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METRIC AND HOMOGENEOUS STRUCTURE OF CLOSED RANGE OPERATORS

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ABSTRACT. Let $\mathcal{CR}$ be the set of all bounded linear operators between Hilbert spaces $\mathcal{H}, \mathcal{K}$ with closed range. This paper is devoted to the study of the topological properties of $\mathcal{CR}$ if certain natural metrics are considered on it. We also define an action of the group $\mathcal{G}_{\mathcal{H}} \times \mathcal{G}_{\mathcal{K}}$ on $\mathcal{CR}$ and determine the orbits of this action. These orbits, which are strongly related to the connected components for the topology defined by the metrics mentioned above, determine a stratification of the set of Fredholm and semi-Fredholm operators. Finally, we calculate the distance, with respect to some of the metrics mentioned above, between different orbits of $\mathcal{CR}$.

KEYWORDS: *Closed range, partial isometry, semi-Fredholm operators, positive operators, orbits, Moore–Penrose inverse.*

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1. INTRODUCTION

Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $\mathcal{CR}$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$ with closed range. This paper is devoted to study a natural homogeneous structure on $\mathcal{CR}$. By this, we mean a topology on $\mathcal{CR}$ and a topological group acting continuously on it. Such structure provides many homeomorphisms on $\mathcal{CR}$ which are of great help in order to understand the topology and, eventually, the geometry of different parts of the set. Many subsets of $\mathcal{CR}$ have been studied from a topological or geometrical viewpoint: idempotents [17], [18], orthogonal projections [46], partial isometries [29], [42], [1], Fredholm and semi-Fredholm operators [6], [26], [39], [40], [49], many classes of invertible operators [4], [5], [19]. Paradoxically, the main obstruction to study $\mathcal{CR}$ as a whole, with the usual norm topology, is that it is a path connected space: the curve $t \mapsto tA$ connects every $A \in \mathcal{CR}$ with the zero operator. Thus, the norm topology is not suitable to separate closed range operators which naturally belong to very different families. Therefore, we study $\mathcal{CR}$ with the metric

$d_R(A,B) = (\|A-B\|^2 + \|P_{R(A)} - P_{R(B)}\|^2)^{1/2}$, where $R(C)$ denotes the range of the operator C and $P_{\mathcal{S}}$ denotes the orthogonal projection onto the closed subspace $\mathcal{S}$ and also another metric related to the nullspaces. These metrics are finer than the one defined by the operator norm and the map $A \to P_{R(A)}$, which is obviously not continuous for the norm topology, is continuous under these metrics. After collecting some notations and preliminary results in Section 2, the third section of the paper is devoted to show many possible choices of equivalent metrics with those properties. Some notions like the reduced minimum modulus of an operator or the Moore–Penrose generalized inverse, naturally enter into the discussion. The fourth section surveys many known (and not so known) topological properties of $\mathcal{CR}$ and its subsets, using the norm topology and the one defined by d_R. The fifth section contains a complete description of the homogeneous structure of $\mathcal{CR}$ by the left action $\mathcal{G}_{\mathcal{H}} \times \mathcal{G}_{\mathcal{K}} \times \mathcal{CR} \to \mathcal{CR}$ which is defined by $((G,H),A) \to GAH^{-1}$, for $G \in \mathcal{G}_{\mathcal{H}}$, $H \in \mathcal{G}_{\mathcal{K}}$, $A \in \mathcal{CR}$. Here $\mathcal{G}_{\mathcal{H}}$ is the group of invertible operators on $\mathcal{H}$, and similarly for $\mathcal{K}$. The orbit $\mathcal{O}_A = \{GAH^{-1} : G \in \mathcal{G}_{\mathcal{H}}, H \in \mathcal{G}_{\mathcal{K}}\}$ is characterized by three cardinal numbers, namely, the nullity $n(A) =$ *dimension of the nullspace* $N(A)$, the rank $r(A) =$ *dimension of* $R(A)$ and the defect $d(A) =$ *dimension of* $R(A)^{\perp}$. This characterization follows the ideas of P. R. Halmos and J. McLaughlin [29]. Even when they did not look at any homogeneous structure for the set $\mathcal{PI}(\mathcal{H},\mathcal{K})$ of all partial isometries from $\mathcal{H}$ to $\mathcal{K}$, they proved that the connected components of $\mathcal{PI}(\mathcal{H},\mathcal{K})$ are determined by the same cardinal numbers, and their proofs show that the connected components coincide with the orbits of the action of $\mathcal{U}_{\mathcal{H}} \times \mathcal{U}_{\mathcal{K}}$, the product of the unitary groups of $\mathcal{H}$ and $\mathcal{K}$, on $\mathcal{PI}(\mathcal{H},\mathcal{K})$. The polar decomposition defines a natural retraction $\mathcal{CR}(\mathcal{H},\mathcal{K}) \to \mathcal{PI}(\mathcal{H},\mathcal{K})$ which is also studied in Section 5. A main result in this section is the computation of the distance, for d_R and d_N, between two different orbits of $\mathcal{CR}$. In the last section, we consider the simpler structure of the subset $\mathcal{CR}_{\mathcal{S}}$ of all $A \in \mathcal{CR}(\mathcal{H},\mathcal{K})$ such that $R(A)$ is a fixed closed subspace $\mathcal{S}$. As it is usual in this type of problems, the existence of continuous local sections of the maps involved, is a relevant question. Its affirmative answer for the map $\mathcal{G}_{\mathcal{H}} \times \mathcal{G}_{\mathcal{K}} \to \mathcal{O}_A$, $(G,H) \to GAH^{-1}$ is a key part in the theorem which exhibits $\mathcal{O}_A$ as a homogeneous space. Some results of [2] are of great help in order to define a local section. Also, the well known geometry of the unitary orbit of an orthogonal projection or the congruence orbit of a closed range positive operator, are useful here. The reader is referred to [46], [18], [15] for details on these matters. We intend to proceed with the differential geometry of $\mathcal{CR}$ elsewhere.

2. PRELIMINARIES

Throughout this paper, $\mathcal{H}, \mathcal{K}$ denote (complex separable) Hilbert spaces and $\mathcal{L}(\mathcal{H},\mathcal{K})$ is the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, with the uniform operator norm. If $\mathcal{H} = \mathcal{K}$ we write $\mathcal{L}(\mathcal{H})$ instead of $\mathcal{L}(\mathcal{H},\mathcal{H})$; $\mathcal{G}_{\mathcal{H}}$ is the group of invertible operators in $\mathcal{L}(\mathcal{H})$, the subgroup of $\mathcal{G}_{\mathcal{H}}$ of all unitary operators is $\mathcal{U}_{\mathcal{H}}$, the cone of positive (respectively, positive invertible) operators on $\mathcal{H}$ is $\mathcal{L}(\mathcal{H})^+$ (respectively, $\mathcal{G}_{\mathcal{H}}^+$). The range or image of $C \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is denoted by $R(C)$ and its nullspace by $N(C)$. A partial isometry from $\mathcal{H}$ to $\mathcal{K}$ is an operator $V \in \mathcal{L}(\mathcal{H},\mathcal{K})$ such that its restriction to $N(V)^\perp$ is an isometry; equivalently, V^*V is idempotent; it follows that V^*V is the orthogonal projection onto $N(V)^\perp$ (the *initial space* of V) and VV^* is the orthogonal projection onto $R(V)$ (the *final space* of V). $\mathcal{PI} = \mathcal{PI}(\mathcal{H},\mathcal{K})$ denotes the set of all partial isometries from $\mathcal{H}$ to $\mathcal{K}$ and $\mathcal{P}_{\mathcal{H}}$ (respectively $\mathcal{P}_{\mathcal{K}}$) the set of orthogonal projections on $\mathcal{H}$ (respectively $\mathcal{K}$). If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ (or $\mathcal{K}$), $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

For $A \in \mathcal{L}(\mathcal{H},\mathcal{K})$, the *reduced minimum modulus* of A is $\gamma(A) = \inf\{\|Ax\| : x \in N(A)^\perp, \|x\| = 1\}$. It is well known that $\gamma(A) > 0$ if and only if $R(A)$ is closed. It holds $\gamma(|A|)^2 = \gamma(A)^2 = \gamma(AA^*) = \gamma(A^*A) = \gamma(A^*)^2 = \gamma(|A^*|)^2$. Also if $R(A)$ is closed and $A^\dagger$ is the Moore–Penrose inverse of A, then $\gamma(A) = \|A^\dagger\|^{-1}$. Recall the definition and main properties of $A^\dagger$, because this notion plays a central role in all what follows. R. Penrose [44] proved that $A^\dagger$ is uniquely determined by the four identities $AA^\dagger A = A$, $A^\dagger AA^\dagger = A^\dagger$, $AA^\dagger = P_{R(A)}$, $A^\dagger A = P_{R(A^*)}$; the range of $A^\dagger$ is $N(A)^\perp = R(A)^*$ and its nullspace is $R(A)^\perp = N(A^*)$.

The following remark will be useful in several proofs of the next sections.

