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A SIMPLE C^*-ALGEBRA WITH PERFORATION AND THE CORONA FACTORIZATION PROPERTY

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ABSTRACT. The corona factorization property is connected with the KK-theory and the behaviour of stability for C^*-algebras. We show that there exists a simple C^*-algebra with perforation and the corona factorization property.

KEYWORDS: *C^*-algebra, Villadsen algebra, corona factorization.*

MSC (2000): 46L05, 46L05, 19K35.

1. INTRODUCTION

The corona factorization property characterizes stable separable C^*-algebras B whose Kasparov $KK^1(A, B)$ groups can be described in a certain elegant form, and moreover implies good behaviour with respect to formation of extensions by stable separable C^*-algebras.

A quarter-century ago, Kasparov showed [9] that the now well-known and important $KK^1(A, B)$ group — a bivariant form of K-theory for nuclear separable C^*-algebras — could be written as the group of absorbing full extensions of A by B, modulo a suitable form of unitary equivalence. (Recall that an extension $\tau : A \longrightarrow \mathcal{M}(B)/B$ is defined to be *absorbing* if $\tau + \sigma$ is unitarily equivalent to τ for all trivial extensions σ.) Kasparov's theorem is useful in proportion to our ability to decide which extensions of a given A by a given B are absorbing. It is generally very difficult to decide this problem by direct application of the definition of the absorption property.

Elliott and Kucerovsky [6] gave the first algebraic criterion for a given extension to be absorbing, but this criterion was applicable to a single (nuclear) extension at a time. We should mention that Kasparov's original definition of $KK^1(A, B)$ has been slightly modified by Skandalis by shifting the nuclearity condition from the algebras to the Busby maps, so that the proof of Kasparov's theorem needs to be modified slightly. With or without this modification, Kasparov's theorem becomes maximally useful in the case that all full (nuclear) extensions of

A by B are absorbing. An injective extension is said to be full if the image does not nontrivially intersect any ideal of the corona. Kucerovsky and Ng [15] studied the case of Rørdam's group $KL^1(A,B)$, and proved the following equivalence of three conditions:

THEOREM 1.1. *Let A and B be separable nuclear C^*-algebras, and let B be stable. The following are equivalent:*

(i) *Every norm-full projection in $\mathcal{M}(B)$ is Murray–von Neumann equivalent to $1_{\mathcal{M}(B)}$.*

(ii) *$KK^1(A,B)$ is equal to the set of full extensions $\tau : A \longrightarrow \mathcal{M}(B)/B$ modulo unitary equivalence.*

(iii) *$KL^1(A,B)$ is equal to the set of full extensions $\tau : A \longrightarrow \mathcal{M}(B)/B$ modulo a form of approximate unitary equivalence.*

The implications (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) are easy applications of the abovementioned Elliott–Kucerovsky result, and the implication (iii) $\Longrightarrow$ (i) rests on a careful study of the class of the identity in KL^1 and some results of Schochet's. Property (i) in the above list is defined to be the *corona factorization* property.

The above theorem can be viewed as giving an algebraic characterization of the basically topological property of having a "nice" KK-group (in the sense of property (ii) of the theorem). We moreover found a connection between the KK-theoretical corona factorization property and the C^*-algebraic property of stability. Recall that a C^*-algebra is said to be *stable* if B is isomorphic to $B \otimes \mathcal{K}$, where $\mathcal{K}$ is a copy of the usual compact operators on an infinite-dimensional separable Hilbert space. It is known that, due to the pioneering work of Rørdam [22], [21], that if $M_n(B)$ is stable, B need not be stable, and that if A and B in a short exact sequence $0 \longrightarrow B \longrightarrow C \longrightarrow A \longrightarrow 0$ are both stable, the extension algebra C need not be stable. We have found that if B has the corona factorization property, then neither of these problems occur.

