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COMPACTNESS OF HANKEL OPERATORS ON HARDY–SOBOLEV SPACES OF THE POLYDISK

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ABSTRACT. We show that a big Hankel operator defined on certain Hardy–Sobolev spaces of the polydisk $\mathbb{D}^n$, $n > 1$, cannot be compact unless it is the zero operator. This result was obtained by Cotlar and Sadosky in 1993 for the classical Hardy space, but our approach here is much different and our result is more general.

KEYWORDS: *Hankel operators, Hardy spaces, polydisk.*

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1. INTRODUCTION

Let $\mathbb{D}^n$ be the polydisk in $\mathbb{C}^n$ and let $\mathbb{T}^n$ be the Shilov boundary of $\mathbb{D}^n$. The normalized Lebesgue measure on $\mathbb{T}^n$ will be denoted by $\mathrm{d}\sigma$. Thus for $f \in L^1(\mathbb{T}^n, \mathrm{d}\sigma)$ and $z = (\mathrm{e}^{\mathrm{i}t_1}, \dots, \mathrm{e}^{\mathrm{i}t_n})$ in $\mathbb{T}^n$ we have

$$\int_{\mathbb{T}^n} f(z)\,\mathrm{d}\sigma(z) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t_1}, \dots, \mathrm{e}^{\mathrm{i}t_n})\,\mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

The Hardy space H^2 consists of holomorphic functions f in $\mathbb{D}^n$ such that

$$\|f\|^2 = \sup_{0<r<1} \int_{\mathbb{T}^n} |f(rz)|^2\,\mathrm{d}\sigma(z) < \infty.$$

It is well known that for every function $f \in H^2$, the radial limit

$$\lim_{r\to 1^-} f(rz)$$

exists for almost every $z \in \mathbb{T}^n$. If we denote the above radial limit by $f(z)$, then the Hardy space H^2 can be regarded (isometrically) as a closed subspace of $L^2 = L^2(\mathbb{T}^n, \mathrm{d}\sigma)$.

Every function $f \in L^2$ admits a Fourier expansion

$$f(z) = \sum_J a_J z^J, \quad z \in \mathbb{T}^n,$$

where $J = (j_1, \ldots, j_n)$ runs over all n-tuples of integers (not just nonnegative integers) and

$$z^J = z_1^{j_1} \cdots z_n^{j_n}.$$

The Fourier coefficients $\{a_J\}$ of $f \in L^2$ satisfy

$$\|f\|^2 = \sum_J |a_J|^2 < \infty.$$

Moreover, a function $f \in L^2$ is the radial limit function of some $f \in H^2$ if and only if $a_J = 0$ whenever at least one of the j_k's is negative. In particular, the set of all monomials on $\mathbb{T}^n$ form an orthonormal basis for L^2, and the set of all holomorphic monomials on $\mathbb{T}^n$ form an orthonormal basis for H^2.

When H^2 is considered as a closed subspace of the Hilbert space L^2, there exists an orthogonal projection

$$P : L^2 \to H^2.$$

It is well known that the orthogonal projection above admits the following integral representation

$$Pf(z) = \int_{\mathbb{T}^n} C(z, \zeta) f(\zeta) \, d\sigma(\zeta), \quad z \in \mathbb{D}^n, \tag{1.1}$$

where

$$C(z, \zeta) = \prod_{j=1}^n \frac{1}{1 - z_j \overline{\zeta}_j}$$

is the Cauchy–Szëgo kernel of $\mathbb{D}^n$. It is clear from a pointwise approximation argument that the domain of the integral operator in (1.1) can be extended to $L^1(\mathbb{T}^n, d\sigma)$, although the resulting holomorphic function Pf is generally not in H^2 when $f \in L^1(\mathbb{T}^n, d\sigma)$.

If f is a function in L^2, we can densely define a linear operator

$$H_f : H^2 \to L^2(\mathbb{T}^n, d\sigma)$$

as follows:

$$H_f(g) = (I - P)(fg) = fg - P(fg).$$

In particular, the above definition makes sense for all polynomials g which form a dense subspace of H^2.