REMARK 2.1. In Sections 3 and 4 we study the continuity of the mapping $A \to A^\dagger$ where different metrics are considered on the set $\mathcal{CR}(\mathcal{H},\mathcal{K})$. In order to do this it is necessary to estimate $\|A^\dagger - B^\dagger\|$. The next identity is due to G.W. Stewart [50], and has been used several times in works concerning perturbations of the Moore–Penrose inverse:

$$A^\dagger - B^\dagger = -A^\dagger(A-B)B^\dagger + A^\dagger A^{*\dagger}(A^*-B^*)(I-BB^\dagger) + (I-A^\dagger A)(A^*-B^*)B^{*\dagger}B^\dagger.$$

Therefore

$$\|A^\dagger - B^\dagger\| \leqslant (\|A^\dagger\|\|B^\dagger\| + \|A^\dagger\|^2 + \|B^\dagger\|^2)\|A-B\|.$$

Next, we review a notion of angle between closed subspaces. Let $\mathcal{M}$ and $\mathcal{N}$ be closed subspaces of a Hilbert space $\mathcal{H}$. Define

$$c_0(\mathcal{M},\mathcal{N}) = \sup\{|\langle x,y\rangle| : x \in \mathcal{M}, y \in \mathcal{N}, \|x\| = \|y\| = 1\},$$

$$c(\mathcal{M},\mathcal{N}) = \sup\{|\langle x,y\rangle| : x \in \mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp, y \in \mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp, \|x\| = \|y\| = 1\}.$$

The *angle* $\alpha(\mathcal{M},\mathcal{N})$ is the number $\alpha \in [0, \frac{\pi}{2}]$ such that $c(\mathcal{M},\mathcal{N}) = \cos\alpha$. It holds that $c(\mathcal{M},\mathcal{N}) = \|P_{\mathcal{M}} - P_{\mathcal{N}^\perp}\|$. Observe that $c(\mathcal{M},\mathcal{N}) = c_0(\mathcal{M}\cap(\mathcal{M}\cap\mathcal{N})^\perp, \mathcal{N}\cap(\mathcal{M}\cap\mathcal{N})^\perp)$. It holds $c_0(\mathcal{M},\mathcal{N}) < 1$ if and only if $\mathcal{M}+\mathcal{N}$ is closed and $\mathcal{M}\cap\mathcal{N} = \{0\}$. Also, $\mathcal{M}+\mathcal{N}$ is closed if and only if $c(\mathcal{M},\mathcal{N}) < 1$, or equivalently, if $\|P_{\mathcal{M}} - P_{\mathcal{N}^\perp}\| < 1$, see [20], and also the paper by D. Buckholtz [10]. The next results will be useful in the main theorem of Section 4.

PROPOSITION 2.2. *If $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces of $\mathcal{H}$ then $\mathcal{H} = \mathcal{M}+\mathcal{N}$ if and only if $c_0(\mathcal{M}^\perp,\mathcal{N}^\perp) < 1$.*

Proof. If $\mathcal{H} = \mathcal{M} + \mathcal{N}$, then $\mathcal{M} + \mathcal{N}$ is obviously closed; then $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (see Theorem 2.13 of [20]; or Theorem 4.8 of [33]); but also, $\mathcal{M}^\perp \cap \mathcal{N}^\perp = (\mathcal{M} + \mathcal{N})^\perp = \{0\}$, so that $c_0(\mathcal{M}^\perp, \mathcal{N}^\perp) < 1$. Conversely, if $c_0(\mathcal{M}^\perp, \mathcal{N}^\perp) < 1$ then $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed, and then $\mathcal{M} + \mathcal{N}$ is closed; also $\mathcal{M}^\perp \cap \mathcal{N}^\perp = \{0\}$. Then $\mathcal{H} = (\mathcal{M}^\perp \cap \mathcal{N}^\perp)^\perp = \mathcal{M} + \mathcal{N}$ (the last equality holds precisely because $\mathcal{M} + \mathcal{N}$ is closed; see Lemma 2.11 of [20], or Theorem 4.8 of [33]. ∎

PROPOSITION 2.3. *Given two operators B, $C \in \mathcal{CR}$ the following conditions are equivalent:*

(i) $\|P_{N(B)} - P_{N(C)}\| < 1$;
(ii) $\mathcal{H} = N(C) + R(B^\dagger)$;
(iii) $c_0(N(B), R(C^\dagger)) < 1$;
(iv) $N(B) = P_{N(B)}(N(C))$.

Proof. (i)$\Longrightarrow$(ii): If (i) holds then $G = I - P_{N(B)} + P_{N(C)}$ is invertible so that $\mathcal{H} = R(G) = N(C) + R(B^\dagger)$.

(ii)$\Longrightarrow$(i): If $\mathcal{H} = N(C) + R(B^\dagger)$ then $N(C) + R(B^\dagger)$ is closed, so that $c(N(C), R(B^\dagger)) = \|P_{N(C)} - P_{R(B^\dagger)^\perp}\| < 1$ or equivalently $\|P_{N(C)} - P_{N(B)}\| < 1$.

(ii)$\Longleftrightarrow$(iii) is a corollary of the above proposition.

(ii)$\Longleftrightarrow$(iv): If $\mathcal{H} = N(C) + R(B^\dagger)$ then $N(C) = P_{N(B)}(N(C) + R(B^\dagger)) = P_{N(B)}(N(C))$. The converse is similar. ∎

3. THE NORM TOPOLOGY ON $\mathcal{CR}$

In this section we collect several known results about the norm topology on $\mathcal{CR}$ and include a new result (Theorem 3.8). Recall, from the introduction, that $\mathcal{CR}$ is a path connected space. We define two different metrics on $\mathcal{CR}$ which will be the main tools for the study of the continuity properties of the Moore–Penrose inverse mapping and other related mappings, in this and the next sections.

Given $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$, define

$$d_R(A,B) = (\|P_{\overline{R(A)}} - P_{\overline{R(B)}}\|^2 + \|A - B\|^2)^{1/2},$$
$$d_N(A,B) = (\|P_{N(A)} - P_{N(B)}\|^2 + \|A - B\|^2)^{1/2}.$$

From now on, we write d_X whenever a result is valid for both d_N and d_R.

REMARK 3.1. Observe that d_R and d_N are metrics in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ such that:
(i) $d_N(A^*, B^*) = d_R(A, B)$ and $d_N(A, B) = d_R(A^*, B^*)$;
(ii) $d_N(A, B) \leqslant \|P_{\overline{R(A^*)}} - P_{\overline{R(B^*)}}\| + \|A - B\|$.

The following lemma relates the reduced minimum moduli of two operators with the distances d_N and d_R between them. It plays a key role in many computations of the following section.

LEMMA 3.2. *Consider $A, B \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. Then*

$$\gamma(B) \leqslant \sqrt{1+\gamma(B)^2}\, d_X(A,B) + \gamma(A).$$

Proof. If $\gamma(B) = 0$ both inequalities are trivial. Suppose that $\gamma(B) > 0$. Consider first the case $X = N$. Let $u \in N(A)^{\perp}$. Observe that γ satisfies $\gamma(B)\|x\| \leqslant \|Bx\|$, for $x \in N(B)^{\perp}$. The inequality holds, in particular, for $v = (I - P_{N(B)})u \in N(B)^{\perp}$. Then

$$\begin{aligned}
\gamma(B)\|u\| &\leqslant \gamma(B)\|u-v\| + \gamma(B)\|v\| \leqslant \gamma(B)\|u-v\| + \|Bv\| \\
&\leqslant \gamma(B)\|u-v\| + \|Au - Bv\| + \|Au\| \\
&\leqslant (\gamma(B)\|P_{N(A)} - P_{N(B)}\| + \|A-B\|)\|u\| + \|Au\| \\
&\leqslant \sqrt{1+\gamma(B)^2}(\|P_{N(A)} - P_{N(B)}\|^2 + \|A-B\|^2)^{1/2}\|u\| + \|Au\| \\
&= \sqrt{1+\gamma(B)^2}\, d_N(A,B)\|u\| + \|Au\|.
\end{aligned}$$

Therefore $\gamma(B) \leqslant \sqrt{1+\gamma(B)^2}\, d_N(A,B) + \gamma(A)$ and the inequality holds in the case $X = N$.

For the case $X = R$, observe that, by Remark 3.1 it holds

$$\gamma(B) = \gamma(B^*) \leqslant \sqrt{1+\gamma(B)^2}\, d_N(A^*, B^*) + \gamma(A^*),$$

so that $\gamma(B) \leqslant \sqrt{1+\gamma(B)^2}\, d_R(A,B) + \gamma(A)$, which ends the proof. ∎

COROLLARY 3.3. *Let $B \in \mathcal{CR}$ and consider $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $d_X(A,B) < \frac{1}{2\sqrt{1+\|B^\dagger\|^2}}$ then $A \in \mathcal{CR}$ and $\|A^\dagger\| \leqslant 2\|B^\dagger\|$.*

Proof. If $B \in \mathcal{CR}$ and $d_X(A,B) < \frac{1}{2\sqrt{1+\|B^\dagger\|^2}}$ then, since $\gamma(B) = \|B^\dagger\|^{-1}$, $d_X(A,B)\sqrt{1+\gamma(B)^2} < \frac{\gamma(B)}{2}$. Thus, applying Lemma 3.2, it follows that $\gamma(A) > 0$. Therefore $A \in \mathcal{CR}$ and $\gamma(A) = \|A^\dagger\|^{-1}$. In this case, also from Lemma 3.2, $1 \leqslant \sqrt{1+\|B^\dagger\|^2}\, d_X(A,B) + \frac{\|B^\dagger\|}{\|A^\dagger\|} \leqslant \frac{1}{2} + \frac{\|B^\dagger\|}{\|A^\dagger\|}$, then $\|A^\dagger\| \leqslant 2\|B^\dagger\|$. ∎

The following inequality is similar to that of Lemma 3.2, but it is symmetric in A and B:

COROLLARY 3.4. *If $A, B \in \mathcal{CR}$, then*

$$|\gamma(B) - \gamma(A)| \leqslant \sqrt{1+\gamma(B)^2}\, \sqrt{1+\gamma(A)^2}\, d_X(A,B).$$

Proof. By Lemma 3.2, it follows that $\gamma(B) - \gamma(A) \leqslant \sqrt{1+\gamma(B)^2}\, d_X(A,B)$ and, multiplying by $1 \leqslant \sqrt{1+\gamma(A)^2}$, we get that

$$\gamma(B) - \gamma(A) \leqslant \sqrt{1+\gamma(B)^2}\, \sqrt{1+\gamma(A)^2}\, d_X(A,B).$$

The result follows by changing the roles of A and B. ∎

PROPOSITION 3.5. *Consider $B \in \mathcal{CR}$ and $B' \in \mathcal{L}(\mathcal{K},\mathcal{H})$ such that $BB'B = B$. Then:*

(1) $\gamma(B) \geqslant \frac{1}{\|B'\|}$.

