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UNIQUE ERGODICITY OF FREE SHIFTS AND SOME OTHER AUTOMORPHISMS OF C^*-ALGEBRAS

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ABSTRACT. A notion of unique ergodicity relative to the fixed-point subalgebra is defined for automorphisms of unital C^*-algebras. It is proved that the free shift on any reduced amalgamated free product C^*-algebra is uniquely ergodic relative to its fixed-point subalgebra, as are automorphisms of reduced group C^*-algebras arising from certain automorphisms of groups. A generalization of Haagerup's inequality, yielding bounds on the norms of certain elements in reduced amalgamated free product C^*-algebras, is proved.

KEYWORDS: *Unique ergodicity, ergodic averages, free shift, Haagerup inequality, property (RD).*

MSC (2000): 46L54, 46L55.

1. INTRODUCTION

Let Ω be a compact Hausdorff space and T a homeomorphism of Ω onto itself. In the terminology of [11], (see also [9] and [3], where slightly different terminology is used), T is called *uniquely ergodic* if there is a unique T-invariant Borel probability measure μ on Ω, (with respect to which T is then necessarily ergodic). Oxtoby shows ([11], 5.1) that if T is uniquely ergodic, then the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$$

converge uniformly to the constant $\int f\,\mathrm{d}\mu$, as $n \to \infty$.

The homeomorphisms of Ω are in 1–1 correspondence with the automorphisms of the C^*-algebra $C(\Omega)$ of all continuous, complex-valued functions on Ω and the Borel probability measures on Ω are by Riesz's Theorem in 1–1 correspondence with the states of $C(\Omega)$. There is a natural noncommutative version of unique ergodicity. Let A be a unital C^*-algebra and let α be an automorphism of A. An α-invariant state of A always exists, and can be found, for example, by taking a weak limit of averages

$$\frac{1}{n}\sum_{k=0}^{n-1}\phi\circ\alpha^k$$

of any state ϕ. We say α is *uniquely ergodic* if there is a unique α-invariant state of A. It is not difficult to show (based on Oxtoby's argument ([11], 5.1) that α is uniquely ergodic if and only if for every $a\in A$ the ergodic averages

$$\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(a) \tag{1.1}$$

converge in norm to a scalar multiple of the identity as $n\to\infty$ and, in this case, the invariant state evaluated at a is equal to this limit. (A more general result is proved in Theorem 3.2 below.)

Our interest in these topics grew out of a question asked by David Kerr [8]: Is the free shift on $C^*_r(\mathbb{F}_\infty)$ uniquely ergodic? A positive answer to Kerr's question follows from Haagerup's inequality [4]. This argument is described in Section 2 below.

In considering more general free shift automorphisms, we were motivated to consider a broader notion of unique ergodicity. If A is a unital C^*-algebra and α an automorphism of A, consider the fixed-point subalgebra

$$A^\alpha=\{a\in A:\alpha(a)=a\}. \tag{1.2}$$

We say that α is *uniquely ergodic relative to its fixed-point subalgebra* if every state of A^α has a unique α-invariant state extension to A. In the case when A^α consists only of scalar multiples of the identity element, this reduces to the usual notion of unique ergodicity. In Section 3, we give some alternative characterizations of unique ergodicity relative to the fixed-point subalgebra. It turns out to be equivalent to norm convergence of the ergodic averages (1.1) as $n\to\infty$ for all $a\in A$. Thus, unique ergodicity relative to the fixed-point subalgebra implies (by taking the limit of the ergodic averages) existence of a unique α-invariant conditional expectation from A onto A^α. However (see Question 3.4) we do not know whether the converse direction holds.

After seeing that the free shift on $C^*_r(\mathbb{F}_\infty)$ is uniquely ergodic, it is natural to ask whether free shifts on other reduced free product C^*-algebras and even on reduced amalgamated free product C^*-algebras are uniquely ergodic relative to their fixed point subalgebras. We give an affirmative answer in Theorem 6.1.

A technical result that we use is an extension of Haagerup's inequality to the setting of reduced amalgamated free product C^*-algebras. Haagerup's inequality says that the operator norm of an element of $C^*_r(\mathbb{F}_\infty)$ that is supported on words of length n is no greater than $n+1$ times the ℓ^2-norm. It is a fundamental inequality, and has been generalized in several different directions; see, for example, [5], [6], [1], [7], [13]. One such generalization is 3.3 of [1], in the context of reduced free product C^*-algebras with amalgamation over the scalars, which applies to all finite linear combinations of words of fixed block length n. A strong generalization, due to Ricard and Xu [13], is in the context of reduced amalgamated free product C^*-algebras; they prove bounds on operator norms that apply to all matrices over all finite linear combinations of words of fixed block length n. In Proposition 5.1, we prove a generalization of Haagerup's inequality in the setting of reduced amalgamated free product C^*-algebras. Our bound on the operator norm applies only to certain linear combinations of words of block length n, but our bound has a rather nice form. In fact, as Eric Ricard kindly showed us, our Proposition 5.1 follows from the results of Ricard and Xu. However, we nonetheless present our direct proof here, as it is slightly simpler (for being a more specific result).

To summarize the contents: Section 2 contains the proof of unique ergodicity of the free shift on $C_r^*(\mathbb{F}_\infty)$; Section 3 gives alternative characterizations of unique ergodicity relative to the fixed-point subalgebra, and contains a generalization of the argument from the previous section to groups with property (RD) of Jolissaint; Section 4 recalls the construction of the reduced amalgamated free product of C^*-algebras; Section 5 contains a generalization of Haagerup's inequality to reduced amalgamated free product C^*-algebras; Section 6 proves that free shifts are uniquely ergodic relative to their fixed-point subalgebras.