It is clear that if f is bounded, then H_f is a bounded linear operator from H^2 into L^2 (from now on we shall simply say that H_f is bounded on H^2). But it is easy to see that there exist unbounded symbol functions f in L^2 such that the Hankel operator H_f extends to a bounded linear operator on H^2. For example, if f happens to be in H^2, then it can be checked that $P(fg) = fg$ for all $g \in H^2$, so the resulting Hankel operator H_f is the zero operator, which is of course bounded on H^2. On the other hand, if $H_f = 0$, then

$$H_f(1) = f - P(f) = 0,$$

so $f = P(f) \in H^2$. Therefore, $H_f = 0$ if and only if $f \in H^2$.

More generally, it was shown in [1] and [6] that, for $f \in L^2$, the Hankel operators H_f and $H_{\overline{f}}$ are simultaneously bounded on H^2 if and only if f belongs to BMO, the space of functions with uniformly bounded mean oscillation in each variable.

In the case when $n = 1$, it is a classical theorem of Nehari (see [7] for example) that the Hankel operator H_f can be extended to a bounded operator on H^2 if and only if there exists a bounded function g such that $H_f = H_g$. Combining this with the earlier remark about zero Hankel operators, we see that H_f is bounded if and only if $f = g + h$, where $g \in H^2$ and $h \in L^\infty$. It was shown in [1], [2], [5] that a direct analogue of Nehari's result does not hold for higher dimensions. Namely, when $n > 1$, there exist functions $f \in L^2$ such that H_f is bounded on H^2 but no decomposition of the form $f = g + h$, where $g \in H^2$ and $h \in L^\infty$, is possible. So in higher dimensions, Nehari's theorem fails in the sense that there are more bounded Hankel operators than those induced by bounded symbols.

In the case when $n = 1$ again, it is a classical theorem of Hartman (see [7] again) that the Hankel operator H_f can be extended to a compact operator on H^2 if and only if $f = g + h$, where $g \in H^2$ and h is continuous on $\mathbb{T}$. The purpose of this note is to show that Hartman's theorem fails in higher dimensions, but in a completely different way. It's not that we get more compact Hankel operators than those induced by continuous symbols. Instead, there exist no compact Hankel operators at all on H^2 in dimensions greater than 1.

We mention in passing that it was shown in [3] that there exists no nonconstant holomorphic symbol function f such that the corresponding Hankel operator $H_{\overline{f}}$ defined on the Bergman space of the polydisk $\mathbb{D}^n$, $n > 1$, is compact. But in that context, there exist a lot of nontrivial compact Hankel operators on the Bergman space of $\mathbb{D}^n$ whose symbols are not conjugate holomorphic. The surprise for the Hardy space case is that even for general symbols, the Hankel operator H_f cannot be compact unless it is the zero operator.

2. THE CASE OF THE HARDY SPACE

In this section we show that if a Hankel operator on the Hardy space of the polydisk is compact, then it must be zero. This result was obtained in [4], but our approach here is different and more transparent.

The key idea of our analysis is to study the action of rotations on the space $L^2 = L^2(\mathbb{T}^n, d\sigma)$. Thus for any $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{T}^n$ we define an operator

$$U_\zeta : L^2(\mathbb{T}^n, d\sigma) \to L^2(\mathbb{T}^n, d\sigma)$$

by

$$U_\zeta(f)(z_1, \dots, z_n) = f(\zeta_1 z_1, \dots, \zeta_n z_n).$$

It is obvious that U_ζ is a unitary operator on L^2. We are going to call U_ζ the operator of rotation by ζ. It is easy to check that $U_\zeta^* = U_{\overline{\zeta}}$, so U_ζ^* is just the operator of rotation by $\overline{\zeta}$. In particular, H^2 is invariant under the actions of U_ζ and U_ζ^*. In other words, H^2 is a reducing subspace for the operators U_ζ, so each U_ζ commutes with the Cauchy–Szëgo projection P that is used in the definition of Hankel operators.