(2) $\|B^\dagger\| \leqslant \|B'\|$.

(3) *If $A \in \mathcal{CR}$ and $A' \in \mathcal{L}(\mathcal{K},\mathcal{H})$ satisfy $AA'A = A$ then:*

(i) $\|B^\dagger B - A^\dagger A\| \leqslant \|B'B - A'A\|$;

(ii) $\|BB^\dagger - AA^\dagger\| \leqslant \|BB' - AA'\|$.

Proof. (1) Consider $u \in \mathcal{H}$ such that $u \notin N(B)$ and write $u = (I - B'B)u + B'Bu$. Then $d(u, N(B)) = d(B'Bu, N(B)) \leqslant \|B'\|\|Bu\|$, so that $\frac{1}{\|B'\|} \leqslant \frac{\|Bu\|}{d(u,N(B))} = \gamma(B)$, for every $u \notin N(B)$. Then $\frac{1}{\|B'\|} \leqslant \gamma(B)$.

(2) Since $\gamma(B) = \frac{1}{\|B^\dagger\|}$, it follows, from (1), that $\frac{1}{\|B^\dagger\|} \geqslant \frac{1}{\|B'\|}$, or $\|B^\dagger\| \leqslant \|B'\|$.

(3) Both inequalities are particular cases of the following result of Mbekhta ([38], 1.10): if $\mathcal{S}, \mathcal{T}$ are closed subspaces of $\mathcal{H}$, P is a projection onto $\mathcal{S}$ and Q is a projection onto $\mathcal{T}$, then $\|P - Q\| \geqslant \|P_\mathcal{S} - Q_\mathcal{T}\|$. ∎

We introduce a subset R_k of $\mathcal{CR}$ which has nice properties, in the norm topology, with respect to the Moore–Penrose inverse operation.

For any positive integer k define

$$\mathcal{R}_k = \mathcal{R}_k(\mathcal{H},\mathcal{K}) = \Big\{A \in \mathcal{L}(\mathcal{H},\mathcal{K}) : \gamma(A) \geqslant \frac{1}{k}\Big\}.$$

It is easy to prove the following properties:

(1) $\mathcal{CR} = \bigcup\{\mathcal{R}_k : k \in \mathbb{N}\}$.

(2) $A \in \mathcal{R}_k$ if and only if $A^* \in \mathcal{R}_k$.

(3) For every $k \in \mathbb{N}$, the set $\mathcal{R}_k$ is closed.

Define $\mathcal{M} = \mathcal{M}(\mathcal{H},\mathcal{K}) = \{A \in \mathcal{CR} : N(A) = 0 \text{ or } R(A) = \mathcal{K}\}$, i.e., $\mathcal{M}$ consists of all injective operators with closed range and of all surjective operators.

THEOREM 3.6. *The set $\mathcal{M}$ consists of all operators which belong to the interior of some $\mathcal{R}_k$:*

$$\mathcal{M} = \bigcup\{\operatorname{int}\mathcal{R}_\mathrm{k} : \mathrm{k} \in \mathbb{N}\}.$$

Proof. If $A \notin \mathcal{M}$ then there exist $u \in N(A)$ and $v \in N(A^*)$ such that $\|u\| = \|v\| = 1$. Define $A_n = A + \frac{1}{n} u \otimes v$. Then $\|A_n - A\| = \frac{1}{n}$ and $\gamma(A_n) \leqslant \frac{1}{n}$; therefore, $A \notin \bigcup\{\operatorname{int}\mathcal{R}_\mathrm{k}\}$. Recall that the set of all surjective bounded linear operators and the set of all injective operators with closed range are both open with the norm topology. Then, it is easy to prove that $\gamma : \mathcal{M} \to \mathbb{R}^+$ is continuous (the reader will find a more general treatment about the continuity points of γ in the next section). Therefore, given $A \in \mathcal{M}$ there exist $\delta > 0$ and $k_0 \in \mathbb{N}$ such that $\gamma(B) \geqslant \frac{1}{k_0}$ for all $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$ with $\|A - B\| < \delta$. Thus $A \in \operatorname{int}\mathcal{R}_{\mathrm{k}_0}$. ∎

REMARK 3.7. By a known result of perturbation theory ([26], [40]), the interior of the set $\mathcal{CR}$ in $\mathcal{L}(\mathcal{H},\mathcal{K})$ is the set of all semi-Fredholm operators, a class which is much larger than $\mathcal{M}$.

LEMMA 3.8. *For every $A, B \in \mathcal{R}_k$ it holds:*
(1) $\|A^\dagger A - B^\dagger B\| \leqslant k\|A - B\|$;
(2) $\|AA^\dagger - BB^\dagger\| \leqslant k\|A - B\|$;
(3) *if* $\|A - B\| < \frac{1}{k}$ *then* $|\gamma(A) - \gamma(B)| \leqslant \|A - B\|$;

The proof of these facts can be found in [38].

COROLLARY 3.9. *For every $A, B \in \mathcal{R}_k$ it holds $\|A - B\| \leqslant d_X(A, B) \leqslant (1 + k^2)^{1/2}\|A - B\|$.*

For $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$ consider the polar decomposition $B = V_B|B|$, where $|B| = (B^*B)^{1/2}$ and V_B is the partial isometry such that $N(V_B) = N(B)$ and $R(V_B)$ is the closure of $R(B)$. It holds that V_B is uniquely determined by these properties. We also consider the reverse polar decomposition $B = |B^*|W$, where W is a partial isometry which is uniquely determined by the conditions $N(W) = N(B)$ and $R(W) = \overline{R(B)}$. It turns out that $W = V_B$ (see [48]). We shall study now the continuity properties of the mappings $\alpha : \mathcal{L}(\mathcal{H},\mathcal{K}) \to \mathcal{L}(\mathcal{K})^+$, $\alpha(B) = |B|$, $v : \mathcal{L}(\mathcal{H},\mathcal{K}) \to \mathcal{CR}$, $v(B) = V_B$, and $\mu : \mathcal{CR} \to \mathcal{CR}(\mathcal{K},\mathcal{H})$, $\mu(C) = C^\dagger$.

LEMMA 3.10. *For every $A, B \in \mathcal{R}_k$ it holds $\|A^\dagger - B^\dagger\| \leqslant 3k^2\|A - B\|$. In particular, the function $\mu : \mathcal{R}_k \to \mathcal{CR}(\mathcal{K},\mathcal{H})$ is Lipschitz.*

Proof. Since $\gamma(C) = \|C^\dagger\|^{-1}$, it holds $\|A^\dagger\| \leqslant k$ and $\|B^\dagger\| \leqslant k$. The inequality follows immediately from Remark 2.1. ∎

The following results establish the continuity points of μ and v. The next theorem is due to Labrousse and Mbekhta ([35], 2.19):

THEOREM 3.11. *The mapping $\mu : \mathcal{CR} \to \mathcal{CR}(\mathcal{K},\mathcal{H})$ is continuous at B if and only if B is injective or surjective.*

THEOREM 3.12. (i) *Let $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$. If $v : \mathcal{L}(\mathcal{H},\mathcal{K}) \to \mathcal{CR}$ is continuous at B then B has a closed range.*
(ii) *If $B \in \mathcal{CR}$, then $v : \mathcal{CR} \to \mathcal{CR}$ is continuous at B if and only if $\mu : \mathcal{CR} \to \mathcal{CR}(\mathcal{K},\mathcal{H})$ is continuous at B.*
(iii) *If $B \in \mathcal{CR}$, then $v : \mathcal{CR} \to \mathcal{CR}$ is continuous at B if and only if B is injective or surjective.*

Proof. (i) Suppose that $B \in \mathcal{L}(\mathcal{H},\mathcal{K})$, and $R(B)$ is not closed. In this case $\gamma(B) = 0$, so that, given $\varepsilon > 0$ there exists $x_0 \in N(B)^\perp$ such that $\|x_0\| = 1$ and $\|Bx_0\| < \varepsilon$. Consider the orthogonal projection P onto the subspace spanned by x_0, $Px = \langle x, x_0\rangle x_0$, for $x \in \mathcal{H}$. If $B = |B^*|V_B$, define $W = V_B(I - 2P)$ and $\widetilde{B} = |B^*|W$. It is easy to see that W is a partial isometry such that $R(W) = R(V_B) = \overline{R(B)}$ so that $V_{\widetilde{B}} = W$; also $\|B - \widetilde{B}\| = \||B^*|(V_B - W)\| = \||B^*|(2V_BP)\| =$

$2\|BP\| \leqslant 2\varepsilon$. But $\|V_B - W\| = 2\|2V_BP\| = 2$ which proves that v is not continuous at B.