In [14], we prove the following:

THEOREM 1.2. *Suppose that B is a separable, stable C^*-algebra. Then the following are equivalent:*

(i) *B has the corona factorization property.*

(ii) *Suppose that $\mathcal{D}$ is a full, hereditary subalgebra of B. Suppose that there is an integer $n \geqslant 1$ such that $\mathbb{M}_n(\mathcal{D})$ is stable. Then $\mathcal{D}$ itself is stable.*

THEOREM 1.3. *Suppose that $J, \mathcal{E}$ and A are separable C^*-algebras, such that $J \otimes \mathcal{K}$ has the corona factorization property. Suppose that we have an exact sequence of the form*

$$0 \to J \to \mathcal{E} \to A \to 0.$$

Then $\mathcal{E}$ is stable if and only if J and A are stable.

If X is a finite-dimensional, compact metric space, then $C(X) \otimes \mathcal{K}$ has the corona factorization property. More generally, if B_0 is a separable C^*-algebra with finite decomposition rank, then $B_0 \otimes \mathcal{K}$ has the corona factorization property.

Finally, we mention Rørdam's stronger notion of regularity, which fits naturally into the subject — though many questions remain.

DEFINITION 1.4. Let B be a separable, stable C^*-algebra. Then B is said to be *regular*, if whenever $\mathcal{D}$ is a full, hereditary subalgebra of B with no nonzero unital quotients and no nonzero bounded traces, $\mathcal{D}$ is stable.

Presently, all examples of C^*-algebras that are known to have the corona factorization property also have regularity, and vice versa.

Among other things, there is the following result:

PROPOSITION 1.5. *Suppose that B is a separable, stable C^*-algebra that is regular. Then*

(i) *B has the corona factorization property.*
(ii) *B is either stably finite or purely infinite.*

Part (ii) is due to Rørdam, and part (i) is straightforward.

In view of these results, it is natural to wonder if C^*-algebras with the corona factorization property can have perforation.

In this paper, we prove the following:

THEOREM 1.6. *There exists a simple C^*-algebra with perforation and the corona factorization property.*

2. A SIMPLE C^*-ALGEBRA WITH PERFORATION AND CORONA FACTORIZATION

The C^*-algebra that we work with is a form of Villadsen's example of a simple, unital AH-algebra with higher stable rank [26]. His algebra is in fact an inductive limit of blocks of the form $p_i(C(X_i) \otimes \mathcal{K})p_i$, where X_i is a connected finite CW-complex and where p_i is a (necessarily constant rank) projection in $C(X_i) \otimes \mathcal{K}$. It is well-known that such building blocks can also be regarded as an algebra generated by sections of a vector bundle, with the vector bundle being trivial if and only if p_i is equivalent to a constant projection in $C(X_i) \otimes \mathcal{K}$. More generally, stable isomorphism classes of vector bundles correspond to K_0-classes of projections in $C(X_i) \otimes \mathcal{K}$. The connecting maps $\phi_{i,i+1} : p_i(C(X_i) \otimes \mathcal{K})p_i \longrightarrow p_{i+1}(C(X_{i+1}) \otimes \mathcal{K})p_{i+1}$ have the property that they map a projection of rank k to a projection of rank $k \cdot (i+2)$. The X_i are in our case the topological spaces

$$I^2 \times \mathbb{C}P^{1\cdot 1!} \times \mathbb{C}P^{2\cdot 2!} \times \cdots \times \mathbb{C}P^{i\cdot i!},$$

where I is the unit interval (hence, I^2 is the unit square) and $\mathbb{C}P^k$ denotes complex projective k-space. Thus X_i has dimension $2(i+1)!$. Finally, the map π^1_{i+1} is simply the natural projection from X_{i+1} to X_i.

One may wonder why complex projective spaces appear. In order to obtain interesting behaviour, it is desirable to use vector bundles with a great deal of twist. It is known from the study of Grassmannians in differential topology that the tautological line bundle γ over $\mathbb{C}P^\infty$ is the maximally twisted line bundle [2]: that is, every line bundle over a manifold of finite type is the pullback of a line bundle over $\mathbb{C}P^\infty$. This point of view is used effectively in, for example, the counterexample to Conjecture 7 of [13].

We now give more detail, for the reader's convenience, on the construction of Villadsen's algebra [25], [26].