LEMMA 2.1. *If $f \in L^2$ and H_f is a bounded Hankel operator on H^2, then, for every $\zeta \in \mathbb{T}^n$,*

$$U_\zeta H_f U_\zeta^* = H_{U_\zeta f}.$$

Proof. This is a simple consequence of the definition of Hankel operators and the fact that each U_ζ commutes with the projection P. ∎

The following result shows that the unitary action of U_ζ on Hankel operators is continuous in the strong operator topology.

LEMMA 2.2. *Suppose $f \in L^2$ and H_f is bounded on H^2. Then for every $g \in H^2$, the mapping $\zeta \mapsto H_{U_\zeta f}(g)$ is continuous from $\mathbb{T}^n$ to L^2.*

Proof. Fix $\eta \in \mathbb{T}^n$. For $\zeta \to \eta$ in $\mathbb{T}^n$ we consider

$$\begin{aligned} H_{U_\zeta f} - H_{U_\eta f} &= U_\zeta H_f U_\zeta^* - U_\eta H_f U_\eta^* = U_\zeta H_f (U_\zeta^* - U_\eta^*) + (U_\zeta - U_\eta) H_f U_\eta^* \\ &= U_\zeta H_f (U_{\overline{\zeta}} - U_{\overline{\eta}}) + (U_\zeta - U_\eta) H_f U_{\overline{\eta}}. \end{aligned}$$

If $g \in H^2$, then

$$\|(H_{U_\zeta f} - H_{U_\eta f})(g)\| \leqslant \|H_f\| \|(U_{\overline{\zeta}} - U_{\overline{\eta}})g\| + \|(U_\zeta - U_\eta) H_f U_{\overline{\eta}} g\|.$$

If a_J are the Fourier coefficients of g, then

$$\|(U_{\overline{\zeta}} - U_{\overline{\eta}})g\|^2 = \sum_{J \geqslant 0} |a_J|^2 |\zeta^J - \eta^J|^2,$$

where $J = (j_1, \dots, j_n)$ is an n-tuple of integers, $J \geqslant 0$ means that each of its components is nonnegative, and $\zeta^J = \zeta_1^{j_1} \cdots \zeta_n^{j_n}$. Since

$$\sum_{J \geqslant 0} |a_J|^2 < \infty, \quad |\zeta^J - \eta^J|^2 \leqslant 4,$$

an application of the dominated convergence theorem shows that

$$\lim_{\zeta \to \eta} \|(U_{\overline{\zeta}} - U_{\overline{\eta}})g\|^2 = 0.$$

Similarly, using the Fourier expansion of the function $H_f U_{\overline{\eta}} g$ (recall that η is fixed) in L^2, we can show that

$$\lim_{\zeta\to\eta} \|(U_\zeta - U_\eta) H_f U_{\overline{\eta}} g\|^2 = 0.$$

This proves the following as desired:

$$\lim_{\zeta\to\eta} \|H_{U_\zeta f}(g) - H_{U_\eta f}(g)\| = 0. \quad \blacksquare$$

COROLLARY 2.3. *If $f \in L^2$ and H_f is bounded on H^2, then for any $g \in H^2$ the mapping $\zeta \mapsto \|H_{U_\zeta f}(g)\|$ is a continuous function from $\mathbb{T}^n$ to $[0, \infty)$.*

We need the following special case before we deal with the general case.

LEMMA 2.4. *Suppose $n > 1$ and*

$$f(z) = z^J = z_1^{j_1} z_2^{j_2} \cdots z_n^{j_n}$$

is a non-holomorphic monomial on $\mathbb{T}^n$. Then H_f is not compact on H^2.

Proof. Since f is not holomorphic, at least one of the j_k's must be negative. Without loss of generality we may assume that the exponent j_1 is negative. If g is any function in H^2 that is independent of the first variable z_1, then

$$fg = \overline{z}_1^{|j_1|} h(z_2, \ldots, z_n)$$

for some function h, and it follows from the product structure of the Cauchy–Szëgo projection that $P(fg) = 0$. So for such g we have

$$H_f g = fg, \quad \|H_f g\| = \|g\|.$$

The space of functions $g \in H^2$ that is independent of z_1 is infinite dimensional, and H_f acts on this space isometrically. So H_f cannot be compact. $\blacksquare$

We can now prove the main result of this section.