(ii) Suppose first that $v : \mathcal{CR} \to \mathcal{CR}$ is continuous at B and let $\lim_{n\to\infty} \|B_n - B\| = 0$. Then, by hypothesis, $\lim_{n\to\infty}\|V_{B_n} - V_B\| = 0$, and also we have that $\lim_{n\to\infty}\|V_{B_n}^* V_{B_n} - V_B^* V_B\| = 0$. But $V_{B_n}^* V_{B_n} = I - P_{N(B_n)}$ and $V_B^* V_B = I - P_{N(B)}$, so that $\lim_{n\to\infty} d_N(B_n, B) = 0$. Applying Remark 2.1 we get $\|B_n^\dagger - B^\dagger\| \leqslant (\|B_n^\dagger\|\|B^\dagger\| + \|B_n^\dagger\|^2 + \|B^\dagger\|^2)\|B_n - B\|$. Moreover, since $d_N(B_n, B) \to 0$, applying Corollary 3.3, consider n such that $d_N(B_n, B) < \frac{1}{2\sqrt{1+\|B^\dagger\|^2}}$, then $\|B_n^\dagger\| \leqslant 2\|B^\dagger\|$ so that $\|B_n^\dagger - B^\dagger\| \leqslant K\|B^\dagger\|\|B_n - B\| \xrightarrow{n\to\infty} 0$. The converse is obvious from the identity $v(B) = V_B = (B^*)^\dagger |B|$.

(iii) It suffices to combine (ii) with the theorem by Labrousse and Mbekhta ([35], Theorem 2.19). ∎

REMARK 3.13. Many perturbation results on the Moore–Penrose inverse can be found in the papers by P.O. Wedin [52], G.W. Stewart [50] and S. Izumino [32]. See also the book by Ben-Israel and Greville [7].

4. $\mathcal{CR}$ WITH THE d_X TOPOLOGY

In this section we study the topological properties of $\mathcal{CR}$ with the metrics d_X and the continuity of μ, α and v with the topology induced by them. We also state several equivalent conditions to the convergence of a sequence B_n with d_X.

As a corollary of Lemma 3.2 we have that

PROPOSITION 4.1. *The set $\mathcal{CR}$ is open in $(\mathcal{L}(\mathcal{H}, \mathcal{K}), d_X)$.*

Proof. Let $B \in \mathcal{CR}$; consider $A \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that $d_X(A, B) < \frac{\gamma(B)}{2\sqrt{1+\gamma(B)^2}}$; then, applying Lemma 3.2, $\gamma(A) \geqslant \frac{\gamma(B)}{2} > 0$, so that $A \in \mathcal{CR}$. ∎

We start the study of the continuity properties of μ, v and α with the metrics d_X. Observe that the continuity of α is obvious for both the norm topology and the topology induced by d_X.

THEOREM 4.2. *The mapping $\mu : (\mathcal{CR}, d_X) \to (\mathcal{CR}, \|\cdot\|)$ is continuous.*

Proof. From Remark 2.1 $\|A^\dagger - B^\dagger\| \leqslant (\|A^\dagger\|\|B^\dagger\| + \|A^\dagger\|^2 + \|B^\dagger\|^2)\|A - B\|$. Then, from Corollary 3.3, it follows that, if $d_X(A, B) < \frac{1}{2\sqrt{1+\|B^\dagger\|^2}}$ then $\|A^\dagger\| \leqslant 2\|B^\dagger\|^2$. Thus, $\|A^\dagger - B^\dagger\| \leqslant K\|A - B\|$, for a constant K that depends only on $\|B^\dagger\|$. ∎

COROLLARY 4.3. *The mapping $\gamma : (\mathcal{CR}, d_X) \to \mathbb{R}^+$ is continuous.*

Proof. The well known formula $\gamma(T) = \|T^\dagger\|^{-1}$, combined with the theorem above, proves the assertion. ∎

REMARK 4.4. The mapping $\gamma : (\mathcal{CR}, \|\cdot\|) \to \mathbb{R}^+$ is upper semicontinuous and γ is continuous at $B \in \mathcal{CR}(\mathcal{H},\mathcal{K})$ if and only if B is surjective or injective. A proof of these facts can be found in [33], [29] and [30].

THEOREM 4.5. *The mapping* $v : (\mathcal{CR}, d_X) \to (\mathcal{CR}, \|\cdot\|)$ *is continuous.*

Proof. By the properties of the polar decomposition, $v(B) = V_B = (B^*)^\dagger|B|$ so that the continuity of v follows from the continuity of α mentioned before and that of μ proved in Theorem 4.2. ∎

As a corollary we obtain the equivalence between d_R and d_N:

COROLLARY 4.6. *The identity map* id : $(\mathcal{CR}, d_X) \to (\mathcal{CR}, d_Y)$ *is continuous.*

Proof. Suppose $X=R$ and $Y=N$. By Theorem 4.2, $\mu : (\mathcal{CR}, d_R) \to (\mathcal{CR}, \|\cdot\|)$ is continuous. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P_{N(A)} - P_{N(B)}\| = \|A^\dagger A - B^\dagger B\| \leqslant \|A^\dagger\|\|A-B\| + \|A^\dagger - B^\dagger\|\|B\| < \varepsilon$ if $d_R(A,B) < \delta$. The case $X = N$ and $Y = R$ is analogous. ∎

COROLLARY 4.7. *The mapping* $\mu : (\mathcal{CR}, d_X) \to (\mathcal{CR}(\mathcal{K},\mathcal{H}), d_Y)$ *is continuous.*

In [32], S. Izumino extended several known results on the continuity of the map $\mu : A \mapsto A^\dagger$ on matrices to closed range operators between Hilbert spaces. In particular, he proved that, if $A_n, A \in \mathcal{CR}$ and $\|A_n - A\| \to 0$ then the following conditions are equivalent:

(1) $\|A_n^\dagger - A^\dagger\| \to 0$;
(2) $\|A_nA_n^\dagger - AA^\dagger\| \to 0$;
(3) $\|A_n^\dagger A_n - A^\dagger A\| \to 0$;
(4) $\sup \|A_n^\dagger\| < \infty$.

These results have been rediscovered many times and several authors have found other equivalent conditions. As a sample, let us mention two, one discovered in [39] (condition (5)) and the other found in [12]:

(5) $\gamma(A_n) \to \gamma(A)$;
(6) for n large it holds $R(A_n) \cap N(A^\dagger) = 0$.

In the next theorem we collect these and other equivalent conditions.

THEOREM 4.8. *Given B and $\{B_n\}_{n\in\mathbb{N}}$ in $\mathcal{CR}$, then the following conditions are equivalent:*

(i) $\lim_{n\to\infty} d_N(B_n, B) = 0$;
(ii) $\lim_{n\to\infty} d_R(B_n, B) = 0$;
(iii) $\lim_{n\to\infty} d_N(B_n^\dagger, B^\dagger) = 0$;
(iv) $\lim_{n\to\infty} d_R(B_n^\dagger, B^\dagger) = 0$;
(v) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and* $\lim_{n\to\infty} \|B_n^\dagger - B^\dagger\| = 0$;
(vi) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and there exists $M > 0$ such that for n large enough,* $\|B_n^\dagger\| \leqslant M$;

(vii) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and there exist* $M > 0$ *and* $B'_n \in \mathcal{CR}(\mathcal{K},\mathcal{H})$ *such that* $B_nB'_nB_n = B_n$ *and* $\|B'_n\| \leqslant M$*;*

(viii) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and there exists* $K>0$ *such that for n large enough* $\gamma(B_n) \geqslant K$*;*

(ix) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and* $\lim_{n\to\infty} \gamma(B_n) = \gamma(B)$*;*

(x) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and for n large enough,* $\mathcal{H} = N(B_n) + R(B^\dagger)$*;*

(xi) $\lim_{n\to\infty} \|B_n - B\| = 0$*, and for n large enough* $c_0(R(B_n^\dagger), N(B)) < 1$ *;*

(xii) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and for n large enough* $N(B) = (I - B^\dagger B)N(B_n)$*;*

(xiii) $\lim_{n\to\infty} \|B_n - B\| = 0$ *and for n large enough* $R(B_n) \cap N(B^\dagger) = \{0\}$.

Proof. (i) $\iff$ (ii) and (iii) $\iff$ (iv) follow from Corollary 4.6.

(i) $\Rightarrow$ (iii) is a consequence of Corollary 4.7. On the other hand, since $(B^\dagger)^\dagger = B$, we have that (iv) $\rightarrow$ (ii). Then (i), (ii), (iii) and (iv) are equivalent.

(i) $\Rightarrow$ (v) if $d_N(B_N, B) \to 0$ then $\|B_n - B\| \to 0$; also, (i) $\Rightarrow$ (iii) implies that $d_N(B_n^\dagger, B^\dagger) \to 0$, so that $\|B_n^\dagger - B^\dagger\| \to 0$.