2. THE FREE SHIFT ON $C_r^*(\mathbb{F}_\infty)$ IS UNIQUELY ERGODIC

Here, $C_r^*(\mathbb{F}_\infty)$ is the reduced group C^*-algebra of the free group on infinitely many generators $\{g_i\}_{i\in\mathbb{Z}}$ and the free shift is the automorphism α of $C_r^*(\mathbb{F}_\infty)$ arising from the automorphism of the group that sends g_i to g_{i+1}.

The C^*-algebra $C_r^*(\mathbb{F}_\infty)$ is densely spanned by the left translation operators λ_h acting on $\ell^2(\mathbb{F}_\infty)$, ($h \in \mathbb{F}_\infty$). If $h = e$ is the trivial group element, then λ_h is the identity element 1 and

$$\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(1) = 1$$

for all n. If h is a nontrivial element of word length p, then by Haagerup's inequality ([4], 1.4),

$$\Big\|\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(\lambda_h)\Big\| \leqslant (p+1)\Big\|\frac{1}{n}\sum_{k=0}^{n-1}\alpha^k(\lambda_h)\Big\|_2 = \frac{p+1}{\sqrt{n}}, \tag{2.1}$$

where $\|\cdot\|_2$ refers to the norm of the corresponding element in $\ell^2(\mathbb{F}_\infty)$. We conclude that the averages appearing on the left-hand-side of (2.1) tend to zero as $n \to \infty$, and this proves that the free shift is uniquely ergodic and that its unique invariant state is the canonical tracial state τ defined by

$$\tau(\lambda_h) = \begin{cases} 1 & h = e, \\ 0 & h \neq e. \end{cases}$$

3. UNIQUE ERGODICITY RELATIVE TO THE FIXED-POINT SUBALGEBRA

In this section, we prove certain conditions equivalent to unique ergodicity relative to the fixed-point subalgebra.

OBSERVATION 3.1. Let A be a C^*-algebra and let $\phi : A \to \mathbb{C}$ be a self–adjoint linear functional, namely a bounded linear functional such that $\phi(a^*)$ is the complex conjugate of $\phi(a)$. Recall (see 3.2.5 of [12]) that the Jordan decomposition of ϕ is the unique pair ϕ_+ and ϕ_- of positive linear functionals such that $\phi = \phi_+ - \phi_-$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\|$. Suppose $\alpha \in \mathrm{Aut}(A)$ and ϕ is α-invariant. Then $\phi = \phi \circ \alpha = \phi_+ \circ \alpha - \phi_- \circ \alpha$ and $\|\phi\| = \|\phi_+\| + \|\phi_-\| = \|\phi_+ \circ \alpha\| + \|\phi_- \circ \alpha\|$. By uniqueness, it follows that ϕ_+ and ϕ_- are both α-invariant.

Recall that a *conditional expectation* from a C^*-algebra A onto a C^*-subalgebra B is a projection E of norm 1 from A onto B. A classical result of Tomiyama [14] is that such a projection E is automatically completely positive and satisfies the conditional expectation property.

THEOREM 3.2. *Let α be an automorphism of a unital C^*-algebra A and let A^α be its fixed-point subalgebra as in* (1.2). *Then the following five statements are equivalent:*

(i) *Every bounded linear functional on A^α has a unique bounded, α-invariant linear extension to A.*

(ii) *Every state of A^α has a unique α-invariant state extension to A.*

(iii) *The subspace $A^\alpha + \{a - \alpha(a) : a \in A\}$ is dense in A.*

(iv) *For all $a \in A$, the following sequence of ergodic averages converges in norm as $n \to \infty$:*

$$\frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a). \tag{3.1}$$

(v) *We have the following where the closure is with respect to the norm topology:*

$$A^\alpha + \overline{\{a - \alpha(a) : a \in A\}} = A.$$

These five statements imply the following statement:

(vi) *There exists a unique α-invariant conditional expectation E from A onto A^α.*

Furthermore, if (i)–(v) *hold, then the conditional expectation E in* (vi) *is given by the formula*

$$E(a) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a). \tag{3.2}$$

DEFINITION 3.3. We say α is *uniquely ergodic relative to its fixed-point subalgebra* if the equivalent statements (i)–(v) hold.

Proof of Theorem 3.2. (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (iii): Suppose, to obtain a contradiction, that (ii) holds but $x \in A$ and

$$x \notin \overline{A^\alpha + \{a - \alpha(a) : a \in A\}}.$$

By the Hahn–Banach Theorem, there is a bounded linear functional $\phi : A \to \mathbb{C}$ such that $\phi(x) \neq 0$, $\phi(A^\alpha) = \{0\}$ and $\phi \circ \alpha = \phi$. Taking the real and imaginary parts, we may without loss of generality assume that ϕ is self-adjoint. Let $\phi = \phi_+ - \phi_-$ be the Jordan decomposition of ϕ. Then ϕ_+ and ϕ_- are α-invariant, by Observation 3.1. Moreover, ϕ_+ and ϕ_- agree on A^α. Either both restrict to zero on A^α, in which case $\phi_\pm(1) = 0$ and $\phi_\pm = 0$, or $\phi_\pm$ are nonzero multiples of states on A and by statement (ii), ϕ_+ and ϕ_- must agree on all of A. This contradicts $\phi(x) \neq 0$.

(iii) $\Longrightarrow$ (iv): Let $a \in A$ and $\varepsilon > 0$. Let $c \in A^\alpha$ and $b \in A$ be such that

$$\|a - (c + b - \alpha(b))\| < \varepsilon.$$

If $n \geqslant m$, then

$$\begin{aligned} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) - \frac{1}{m} \sum_{k=0}^{m-1} \alpha^k(a) \Big\| &< 2\varepsilon + \Big\| \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(b - \alpha(b)) - \frac{1}{m} \sum_{k=0}^{m-1} \alpha^k(b - \alpha(b)) \Big\| \\ &= 2\varepsilon + \Big\| \frac{1}{n}(b - \alpha^n(b)) + \frac{1}{m}(b - \alpha^m(b)) \Big\| \\ &\leqslant 2\varepsilon + \frac{4\|b\|}{m}. \qquad (3.3) \end{aligned}$$

Taking m large enough, the upper bound (3.3) can be made $< 3\varepsilon$. This shows that the sequence of ergodic averages (3.1) is Cauchy.

