^2-\psi^{(\ell)}(e)\|<\frac{\varepsilon}{2}$;
(4) $\|\psi^{(kn+\ell)}(v)^*\psi^{(kn+\ell)}(v)-\psi^{(kn+\ell)}(v^*v)\|<\frac{1}{12}$;
(5) $\|\psi^{(kn+\ell)}(v)\psi^{(kn+\ell)}(v)^*-\psi^{(kn+\ell)}(vv^*)\|<\frac{1}{12}$;
(6) $\|\psi^{(kn+\ell)}(vv^*)\psi^{(kn+\ell)}(nq\oplus e)-\psi^{(kn+\ell)}(vv^*)\|<\frac{1}{12}$;
(7) $\|\psi^{(kn+\ell)}(vv^*)^2-\psi^{(kn+\ell)}(vv^*)\|<\frac{1}{12}$.

By Lemma 2.5.5 in [15], there exist projections p_0, q_0 in $\mathrm{M}_k(\mathcal{B})$, e_0 in $\mathrm{M}_\ell(\mathcal{B})$, and r in $\mathrm{M}_{nk+\ell}(\mathcal{B})$ such that:

(1) $\|p_0-\psi^{(k)}(p)\|<\varepsilon$ and $\|q_0-\psi^{(k)}(q)\|<\varepsilon$;
(2) $\|e_0-\psi^{(\ell)}(e)\|<\varepsilon$ and $\|r-\psi^{(kn+\ell)}(vv^*)\|<\frac{2}{12}$.

Set $y = r\psi^{(kn+\ell)}(v)(np_0 \oplus e_0)$. Then an easy computation shows that

$$\|y^*y - np_0 \oplus e_0\| < 1 \quad \text{and} \quad \|yy^* - r\| < 1.$$

Thus, $np_0 \oplus e_0$ is Murray–von Neumann equivalent to r.

Another easy computation shows that

$$\|r(nq_0 \oplus e_0) - r\| < 1.$$

Hence, by Lemma 2.5.2 of [15], r is Murray–von Neumann equivalent to a subprojection of $nq_0 \oplus e_0$. ∎

THEOREM 4.6. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra with the* SLF *property. For every positive integer k and for all projections p and q in* $\mathrm{M}_k(\mathcal{A})$, *if* $\tau(p) < \tau(q)$ *for all tracial states of $\mathcal{A}$, then p is Murray–von Neumann equivalent to a subprojection of q.*

Consequently, $K_0(\mathcal{A})$ is weakly unperforated, i.e. if $nx > 0$ for some positive integer n, then $x \geqslant 0$ in $K_0(\mathcal{A})$.

Proof. If p is zero, then we are done. Therefore, we may assume p is a nonzero projection. Let $\alpha = \inf\{\tau(q) - \tau(p) : \tau \in T(\mathcal{A})\}$. Since $T(\mathcal{A})$ is compact, α is a nonzero positive number. Since $\mathrm{M}_k(\mathcal{A})$ has real rank zero (cf. Theorem 4.3), there exists a nonzero projection r in $q\mathrm{M}_k(\mathcal{A})q$ such that $\tau(r) < \alpha$ for all τ in $T(\mathcal{A})$. Thus $\tau(p) < \tau(q - r)$ for all τ in $T(\mathcal{A})$. We now use a result of Goodearl and Handelman (Corollary 4.3 in [14]) and Theorem 4.1 to get positive integers n, ℓ and a projection e in $\mathrm{M}_\ell(\mathcal{A})$ such that $np \oplus e$ is Murray–von Neumann equivalent to a subprojection of $n(q - r) \oplus e$.

Let δ be the positive number and let $\mathcal{F}$ be the finite subset of $\mathcal{A}$ provided by Lemma 4.5 corresponding to $p, q - r, e$, and $\varepsilon = \frac{1}{3}$. Let δ_0 be a positive number less than $\min\{\delta, \frac{1}{36k^2}\}$ and let r_0 be a nonzero projection in $\mathcal{A}$ such that kr_0 is Murray–von Neumann equivalent to a subprojection of r. Since $\mathcal{A}$ has the SLF property, there exist unital $\mathcal{F}$-δ_0-multiplicative completely positive contractive maps

(1) $\varphi : \mathcal{A} \to r_0 A r_0$,
(2) $\Phi : \mathcal{A} \to (1_\mathcal{A} \oplus r_0)\mathrm{M}_2(\mathcal{A})(1_\mathcal{A} \oplus r_0)$,

such that the range of Φ is contained in a finite dimensional sub-C^*-algebra $\mathcal{D}$ of $(1_\mathcal{A} \oplus r_0)\mathrm{M}_2(\mathcal{A})(1_\mathcal{A} \oplus r_0)$ and

$$\|x \oplus \varphi(x) - \Phi(x)\| < \delta_0$$

for all x in $\mathcal{F}$.