(v) $\Rightarrow$ (i) Observe that $\|P_{N(B_n)} - P_{N(B)}\| = \|B_n^\dagger B_n - B^\dagger B\| \leqslant \|B_n^\dagger\|\|B_n - B\| + \|B_n^\dagger - B^\dagger\|\|B\|$ which tends to zero if $B_n \to B$ and $B_n^\dagger \to B^\dagger$.

(v) $\Rightarrow$ (vi) Since $\|B_n^\dagger - B^\dagger\| \to 0$ there exists $M > 0$ such that $\|B_n^\dagger\| \leqslant M$.

(vi) $\Rightarrow$ (v) follows from the proof of Theorem 4.2.

(vi) $\Rightarrow$ (vii) Take $B'_n = B_n^\dagger$.

(vii) $\Rightarrow$ (viii) If B'_n satisfies $B_nB'_nB_n = B_n$ and $\|B'_n\| \leqslant M$, for $M > 0$, applying (1) of Proposition 3.5, $\gamma(B_n) \geqslant \frac{1}{\|B'_n\|} \geqslant \frac{1}{M}$.

(vi) $\iff$ (viii) because $\gamma(B) = \|B^\dagger\|^{-1}$.

(v) $\Rightarrow$ (ix) follows from the continuity of $\|\cdot\|$ and $f(x) = x^{-1}$ in $\mathbb{R} -> 0$.

(ix) $\Rightarrow$ (viii) If $\gamma(B_n) \to \gamma(B)$, let $M > 0$ such that $\gamma(B) > M$, then $\gamma(B_n) > \frac{M}{2}$ for n large enough.

Then, (i), (v), (vi), (vii), (viii) and ix) are equivalent.

The equivalence between (i), (x), (xi) and (xii) follows from Proposition 2.4.

(xii) $\Rightarrow$ (xiii) Suppose that (xii) holds and consider $y \in R(B_n) \cap N(B^\dagger)$. Then $y = B_nx$, for $x \in \mathcal{H}$, and $B^\dagger y = B^\dagger B_n x = 0$, so that $B(I + B^\dagger(B_n - B))x = Bx + BB^\dagger(B_n - B)x = BB^\dagger B_n x = 0$. Therefore, $(I + B^\dagger(B_n - B))x \in N(B)$. Since $N(B) = (I - B^\dagger B)(N(B_n))$, there exists $w \in N(B_n)$ such that $(I + B^\dagger(B_n - B))x = (I - B^\dagger B)w = [I + B^\dagger(B_n - B)]w$. But, for n large enough, $I + B^\dagger(B_n - B)$ is invertible and then $x = w \in N(B_n)$. In this case $y = B_nx = 0$ so that (xiii) holds.

(xiii) $\Rightarrow$ (vii) If $B_n \to B$ then, for n large enough, the operators $G_1 = I + B^\dagger(B_n - B)$ and $G_2 = I + (B_n - B)B^\dagger$ are invertible. Set $A_n = G_1^{-1}B^\dagger = B^\dagger G_2^{-1}$. Then $N(A_n) = N(B^\dagger)$, $R(A_n) = R(B^\dagger)$ and $\|A_n\| \leqslant 2\|B^\dagger\|$, if $\|B_n - B\| < \frac{1}{2\|B^\dagger\|}$. Since $B^\dagger BG_1 = B^\dagger B_n$, we have that $B^\dagger B = B^\dagger B_n G_1^{-1}$. Therefore, $B^\dagger = B^\dagger BB^\dagger = B^\dagger B_n G_1^{-1} B^\dagger = B^\dagger B_n A_n$. Hence, $A_nB_nA_n = G_1^{-1}B^\dagger B_n A_n = G_1^{-1}B^\dagger = A_n$ and then $A_nB_nA_n = A_n$. On the other hand, if $x \in \mathcal{H}$, $y_n = (B_n - B_nA_nB_n)x = B_n(I - A_nB_n) \in R(B_n)$ and $y_n = (I - B_nA_n)B_nx \in N(A_n) = N(B^\dagger)$. Then

$y_n \in R(B_n) \cap N(B^\dagger) = \{0\}$ so that $B_n = B_n A_n B_n$ and A_n is a generalized inverse of B_n, such that $\|A_n\| \leqslant 2\|B^\dagger\|$. ∎

REMARK 4.9. By interchanging N with R, many other equivalent conditions can be added. Observe also that the angle condition can be stated in a uniform way, in the sense that there exists c, $0 \leqslant c < 1$ such that $c_0(N(B), R(B_n^\dagger)) \leqslant c$ for n large enough. On the other hand, the hypothesis of the theorem can be relaxed using the fact that $(\mathcal{CR}, d_X)$ is an open set of $(\mathcal{L}(\mathcal{H},\mathcal{K}), d_X)$. In fact if $d_X(B_n, B) \to 0$ and B has closed range, then every B_n, for n large enough, has also a closed range.

5. $\mathcal{CR}$ AS A HOMOGENEOUS SPACE

This section is devoted to study a homogeneous structure on $\mathcal{CR}$. For this, consider the action $L : \mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H} \times \mathcal{CR} \to \mathcal{CR}$ defined by the following, where $G \in \mathcal{G}_\mathcal{K}$, $H \in \mathcal{G}_\mathcal{H}$, and $A \in \mathcal{CR}$:

$$L((G,H), A) = L_{(G,H)}A = GAH^{-1}.$$

For any $A \in \mathcal{CR}$, the *orbit* of A by the action L is

$$\mathcal{O}_A = \{GAH^{-1} : G \in \mathcal{G}_\mathcal{K}, H \in \mathcal{G}_\mathcal{H}\}.$$

Observe that $\mathcal{O}_A = \mathcal{O}_B$ if $B \in \mathcal{O}_A$, because each orbit is an equivalence class: two operators are equivalent if they belong to the same orbit. By elementary spectral theory, the groups $\mathcal{G}_\mathcal{H}$ and $\mathcal{G}_\mathcal{K}$ are connected, moreover they are path connected. (Indeed, they are contractible; however, we do not need this deep result due to N. Kuiper [34].) Therefore, each orbit $\mathcal{O}_A$ is path connected. We are going to prove that $\mathcal{O}_A$ is the connected component of A in $(\mathcal{CR}, d_X)$.

The group $\mathcal{U}_\mathcal{K} \times \mathcal{U}_\mathcal{H}$ acts on $\mathcal{PI}$ by restriction of the action L. More precisely, $L' : \mathcal{U}_\mathcal{K} \times \mathcal{U}_\mathcal{H} \times \mathcal{PI} \to \mathcal{PI}$, defined by

$$L'((U,W), V) = UVW^*, \ U, \in \mathcal{U}_\mathcal{K},\ W \in \mathcal{U}_\mathcal{H},\ V \in \mathcal{PI},$$

is a left action on $\mathcal{PI}$. The orbits for this action are called *unitary orbits*. Thus, the *unitary orbit* of $V \in \mathcal{PI}$ is the set

$$\mathcal{UO}_V = \{UVW^* : U \in \mathcal{U}_\mathcal{K},\ W \in \mathcal{U}_\mathcal{H}\}.$$

The next results characterize the orbits of $\mathcal{CR}$ and $\mathcal{PI}$. For $k, \ell, m \in \mathbb{N} \cup \{0, \infty\}$ such that $k + \ell = \infty$ and $\ell + m = \infty$ define the sets:

$$\mathcal{A}_{k,\ell,m} = \{A \in \mathcal{CR} : \dim N(A) = k, \dim R(A) = \ell,\ \operatorname{codim} R(A) = m\},$$
$$\mathcal{V}_{k,\ell,m} = \{V \in \mathcal{PI} : \dim N(V) = k, \dim R(V) = \ell,\ \operatorname{codim} R(V) = m\}.$$

THEOREM 5.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional separable Hilbert spaces, and let $A \in \mathcal{A}_{k,\ell,m}$ and $V \in \mathcal{V}_{k,\ell,m}$. Then $\mathcal{O}_A = \mathcal{A}_{k,\ell,m}$ and $\mathcal{UO}_V = \mathcal{V}_{k,\ell,m}$.*

Proof. Consider $A, B \in \mathcal{A}_{k,\ell,m}$. Therefore, $\dim R(A) = \dim R(B)$ so that $\dim N(A)^\perp = \dim N(B)^\perp$ so that there exists an isomorphism $U : N(A)^\perp \to N(B)^\perp$. Consider $W : R(A) \to R(B)$ defined by $W = BUA^{-1}$ where $A^{-1} = \{A|_{N(A)^\perp}\}^{-1} : R(A) \to N(A)^\perp$. Then W is an isomorphism. Since $\operatorname{codim} R(A) = \operatorname{codim} R(B)$, there exists an isomorphism $V' : R(A)^\perp \to R(B)^\perp$. Define $G = WP_{R(A)} + V'(I - P_{R(A)})$; it holds $G \in \mathcal{G}_{\mathcal{K}}$. In the same way, since $\dim N(A) = \dim N(B)$, there exists an isomorphism $U' : N(A) \to N(B)$, and if $H = UP_{R(A^*)} + U'(I - P_{R(A^*)})$, then $H \in \mathcal{G}_{\mathcal{H}}$. Finally, $GAx = WP_{R(A)}Ax = WAx = BUA^{-1}Ax = BUP_{R(A^*)}x = B(UP_{R(A^*)}x + U'(I - P_{R(A^*)}x) = BHx$, because $U'(I - P_{R(A^*)})x \in N(B)$, for all $x \in \mathcal{H}$. Therefore $GA = BH$, or, $GAH^{-1} = B$, as claimed.