(iv) $\Longrightarrow$ (vi)+(3.2): Let E be defined by the formula (3.2). Clearly, E restricts to the identity map on A^α. One easily shows $\|E\| = 1$ and $E \circ \alpha = \alpha \circ E = E$. So E is an α-invariant conditional expectation from A onto A^α. If $E' : A \to A^\alpha$ is any α-invariant conditional expectation onto A^α, then

$$E'(a) = \frac{1}{n} \sum_{k=0}^{n-1} E'(\alpha^k(a)) = E'\Big(\frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) \Big).$$

Taking the limit as $n \to \infty$ gives

$$E'(a) = E'(E(a)) = E(a).$$

(iv)+(vi)+(3.2) $\Longrightarrow$ (i): Let $\tau : A^\alpha \to \mathbb{C}$ be a bounded linear functional. Then $\tau \circ E$ is an α-invariant extension of τ to A. To show uniqueness, suppose $\phi : A \to \mathbb{C}$ is any bounded, α-invariant, linear extension of τ. Then

$$\phi(a) = \frac{1}{n} \sum_{k=0}^{n-1} \phi(\alpha^k(a)) = \phi\Big(\frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) \Big).$$

Taking the limit as $n \to \infty$ gives

$$\phi(a) = \phi(E(a)) = \tau(E(a)),$$

so $\phi = \tau \circ E$.

We have now proved the equivalence of (i)–(iv), and that these imply (vi) and (3.2).

(i)+(vi) $\Longrightarrow$ (v): Since $A = A^\alpha + \ker E$, it will suffice to show

$$\ker E \subseteq \overline{\{a - \alpha(a) : a \in A\}}.$$

Suppose, to obtain a contradiction, $x \in \ker E$ but $x \notin \overline{\{a - \alpha(a) : a \in A\}}$. By the Hahn–Banach Theorem, there is a bounded linear functional $\phi : A \to \mathbb{C}$ such that $\phi(x) \neq 0$ and $\phi \circ \alpha = \phi$. By (i), we must have $\phi = \phi \circ E$, so $\phi(x) = 0$, a contradiction.

(v) $\Longrightarrow$ (iii) is clear. ∎

QUESTION 3.4. In Theorem 3.2, is (vi) equivalent to (i)–(v)?

Note that if $A^\alpha = \mathbb{C}1$, then (vi) $\Longrightarrow$ (ii) is immediate, and that this implication also holds when A^α is finite dimensional. Indeed, suppose that A^α is finite dimensional and (vi) holds. Suppose there is a state ϕ of A^α that has two distinct α-invariant extensions ψ_1 and ψ_2 to A. By taking a convex combination with a faithful state on A^α, we may without loss of generality assume ϕ is faithful on A^α. Now it is easy to construct conditional expectations $E_i : A \to A^\alpha$ with the property that $\phi \circ E_i = \psi_i$ $(i = 1, 2)$. By assumption, $E_1 = E_2$, which contradicts $\psi_1 \neq \psi_2$.

It was kindly pointed out to us by Thierry Fack that the argument used in Section 2 applies more generally. Indeed, as the following proposition shows, the argument carries over to groups with property (RD), as defined by Jolissaint in [6]. Note that by [5] this includes the case of Gromov's hyperbolic groups.

PROPOSITION 3.5. *Let G be a group with property* (RD) *for a length function L and let β be an L-preserving automorphism of G such that all orbits of β are either singletons or infinite. Let $H = \{h \in G : \beta(h) = h\}$. Then the automorphism α induced by β on $C^*_{\mathrm{r}}(G)$ is uniquely ergodic relative to its fixed-point subalgebra, which is the canonical copy of $C^*_{\mathrm{r}}(H)$ in $C^*_{\mathrm{r}}(G)$.*

Proof. If $g \in G$ is such that $\beta(g) \neq g$, then by Remark 1.2.2 of [6] there exist positive numbers C and s such that

$$\Big\|\frac{1}{n}\sum_{0}^{n-1} \alpha^k(\lambda_g)\Big\| \leqslant C\Big\|\frac{1}{n}\sum_{0}^{n-1} \alpha^k(\lambda_g)\Big\|_{2,s,L} = \frac{C}{\sqrt{n}}(1 + L(g))^s,$$

and this upper bound approaches zero as n goes to ∞. If $\beta(g) = g$, then

$$\frac{1}{n}\sum_{0}^{n-1} \alpha^k(\lambda_g) = \lambda_g$$

for all n. Now one easily sees that condition (iv) of Theorem 3.2 holds and that $C^*_{\mathrm{r}}(H)$ is the fixed-point subalgebra for α. ∎

4. THE CONSTRUCTION OF REDUCED AMALGAMATED FREE PRODUCT C^*-ALGEBRAS

In this section we will review in some detail and thereby set some notation for the reduced amalgamated free product of C^*-algebras, which was invented by Voiculescu [15].