THEOREM 2.5. *Suppose $f \in L^2(\mathbb{T}^n, \mathrm{d}\sigma)$ and $n > 1$. If H_f is compact on H^2, then $H_f = 0$, that is, $f \in H^2$.*

Proof. Suppose

$$f(z) = \sum_J a_J z^J$$

is the Fourier expansion of f on $\mathbb{T}^n$, where $J = (j_1, \ldots, j_n)$ runs over all n-tuples of integers. Fix any J such that $a_J \neq 0$ and write $f_J = a_J z^J$. It is easy to see that

$$f_J(z) = \int_{\mathbb{T}^n} (U_\zeta f)(z)\, \overline{\zeta}^J \,\mathrm{d}\sigma(\zeta).$$

An application of Fubini's theorem shows that

$$\langle H_{f_J}(g), h\rangle = \int_{\mathbb{T}^n} \langle H_{U_\zeta f}(g), h\rangle\, \overline{\zeta}^J \,\mathrm{d}\sigma(\zeta),$$

where $g \in H^2$ and $h \in L^2$. For the careful reader, this should be done in two steps: first check the identity for $g \in H^\infty(\mathbb{T}^n)$ and $h \in L^\infty(\mathbb{T}^n)$, which are dense subspaces of H^2 and L^2 respectively, then obtain the general case by an approximation argument.

If h is a unit vector in L^2, then we have

$$|\langle H_{f_J}(g), h\rangle| \leqslant \int_{\mathbb{T}^n} |\langle H_{U_\zeta f}(g), h\rangle| \, d\sigma(\zeta) \leqslant \int_{\mathbb{T}^n} \|H_{U_\zeta f}(g)\| \, d\sigma(\zeta).$$

Here we used Lemma 2.2 and Corollary 2.3 to make sure that all integrals above are well-defined. Taking the supremum on the left-hand side of the above inequality over all unit vectors h, we obtain the following where g is any function in H^2:

$$\|H_{f_J}(g)\| \leqslant \int_{\mathbb{T}^n} \|H_{U_\zeta f}(g)\| \, d\sigma(\zeta). \tag{2.1}$$

Since H_f is compact, it follows from Lemma 2.1 that each Hankel operator $H_{U_\zeta f}$ is compact. If $\{g_k\}$ is any sequence in H^2 that converges to 0 weakly, then the compactness of $H_{U_\zeta f}$ implies that

$$\lim_{k\to\infty} \|H_{U_\zeta f}(g_k)\| = 0. \tag{2.2}$$

Every weakly convergent sequence in H^2 is bounded, and the Hankel operators $H_{U_\zeta f}$ are uniformly bounded. Therefore, we can find a positive constant C such that

$$\|H_{U_\zeta f}(g_k)\| \leqslant C$$

for all k and all ζ. In view of (2.1), (2.2), and the theorem of dominated convergence, we conclude that

$$\lim_{k\to\infty} \|H_{f_J}(g_k)\| = 0.$$

Since $\{g_k\}$ is arbitrary, we have shown that H_{f_J} is compact.

Now f_J is a nonzero monomial on $\mathbb{T}^n$. We deduce from Lemma 2.4 and the compactness of H_{f_J} that f_J must be a holomorphic monomial. But f_J was an arbitrary term in the Fourier expansion of f, so the function f must be holomorphic. This completes the proof of the theorem. ∎

3. GENERALIZATION TO HARDY–SOBOLEV SPACES

In this section we extend the results of the previous section to a class of Hankel type operators on certain Hardy–Sobolev spaces on the polydisk.

Fix a real weight parameter α and define a weighted L^2 space on the unit circle $\mathbb{T}$ as follows.

$$L^{2,\alpha}(\mathbb{T}) = \left\{ \sum_{k=-\infty}^{\infty} a_k z^k : \sum_{k=-\infty}^{\infty} (|k|+1)^\alpha |a_k|^2 < \infty \right\}.$$

In general, elements in $L^{2,\alpha}(\mathbb{T})$ should be thought of as formal power series or as distributions. In particular, algebraic operations in $L^{2,\alpha}(\mathbb{T})$ will be performed in the sense of formal power series or distributions. In the case when $\alpha \geqslant 0$, the space $L^{2,\alpha}(\mathbb{T})$ is indeed a space of functions on $\mathbb{T}$.