For each integer $k \geqslant 1$, let $\sigma(k) := k \cdot k!$. Let, $X_0 = I^2$, and $X_{i+1} = X_i \times \mathbb{C}P^{\sigma(i+1)}$. Let

$$\pi^1_{i+1} : X_{i+1} \to X_i, \quad \pi^2_{i+1} \to \mathbb{C}P^{\sigma(i+1)}$$

denote the natural coordinate projections. Let γ_k be the universal line bundle over $\mathbb{C}P^k$ and let $\zeta_i := \pi_i^{2*}(\gamma_{\sigma(i)})$. Choose a dense sequence $\{s_i^l\}_{l=1}^\infty$ in X_i, and choose for each $j = 1, 2, \ldots, i+1$ a point $t_{i,j} \in X_i$ such that $t_{i,i+1} = s_i^1, t_{i,i} = s_i^2$, and $\pi^1_{j+1} \circ \pi^1_j \circ \cdots \circ \pi^1_i(t_{i,j}) = s_j^{i-j+2}$ for $1 \leqslant j \leqslant i-1$. The sequences $\{s_i^l\}_{l=1}^\infty$ ($1 \leqslant i < \infty$) have to be chosen simultaneously to satisfy both a density condition and the appropriate compatibility conditions, used to obtain simplicity of the algebra constructed. Now for each i, let q_i be a projection in $C(X_i) \otimes \mathcal{K}$ corresponding to the vector bundle ζ_i. Let $q_{i,1}, q_{i,2}, \ldots, q_{i,i}$ be pairwise orthogonal projections in $C(X_i) \otimes \mathcal{K}$, such that for each j, $q_{i,j}$ is Murray–von Neumann equivalent to q_i in $C(X_i) \otimes \mathcal{K}$. Now let $\widetilde{\phi}_i : C(X_i) \otimes \mathcal{K} \to C(X_{i+1}) \otimes \mathcal{K}$ be the $*$-homomorphism given by

$$\widetilde{\phi}_i := (\mathrm{id}_{C(X_{i+1})} \otimes \alpha) \circ (\widetilde{\widetilde{\phi}}_i \otimes \mathrm{id}_{\mathcal{K}}),$$

where α is some isomorphism of $\mathcal{K} \otimes \mathcal{K}$ with $\mathcal{K}$ and

$$\widetilde{\widetilde{\phi}}_i : C(X_i) \to C(X_{i+1}) \otimes \mathcal{K} : f \mapsto (f \circ \pi^1_{i+1})\theta_{i+1} + \sum_{j=1}^{i+1} f(t_{i,j}) q_{i+1,j}.$$

Here θ_{i+1} is a projection in $C(X_{i+1}) \otimes \mathcal{K}$ corresponding to the (complex) one-dimensional trivial bundle over X_{i+1}, such that θ_1 is orthogonal to $q_{i+1,j}$ for every j.

Now let $p_0 \in C(X_0) \otimes \mathcal{K}$ be a projection which corresponds to the (complex) one-dimensional trivial bundle; and for each $i \geqslant 1$, let $p_i := \widetilde{\phi}_{i,0}(p_0)$. For each $i \geqslant 0$, let $A_i := p_i(C(X_i) \otimes \mathcal{K})p_i$ and let $\phi_{i,i+1} : A_i \to A_{i+1}$ be the restriction of $\widetilde{\phi}_i$. Now let A denote the inductive limit C^*-algebra $A := \varinjlim(A_i, \phi_{i,i+1})$.

In [26], Villadsen has proven that:

THEOREM 2.1. *A is a unital, simple* AH*-algebra with perforation and stable rank* 2.

Let $B := A \otimes \mathcal{K}$ be the stabilization of A. We first give the properties of A and B that we shall need.

(i) The algebra A is simple, unital, and has a unique tracial state, τ_A. Thus, B has a unique (up to a scalar multiple) semicontinuous trace, $\tau_A \otimes \tau_{\mathcal{K}}$. Also, τ_A

can be extended, by a construction with approximate units, to τ on the positive elements of the multiplier algebra $\mathcal{M}(A \otimes \mathcal{K})$. Since on positive elements these traces are in fact the same except for domain (if we regard A as the e_{11} corner inside B), we may as well denote all of them by τ.

(ii) The dimension of X_i is $2(i+1)!$.