We first set some notation concerning a right Hilbert C^*-module E over a C^*-algebra B (see [10] for a general reference on Hilbert C^*-modules). If $x \in E$, then we let

$$|x| = \langle x, x\rangle^{1/2} \in B$$

and the norm of x is defined by

$$\|x\|_E = \|\, |x|\, \|_B.$$

Let B be a unital C^*-algebra, let I be a set with at least two elements and for every $i \in I$ let A_i be a unital C^*-algebra containing a copy of B as a unital C^*-subalgebra and having a conditional expectation $\phi_i : A_i \to B$ such that for each $a_i \in A_i$ there exists $x \in A_i$ for which $\phi_i(x^*a_i^*a_ix) \neq 0$. We denote by $E_i = L^2(A_i, \phi_i)$ the right Hilbert C^*-module over B obtained by separation and completion of A_i with respect to the inner product $\langle x, y\rangle = \phi_i(x^*y)$. For $a_i \in A_i$, we denote by $\widehat{a}_i$ the image of a_i in E_i under the canonical map. There is a faithful $*$-representation π_i of A_i on E_i by adjointable operators given by

$$\pi_i(x)(\widehat{y}) = (xy)\widehat{\ },$$

for $x, y \in A_i$. We will often omit the reference to π_i and write simply av to denote $\pi_i(a)(v)$, for $a \in A_i$ and $v \in E_i$.

This inclusion $B \subseteq A_i$ yields a copy of B as a complemented Hilbert C^*-submodule of E_i, and we write $E_i = B \oplus E_i^\circ$ and let $H_i : E_i \to E_i^\circ$ be the orthogonal projection onto E_i°. So, for example, we have

$$H_i(\widehat{a}) = (a - \phi_i(a))\widehat{\ }, \quad (a \in A_i).$$

Since $\pi_i(b)$ sends E_i° into E_i° whenever $b \in B$, we regard E_i° as equipped with a left B-action via π_i. We consider the right Hilbert B-module

$$E = B \oplus \bigoplus_{m\in\mathbb{N},\, i_1,\ldots,i_m\in I,\, i_j\neq i_{j+1}} E_{i_1}^\circ \otimes_B E_{i_2}^\circ \otimes_B \cdots \otimes_B E_{i_m}^\circ, \tag{4.1}$$

where the tensor products are with respect to the right Hilbert B-module structures and the left actions of B described above, and where the summand B in (4.1) denotes the C^*-algebra B with its usual Hilbert C^*-module structure over itself.

There is a faithful $*$-representation of A_i by adjointable operators on E, which is denoted by $a \mapsto \lambda^i_a$ and which can be defined by

$$\lambda^i_a(b) = \phi_i(ab) + H_i((ab)\widehat{\ }) \in B \oplus E^\circ_i, \quad (b \in B) \tag{4.2}$$

and, considering a simple tensor

$$x_1 \otimes \cdots \otimes x_m \tag{4.3}$$

where $m \geqslant 1$, $x_j \in E^\circ_{i_j}$, $i_1, \dots, i_m \in I$ and $i_j \neq i_{j+1}$ for all $j = 1, \dots, m-1$, by

$$\lambda^i_a(x_1 \otimes \cdots \otimes x_m) = \begin{cases} H_i(\widehat{a}) \otimes x_1 \otimes \cdots \otimes x_m \\ \quad + \phi_i(a) x_1 \otimes x_2 \otimes \cdots \otimes x_m & i \neq i_1, \\ H_i(ax_1) \otimes x_2 \otimes \cdots \otimes x_m \\ \quad + \langle (a^*)\widehat{\ }, x_1 \rangle x_2 \otimes \cdots \otimes x_m & i = i_1. \end{cases} \tag{4.4}$$

Note that for $b \in B$, λ^i_b is the same for all i. We will write λ_a or simply a instead of λ^i_a, when no confusion will result.

The reduced amalgamated free product C^*-algebra

$$(A, \phi) = (*_B)_{i \in I}(A_i, \phi_i)$$

consists of the C^*-algebra A generated in $\mathcal{L}(E)$ by the set $\{\lambda^i_a : a \in A_i,\ i \in I\}$ and the conditional expectation $\phi : A \to B$ defined by

$$\phi(a) = \langle a 1_B, 1_B \rangle, \quad (a \in A).$$

We write $A^\circ_k = A_k \cap \ker \phi_k$. Thus, the C^*-algebra A is the closed span of B together with the set of all words of the form

$$w = a_1 \cdots a_n \tag{4.5}$$

where $a_i \in A^\circ_{k(i)}$, $k(1), \dots, k(n) \in I$ and $k(i) \neq k(i+1)$ for all $i \in \{1, \dots, n-1\}$.

5. SOME NORM ESTIMATES IN REDUCED AMALGAMATED FREE PRODUCT C^*-ALGEBRAS

The main result of this section is the following norm estimate, which applies to certain linear combinations of words of length n in reduced amalgamated free product C^*-algebras. It is a version of the Haagerup inequality.

PROPOSITION 5.1. *Suppose $n \geqslant 1$ and consider*

$$f = \sum_{k \in \mathcal{K}} a_{k,1} a_{k,2} \cdots a_{k,n} \in A,$$

where $\mathcal{K}$ is a finite subset of I^n such that for all $k = (k(1), \dots, k(n)) \in \mathcal{K}$ we have $k(i) \neq k(i+1)$ for all $i \in \{1, \dots, n-1\}$ and where $a_{k,i} \in A^\circ_{k(i)}$ for all $k \in \mathcal{K}$ and $i \in \{1, \dots, n\}$. Suppose, furthermore, that

$$\text{if } k, k' \in \mathcal{K} \text{ and } k \neq k', \text{ then } k(1) \neq k'(1) \text{ and } k(n) \neq k'(n). \tag{5.1}$$

Then

$$\|f\| \leqslant (2n+1)\Big(\sum_{k\in\mathcal{K}}\prod_{i=1}^{n}\|a_{k,i}\|^2\Big)^{1/2}. \tag{5.2}$$

Before we get to the proof, we consider some preliminary constructions and results. Let us define some elementary adjointable operators on E, in terms of which we will later describe the action of a word w as in (4.5) on a tensor $x_1 \otimes \cdots \otimes x_m$ in (4.3).