By Lemma 4.5, there exist projections p_0, q_0 in $\mathrm{M}_k(\mathcal{D})$ and e_0 in $\mathrm{M}_\ell(\mathcal{D})$ such that

(1) $\|p_0 - \Phi^{(k)}(p)\| < \frac{1}{3}$,
(2) $\|q_0 - \Phi^{(k)}(q - r)\| < \frac{1}{3}$,
(3) $\|e_0 - \Phi^{(\ell)}(e)\| < \frac{1}{3}$,

such that $np_0 \oplus e_0$ is Murray–von Neumann equivalent to a subprojection of $nq_0 \oplus e_0$. Since $\mathcal{D}$ is a finite dimensional C^*-algebra, p_0 is Murray–von Neumann equivalent to a subprojection of q_0.

Since $\|\varphi^{(k)}(p)^2 - \varphi^{(k)}(p)\| < \frac{1}{6}$ and $\|\varphi^{(k)}(q-r)^2 - \varphi^{(k)}(q-r)\| < \frac{1}{6}$, there exist projections p_1 and q_1 in $(kr_0)\mathrm{M}_k(\mathcal{A})(kr_0)$ such that

$$\|p_1 - \varphi^{(k)}(p)\| < \frac{1}{3} \quad \text{and} \quad \|q_1 - \varphi^{(k)}(q-r)\| < \frac{1}{3}.$$

Note that

$$\|p\oplus p_1 - p_0\| \leqslant \|p\oplus p_1 - p\oplus\varphi^{(k)}(p)\| + \|p\oplus\varphi^{(k)}(p) - \Phi^{(k)}(p)\| + \|\Phi^{(k)}(p) - p_0\| < 1.$$

Similarly, $\|(q-r)\oplus q_1 - q_0\| < 1$. Hence, $p \oplus p_1$ is Murray–von Neumann equivalent to p_0 and $(q-r)\oplus q_1$ is Murray–von Neumann equivalent to q_0.

Since kr_0 is Murray–von Neumann equivalent to a subprojection of r, and since q_1 is in $(kr_0)\mathrm{M}_k(\mathcal{A})(kr_0)$, we have that q_1 is Murray–von Neumann equivalent to a subprojection of r. Therefore, $(q-r)\oplus q_1$ is Murray–von Neumann equivalent to a subprojection of $(q-r)\oplus r$. Since $(q-r)\oplus r$ is Murray–von Neumann equivalent to q and p is Murray–von Neumann equivalent to a subprojection of $p\oplus p_1$, we have that p is Murray–von Neumann equivalent to a subprojection of q. ∎

The proof of the next theorem is contained in the proof of Theorem 4.3.12 of [1]. We will include the proof for the convenience of the reader.

THEOREM 4.7. *Let $\mathcal{A}$ be a unital simple separable stably finite C^*-algebra with real rank zero. Suppose for all k and for all projections p and q in $\mathrm{M}_k(\mathcal{A})$, we have that p is Murray–von Neumann equivalent to a subprojection of q if $\tau(p) < \tau(q)$ for all quasi-traces τ of $\mathcal{A}$. Then every unital hereditary sub-C^*-algebra of $\mathcal{A}$ has stable rank less than or equal to two and $K_0(\mathcal{A})$ is weakly unperforated.*

Proof. Suppose $n([q]-[p]) > 0$ for some positive integer n. Then $\tau(q) > \tau(p)$ for all quasi-trace τ of $\mathcal{A}$. Hence, p is Murray–von Neumann equivalent to a subprojection of q. Therefore, $[q]-[p] \geqslant 0$ in $K_0(\mathcal{A})$.

We will now show that every unital hereditary sub-C^*-algebra of $\mathcal{A}$ has stable rank less than or equal to two. Since the properties of $\mathcal{A}$ in the hypothesis are all inherited by unital hereditary sub-C^*-algebras of $\mathcal{A}$ (by the definitions of those properties), it is enough to show that the stable rank of $\mathcal{A}$ is less than or equal to two.

Let a_1 and a_2 be elements of $\mathcal{A}$ and let ε be a positive number. Let

$$a = \begin{pmatrix} a_1 & 0 \\ a_2 & 0 \end{pmatrix}.$$

Since the real rank of $\mathcal{A}$ is zero, by Proposition 4.3.4 in [1], there exist a projection p in $\mathrm{M}_2(\mathcal{A})$ and

$$b = \begin{pmatrix} b_1 & 0 \\ b_2 & 0 \end{pmatrix}$$

such that $bp = b$, b^*b is invertible in $p\mathrm{M}_2(\mathcal{A})p$, and

$$\|a - b\| < \frac{\varepsilon}{2}.$$

Let $u = b(b^*b)^{-1/2}$. Then $b = u(b^*b)^{1/2}$ and $u^*u = p$.