(iii) The rank of p_i is $(i+1)!$.

We shall use a couple of results from topological K-theory. Recall that one of the main notions of this theory is stable equivalence of vector bundles: Two vector bundles over a compact metric space X are said to be *stably equivalent* or *stably isomorphic*, if they become isomorphic after addition (to both) of some trivial vector bundle. The following result is from Section 8.1 of [8], Theorem 1.2 and Theorem 1.5:

LEMMA 2.2. *Let ξ_2 and ξ_2 be stably isomorphic complex vector bundles, of fibre dimension r, over some finite CW complex of topological dimension d. Let $k := \left\lceil \frac{d}{2} \right\rceil$ (the least integer greater than or equal to $\frac{d}{2}$). Then:*

(i) *ξ_i is isomorphic to $\eta_i \oplus \theta_i$ where η_i has fibre dimension at most k and θ_i is trivial.*

(ii) *If $r \geqslant k$, then ξ_1 and ξ_2 are actually isomorphic.*

Lemma 2.2 implies a sharpened version (with estimates) of Swan's Theorem. We provide a short proof for the convenience of the reader.

LEMMA 2.3. *Let X be a connected finite CW complex with topological dimension d. Let $k := \left\lceil \frac{d}{2} \right\rceil$ (the least integer greater than or equal to $\frac{d}{2}$). Let ξ be a vector bundle of constant fibre dimension r over X. Then there is a vector bundle γ over X, with fibre dimension k, such that $\xi \oplus \gamma$ is a trivial vector bundle.*

Proof. By the properties of K-theory, there is a vector bundle γ' over X, with fibre dimension greater than k, such that $\xi \oplus \gamma'$ is a trivial bundle. By Lemma 2.2 (i), $\gamma' = \gamma \oplus \theta$ where γ is a vector bundle with fibre dimension k and θ is a trivial vector bundle. Then $\xi \oplus \gamma$ is a vector bundle over X which is stably isomorphic to a trivial bundle. But $\xi \oplus \gamma$ has fibre dimension greater than or equal to k. Hence by Lemma 2.2 (ii), $\xi \oplus \gamma$ is a trivial bundle. ∎

Villadsen points out that his algebra A cannot have the strict comparability property for projections even though there is a unique trace. However, the above results on stable isomorphism yield the following very weak form of comparability for projections:

LEMMA 2.4. *Suppose that p is a projection in B. Then there is a positive real number L, dependent only on p, such that for any projection $q \in B$, whenever $L \leqslant \tau(q)$ then p is Murray–von Neumann equivalent in B to a subprojection of q.*

Proof. We will show that we can take $L = \tau(p) + 2$.

Suppose that $\tau(q) \geqslant \tau(p) + 2$. First perturb p and q into a block. Since projections which are close in norm are Murray–von Neumann equivalent, we can regard p and q as elements of $A_n \otimes M_m$ for some n and m. Let d be the topological dimension of the base space X_n of A_n and let $k := \left\lceil \frac{d}{2} \right\rceil$, the least integer greater than or equal to $\frac{d}{2}$.

By Lemma 2.3, there is a projection $s \in A_n \otimes \mathcal{K} \cong C(X_n) \otimes \mathcal{K}$, such that $s \perp p$, s has rank k, and $p \oplus s$ corresponds to a trivial vector bundle over X_n. The fact that A_n and $C(X_n)$ are stably isomorphic comes from work of Dixmier and Douady, see for example [19] for an exposition. Also notice that $p \oplus s$ corresponds to a trivial vector bundle with fibre dimension $\mathrm{rank}(p) + k$.