NOTATION 5.2. Let P_0 denote the orthogonal projection of E onto the summand $B \subseteq E$ and for $m \geqslant 1$ let P_m denote the orthogonal projection of E onto

$$\bigoplus_{i_1,\ldots,i_m\in I,\, i_j\neq i_{j+1}} E^{\circ}_{i_1}\otimes_B E^{\circ}_{i_2}\otimes_B\cdots\otimes_B E^{\circ}_{i_m}.$$

NOTATION 5.3. For $k \in I$, let Q_k denote the orthogonal projection of E onto

$$\bigoplus_{m\geqslant 1,\, i_1,\ldots,i_m\in I,\, i_j\neq i_{j+1},\, i_1=k} E^{\circ}_{i_1}\otimes_B E^{\circ}_{i_2}\otimes_B\cdots\otimes_B E^{\circ}_{i_m}.$$

Note that Q_k and P_m commute.

NOTATION 5.4. Given $k\in I$ and $y\in E^{\circ}_k$, let $\psi(y)=\psi_k(y)\in\mathcal{L}(E)$ be given by

$$\psi(y)b = (yb)\hat{}\in E^{\circ}_k, \quad (b\in B) \tag{5.3}$$

and, for $x_1\otimes\cdots\otimes x_m$ as in (4.3),

$$\psi(y)(x_1\otimes\cdots\otimes x_m) = \begin{cases} 0 & i_1 = k,\\ y\otimes x_1\otimes\cdots\otimes x_m & i_1\neq k.\end{cases} \tag{5.4}$$

Therefore, we have $\psi(y) = Q_k\psi(y)(1-Q_k)$ and $\psi(y)^*\psi(y) = |y|^2(1-Q_k)$, and also:

$$\psi(y)^* b = 0, \quad (b\in B); \tag{5.5}$$

$$\psi(y)^*(x_1\otimes\cdots\otimes x_m) = \begin{cases} 0 & i_1\neq k,\\ \langle y, x_1\rangle & i_1 = k,\ m=1,\\ \langle y, x_1\rangle x_2\otimes x_3\otimes\cdots\otimes x_m & i_1=k,\ m>1;\end{cases} \tag{5.6}$$

$$\|\psi(y)\| = \|y\|. \tag{5.7}$$

NOTATION 5.5. For $k\in I$ and $a\in A_k$, we let $\rho(a) = \rho_k(a)\in\mathcal{L}(E)$ be defined by

$$\rho(a)b = 0, \quad (b\in B) \tag{5.8}$$

and, for $x_1\otimes\cdots\otimes x_m$ as in (4.3),

$$\rho(a)(x_1\otimes\cdots\otimes x_m) = \begin{cases} (H_k(ax_1))\otimes x_2\otimes\cdots\otimes x_m & i_1 = k,\\ 0 & i_1\neq k.\end{cases} \tag{5.9}$$

(Recall that $H_k : E_k \to E_k^\circ$ is the orthogonal projection.) Therefore, we have $\rho(a) = Q_k\rho(a)Q_k$ and

$$\|\rho(a)\| \leqslant \|a\|. \tag{5.10}$$

To ease notation, for $a \in A_k$ we let

$$\widehat{a}^\dagger = (a^*)\widehat{\ } \in E_k.$$

The following lemma describes how a word $w = a_1 \cdots a_n$ as in (4.5) can act on a tensor $x_1 \otimes \cdots \otimes x_m$ as in (4.3). What can happen is: w can first devour some initial string $x_1 \otimes \cdots \otimes x_q$ of the tensor. Then it can either push some more stuff onto the tensor from the left, or it can instead act on the next letter x_{q+1} and then push some more stuff onto the tensor from the left. This is all that can happen, because neighboring letters in w and neighboring x_j in $x_1 \otimes \cdots \otimes x_m$ are constrained to come from different A_k°, respectively different E_i°. It's not too difficult to see this by considering some examples. We give a more precise statement and a rigorous proof below.

LEMMA 5.6. *Let $n \geqslant 1$ and let $k = (k(1), \dots, k(n)) \in I^n$ be such that $k(i) \neq k(i+1)$ for all $i \in \{1, \dots, n-1\}$. Let $w = a_1 \cdots a_n$, where $a_i \in A_{k(i)}^\circ$ for all $i \in \{1, \dots, n\}$. Let $m, r \geqslant 0$ be integers.*

(i) *If $r > m + n$ or $r < |m - n|$, then $P_r w P_m = 0$.*

(ii) *If $r = m + n - 2s$ with $s \in \{0, 1, \dots, \min(m, n)\}$, then*

$$P_r w P_m = \psi(\widehat{a}_1)\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s}) \cdot \psi(\widehat{a}_{n-s+1}^\dagger)^*\psi(\widehat{a}_{n-s+2}^\dagger)^*\cdots\psi(\widehat{a}_n^\dagger)^* P_m. \tag{5.11}$$

(iii) *If $r = m + n - 2s + 1$ with $s \in \{1, 2, \dots, \min(m, n)\}$, then*

$$P_r w P_m = \psi(\widehat{a}_1)\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s}) \cdot \rho(a_{n-s+1})\psi(\widehat{a}_{n-s+2}^\dagger)^*\psi(\widehat{a}_{n-s+3}^\dagger)^*\cdots\psi(\widehat{a}_n^\dagger)^* P_m.$$