Note that b^*b is an element of $(1_{\mathcal{A}} \oplus 0)\mathrm{M}_2(\mathcal{A})(1_{\mathcal{A}} \oplus 0)$. Thus, $p = \mathrm{diag}(p_0, 0)$ for some projection p_0 in $\mathcal{A}$. Since $\tau(p) = \tau(uu^*)$ and $\tau((1_{\mathcal{A}} - p_0) \oplus 0) < \tau(1_{\mathrm{M}_2(\mathcal{A})} - uu^*)$ for all quasi-traces τ of $\mathcal{A}$, there exists a partial isometry $v = \begin{pmatrix} v_1 & 0 \\ v_2 & 0 \end{pmatrix}$ such that

$$v^*v = p \quad \text{and} \quad vv^* \leqslant 1_{\mathrm{M}_2(\mathcal{A})} - uu^*.$$

Set $y_i = b_i + \frac{\varepsilon}{2}v_i$. Then $y_1^*y_1 + y_2^*y_2$ is invertible in $\mathcal{A}$ and $\|y_i - a_i\| < \varepsilon$. ∎

COROLLARY 4.8. *Let $\mathcal{A}$ be a unital separable simple C^*-algebra with the* SLF *property. Then the stable rank of $\mathcal{A}$ is one.*

Proof. By Theorem 4.3, Proposition 4.4, and Theorem 4.6, $\mathcal{A}$ satisfies all the assumption in Theorem 4.7. Therefore, by Theorem 4.2.2 in [1], $\mathcal{A}$ has cancellation of projections. By Corollary 4.3.7 in [1], $\mathcal{A}$ has stable rank one. ∎

5. COMMUTATIVE SUB-C^*-ALGEBRAS OF SLF C^*-ALGEBRAS

In this section, we will show that separable simple unital SLF C^*-algebras have the TR0 property for commutative sub-C^*-algebras; in other words, we will show that any commutative sub-C^*-algebra of a separable simple unital SLF C^*-algebra can be tracially approximated by finite dimensional sub-C^*-algebras.

The following is Lemma 6.2.7 of [15].

LEMMA 5.1 (Lin). *Let X be a compact metric space. For any finite subset $\mathcal{F} \subset C(X)$ and any $\varepsilon > 0$, there exist a finite subset $\mathcal{G} \subset C(X)$ and $\delta > 0$ such that for any full matrix algebra $\mathcal{D}$ and any completely positive unital map $\phi : C(X) \to \mathcal{D}$ which is $\mathcal{G}$-δ-multiplicative, there exist a projection $p \in \mathcal{D}$, a completely positive unital map $\psi : C(X) \to p\mathcal{D}p$ which is $\mathcal{F}$-δ-multiplicative, and a unital $*$-homomorphism $h : C(X) \to (1-p)\mathcal{D}(1-p)$ such that*

$$\|\phi(f) - (\psi \oplus h)(f)\| < \varepsilon \quad \textit{for any } f \in \mathcal{F},$$

and $\mathrm{tr}(p) < \varepsilon$ *where* tr *is the canonical normalized trace of $\mathcal{D}$.*

The next lemma follows from Lemma 2.9 of [11] and [13] (see also [8]):

LEMMA 5.2. *Let X be a connected finite CW complex. For any $\varepsilon > 0$, and any finite subset $\mathcal{F} \subseteq C(X)$, there exist integers N and n, a unital $*$-homomorphism $\psi : C(X) \to \mathrm{M}_N(C(X))$, $x_i \in X$, and pairwise orthogonal projections $p_i \in \mathrm{M}_{N+1}(C(X))$, $i = 1, \ldots, n$ such that, for any $f \in \mathcal{F}$,*

$$\left\| (\mathrm{id} \oplus \psi)(f) - \sum_{i=1}^{n} f(x_i) p_i \right\| < \varepsilon.$$

Finally, we need the following perturbation result which can be found in Lemma 14.1.5 and Theorem 14.2.2 of [19]:

LEMMA 5.3. *Let $\mathcal{D}$ be a finite dimensional C^*-algebra. Then for every $\varepsilon > 0$ and for every finite subset $\mathcal{F} \subset \mathcal{D}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset \mathcal{D}$ such that the following holds:*

If $\mathcal{A}$ is a unital C^*-algebra and if $\phi : \mathcal{D} \to \mathcal{A}$ is a completely positive unital $\mathcal{G}$-δ-multiplicative map, then there exists a unital $*$-homomorphism $\psi : \mathcal{D} \to \mathcal{A}$ such that, for all $a \in \mathcal{F}$,

$$\|\phi(a) - \psi(a)\| < \varepsilon.$$

THEOREM 5.4. *Let $\mathcal{A}$ be a separable simple unital SLF C^*-algebra and let X be a finite CW complex. Let ϕ be a unital injective $*$-homomorphism from $C(X)$ to $\mathcal{A}$. For any finite subset $\mathcal{F} \subset C(X)$ and any $\varepsilon > 0$, there exists a commutative finite dimensional sub-C^*-algebra $\mathcal{D}$ of $\mathcal{A}$ such that for any $f \in \mathcal{F}$, one has that:*