Three special cases are worth mentioning. First, if $\alpha = 0$, $L^{2,\alpha}(\mathbb{T})$ becomes the standard L^2 space of the unit circle with respect to Lebesgue measure. Second, if $\alpha = 1$, then $L^{2,\alpha}(\mathbb{T})$ is the diagonal Besov space B_2. Equivalently, $f \in L^{2,1}(\mathbb{T})$ if and only if the Cauchy transforms of f and $\overline{f}$ both belong to the Dirichlet space. Finally, if $\alpha = -1$, then $L^{2,-1}(\mathbb{T})$ can be thought of as the boundary distributions of harmonic Bergman functions. Here we say that a harmonic function h in the unit disk is in the Bergman space if

$$\int_{\mathbb{D}} |h(z)|^2 \, \mathrm{d}A(z) < \infty,$$

where $\mathrm{d}A$ is area measure on $\mathbb{D}$.

It is clear that $L^{2,\alpha}(\mathbb{T})$ is a Hilbert space with the following inner product:

$$\langle f, g \rangle_\alpha = \sum_{k=-\infty}^{\infty} (|k|+1)^\alpha a_k \overline{b}_k,$$

where

$$f(z) = \sum_{k=-\infty}^{\infty} a_k z^k, \quad g(z) = \sum_{k=-\infty}^{\infty} b_k z^k,$$

are elements in $L^{2,\alpha}(\mathbb{T})$. The coefficients $\{a_k\}$ associated with an element $f \in L^{2,\alpha}(\mathbb{T})$ will be called the Fourier coefficients of f. From now on elements of $L^{2,\alpha}(\mathbb{T})$ will be called functions as well.

Let $H^{2,\alpha}(\mathbb{T})$ denote the closed subspace of $L^{2,\alpha}(\mathbb{T})$ consisting of functions whose Fourier coefficients $\{a_k\}$ satisfy $a_k = 0$ for all $k < 0$. Intuitively, $H^{2,\alpha}(\mathbb{T})$ is the subspace of $L^{2,\alpha}(\mathbb{T})$ consisting of "analytic functions". Obviously, $H^{2,\alpha}(\mathbb{T})$ is a closed subspace of $L^{2,\alpha}(\mathbb{T})$.

Let

$$P_\alpha : L^{2,\alpha}(\mathbb{T}) \to H^{2,\alpha}(\mathbb{T})$$

denote the orthogonal projection. Given a symbol function $\varphi \in L^{2,\alpha}(\mathbb{T})$ we define two operators

$$T_\varphi : H^{2,\alpha}(\mathbb{T}) \to H^{2,\alpha}(\mathbb{T}) \quad \text{and} \quad H_\varphi : H^{2,\alpha}(\mathbb{T}) \to L^{2,\alpha}(\mathbb{T})$$

as follows, where I is the identity operator on $L^{2,\alpha}(\mathbb{T})$:

$$T_\varphi(f) = P_\alpha(\varphi f), \quad H_\varphi(f) = (I - P_\alpha)(\varphi f).$$

These operators will be called Toeplitz and Hankel operators respectively, and they are at least densely defined. For example, if f is a finite power series, then both $T_\varphi(f)$ and $H_\varphi(f)$ are well defined. It is easy to see that the set of all finite power series is dense in $L^{2,\alpha}(\mathbb{T})$, and the set of all finite analytic power series is dense in $H^{2,\alpha}(\mathbb{T})$.

A natural and fundamental problem is to determine the symbol functions φ such that the operators T_φ and/or H_φ are bounded or compact. Before we turn out attention to the polydisk, we mention two elementary examples below and will cite two known results concerning the boundedness of Toeplitz and Hankel operators on $H^{2,\alpha}(\mathbb{T})$.