Conversely, if $B \in \mathcal{O}_A$, then there exists $G \in \mathcal{G}_{\mathcal{K}}$ and $H \in \mathcal{G}_{\mathcal{H}}$ such that $GA = BH$. Then $G(R(A)) = R(B)$ and $H(N(A)) = N(B)$. Also $A^*G^* = H^*B^*$, so that $R(B)^\perp = {G^*}^{-1}(R(A)^\perp)$. The proof for the partial isometries is analogous. ∎

An operator $B \in \mathcal{CR}$ is called *semi-Fredholm* if $\dim N(B)$ is finite or codim $R(B)$ is finite. Denote $SF_+ = \{T \in \mathcal{CR} : \dim N(T) < \infty\}$ and $SF_- = \{T \in \mathcal{CR} : \operatorname{codim} R(T) < \infty\}$. For $k < \infty$ or $m < \infty$, denote $SF_{k,m} = \{B \in \mathcal{CR} : \dim N(B) = k, \operatorname{codim} R(B) = m\}$.

For $B \in SF$, the set of all semi-Fredholm operators in $L(\mathcal{H}, \mathcal{K})$, define the index of B

$$\operatorname{ind}(B) = \dim N(B) - \operatorname{codim} R(B).$$

As it was pointed out in Remark 3.7 the interior of the set $\mathcal{CR}$ with the norm topology, in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, is exactly SF. On the other hand, the set SF is dense in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, with the norm topology: in fact, the set $\mathcal{M}$, defined in Section 3, verifies $\mathcal{M} \subset SF \subset \mathcal{L}(\mathcal{H}, \mathcal{K})$ and $\mathcal{M}$ is dense in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, (see [28]). Observe that, a fortiori, $\mathcal{CR}$ is dense in $\mathcal{L}(\mathcal{H}, \mathcal{K})$.

The connected components of SF are $\mathcal{F}_n = \{B \in SF : \operatorname{ind}(B) = n\}$, with $n \in \mathbb{Z} \cup \{-\infty, +\infty\}$, (see [11]). Moreover, the boundary of $\mathcal{F}_n$ in $\mathcal{L}(\mathcal{H}, \mathcal{K})$, $\partial\mathcal{F}_n$, does not depend on n. In fact, it coincides with $\mathcal{L}(\mathcal{H}, \mathcal{K}) \setminus SF$, see [39].

REMARK 5.2. If $A \in SF_{k,m}$ then $\mathcal{O}_A = SF_{k,m}$.

The next two results provide other characterization of $\mathcal{O}_A$. Both are based in techniques used in [14] and [15], where the main goal is the study of the congruence orbit of a positive operator.

PROPOSITION 5.3. *Let $A, B \in \mathcal{CR}$; consider the (reverse) polar decompositions of A and B, $A = |A^*|V_A$, $B = |B^*|V_B$. Then the following statements are equivalent:*

(i) $B \in \mathcal{O}_A$;

(ii) $P_{R(B)} \in \mathcal{UO}_{P_{R(A)}}$ *and* $P_{R(B^*)} \in \mathcal{UO}_{P_{R(A^*)}}$;

(iii) $V_B \in \mathcal{UO}_{V_A}$.

Proof. (i)$\Longrightarrow$ (ii) If $B \in \mathcal{O}_A$ then there exist $G \in \mathcal{G}_{\mathcal{K}}, H \in \mathcal{G}_{\mathcal{H}}$ such that $B = GAH^{-1}$. Then $R(B) = GR(A)$. Applying Theorem 3.1 of [25], there exists $U \in \mathcal{U}_{\mathcal{K}}$ such that $R(B) = UR(A)$ (and then $R(B)^{\perp} = U(R(A)^{\perp})$). Let $Q = UP_{R(A)}U^*$; then $Q \in \mathcal{L}(\mathcal{K})$ is the orthogonal projection onto $R(B)$, i.e., $Q = P_{R(B)} \in \mathcal{UO}_{P_{R(B)}}$. In a similar way, since $B^* = H^{-1*}A^*G^*$ there exists $W \in \mathcal{U}_{\mathcal{H}}$ such that $R(B^*) = WR(A^*)$. Then $P_{R(B^*)} = WP_{R(A^*)}W^*$ so that $P_{R(B^*)} \in \mathcal{UO}_{P_{R(A^*)}}$.

(ii) $\Longrightarrow$ (i) Conversely, suppose that, for $B \in \mathcal{CR}$, $P_{R(B)} \in \mathcal{UO}_{P_{R(A)}}$ and $P_{R(B^*)} \in \mathcal{UO}_{P_{R(A^*)}}$. Then there exist $U \in \mathcal{U}_{\mathcal{K}}$, $W \in \mathcal{U}_{\mathcal{H}}$ such that $UP_{R(A)}U^* = P_{R(B)}$ and $WP_{R(A^*)}W^* = P_{R(B^*)}$. Consider $G = A^{\dagger}U^*B + (I - P_{R(A^*)})W^*$; it holds $UAG = UP_{R(A)}U^*B = P_{R(B)}B = B$. It is easy to see that if $H = B^{\dagger}UA + W(I - P_{R(A^*)})$ then $H = G^{-1}$. Therefore, $B \in \mathcal{O}_A$.

(i) $\Longleftrightarrow$ (iii) In the same way, it is easy to see that if $V, V_0 \in \mathcal{PI}$, then $V \in \mathcal{UO}_{V_0}$ if and only if $P_{R(V)} \in \mathcal{UO}_{P_{R(V_0)}}$ and $P_{R(V^*)} \in \mathcal{UO}_{P_{R(V_0^*)}}$. But $P_{R(V_B)} = P_{R(B)}$ and $P_{R(V_B^*)} = P_{R(B^*)}$. Using again part (i) it follows that $B \in \mathcal{O}_A$ if and only if $V_B \in \mathcal{UO}_{V_A}$. ∎

COROLLARY 5.4. *If $A \in \mathcal{CR}$ has polar decomposition $A = |A^*|V_A$ then $\mathcal{O}_A = \mathcal{O}_{V_A}$.*

Proof. Consider $G = |A^{\dagger^*}| + I - P_{R(A)}$. Then $G \in \mathcal{G}_{\mathcal{K}}$, $G^{-1} = |A^*| + I - P_{R(A)}$ and also $GA = V_A$; therefore, $V_A \in \mathcal{O}_A$, so that $\mathcal{O}_A = \mathcal{O}_{V_A}$. ∎

For a fixed $A \in \mathcal{CR}$, consider the mapping $\varphi : \mathcal{CR} \to \mathcal{P}_{\mathcal{K}} \times \mathcal{P}_{\mathcal{H}}$ defined by

$$\varphi(B) = (\varphi_1(B),\ \varphi_2(B)) = (BB^{\dagger}, B^{\dagger}B) = (P_{R(B)}, P_{R(B^*)}).$$

Then we have the following fact:

PROPOSITION 5.5. *The image of φ is the product $\mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}$.*

Proof. By the above proposition $\varphi(\mathcal{O}_A) \subset \mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}$. Conversely, if $(P, Q) \in \mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}$ there exist $U \in \mathcal{U}_{\mathcal{K}}, W \in \mathcal{U}_{\mathcal{H}}$ such that $P = UP_{R(A)}U^*$ and $Q = WP_{R(A^*)}W^*$. Let $B = UAW^*$; then $B \in \mathcal{O}_A$, $B^{\dagger} = WA^{\dagger}U^*$, $P_{R(B)} = P$ and $P_{R(B^*)} = Q$. Therefore, $(P, Q) = \varphi(B)$. ∎

Consider also the mappings

$\pi_A : \mathcal{G}_{\mathcal{K}} \times \mathcal{G}_{\mathcal{H}} \to \mathcal{O}_A,\ \pi_A(G, H) = L_{(G,H)}A = AGH^{-1}, \quad G \in \mathcal{G}_{\mathcal{K}},\ H \in \mathcal{G}_{\mathcal{H}},$

$\Pi_A : G_{\mathcal{K}} \times G_{\mathcal{H}} \to \mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}},\ \Pi_A(G, H) = (P_{G(R(A))}, P_{H(N(A))^{\perp}}), G \in \mathcal{G}_{\mathcal{K}}, H \in \mathcal{G}_{\mathcal{H}}.$

It is apparent that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{G}_{\mathcal{K}} \times \mathcal{G}_{\mathcal{H}} \arrow[r, "\pi_A"] \arrow[dr, "\Pi_A"'] & \mathcal{O}_A \arrow[d, "\varphi"] \\
& \mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}
\end{tikzcd}$$

Notice that with the norm topology on $\mathcal{O}_A$, the mapping φ is not continuous and π_A does not have continuous local sections. However, the following result permits a finer understanding of the structure of each orbit.