(i) $\|1_{\mathcal{D}}\phi(f) - \phi(f)1_{\mathcal{D}}\| < \varepsilon$;

(ii) $1_{\mathcal{D}}\phi(f)1_{\mathcal{D}}$ *is within ε of an element of $\mathcal{D}$; and*

(iii) $\tau(1_{\mathcal{A}} - 1_{\mathcal{D}}) < \varepsilon$ *for any tracial state τ of $\mathcal{A}$.*

Proof. Since ϕ is injective, we may consider $C(X)$ as a unital sub-C^*-algebra of $\mathcal{A}$. We may also assume that $\mathcal{F}$ is in the unit ball of $C(X)$ and that $\mathcal{F}$ contains the unit $1_{C(X)}$.

Plug $\frac{\varepsilon}{100}$ and $\mathcal{F}$ into Lemma 5.2 to get integers N, m, a unital $*$-homomorphism $\psi' : C(X) \to \mathrm{M}_N(C(X))$, points $y_j \in X$ and pairwise orthogonal projections $q'_j \in \mathrm{M}_{N+1}(C(X))$, $j = 1, 2, \ldots, m$. Let $\mathcal{D}_{00}$ be the commutative sub-C^*-algebra of $\mathrm{M}_{N+1}(C(X))$ that is generated by the q'_js. Also, let $\Psi : C(X) \to \mathcal{D}_{00}$ be the unital $*$-homomorphism given by $\Psi : f \mapsto \sum_{j=1}^{m} f(y_j) q'_j$.

Now plug $\frac{\varepsilon}{100}$, $\mathcal{D}_{00}$ and the set $\{q'_j : 1 \leqslant j \leqslant m\} \cup \left\{ \sum_{j=1}^{m} f(y_j) q'_j : f \in \mathcal{F} \right\}$ into Lemma 5.3 to get a finite subset $\mathcal{G}'_{00} \subset \mathcal{D}_{00} \subset \mathrm{M}_{N+1}(C(X))$ and a positive real number $\delta'_{00} > 0$. Also, let $\mathcal{G}_{00}$ be a finite subset of $C(X)$ and let $\delta_{00} > 0$ be a positive real number such that for any unital C^*-algebra $\mathcal{C}$, if $\gamma : C(X) \to \mathcal{C}$ is a completely positive unital $\mathcal{G}_{00}$-δ_{00}-multiplicative map then the induced map $\gamma^{(N+1)} = \gamma \otimes \mathrm{id} : \mathrm{M}_{N+1}(C(X)) \to \mathrm{M}_{N+1}(\mathcal{C})$ is $\mathcal{G}'_{00}$-δ'_{00}-multiplicative. Contracting δ_{00} if necessary, we may assume that the elements of $\mathcal{G}_{00}$ all have norm less than or equal to one and that $\delta_{00} < \frac{\varepsilon}{10}$.

Since $\mathcal{F}$ is a finite subset of $C(X)$, there exists $\delta_0 > 0$ such that $|f(x) - f(y)| < \frac{\varepsilon}{100}$ whenever $d(x,y) < \delta_0$ for any $f \in \mathcal{F}$ and for every $x, y \in X$. (Here, $d(\cdot,\cdot)$ is a metric on X which induces the topology.) Contracting δ_0 if necessary, we may assume that $\delta_0 < \frac{\varepsilon}{10}$. Pick a finite δ_0-dense subset $\{x_1, \ldots, x_n\}$ of X (i.e., every point in X is within a d-distance δ_0 of a point in $\{x_1, \ldots, x_n\}$). For simplicity, we may assume that any two points in $\{x_1, \ldots, x_n\}$ are at least a d-distance $2\frac{\delta_0}{3}$ apart. For each x_i, choose a continuous function $g_i \in C(X)$ such that $0 \leqslant g_i \leqslant 1$ and

$$g_i(x) = \begin{cases} 1 & \text{if } d(x, x_i) < \frac{\delta_0}{8}, \\ 0 & \text{if } d(x, x_i) > \frac{\delta_0}{4}. \end{cases}$$

Let $\mathcal{F}'$ be the set consisting of all the g_is. Since $\mathcal{A}$ is simple and unital, there is a positive real number $\delta' > 0$ such that for all i,

$$\tau(g_i) > \delta' \quad \text{for any } \tau \in T(A).$$

(Here, we view $C(X)$ as a sub-C^*-algebra of $\mathcal{A}$, since the $*$-homomorphism ϕ is injective.)