Recall that $1_{A_n} \otimes e_{1,1}$ is a projection with rank $(n+1)!$ in $C(X_n) \otimes \mathcal{K}$, and X_n has topological dimension $2(n+1)! \geqslant 2k$. Hence, since $\tau(q) \geqslant \tau(p) + 2 = \tau(p) + 2\tau(1_{A_n} \otimes e_{1,1})$, q corresponds to a vector bundle over X_n with fibre dimension greater than or equal to $\mathrm{rank}(p) + 2k$. Hence, by Lemma 2.2 (i), q can be realized as an orthogonal direct sum $q = q_1 \oplus q_2$, such that q_1 is a projection of rank k in $C(X_n) \otimes \mathcal{K}$, and such that q_2 corresponds to a trivial vector bundle (over X_n) with fibre dimension at least $\mathrm{rank}(p) + k$. From this and the previous paragraph, we have that $p \oplus s$ is Murray–von Neumann equivalent (in B) to a subprojection of q. Hence, p is Murray–von Neumann equivalent in B to a subprojection of q as required. ∎

Rørdam has proven that many Villadsen algebras have the SP property — every hereditary subalgebra contains a projection. For this particular algebra A, we can do somewhat better.

LEMMA 2.5. *Let c be a positive element of $A \otimes M_n$, with norm 1. The hereditary subalgebra $\overline{c(A \otimes M_n)c}$ of $A \otimes M_n$ contains a projection, q, and we can choose q so that $\tau(q)$ is arbitrarily close to $\lim\limits_{k\to\infty} \tau(c^{1/k})$.*

Proof. Note that since $A \otimes M_n$ is unital, $\lim\limits_{k\to\infty} \tau(c^{1/k})$ is a finite (nonnegative) real number. Choose l so large that $\tau(c^{1/l})$ is within δ of $\lim\limits_{k\to\infty} \tau(c^{1/k})$. There exists a positive element b in the algebraic direct limit such that $\|b\| = 1$ and b is within δ of $c^{1/l}$. A factoring argument, (for example, apply Lemma 2.2 in [12]) shows that $(b-\delta)_+ = rc^{1/l}r^*$ for some contraction r, and then in particular, $(b-\delta)_+$ is Murray–von Neumann equivalent (in a generalized sense) to some element of $\overline{c(A \otimes M_n)c}$. Now since $(b-\delta)_+$ is within δ of b, and since b is within δ of $c^{1/l}$, $(b-\delta)_+$ is within 2δ of $c^{1/l}$. Hence, $\tau((b-\delta)_+)$ is within $2n\delta$ of $\tau(c^{1/l})$. Hence $\tau((b-\delta)_+)$ is within $2n\delta + \delta$ of $\lim\limits_{k\to\infty} \tau(c^{1/k})$. Of course, n is fixed and δ is arbitrary.

Hence, we have reduced to the case where the generator, which we again denote c, is in the algebraic direct limit. Let $\varepsilon > 0$ be given. Choose $l > 0$ such that $\tau(c^{1/l})$ is within ε of $\lim_{k\to\infty} \tau(c^{1/k})$. We may write $c^{1/l} = \phi_{i,\infty}(e)$ $(= (\phi_{i,\infty} \otimes \mathrm{Id}_{M_n})(e))$, where e is a positive element with norm one contained in the building block $A_i \otimes M_n$. Moreover, by increasing i if necessary, we can suppose that e is nowhere zero, as pointed out by Villadsen in [25], page 111, as part of his proof that his algebras are simple. Thus, we can suppose that none of the $e(t_{ij})$ are zero.

Applying the connecting map N times, we notice that $\phi_{i,i+N}(e)$ $(= (\phi_{i,i+N} \otimes \mathrm{id}_{M_n})(e))$ must have the form

$$(e(\pi_i^1 \pi_{i+1}^1 \cdots \pi_{i+N}^1(x)) \otimes \theta) \oplus (e(t_{i,1}) \otimes q_1) \oplus (e(t_{i,2}) \otimes q_2) \oplus \cdots \oplus (e(t_{i,K(N)}) \otimes q_{K(N)})$$

where $\theta, q_1, q_2, \ldots, q_{K(N)}$ are pairwise orthogonal projections with equal rank in $A_{i+N} \otimes M_n$. Moreover, as $N \to \infty$, $K(N) \to \infty$. Hence, we may assume that N is sufficiently large so that $\frac{1}{K(N)+1} < \varepsilon$.

Next, note that since $A = \varinjlim A_k$ is an inductive limit where the connecting maps are unital, the tracial state space $T(A)$ can be expressed as $T(A) = \varprojlim T(A_k)$, an inverse limit.