Proof. The following equation is equivalent to parts (i)–(iii) of the Lemma 5.6 taken together:

$$\begin{aligned} wP_m = &\sum_{s=0}^{\min(m,n)} P_{n+m-2s}\psi(\widehat{a}_1)\cdots\psi(\widehat{a}_{n-s})\cdot\psi(\widehat{a}_{n-s+1}^\dagger)^*\cdots\psi(\widehat{a}_n^\dagger)^* P_m + \\ &\sum_{s=1}^{\min(m,n)} P_{n+m-2s+1}\psi(\widehat{a}_1)\cdots\psi(\widehat{a}_{n-s})\rho(a_{n-s+1})\cdot\psi(\widehat{a}_{n-s+2}^\dagger)^*\cdots\psi(\widehat{a}_n^\dagger)^* P_m. \end{aligned} \tag{5.12}$$

We will prove (5.12) by induction on n. For $n = 1$, taking first $m \geqslant 1$ and using the fact that $\phi_{k(1)}(a_1) = 0$ together with (4.4), (5.4), (5.6), and (5.9), we find

$$a_1 P_m = (\psi(\widehat{a}_1) + \rho(a_1) + \psi(\widehat{a}_1^\dagger)^*)P_m \tag{5.13}$$

$$= P_{m+1}\psi(\widehat{a}_1)P_m + P_m\rho(a_1)P_m + P_{m-1}\psi(\widehat{a}_1^\dagger)^*P_m, \tag{5.14}$$

while in the case $m = 0$, using (4.2), (5.3), (5.5), and (5.8), we find

$$a_1 P_0 = \psi(\widehat{a}_1)P_0 = P_1\psi(\widehat{a}_1)P_0. \tag{5.15}$$

Thus, (5.12) is proved in the case $n = 1$.

Now let $n \geqslant 2$ and set $w' = a_2 a_3 \cdots a_n$. By the induction hypothesis, we have

$$w'P_m = \sum_{s=0}^{\min(m,n-1)} P_{n+m-2s-1}\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s})\cdot\psi(\widehat{a}^{\dagger}_{n-s+1})^{*}\cdots\psi(\widehat{a}^{\dagger}_n)^{*}P_m + \tag{5.16}$$

$$\sum_{s=1}^{\min(m,n-1)} P_{n+m-2s}\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s})\rho(a_{n-s+1})\cdot\psi(\widehat{a}^{\dagger}_{n-s+2})^{*}\cdots\psi(\widehat{a}^{\dagger}_n)^{*}P_m. \tag{5.17}$$

Now we multiply both sides of (5.16) and (5.17) on the left by a_1, and use (5.14) and (5.15), as needed. For example, from (5.16) consider

$$a_1 P_{n+m-2s-1}\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s})\psi(\widehat{a}^{\dagger}_{n-s+1})^{*}\cdots\psi(\widehat{a}^{\dagger}_n)^{*}P_m. \tag{5.18}$$

If $s < n-1$, then the initial part of (5.18) is

$$\begin{aligned} a_1 P_{n+m-2s-1}\psi(\widehat{a}_2) &= P_{n+m-2s}\psi(\widehat{a}_1)P_{n+m-2s-1}\psi(\widehat{a}_2) + P_{n+m-2s-1}\rho(a_1)P_{n+m-2s-1}\psi(\widehat{a}_2) \\ &\quad + P_{n+m-2s-2}\psi(\widehat{a}^{\dagger}_1)^{*}P_{n+m-2s-1}\psi(\widehat{a}_2) \\ &= P_{n+m-2s}\psi(\widehat{a}_1)P_{n+m-2s-1}\psi(\widehat{a}_2) = P_{n+m-2s}\psi(\widehat{a}_1)\psi(\widehat{a}_2), \end{aligned}$$

where, noting that every P_r and Q_s commute, we have used

$$\begin{aligned} \rho(a_1)P_{n+m-2s-1}\psi(\widehat{a}_2) &= \rho(a_1)Q_{k(1)}P_{n+m-2s-1}Q_{k(2)}\psi(\widehat{a}_2) = 0, \\ \psi(\widehat{a}^{\dagger}_1)^{*}P_{n+m-2s-1}\psi(\widehat{a}_2) &= \psi(\widehat{a}^{\dagger}_1)^{*}Q_{k(1)}P_{n+m-2s-1}Q_{k(2)}\psi(\widehat{a}_2) = 0. \end{aligned}$$

If $s = n-1 < m$, then the initial part of (5.18) is

$$\begin{aligned} a_1 P_{m-s}\psi(\widehat{a}^{\dagger}_2)^{*} &= P_{m-s+1}\psi(\widehat{a}_1)P_{m-s}\psi(\widehat{a}^{\dagger}_2)^{*} + P_{m-s}\rho(a_1)P_{m-s}\psi(\widehat{a}^{\dagger}_2)^{*} + P_{m-s-1}\psi(\widehat{a}^{\dagger}_1)^{*}P_{m-s}\psi(\widehat{a}^{\dagger}_2)^{*} \\ &= P_{m-s+1}\psi(\widehat{a}_1)\psi(\widehat{a}^{\dagger}_2)^{*} + P_{m-s}\rho(a_1)\psi(\widehat{a}^{\dagger}_2)^{*} + P_{m-s-1}\psi(\widehat{a}^{\dagger}_1)^{*}\psi(\widehat{a}^{\dagger}_2)^{*}, \end{aligned}$$

while if $s = n-1 = m$, then the initial part of (5.18) is

$$a_1 P_0 \psi(\widehat{a}^{\dagger}_2)^{*} = P_1\psi(\widehat{a}_1)P_0\psi(\widehat{a}^{\dagger}_2)^{*} = P_1\psi(\widehat{a}_1)\psi(\widehat{a}^{\dagger}_2)^{*}.$$

Turning now to (5.17), we consider

$$a_1 P_{n+m-2s}\psi(\widehat{a}_2)\cdots\psi(\widehat{a}_{n-s})\rho(a_{n-s+1})\psi(\widehat{a}^{\dagger}_{n-s+2})^{*}\cdots\psi(\widehat{a}^{\dagger}_n)^{*}P_m. \tag{5.19}$$