First, if φ itself has a finite power series, then it is easy to check that both T_φ and H_φ are bounded. Actually, H_φ is compact in this case; see Corollary 3.6. Second, if $\varphi \in H^{2,\alpha}(\mathbb{T})$, then $H_\varphi = 0$, and so is bounded. On the other hand, if $H_\varphi = 0$, then

$$H_\varphi(1) = \varphi - P_\alpha(\varphi) = 0,$$

so that $\varphi = P_\alpha(\varphi)$ belongs to $H^{2,\alpha}(\mathbb{T})$. Thus $H_\varphi = 0$ if and only if $\varphi \in H^{2,\alpha}(\mathbb{T})$.

It turns out that the most interesting case is when $0 < \alpha < 1$. In this case, the spaces $H^{2,\alpha}(\mathbb{T})$ are between the Hardy space $H^2(\mathbb{T})$ and the classical Dirichlet space, so elements in $H^{2,\alpha}(\mathbb{T})$ are analytic functions in the unit disk. Moreover, it is easy to see that an analytic function f in $\mathbb{D}$ belongs to $H^{2,\alpha}(\mathbb{T})$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^{1-\alpha}\,\mathrm{d}A(z) < \infty.$$

Such spaces are usually called Dirichlet type spaces and they have been studied extensively in recent years. For example, closely related (but differently defined) Hankel and Toeplitz operators on Dirichlet type spaces are studied in [8], [10], [12]. We warn the reader that the space $L^{2,\alpha}$ defined here differs in an essential way from the space $L^{2,\alpha}$ defined in [8].

A class of spaces introduced by Wu (see [10], [11], [12]), the analog of BMOA in the context of Dirichlet spaces, play an important role in all these studies. More specifically, for any $\alpha \in (0,1)$ let W_α denote the space of analytic functions f in $\mathbb{D}$ with the property that there exists a positive constant $C = C_f$ such that

$$\int_{\mathbb{D}} |g(z)|^2|f'(z)|^2(1-|z|^2)^{1-\alpha}\,\mathrm{d}A(z) \leqslant C\|g\|_\alpha^2$$

for all $g \in H^{2,\alpha}(\mathbb{T})$, where $\|g\|_\alpha$ is the norm of g in $H^{2,\alpha}(\mathbb{T})$. In other words, an analytic function f in $\mathbb{D}$ belongs to W_α if and only if the measure

$$\mathrm{d}\mu(z) = |f'(z)|^2(1-|z|^2)^{1-\alpha}\,\mathrm{d}A(z)$$

is a Carleson measure for the Dirichlet type space $H^{2,\alpha}(\mathbb{T})$. Geometric conditions in terms of a certain capacity are obtained in [9] and [13] that characterize Carleson measures for Dirichlet type spaces. Note that our usage of the parameter α is different from that in the papers cited above.

The following result can be found in [13]. See [14] as well.

PROPOSITION 3.1. *Suppose* $\varphi \in H^{2,\alpha}(\mathbb{T})$ *and* $0 < \alpha < 1$. *Then the following conditions are equivalent:*

(i) *The function* φ *is a pointwise multiplier of* $H^{2,\alpha}(\mathbb{T})$.

(ii) *The Toeplitz operator $T_{\overline{\varphi}}$ is bounded on $H^{2,\alpha}(\mathbb{T})$.*
(iii) *The function φ belongs to $H^\infty \cap W_\alpha$.*

If φ is analytic, then it is easy to check that our Hankel operator $H_{\overline{\varphi}}$ is bounded on $H^{2,\alpha}(\mathbb{T})$ if and only if there exists a positive constant C such that

$$|\langle \varphi, fg\rangle_\alpha| \leqslant C\|f\|_\alpha\|g\|_\alpha$$

for all analytic polynomials f and g. It can be checked that this condition is also equivalent to the boundedness of the small Hankel operator $h_{\overline{\varphi}}$ defined in [8].