Note also that for each j, $e(t_{i,j})$ is a finite linear combination of pairwise orthogonal projections in $p_i \mathcal{K} p_i \otimes M_n$, where the scalars are nonnegative real numbers. Fix j. Let

$$e(t_{i,j}) = r_{j,1} p_{j,1} + r_{j,2} p_{j,2} + \cdots + r_{j,m(j)} p_{j,m(j)}$$

where each $r_{j,s}$ is a strictly positive real number (with absolute value less than or equal to one) and the $p_{j,s}$ are pairwise orthogonal nonzero projections. Now consider the projection $q := \sum_{j=1}^{K(N)} \sum_{s=1}^{m(j)} p_{j,s} \otimes q_j$ in $M_n \otimes A_{i+N} \subseteq M_n \otimes A$. q is in the hereditary subalgebra of A which is generated by c. Hence,

$$\tau(c^{1/l}) \leqslant \tau(q) + \varepsilon n \leqslant \lim_{k\to\infty} \tau(c^{1/k}) + \varepsilon n.$$

The second inequality is since q is a projection in the hereditary subalgebra generated by c. But $\tau(c^{1/l})$ is within ε of $\lim_{k\to\infty} \tau(c^{1/k})$. Hence, $\tau(q)$ is within $\varepsilon + 2\varepsilon n$ of $\lim_{k\to\infty} \tau(c^{1/k})$. Since n is fixed and since ε is arbitrary, we are done. ∎

The next lemma is implicit in the proof of Theorem 3.1 in [5]. We present the short argument for the convenience of the reader.

LEMMA 2.6. *Let A be a separable unital C^*-algebra. Let c be an element of $\mathcal{M}(A \otimes \mathcal{K})$. Then there are diagonal elements c_0, c_1 and c_2 of $\mathcal{M}(A \otimes \mathcal{K})$, and there is an element b of $A \otimes \mathcal{K}$, such that*

$$c = c_0 + c_1 + c_2 + b.$$

The diagonal elements are each with respect to possibly different approximate units of $1 \otimes \mathcal{K}$.

Moreover, if c is positive, then the c_i can all be taken to be positive.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be a countable approximate unit for $A \otimes \mathcal{K}$, consisting of an increasing sequence of projections. Let $1_{\mathcal{M}(A\otimes\mathcal{K})}$ be the unit of the multiplier algebra $A \otimes \mathcal{K}$. Note that for any projection p in $A \otimes \mathcal{K}$, the norm $\|(1_{\mathcal{M}(A\otimes\mathcal{K})} - e_m)cp\|$ approaches zero as m approaches infinity. Hence, replacing $\{e_n\}_{n=1}^{\infty}$ by a subsequence if necessary, we may assume that

$$\sum_{n=1}^{\infty} \|(1_{\mathcal{M}(A\otimes\mathcal{K})} - e_{n+1})c(e_n - e_{n-1})\|$$

is finite, and similarly for

$$\sum_{n=1}^{\infty} \|(1_{\mathcal{M}(A\otimes\mathcal{K})} - e_{n+1})c^*(e_n - e_{n-1})\|.$$

For each $n \geqslant 1$, let $f_n := e_n - e_{n-1}$. Hence,

$$c = c_0 + c_1 + c_2 + b,$$

where for $i = 0, 1, 2,$

$$c_i := \sum_{n=0}^{\infty} (f_{3n+i-1} + f_{3n+i} + f_{3n+i+1})cf_{3n+i}.$$

This completes the first part of the lemma.

Now suppose that c is positive. As in the first part of this argument, decompose $c^{1/2} = c_0 + c_1 + c_2 + b$, where the c_i's are diagonal and where b is an element of $A \otimes \mathcal{K}$. Note that by our construction above, $c_i c_j^* = 0$ for $i \neq j$. Hence, $c = c_0c_0^* + c_1c_1^* + c_2c_2^* + b$, where b is an element of $A \otimes \mathcal{K}$. Now each $c_ic_i^*$ is a positive diagonal element of $\mathcal{M}(A \otimes \mathcal{K})$, as required. ∎