We find that the initial part of (5.19) is

$$a_1 P_{n+m-2s}\psi(\widehat{a}_2) = \begin{cases} P_{n+m-2s+1}\psi(\widehat{a}_1)\psi(\widehat{a}_2) & s < n-1, \\ P_{m-s+2}\psi(\widehat{a}_1)\rho(a_2) & s = n-1. \end{cases}$$

Taking all of these cases into account, we prove (5.12). ∎

LEMMA 5.7. *Let f be as in Proposition 5.1. Let m, r be nonnegative integers. Then*

$$\|P_r f P_m\|^2 \leqslant \sum_{k\in\mathcal{K}} \prod_{i=1}^{n} \|a_{k,i}\|^2. \tag{5.20}$$

Proof. If $r < |m-n|$ or $r > m+n$, then by Lemma 5.6(i), we have $P_r f P_m = 0$.

Case I. Suppose $r = m+n-2s$ for $s \in \{0, 1, \dots, \min(m,n)\}$ and with $s < n$.

By Lemma 5.6(ii), we have

$$P_r f P_m = \sum_{k \in \mathcal{K}} \psi(\widehat{a}_{k,1}) \cdots \psi(\widehat{a}_{k,n-s}) \psi(\widehat{a}^\dagger_{k,n-s+1})^* \cdots \psi(\widehat{a}^\dagger_{k,n})^* P_m \tag{5.21}$$

and

$$(P_r f P_m)^*(P_r f P_m) = \sum_{k,k' \in \mathcal{K}} P_m \psi(\widehat{a}_{k,n}) \cdots \psi(\widehat{a}_{k,n-s+1}) \psi(\widehat{a}^\dagger_{k,n-s})^* \cdots \psi(\widehat{a}^\dagger_{k,1})^*$$
$$\cdot \psi(\widehat{a}_{k',1}) \cdots \psi(\widehat{a}_{k',n-s}) \psi(\widehat{a}^\dagger_{k',n-s+1})^* \cdots \psi(\widehat{a}^\dagger_{k',n})^* P_m.$$

By the hypothesis (5.1), if $k \neq k'$, then $k(1) \neq k'(1)$ and, consequently,

$$\psi(\widehat{a}^\dagger_{k,1})^* \psi(\widehat{a}_{k',1}) = \psi(\widehat{a}^\dagger_{k,1})^* Q_{k(1)} Q_{k'(1)} \psi(\widehat{a}_{k',1}) = 0.$$

Therefore, using also (5.7), we get

$$\|P_r f P_m\|^2 \leqslant \sum_{k \in \mathcal{K}} \prod_{i=1}^n \|\widehat{a}_{k,i}\|^2 \leqslant \sum_{k \in \mathcal{K}} \prod_{i=1}^n \|a_{k,i}\|^2. \tag{5.22}$$

Case II. Suppose $r = m+n-2s$ for $s = n \leqslant m$.

Then (5.21) becomes

$$P_r f P_m = \sum_{k \in \mathcal{K}} \psi(\widehat{a}^\dagger_{k,1})^* \cdots \psi(\widehat{a}^\dagger_{k,n})^* P_m \tag{5.23}$$

and we have

$$(P_r f P_m)(P_r f P_m)^* = \sum_{k,k' \in \mathcal{K}} \psi(\widehat{a}^\dagger_{k,1})^* \cdots \psi(\widehat{a}^\dagger_{k,n})^* P_m \psi(\widehat{a}_{k',n}) \cdots \psi(\widehat{a}_{k',1}).$$

Again, by the hypothesis (5.1), if $k \neq k'$, then $k(n) \neq k'(n)$ and, consequently,

$$\psi(\widehat{a}^\dagger_{k,n})^* P_m \psi(\widehat{a}_{k',n}) = \psi(\widehat{a}^\dagger_{k,n})^* Q_{k(1)} P_m Q_{k'(1)} \psi(\widehat{a}_{k',n}) = 0.$$

Using again (5.7), we get (5.22) also in this case.

Case III. Suppose $r = m+n-2s+1$ for $s \in \{1, \dots, \min(m,n)\}$.

Then using Lemma 5.6(iii) we get

$$P_r f P_m = \sum_{k \in \mathcal{K}} \psi(\widehat{a}_{k,1}) \cdots \psi(\widehat{a}_{k,n-s}) \rho(a_{k,n-s+1}) \psi(\widehat{a}^\dagger_{k,n-s+2})^* \cdots \psi(\widehat{a}^\dagger_{k,n})^* P_m.$$

As in case I, $k \neq k'$ implies that $\psi(\widehat{a}^\dagger_{k,1})^* \psi(\widehat{a}_{k',1}) = 0$. Therefore we have

$$\|P_r f P_m\|^2 = \|(P_r f P_m)^*(P_r f P_m)\|$$
$$\leqslant \sum_{k \in \mathcal{K}} \|a_{k,n-s+1}\|^2 \prod_{1 \leqslant i \leqslant n i \neq n-s+1} \|\widehat{a}_{k,i}\|^2 \leqslant \sum_{k \in \mathcal{K}} \prod_{i=1}^n \|a_{k,i}\|^2. \tag{5.24}$$

∎

Then using Lemma 5.6(iii) and proceeding similarly to above, we obtain the estimate

$$\|P_r f P_m\|^2 \leqslant \sum_{k\in\mathcal{K}} \|a_{k,n-s+1}\|^2 \prod_{1\leqslant i\leqslant n,\, i\neq n-s+1} \|\widehat{a}_{k,i}\|^2 \leqslant \sum_{k\in\mathcal{K}} \prod_{i=1}^{n} \|a_{k,i}\|^2. \tag{5.25}$$

REMARK 5.8. The left-hand inequalities in (5.22) and (5.25) are better than required in (5.20). In fact, (5.22) and (5.25) seem to be quite close in spirit to the inequality obtained in 3.3 of [1], which applied to free products with amalgamation over the scalars.