PROPOSITION 5.6. *The mapping* $\varphi : (\mathcal{O}_A, d_R) \to \mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}$ *is continuous.*

Proof. The result follows from the equivalence of d_R and d_N stated in Corollary 4.6. ∎

PROPOSITION 5.7. *The map* $\pi_A : (\mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H}, \|\cdot\|) \to (\mathcal{O}_A, d_R)$ *is continuous and it admits continuous local cross sections.*

Proof. Observe that the continuity of $\pi_A : (\mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H}, \|\cdot\|) \to (\mathcal{O}_A, d_R)$ is equivalent to the continuity of $\pi_A : (\mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H}, \|\cdot\|) \to (\mathcal{O}_A, \|\cdot\|)$, which is evident, and that of $\Pi_A : (\mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H}, \|\cdot\|) \to (\mathcal{UO}_{P_{R(A)}} \times \mathcal{UO}_{P_{R(A^*)}}, \|\cdot\|)$. The orthogonal projection onto $G(R(A))$ is given by the formula

$$P_{G(R(A))} = GP_{R(A)}G^{-1}(GP_{R(A)}G^{-1})^*(I - (GP_{R(A)}G^{-1} - (GP_{R(A)}G^{-1})^*)^2)^{-1};$$

this shows that $P_{G(R(A))}$ depends continuously on G, see [1]. In the same way the orthogonal projection onto $H(N(A))^\perp = H^{-1}(N(A)^\perp)$ depends continuously on H; therefore $\Pi_A(G, H) = (P_{G(N(A))}, P_{H(N(A))^\perp})$ is continuous.

In order to prove that π_A admits local cross sections, observe that there exists a neighbourhood $\mathcal{N}$ of A in $\mathcal{O}_A$, such that if $B \in \mathcal{N}$ and

$$\sigma(B) = (BA^\dagger + (I - P_{R(B)})(I - P_{R(A)}), P_{R(B^\dagger)}P_{R(A^\dagger)} + (I - P_{R(B^*)})(I - P_{R(A^*)}))$$

then $\sigma : (\mathcal{N}, d_R) \to \mathcal{G}_\mathcal{K} \times \mathcal{G}_\mathcal{H}$ is continuous. Also $\pi_A(\sigma(B)) = B$, for all $B \in \mathcal{N}$. See 2.1 of [2] for details. Therefore σ is a continuous local cross section of π_A in $\mathcal{N}$. ∎

REMARK 5.8. Suppose that the topological group $\mathcal{G}$ acts over the topological space $\mathcal{X}$ on the left, with the property that each $x_0 \in \mathcal{X}$ has a open neighborhood $\mathcal{W}$ with a continuous section $\sigma : \mathcal{W} \to \mathcal{G}$ of π_{x_0} (here $\pi_{x_0}(G) = G \cdot x_0 = L_G x_0$ for each $G \in \mathcal{G}$). Then every orbit $\mathcal{O}_{x_0} = \{L_G x_0 : G \in \mathcal{G}\}$ is open and closed in $\mathcal{X}$; it is open because of the existence of the local section σ, and if every orbit is open then it is automatically closed. From these comments, the next two results follow easily.

COROLLARY 5.9. *The connected component of* A *in* $(\mathcal{CR}, d_X)$ *is* $\mathcal{O}_A$.

COROLLARY 5.10. *For every* $A \in \mathcal{CR}$, *the orbit* $\mathcal{O}_A$, *with the* d_X*-topology, is a homogeneous space of* $\mathcal{G}_\mathcal{H} \times \mathcal{G}_\mathcal{K}$.

We finish the section with a computation of the distance between different orbits. Mbekhta and Skhiri [42], following the characterization of Halmos and McLaughlin [29] of the components of $\mathcal{PI}(\mathcal{H}, \mathcal{K})$, have computed the distance between the orbits of $\mathcal{PI}$ with the operator norm. Here we follow the same program for the orbits of $\mathcal{CR}$ with the d_R and d_N metrics.

THEOREM 5.11. *Consider $A, B \in \mathcal{CR}$ such that $B \notin \mathcal{O}_A$. Then*

$$d_R(\mathcal{O}_A, \mathcal{O}_B) = \begin{cases} 0 & \text{if } \dim R(A) = \dim R(B) \text{ and } \operatorname{codim} R(A) = \operatorname{codim} R(B), \\ 1 & \text{if } \dim R(A) \neq \dim R(B) \text{ or } \operatorname{codim} R(A) \neq \operatorname{codim} R(B). \end{cases}$$

Proof. First observe that $d_R(\mathcal{O}_A, \mathcal{O}_B) = \inf\{\|P_{R(A')} - P_{R(B')}\|\ A' \in \mathcal{O}_A, B' \in \mathcal{O}_B\}$: in fact, if $d = \inf\{\|P_{R(A')} - P_{R(B')}\|\ A' \in \mathcal{O}_A, B' \in \mathcal{O}_B\}$, it holds $d \leqslant d_R(\mathcal{O}_A, \mathcal{O}_B)$. To prove the converse inequality consider $\varepsilon > 0$, then there exist $A' \in \mathcal{O}_A$ and $B' \in \mathcal{O}_B$ such that $d \leqslant \|P_{R(A')} - P_{R(B')}\| < d + \varepsilon$. Consider

$$A'' = \frac{\varepsilon}{2(\|A'\| + \|B'\|)} A' \quad \text{and} \quad B'' = \frac{\varepsilon}{2(\|A'\| + \|B'\|)} B'.$$

Then $A'' \in \mathcal{O}_A$ and $B'' \in \mathcal{O}_B$; also $d_R^2(A'', B'') = \|A'' - B''\|^2 + \|P_{R(A'')} - P_{R(B'')}\|^2 \leqslant \frac{\varepsilon^2}{4} + (d + \varepsilon)^2 \leqslant d^2 + \varepsilon k$, for a constant k. Therefore $d_R(\mathcal{O}_A, \mathcal{O}_B) \leqslant d$.

Suppose that $\dim R(A) = \dim R(B)$ and $\operatorname{codim} R(A) = \operatorname{codim} R(B)$. Then $\dim N(B) \neq \dim N(A)$. Define $B' \in L(\mathcal{H}, \mathcal{K})$ as follows: $B'|_{N(B)^\perp} : N(B)^\perp \to R(A)$ is an isomorphism, $N(B') = N(B)$. Then $R(B') = R(A)$ so that $B' \in \mathcal{CR}$; moreover $B' \in \mathcal{O}_B$, by its construction, and $P_{R(B')} = P_{R(A)}$. Therefore, $d_R(\mathcal{O}_A, \mathcal{O}_B) = 0$, by the remark at the beginning of the proof.

If there exist $A' \in \mathcal{O}_A$ and $B' \in \mathcal{O}_B$ such that $\|P_{R(A')} - P_{R(B')}\| < 1$ it easily follows that $\dim R(A') = \dim R(B')$ and $\operatorname{codim} R(A') = \operatorname{codim} R(B')$. If $\dim R(A') \neq \dim R(B')$ or $\operatorname{codim} R(A') \neq \operatorname{codim} R(B')$, it holds $\|P_{R(A')} - P_{R(B')}\| = 1$ and the theorem follows. ∎

COROLLARY 5.12. *Consider $A, B \in \mathcal{CR}$ such that $B \notin \mathcal{O}_A$. Then*

$$d_N(\mathcal{O}_A, \mathcal{O}_B) = \begin{cases} 0 & \text{if } \dim N(A) = \dim N(B) \text{ and } \operatorname{codim} N(A) = \operatorname{codim} N(B), \\ 1 & \text{if } \dim N(A) \neq \dim N(B) \text{ or } \operatorname{codim} N(A) \neq \operatorname{codim} N(B). \end{cases}$$

Proof. The result follows easily applying Theorem 5.11 to A^* and B^* and observing that $d_N(A, B) = d_R(A^*, B^*)$. ∎

REMARK 5.13. It is possible to estimate the d_X-distance between unitary orbits of partial isometries, using the results obtained in [42] by Mbekhta and Skhiri to compute the distance between these orbits, with the operator norm.

6. THE SET $\mathcal{CR}_\mathcal{S}$

In this section $\mathcal{S}$ is a fixed closed subspace of $\mathcal{K}$ and $\mathcal{CR}_\mathcal{S}$ denotes the subset of $\mathcal{CR}$ of all operators with range $\mathcal{S}$.

Observe, first, that $\mathcal{CR}_\mathcal{S} = \varphi_1^{-1}(\{P_\mathcal{S}\})$, where $\varphi_1(B) = BB^\dagger$. Also the metric d_R obviously coincides with the metric given by the uniform operator norm on $\mathcal{CR}_\mathcal{S}$ because $R(A) = R(B) = \mathcal{S}$ for every A, $B \in \mathcal{CR}_\mathcal{S}$. In what follows, $\mathcal{G}_\mathcal{S}$ shall be identified with the subgroup of $\mathcal{G}_\mathcal{K}$ consisting of all operators in $\mathcal{L}(\mathcal{K})$ of the form $G'(x+y) = Gx + y$, for $G \in \mathcal{G}_{\mathcal{K}}$ and $x \in \mathcal{S}, y \in \mathcal{S}^{\perp}$. Consider the restriction of the action L, defined in Section 5, where $G \in \mathcal{G}_{\mathcal{S}}$, $H \in \mathcal{G}_{\mathcal{H}}$ and $B \in \mathcal{CR}_{\mathcal{S}}$:

$$L_{\mathcal{S}} : \mathcal{G}_{\mathcal{S}} \times \mathcal{G}_{\mathcal{H}} \times \mathcal{CR}_{\mathcal{S}} \to \mathcal{CR}_{\mathcal{S}},$$
$$((G,H),B) \to GBH^{-1}.$$