Applying Lemma 5.1 to $C(X)$, $\mathcal{F} \cup \mathcal{F}' \cup \mathcal{G}_{00}$ and $\min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$, there exist a finite subset $\mathcal{G} \subset C(X)$ and a positive real number $\delta > 0$ satisfying the statement of Lemma 5.1. Decreasing $\delta > 0$ if necessary, we may assume $\delta < \min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$ and that $\mathcal{G}$ is contained in the unit ball of $C(X)$. We may also assume that $\mathcal{G}$ contains $\mathcal{F} \cup \mathcal{F}' \cup \mathcal{G}_{00}$. By [21], since $\mathcal{A}$ is unital simple separable real rank zero and quasidiagonal, $\mathcal{A}$ has the Popa property. Hence, there is a nonzero finite dimensional sub-C^*-algebra $\mathcal{D}_0$ of $\mathcal{A}$ such that for any $f \in \mathcal{G}$, the following statements hold:

(1) $\|1_{\mathcal{D}_0} f - f 1_{\mathcal{D}_0}\| < \frac{\delta}{10}$; and

(2) $1_{\mathcal{D}_0} f 1_{\mathcal{D}_0}$ is within a distance $\frac{\delta}{10}$ of an element in $\mathcal{D}_0$. (We will abbreviate this by writing $1_{\mathcal{D}_0} f 1_{\mathcal{D}_0} \in_{\delta/10} \mathcal{D}_0$.)

By Arveson's extension theorem, there is a completely positive unital linear map $\eta : 1_{\mathcal{D}_0} \mathcal{A} 1_{\mathcal{D}_0} \to \mathcal{D}_0$ extending the identity map of $\mathcal{D}_0$. Define unital completely positive maps

$$\psi_0 : C(X) \ni g \mapsto \eta(1_{\mathcal{D}_0} \phi(g) 1_{\mathcal{D}_0}) \in \mathcal{D}_0, \quad \psi_1 : C(X) \ni g \mapsto (1_{\mathcal{A}} - 1_{\mathcal{D}_0}) \phi(g) (1_{\mathcal{A}} - 1_{\mathcal{D}_0}) \in \mathcal{A}.$$

One can verify that both ψ_0 and ψ_1 are $\mathcal{G}$-δ-multiplicative. Applying Lemma 5.1 to each simple summand of $\mathcal{D}_0$, there are a $*$-homomorphism $h : C(X) \to \mathcal{D}_0$ and a $\mathcal{F} \cup \mathcal{F}'$-$\delta$-multiplicative completely positive contractive map $\psi_0' : C(X) \to \mathcal{D}_0$ such that $r(\psi_0'(1_{C(X)})) < \min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$ for every normalized trace r on $\mathcal{D}_0$ and

$$\|\psi_0(f) - (\psi_0' \oplus h)(f)\| < \min\left\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\right\} \quad \text{for any } f \in \mathcal{F} \cup \mathcal{F}'.$$

The $*$-homomorphism h has the form

$$h(f) = \sum_{i=1}^{k} f(z_i) p_i$$

where $z_i \in X$ and p_i are pairwise orthogonal projections in $\mathcal{D}_0$.

We claim that we can choose $\mathcal{D}_0$, ψ_0' and the finite dimensional $*$-homomorphism h in such a way that $h(g_j) > 0$ (i.e., strictly positive) holds for any $1 \leqslant j \leqslant n$; that is, for any x_j there is z_i such that $d(z_i, x_j) < \frac{\delta_0}{4}$. We denote this claim by "$(*)$". We now prove claim $(*)$. Proof is by contradiction.

Since $\mathcal{A}$ is separable, there is an increasing sequence $\{\mathcal{F}_i\}_{i=1}^{\infty}$ of finite subsets of $\mathcal{A}$ with dense union. Let $\{\varepsilon_i\}_{i=1}^{\infty}$ be a sequence of strictly positive real numbers decreasing to 0. For each i, let M_i be the positive real number given by $M_i := \max\{1 + \|a\| + \|a\|^2 : a \in \mathcal{F}_i\}$. Note that since the sequence $\{\mathcal{F}_i\}_{i=1}^{\infty}$ is increasing (as sets), the sequence $\{M_i\}_{i=1}^{\infty}$ is also increasing (as real numbers). Since $\mathcal{A}$ has the Popa property, we get a sequence $\{\mathcal{D}_i\}_{i=1}^{\infty}$ of finite dimensional sub-C^*-algebras of $\mathcal{A}$ such that for every i and for all $a \in \mathcal{F}_i$,

(1) $\|1_{\mathcal{D}_i} a - a 1_{\mathcal{D}_i}\| < \frac{\varepsilon_i}{10+M_i}$, and
(2) $1_{\mathcal{D}_i} a 1_{\mathcal{D}_i} \in_{\varepsilon_i/(10+M_i)} \mathcal{D}_i$.