Proof of Proposition 5.1. Let $\sigma : B \to \mathcal{L}(\mathcal{H})$ be a faithful $*$-representation of B on a Hilbert space $\mathcal{H}$. Then the internal tensor product $\widetilde{\mathcal{H}} = E \otimes_\sigma \mathcal{H}$ is a Hilbert space and the $*$-representation $\widetilde{\sigma} : \mathcal{L}(E) \to \mathcal{L}(\widetilde{\mathcal{H}})$ given by $\widetilde{\sigma}(a) = a \otimes \mathrm{id}_{\mathcal{H}}$ is faithful.

Let $v \in \widetilde{\mathcal{H}}$. Then

$$\|\widetilde{\sigma}(f)v\|^2 = \sum_{r=0}^{\infty} \|\widetilde{\sigma}(P_r f)v\|^2. \tag{5.26}$$

Let

$$\gamma = \Big(\sum_{k\in\mathcal{K}} \prod_{i=1}^{n} \|a_{k,i}\|^2\Big)^{1/2}.$$

Then

$$\begin{aligned}
\|\widetilde{\sigma}(P_r f)v\|^2 &= \Big\| \sum_{m=|r-n|}^{r+n} \widetilde{\sigma}(P_r f P_m)v \Big\|^2 && (5.27)\\
&\leqslant \Big(\sum_{m=|r-n|}^{r+n} \|\widetilde{\sigma}(P_r f P_m)v\| \Big)^2 \\
&\leqslant \Big(\sum_{m=|r-n|}^{r+n} \gamma \|\widetilde{\sigma}(P_m)v\| \Big)^2 && (5.28)\\
&\leqslant \Big(\sum_{m=|r-n|}^{r+n} \gamma^2 \Big)\Big(\sum_{m=|r-n|}^{r+n} \|\widetilde{\sigma}(P_m)v\|^2 \Big) && (5.29)\\
&\leqslant (2n+1)\gamma^2 \sum_{m=|r-n|}^{r+n} \|\widetilde{\sigma}(P_m)v\|^2 && (5.30)
\end{aligned}$$

where we used Lemma 5.6(i) to get (5.27), Lemma 5.7 to get (5.28) and the Cauchy–Schwarz inequality to get (5.29). From (5.26) and (5.27)–(5.30), we get

$$\begin{aligned}
\|\widetilde{\sigma}(f)v\|^2 &\leqslant (2n+1)\gamma^2 \sum_{r=0}^{\infty} \sum_{m=|r-n|}^{r+n} \|\widetilde{\sigma}(P_m)v\|^2 = (2n+1)\gamma^2 \sum_{m=0}^{\infty} \sum_{r=|m-n|}^{m+n} \|\widetilde{\sigma}(P_m)v\|^2 \\
&\leqslant (2n+1)^2\gamma^2 \sum_{m=0}^{\infty} \|\widetilde{\sigma}(P_m)v\|^2 = (2n+1)^2\gamma^2\|v\|^2.
\end{aligned}$$

This shows $\|\widetilde{\sigma}(f)\| \leqslant (2n+1)\gamma$, which implies (5.2). ∎

6. FREE SHIFTS

Let D be a unital C^*-algebra, and let $E_B^D : D \to B$ be a conditional expectation onto a unital C^*-subalgebra B. For each integer $i \in \mathbb{Z}$ let (A_i, ϕ_i) be a copy of (D, E_B^D). Let

$$(A, \phi) = (*_B)_{i\in I}(A_i, \phi_i) \tag{6.1}$$

be the reduced amalgamated free product, and let $a \mapsto \lambda_a^i$ denote the embedding of A_i in the free product algebra A arising from the free product construction, as described in Section 4. The free-shift automorphism α on A is the automorphism of A given by $\alpha(\lambda_a^i) = \lambda_a^{i+1}$ for all $a \in A$ and $i \in \mathbb{Z}$.

THEOREM 6.1. *Let α be the free-shift automorphism on the amalgamated free product C^*-algebra A as given in* (6.1) *above. Then B is the fixed-point subalgebra for α and α is uniquely ergodic relative to its fixed-point subalgebra.*

Proof. We will show

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(a) = \phi(a) \tag{6.2}$$

for every $a \in A$. This will imply that B is the fixed-point subalgebra for α and that condition (iv) of Theorem 3.2 holds.

It will suffice to show (6.2) for all $a \in B$ and words a of the form $w = a_1 a_2 \cdots a_p$ for some $p \geqslant 1$ and $a_i \in A^\circ_{k(i)}$, and some $k(i) \in \mathbb{Z}$ with $k(i) \neq k(i+1)$, $i = 1, \dots, p-1$. Since B is α invariant, (6.2) is clear for $a \in B$. So assume $a = w$ as above. Then $\phi(w) = 0$ and $\sum_{k=0}^{n-1} \alpha^k(w)$ is a finite linear combination of words of length p to which Proposition 5.1 applies, and we have

$$\Big\| \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(w) \Big\| \leqslant \frac{1}{n}(2p+1)n^{1/2} \prod_{i=1}^{p} \|a_i\|.$$

Thus, we get

$$\lim_{n\to\infty} \Big\| \frac{1}{n} \sum_{k=0}^{n-1} \alpha^k(w) \Big\| = 0,$$

as required. ∎

Note added in revision: The referee kindly pointed out that the equivalence of (i)–(v) in Theorem 3.2 applies more generally, in that α can be replaced by a unital, positive map. This is also noted (in the completely positive case) by Fidaleo and Mukhamedov in their recent paper [2].

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