LEMMA 2.7. *Let B be a separable, stable C^*-algebra. Suppose that c is a norm-full positive element of $\mathcal{M}(B)/B$, and x is an element of $\mathcal{M}(B)/B$, such that $1_{\mathcal{M}(B)/B} = xcx^*$. Suppose that d is a positive lift of c to $\mathcal{M}(B)$. Then there exists an element y in $\mathcal{M}(B)$ such that $1_{\mathcal{M}(B)} = ydy^*$.*

Proof. Lifting x to $\widetilde{x}$ in $\mathcal{M}(B)$ we have that $\widetilde{x}d\widetilde{x}^* = 1_{\mathcal{M}(B)} + b$ for some $b \in B$. Since b is stable, there is a copy of O_∞ in the multipliers, and in particular, multiplier isometries v_i such that $\sum v_iv_i^*$ converges strictly to 1. Thus, given $\varepsilon > 0$ there exists an i such that $\|v_i^*bv_i\| = \|v_i^*b(v_iv_i^*)v_i\| < \varepsilon$. It follows that for a sufficiently large i, the positive element $v_i^*\widetilde{x}d\widetilde{x}^*v_i = 1_{\mathcal{M}(B)} + v_i^*bv_i$ is invertible. Denoting the inverse by r, let $y := r^{1/2}v_i^*\widetilde{x}$. ∎

We now proceed to prove our result.

THEOREM 2.8. *There exists a simple* AH-*algebra with stable rank two, perforation, and the corona factorization property.*

Proof. Let A be the unital simple AH-algebra described just before Theorem 2.1. We use the same notation as thence.

Let $B := A \otimes \mathcal{K}$ be the stabilization of A.

We are to show that every full element $c \in \mathcal{M}(B)$ is quasi-invertible (i.e. $1 = rcr^*$).

First, we reduce to the case where c is diagonal with respect to some approximate unit of $1 \otimes \mathcal{K}$ and has trace $\tau(c) = \infty$.

Let J_τ be the proper ideal of $\mathcal{M}(B)$ that is generated by τ. This means that J_τ is the smallest ideal in $\mathcal{M}(B)$ that contains all positive elements $a \in \mathcal{M}(B)$ with $\tau(a) < \infty$. See [20].

Since J_τ contains B and since c is norm-full in $M(B)$, it is clear that in the decomposition $c = c_1 + c_2 + c_3 + b$ of Lemma 2.6, not all the c_i can belong to the ideal J_τ. Thus, at least one of the c_i, say c_1, is not contained in J_τ. Hence, $\tau(c_1) = \infty$. To complete the reduction, we show that if $1 = rc_1r^*$, then there is some r' with $1 = r'cr'^*$. Since $c_1 + c_2 + c_3 \geqslant c_1$, we have that $r(c_1 + c_2 + c_3)r^* \geqslant 1$, and thus that $r(c_1 + c_2 + c_3)r^*$ is invertible. It follows that $c = c_1 + c_2 + c_3 + b$ is quasi-invertible in the corona $\mathcal{M}(B)/B$, so that by Lemma 2.7 the original element c is quasi-invertible in the multipliers $\mathcal{M}(B)/B$. This concludes the reduction.

Now fix a nonzero projection θ in B. By Lemma 2.4, let L be a real number such that for a projection $q \in B$, $L \leqslant \tau(q)$ implies that θ is Murray–von Neumann equivalent to a subprojection of q.

We are given that $c := \sum\limits_1^\infty c_i$, where the sum is in the strict topology, and all the c_i are pairwise orthogonal positive elements of B. We are also given that $\tau(c) = \sum \tau(c_i)$ is infinite. Now for $\delta > 0$, let $f_\delta : [0, \|c\| + 1] \to \mathbb{R}$ be $f_\delta(\lambda) := (\lambda - \delta)_+$. Since J_τ is norm-closed, the complement is open, and thus there exists a $\delta' > 0$ such that $f_{\delta'}(c)$ is not an element of J_τ. Hence, $\tau(f_{\delta'}(c)) = \infty$. To simplify notation, let $d := f_{\delta'}(c)$ and let $d_n := f_{\delta'}(c_n)$ for all n. These elements have the following properties:

(i) the d_n are pairwise orthogonal positive elements of B,

(ii) $d = \sum\limits_{n=1}^\infty d_n$, where the sum converges in the strict topology in $\mathcal{M}(B)$,

(iii) $0 \leqslant d \leqslant c$ and $0 \leqslant d_n \leqslant c_n$ for all n, and

(iv) $\tau(d) = \sum\limits_{n=1}^\infty \tau(d_n) = \infty$.