Thus, by the same argument as above, we get the following:

(i.) There are completely positive unital maps $\psi_i : \mathcal{A} \to \mathcal{D}_i$ such that we have the following as $i \to \infty$, for all $a, b \in \mathcal{A}$:

$$\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0.$$

(ii.) As $i \to \infty$ for all $a \in \mathcal{A}$ we have the following:

$$\|1_{\mathcal{D}_i} a 1_{\mathcal{D}_i} - \psi_i(a)\| \to 0.$$

(iii.) There are completely positive contractive maps $\psi_i' : C(X) \to \mathcal{D}_i$ such that, as $i \to \infty$ for all $f, g \in C(X)$,

$$\|\psi_i'(fg) - \psi_i'(f)\psi_i'(g)\| \to 0.$$

(iv.) There are $*$-homomorphisms $h_i : C(X) \to \mathcal{D}_i$ with range orthogonal to ψ_i' such that

$$\|\psi_i(f) - (\psi_i'(f) \oplus h_i(f))\| \to \infty$$

as $i \to \infty$ for all $f \in C(X)$. (We view $C(X)$ as a sub-C^*-algebra of $\mathcal{A}$.)

(v.) $\sup_{r \in T(\mathcal{D}_i)} r(\psi_i'(1_{C(X)})) \to 0$ as $i \to \infty$.

If the claim is not true, then there is g_j and a subsequence $\{i_k\}_{k=1}^{\infty}$ of the positive integers such that for any k, we have that $h_{i_k}(g_j) = 0$. Let r_i be any normalized trace of $\mathcal{D}_i$, and let τ_i denote the state $r_i \circ \psi_i$ of $\mathcal{A}$. It is clear that $\tau_{i_k}(g_j) \to 0$ as $k \to \infty$. Now let τ be a cluster point of $\{\tau_{i_k}\}$ in the compact convex set of states of $\mathcal{A}$ with the weak*-topology. It is easy to see that τ is a tracial state of $\mathcal{A}$. But we then have that

$$\tau(g_j) \leqslant \limsup_k \tau_{i_k}(g_j) = 0$$

which contradicts the simplicity of $\mathcal{A}$. Thus the claim $(*)$ holds.

Let us return to the proof of the theorem. Let $q := 1_{\mathcal{A}} - 1_{\mathcal{D}_0} = \psi_1(1_{C(X)})$. By the definitions of $\mathcal{D}_0$, ψ_0, ψ_0', h and ψ_1 (and viewing $C(X)$ as a sub-C*-algebra of $\mathcal{A}$), we have that

$$\|f - (\psi_1 \oplus \psi_0' \oplus h)(f)| < \delta + \min\left\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\right\} \quad \text{for any } f \in \mathcal{F}.$$

We denote the above statement by "$(**)$".

Since $q\mathcal{A}q$ is SLF, there is a projection $q' \in q\mathcal{A}q$, a $\mathcal{G}$-$\min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$-multiplicative completely positive unital map $\iota : C(X) \to q'\mathcal{A}q'$, and a finite dimensional sub-C*-algebra $\mathcal{E} \subset (q' \oplus q)\mathrm{M}_2(\mathcal{A})(q' \oplus q)$ such that for any $f \in \mathcal{G}$,

(1) $\iota(f) \oplus \psi_1(f)$ is within $\min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$ of an element of $\mathcal{E}$;
(2) $q' \oplus q = \iota(1_{C(X)}) \oplus \psi_1(1_{C(X)}) = 1_{\mathcal{E}}$; and
(3) $(N+1)\tau(q') < \min\{\frac{\varepsilon}{100}, \tau(p_i) : 1 \leqslant i \leqslant k\}$ for any tracial state τ of $\mathcal{A}$.

We have an induced map

$$\iota^{(N+1)} := \iota \otimes \mathrm{id} : \mathrm{M}_{N+1}(C(X)) \to \mathrm{M}_{N+1}(q'\mathcal{A}q').$$

Since ι is $\mathcal{G}$-$\min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$-multiplicative, it follows by the definition of $\mathcal{G}$ that ι is $\mathcal{G}_{00}$-δ_{00}-multiplicative. Hence, by the definitions of $\mathcal{G}_{00}$ and δ_{00}, the induced map $\iota^{(N+1)} = \iota \otimes \mathrm{id}$ is $\mathcal{G}_{00}'$-δ_{00}'-multiplicative. Hence, by the definition of $\mathcal{G}_{00}'$ and δ_{00}' and by Lemma 5.3, there is a unital $*$-homomorphism $\rho : \mathcal{D}_{00} \to \mathrm{M}_{N+1}(q'\mathcal{A}q')$ such that for all $a \in \{q_j' : 1 \leqslant j \leqslant m\} \cup \left\{\sum_{j=1}^{m} f(y_j)q_j' : f \in \mathcal{F}\right\}$,

$$\|\rho(a) - \iota^{(N+1)}(a)\| < \frac{\varepsilon}{100}.$$

Hence, for all $f \in \mathcal{F}$, we must have that

$$\|\rho \circ \Psi(f) - \iota^{(N+1)} \circ (\psi'(f) \oplus f)\| < \frac{\varepsilon}{50}.$$