For $B \in \mathcal{CR}_{\mathcal{S}}$, denote by $\mathcal{O}_{B,\mathcal{S}}$ the orbit of B given by the action $L_{\mathcal{S}}$, i.e., (where obviously, $\mathcal{O}_{B,\mathcal{S}}$ is a subset of $\mathcal{O}_B$):

$$\mathcal{O}_{B,\mathcal{S}} = \{GBH^{-1} : G \in \mathcal{G}_{\mathcal{S}}, H \in \mathcal{G}_{\mathcal{H}}\}.$$

PROPOSITION 6.1. *Consider $B \in \mathcal{CR}_{\mathcal{S}}$. If* $\dim N(B) = k \in \mathbb{N} \cup \{\infty\}$, *then*

$$\mathcal{O}_{B,\mathcal{S}} = \{C \in \mathcal{CR}_{\mathcal{S}} : \dim N(C) = k\}.$$

Proof. If $C \in \mathcal{CR}_{\mathcal{S}}$ and $\dim N(C) = k$ then $C \in \mathcal{O}_B$: in fact $\dim R(C) = \dim S = \dim R(B)$ and $\operatorname{codim} R(C) = \operatorname{codim} \mathcal{S} = \operatorname{codim} R(B)$. Therefore there exist $G \in \mathcal{G}_{\mathcal{K}}$ and $H \in \mathcal{G}_{\mathcal{H}}$ such that $C = GBH^{-1}$. Observe that $G(\mathcal{S}) = R(C) = \mathcal{S}$ and defined if $G' = GP + I - P$, where $P = P_{\mathcal{S}}$, then $G' \in \mathcal{G}_{\mathcal{S}}$ and $C = G'BH^{-1}$, which shows that $C \in \mathcal{O}_{B,\mathcal{S}}$.

Conversely, if $C \in \mathcal{O}_{B,\mathcal{S}}$, it follows that $C \in \mathcal{O}_B$ so that, by Proposition 4.1, $\dim N(C) = k$. ∎

Observe that

$$\sigma(C) = (CB^{\dagger} + I - P, P_{R(C^*)}P_{R(B^*)} + (I - P_{R(C^*)})(I - P_{R(B^*)}))$$

is a continuous local cross section in a neighbourhood of $B \in \mathcal{CR}_{\mathcal{S}}$ (see the proof of Proposition 5.10), because d_N defines the norm topology in $\mathcal{CR}_{\mathcal{S}}$.

In what follows we characterize $\mathcal{CR}_{\mathcal{S}}$ as a product space of two homogeneous spaces; this characterization naturally induces a different structure of homogeneous space on $\mathcal{CR}_{\mathcal{S}}$.

For $A \in \mathcal{L}(\mathcal{K})^+$ the *Thompson component* of A is defined as

$$\mathcal{C}_A = \{B \in \mathcal{L}(\mathcal{K})^+ : A \leqslant \beta B \text{ and } B \leqslant \alpha A, \text{ for } \alpha, \beta > 0\}.$$

This notion, introduced by A.C. Thompson [51], has been extremely useful in the analytical study of cones in Banach spaces. The reader is referred to the paper by R. Nussbaum [43] for many applications of Thompson components.

If $A \in \mathcal{CR}(\mathcal{K})^+$ has closed range, then $\mathcal{C}_A = \{B \in \mathcal{L}(\mathcal{K})^+ : R(B) = R(A)\}$, see [14], [15], so that the component of A only depends on the range of A. Observe that the map μ is continuous on each component $\mathcal{C}_A$.

Denote $\mathcal{PI}_{\mathcal{S}} = \{V \in \mathcal{CR} : VV^* = P\}$ where $P = P_{\mathcal{S}}$, i.e., $\mathcal{PI}_{\mathcal{S}}$ is the set of partial isometries with fixed range $\mathcal{S}$.

PROPOSITION 6.2. *$\mathcal{CR}_{\mathcal{S}}$ is homeomorphic to $\mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$.*

Proof. Let $B \in \mathcal{CR}_{\mathcal{S}}$ and let $B = |B^*|V$ be the reverse polar decomposition of B. Then $R(V) = R(|B^*|) = \mathcal{S}$, so that $V \in \mathcal{PI}_{\mathcal{S}}$.

Define $f : \mathcal{CR}_{\mathcal{S}} \to \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$, $f(B) = (|B^*|, |B^*|^\dagger B)$. Then f is continuous because $|B^*| \in \mathcal{C}_P$ and the Moore–Penrose pseudoinverse is continuous on every Thompson component. Observe that for every $A \in \mathcal{C}_P, V \in \mathcal{PI}_{\mathcal{S}}$ it holds $f^{-1}(A, V) = AV$, which is continuous. Then f is a homeomorphism which allows the identification of both sets. ∎

The subsets $\mathcal{C}_P$ and $\mathcal{PI}_{\mathcal{S}}$ of $\mathcal{CR}(\mathcal{K})$ and $\mathcal{CR}$, respectively, have been both studied as homogeneous spaces of certain subgroups of $\mathcal{G}_{\mathcal{K}}$, and $\mathcal{G}_{\mathcal{H}}$, respectively, see [14], [1]. More precisely, the subgroup $\mathcal{G}_{\mathcal{S}}$ defined before is a subgroup of $\mathcal{G}_{\mathcal{K}}$ acting on $\mathcal{C}_P$: define $L_1 : \mathcal{G}_{\mathcal{S}} \times \mathcal{C}_P \to \mathcal{C}_P$, $L_1(G, B) = GBG^*$, $G \in \mathcal{G}_{\mathcal{S}}$, $B \in \mathcal{C}_P$. The unitary group $\mathcal{U}_{\mathcal{H}}$ acts on $\mathcal{PI}_{\mathcal{S}}$: define $L_2 : \mathcal{U}_{\mathcal{H}} \times \mathcal{PI}_{\mathcal{S}} \to \mathcal{PI}_{\mathcal{S}}$, $L_2(U, V) = VU^*$, $V \in \mathcal{PI}_{\mathcal{S}}$, $U \in \mathcal{U}_{\mathcal{H}}$. The pairs $(\mathcal{G}_{\mathcal{S}}, \mathcal{C}_P)$ and $(\mathcal{U}_{\mathcal{H}}, \mathcal{PI}_{\mathcal{S}})$ are both homogeneous spaces (see [14], [16], [1], [2]).

Then $\mathcal{CR}_{\mathcal{S}}$ admits a natural structure of homogeneous space of $\mathcal{G}_{\mathcal{S}} \times \mathcal{U}_{\mathcal{H}}$: consider the identification of $\mathcal{CR}_{\mathcal{S}}$ with $\mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$ and define the action

$$L' : (\mathcal{G}_{\mathcal{S}} \times \mathcal{U}_{\mathcal{H}}) \times (\mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}) \to \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$$

by

$$L'((G, U), (A, V)) = L'_{(G,U)}(A, V) = (L_1(G, A), L_2(U, V)) = (GAG^*, VU^*),$$

for $G \in \mathcal{G}_{\mathcal{S}}$, $U \in \mathcal{U}_{\mathcal{H}}$, $(A, V) \in \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$. The action L' is locally transitive because L_1 and L_2 are both locally transitive. In fact, since L_1 is transitive on $\mathcal{C}_P$, the orbit of a pair $(B, V) \in \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$ is $\mathcal{C}_P \times \mathcal{O}_V$, where $\mathcal{O}_V$ is the orbit of V by the action L_2. In fact:

PROPOSITION 6.3. *Consider $B \in \mathcal{CR}_{\mathcal{S}}$ with* $\dim N(B) = k$. *Then the orbit $\mathcal{O}'_B$ of B by the action L' coincides with $\mathcal{O}_{B,\mathcal{S}}$.*

Proof. Consider $C \in \mathcal{O}'_B = \mathcal{C}_{|B^*|} \times \mathcal{O}_{V_B}$. Then, there exist $G \in \mathcal{G}_{\mathcal{S}}$ and $U \in \mathcal{U}$ such that $C = G|B^*|G^* V_B U^*$. It is easy to see that $N(C) = UN(B)$, so that $\dim N(C) = \dim N(B) = k$. The converse follows as in Proposition 5.1. ∎

Fix the pair $(P, W) \in \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$ and define, for $G \in \mathcal{G}_{\mathcal{S}}$, $U \in \mathcal{U}_{\mathcal{H}}$,

$$\pi : \mathcal{G}_{\mathcal{S}} \times \mathcal{U}_{\mathcal{H}} \to \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}},\ \pi(G, U) = L'_{(G,U)}(P, W) = (GPG^*, WU^*).$$

The map π admits local cross sections. In fact, let $(B, V) \in \mathcal{C}_P \times \mathcal{PI}_{\mathcal{S}}$; there exists a neighbourhood $\mathbb{N}$ of W in $\mathcal{PI}_{\mathcal{S}}$ such that $\sigma(B, V) = (B^{1/2} + I - P, V^*W + (I - V^*V)(I - W^*W))$, is well defined, $\sigma : \mathcal{C}_P \times \mathbb{N} \to \mathcal{G}_{\mathcal{S}} \times \mathcal{U}_{\mathcal{H}}$ and $\pi(\sigma(B, V)) = (B, V)$, for $(B, V) \in \mathcal{C}_P \times \mathbb{N}$ (see [2] for details).

REMARK 6.4. The homogeneous structure is extremely useful in the differential geometry of the orbits and also of $\mathcal{CR}_{\mathcal{S}}$. This study will be done elsewhere.

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