Because the series $\sum\limits_{n=1}^\infty \tau(d_n)$ diverges, it follows from Lemmas 2.5 and 2.4 that there is a sequence $\{N_l\}_{l=1}^\infty$ of increasing positive integers such that $L + 2 \leqslant \sum\limits_{n=N_i}^{N_{i+1}} \tau(d_n)$, and in particular, θ is equivalent to a projection p_i in the hereditary subalgebra $\mathrm{Her}\Big(\sum\limits_{n=N_i}^{N_{i+1}} d_n\Big)$ of B that is generated by $\sum\limits_{n=N_i}^{N_{i+1}} d_n$. We thus have a sequence p_i of pairwise orthogonal equivalent projections, and it is clear that $P := \sum\limits_1^\infty p_i$ is properly infinite. The projection P is in the hereditary subalgebra

Her(d) of $\mathcal{M}(B)$ that is generated by d. It follows, for example by Cohen's factorization theorem [4], that there exists a positive function g with $g(0) = 0$, such that $P \leqslant g(d)$. Since $g \circ f_{\delta'}$ is a real-valued function that is zero in a neighbourhood of 0, there is some constant $\alpha > 0$ such that $(g \circ f_{\delta'})(\lambda) \leqslant \alpha\lambda$. But then $P \leqslant g(d) \leqslant \alpha c$. From the fact that P is a full and properly infinite projection, it follows that $rPr^* = 1$ for some r. Hence αrcr^* is invertible, and so is rcr^* as was to be shown. ∎

From the proof of the above, we have a corollary indicating that the ideal structure of the multipliers of a Villadsen algebra is partially controlled by the traces:

REMARK 2.9. A positive element of the multiplier algebra of the (stabilized) Villadsen algebra of Theorem 2.8 is full if and only if the (extended) trace is finite on the element.

COROLLARY 2.10. *Suppose that B be the stabilization of the algebra of Theorem* 2.8*. Then B is regular.*

Proof. We freely use the notation of the proof of Theorem 2.8.

Suppose that $\mathcal{D}$ is a full, hereditary subalgebra of B, with no nonzero unital quotients and no nonzero bounded traces.

Let $\{e_{i,j}\}_{1 \leqslant i,j < \infty}$ be a system of matrix units for $\mathcal{K}$. Let $1_{\mathcal{D}} \otimes e_{1,1}$ be the projection in $\mathcal{M}(\mathcal{D} \otimes \mathcal{K})$, such that $1_{\mathcal{D}} \otimes e_{1,1}$ is the strict limit of an (and hence, any) approximate unit for $\mathcal{D} \otimes e_{1,1}$. Hence, $(1_{\mathcal{D}} \otimes e_{1,1})(\mathcal{D} \otimes \mathcal{K})(1_{\mathcal{D}} \otimes e_{1,1}) = \mathcal{D} \otimes e_{1,1} \cong \mathcal{D}$. Hence, since $\mathcal{D} \otimes \mathcal{K}$ is $*$-isomorphic to B, there is a projection P in $\mathcal{M}(B)$, such that PBP is $*$-isomorphic to $\mathcal{D}$ (though not necessarily equal).

Now note that (using notation as in Theorem 2.8) since $\mathcal{D}$ has no nonzero bounded traces, $\tau(P) = \infty$. Hence, since P is a projection, P cannot be an element of J_τ. Hence, using the same argument as that of Theorem 2.8, we can show that P is Murray–von Neumann equivalent to the unit of $\mathcal{M}(B)$. (We note that if A is an element of $\mathcal{M}(B)$ that is not in J_τ, then for every $b \in B$, $A + b$ is also not in J_τ.)

Since P is Murray–von Neumann equivalent to the unit of $\mathcal{M}(B)$, $\mathcal{D} \cong PBP \cong B$. Hence, $\mathcal{D}$ is stable. ∎

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