Note that for all $f \in C(X)$, $\psi'(f) \oplus f = (\psi' \oplus \mathrm{id})(f)$. Also, $\iota^{(N+1)} \circ (\psi' \oplus \mathrm{id}) = (\iota^{(N)} \circ \psi') \oplus \iota$. Let us collectively denote the equations and inequalities in this paragraph by "$(***)$".

Now suppose that there are points $\{w_i\}_{i=1}^{L}$ in X and pairwise orthogonal projections $\{q_i''\}_{i=1}^{L}$ in $\mathrm{M}_{N+1}(q'\mathcal{A}q')$ such that the finite dimensional $*$-homomorphism $\rho \circ \Psi : C(X) \to \mathrm{M}_{N+1}(q'\mathcal{A}q')$ has the form $\rho \circ \Psi : f \mapsto \sum_{i=1}^{L} f(w_i)q_i''$. Since $\{x_1, x_2, \ldots, x_n\}$ is δ_0-dense in X, for each i with $1 \leqslant i \leqslant L$, let j_i be an integer with $1 \leqslant j_i \leqslant n$ such that $d(w_i, x_{j_i}) < \delta_0$. Let $\widetilde{\rho} : C(X) \to \mathrm{M}_{N+1}(q'\mathcal{A}q')$ be the finite dimensional $*$-homomorphism given by $\widetilde{\rho} : f \mapsto \sum_{i=1}^{L} f(x_{j_i})q_i''$. Then by the definition of δ_0 we must have that

$$\|\widetilde{\rho}(f) - \rho \circ \Psi(f)\| < \frac{\varepsilon}{100}$$

for all $f \in \mathcal{F}$. We denote the above statement by "$(* * **)$".

Also, by the claim $(*)$, we have that for each i with $1 \leqslant i \leqslant n$ there is an integer j_i' such that $d(z_{j_i'}, x_i) < \frac{\delta_0}{4}$. Moreover, since the x_is are at least a d-distance $2\frac{\delta_0}{3}$ apart, we must have that the map $i \mapsto j_i'$ is injective. In particular, this implies that $k \geqslant n$. For simplicity, let us assume that for $1 \leqslant i \leqslant n$, $j_i' = i$. Consider the $*$-homomorphism $\widetilde{h} : C(X) \to \mathcal{D}_0$ that is given by $\widetilde{h} : f \mapsto \sum_{i=1}^{n} f(x_i)p_i + \sum_{i=n+1}^{k} f(z_i)p_i$. By the definition of δ_0, we must have that

$$\|h(f) - \widetilde{h}(f)\| < \frac{\varepsilon}{100}$$

for all $f \in \mathcal{F}$. We denote the above statement by "$(* * * * *)$".

Since

$$(N+1)\tau(q') < \min\{\tau(p_i) : 1 \leqslant i \leqslant k\}$$

for any tracial state τ of $\mathcal{A}$ and since $\mathcal{A}$ has stable rank one and weak unperforation (proven in the previous section), we must have that

$$(N+1)q' \preceq p_i$$

for $1 \leqslant i \leqslant k$, where "$\preceq$" means Murray–von Neumann equivalent to a subprojection. So we must have that

$$\sum_{j=1}^{L} q_j'' \preceq p_i$$

for all i. Note that the sum on the left-hand-side is a projection since the projections q_j'' are pairwise orthogonal. Hence, let v be a partial isometry in $\mathcal{A} \otimes \mathcal{K}$ with initial projection $\sum_{i=1}^{L} q_i''$ and range projection contained in $\sum_{i=1}^{n} p_i$ such that for all i, there is an integer $l(i)$ (with $1 \leqslant l(i) \leqslant n$) with $vq_i''v^* \leqslant p_{l(i)}$ and with $x_{l(i)} = x_{j_i}$.

Let $\widetilde{\widetilde{\rho}} : C(X) \to 1_{\mathcal{D}_0}\mathcal{A}1_{\mathcal{D}_0}$ be the finite dimensional $*$-homomorphism given by $\widetilde{\widetilde{\rho}}(f) = v\widetilde{\rho}(f)v^*$ for all $f \in C(X)$. Then it is clear that $\widetilde{\widetilde{\rho}}(1_{C(X)})$ is a projection contained in $1_{\mathcal{D}_0}$ and that $\widetilde{h} - \widetilde{\widetilde{\rho}}$ is a finite dimensional $*$-homomorphism. In particular, there is a finite dimensional $*$-homomorphism $h'' : C(X) \to 1_{\mathcal{D}_0}\mathcal{A}1_{\mathcal{D}_0}$ which is orthogonal to $\widetilde{\widetilde{\rho}}$ such that $h'' + \widetilde{\widetilde{\rho}} = \widetilde{h}$. ($h'' = \widetilde{h} - \widetilde{\widetilde{\rho}}$.) Also, by $(**)$ and $(* * * * *)$, we have that

$$\|f - (\psi_1 \oplus \psi_0' \oplus h'' \oplus \widetilde{\widetilde{\rho}})(f)\| < \delta + \min\left\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\right\} + \frac{\varepsilon}{100}$$

for all $f \in \mathcal{F}$. We denote this inequality by "$(*****\!*)$".

By appropriate shifting of coordinates if necessary, we may assume that $v(q' \oplus 0)$ is a partial isometry with initial projection $q' \oplus 0$ and that $v(0 \oplus 1_{\mathcal{A}}) = 0$, where the first and second coordinates are the first and second coordinates (respectively) in the definition of $\mathcal{E}$. Hence, $v(q' \oplus 0)$ and $0 \oplus q$ are orthogonal in $\mathcal{A} \otimes \mathcal{K}$ and, by the definitions of ι and $\mathcal{E}$, we must have that $(v + (0 \oplus q))\mathcal{E}(v^* + (0 \oplus q)) = (v(q' \oplus 0) + (0 \oplus q))\mathcal{E}((q' \oplus 0)v^* + (0 \oplus q))$ is a finite dimensional sub-C*-algebra of $\mathcal{A}$ such that $(v(q' \oplus 0) + (0 \oplus q))(\iota(f) \oplus \psi_1(f))((q' \oplus 0)v^* + (0 \oplus q)) = v\iota(f)v^* + \psi_1(f)$ is within $\min\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\}$ of an element of $(v + (0 \oplus q))\mathcal{E}(v^* + (0 \oplus q))$ for every $f \in \mathcal{F}$. (Recall that $1_{\mathcal{E}} = q' \oplus q$.) Also, $v\iota(1_{C(X)})v^* + \psi_1(1_{C(X)}) = 1_{(v+(0\oplus q))\mathcal{E}(v^*+(0\oplus q))}$. We denote the contents of this paragraph by "$(*******)$".

But by $(***)$, $(****)$ and the definition of $\tilde{\tilde{\rho}}$, we must have that

$$\|v(\iota^{(N)} \circ \psi')(f)v^* + v\iota(f)v^* - \tilde{\tilde{\rho}}(f)\| < \frac{\varepsilon}{50} + \frac{\varepsilon}{100} = 3\frac{\varepsilon}{100}$$

for every $f \in \mathcal{F}$. Note that the first sum is a sum of orthogonal elements in $\mathcal{A} \otimes \mathcal{K}$; and also, $v(\iota^{(N)} \circ \psi')(1_{C(X)})v^*$ and $v\iota(1_{C(X)})v^*$ are orthogonal projections which sum up to $\tilde{\tilde{\rho}}(1_{C(X)})$. Hence, from this and $(******)$, we have that

$$\begin{aligned}\|f - (\psi_1 \oplus \psi_0' \oplus h'' \oplus v(\iota^{(N)} \circ \psi')v^* \oplus v\iota v^*)(f)\| &< \delta + \min\left\{\frac{\delta_0}{10}, \frac{\delta'}{10}, \frac{\delta_{00}}{10}\right\} + 4\frac{\varepsilon}{100} \\ &< 6\frac{\varepsilon}{100}\end{aligned}$$

for all $f \in \mathcal{F}$. Now note that h'' is a finite dimensional $*$-homomorphism. Also, it follows from $(*******)$ that $\psi_1(f) \oplus v\iota(f)v^*$ is within $\frac{\varepsilon}{100}$ of an element of the finite dimensional C*-algebra $(v + (0 \oplus q))\mathcal{E}(v^* + (0 \oplus q))$ for all $f \in \mathcal{F}$.

Next, from the definition of ψ_0', we have that $\tau(\psi_0'(1_{C(X)})) < \frac{\varepsilon}{100}$ for every tracial state τ of $\mathcal{A}$. Finally, from the definitions of ι and q', we have that

$$\tau(v(\iota^{(N)} \circ \psi')(1_{C(X)})v^*) = \tau((\iota^{(N)} \circ \psi')(1_{C(X)})) \leqslant (N+1)\tau(q') < \frac{\varepsilon}{100}$$

for every tracial state τ of $\mathcal{A}$.

Since $\varepsilon > 0$ is arbitrary, this finishes the proof. ∎

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