For a general symbol $\varphi \in L^{2,\alpha}(\mathbb{T})$ we can use the projection P_α to write $\varphi = \varphi_1 + \overline{\varphi}_2$, where each $\varphi_k \in H^{2,\alpha}(\mathbb{T})$. Since $H_{\varphi_1} = 0$, the following result follows from [8], although the definition of Hankel operators in the two instances are completely different.

PROPOSITION 3.2. *Suppose $\varphi \in L^{2,\alpha}(\mathbb{T})$ and $0 < \alpha < 1$. Then the following conditions are equivalent:*
(i) *The operator H_φ is bounded on $H^{2,\alpha}(\mathbb{T})$.*
(ii) *The function $P_\alpha(\overline{\varphi})$ is in W_α.*
(iii) *The following measure is a Carleson measure for $H^{2,\alpha}(\mathbb{T})$:*

$$|\overline{\partial}\varphi|^2(1-|z|^2)^{1-\alpha}\,\mathrm{d}A(z).$$

As a consequence, the two Hankel operators H_φ and $H_{\overline{\varphi}}$ are both bounded if and only if the measure

$$|\nabla\varphi(z)|^2(1-|z|^2)^{1-\alpha}\,\mathrm{d}A(z)$$

is a Carleson measure for $H^{2,\alpha}(\mathbb{T})$. Similar results can be proved for the compactness of Hankel operators using vanishing Carleson measures. We leave the details to the interested reader.

More generally, for any fixed weight parameter α and any positive integer $n > 1$, we consider the space and its holomorphic subspace, respectively:

$$L^{2,\alpha}(\mathbb{T}^n) = L^{2,\alpha}(\mathbb{T})\otimes\cdots\otimes L^{2,\alpha}(\mathbb{T}),\quad H^{2,\alpha}(\mathbb{T}^n) = H^{2,\alpha}(\mathbb{T})\otimes\cdots\otimes H^{2,\alpha}(\mathbb{T}).$$

For simplicity of notation let $P = P_{\alpha,n}$ denote the orthogonal projection

$$P : L^{2,\alpha}(\mathbb{T}^n) \to H^{2,\alpha}(\mathbb{T}^n).$$

Given a symbol function $\varphi \in L^{2,\alpha}(\mathbb{T}^n)$, we can use the projection P to (densely) define a Toeplitz operator

$$T_\varphi : H^{2,\alpha}(\mathbb{T}^n) \to H^{2,\alpha}(\mathbb{T}^n)$$

and a Hankel operator

$$H_\varphi : H^{2,\alpha}(\mathbb{T}^n) \to L^{2,\alpha}(\mathbb{T}^n)$$

just as before. Although we do not have a characterization of symbols φ that induce bounded Hankel operators on $H^{2,\alpha}(\mathbb{T}^n)$, we can determine the symbols φ that induce compact Hankel operators on $H^{2,\alpha}(\mathbb{T}^n)$ when $n > 1$.

THEOREM 3.3. *Suppose $n > 1$ and $\varphi \in L^{2,\alpha}(\mathbb{T}^n)$. Then the following conditions are equivalent:*

(i) *H_φ is compact on $H^{2,\alpha}(\mathbb{T}^n)$.*

(ii) *$\varphi \in H^{2,\alpha}(\mathbb{T}^n)$.*

(iii) *$H_\varphi = 0$.*

Proof. We have already shown how to prove the equivalence of (ii) and (iii), and it is trivial that (iii) implies (i). So it suffices for us to show that (i) implies (ii). Since this part of the proof is similar to that of Theorem 2.5, we will be a little sketchy and leave the routine details to the interested reader.

Suppose

$$\varphi(z) = \sum_J a_J z^J$$

is the Fourier expansion of φ, where $z \in \mathbb{T}^n$ and $J = (j_1, \dots, j_n)$ runs over all n-tuples of integers. The very definition of $L^{2,\alpha}(\mathbb{T}^n)$ ensures that each element of it has such a Fourier expansion. If H_φ is compact on $H^{2,\alpha}(\mathbb{T}^n)$, then using the unitary actions induced by rotations on $\mathbb{T}^n$ we can show that H_{φ_J} is compact on $H^{2,\alpha}(\mathbb{T}^n)$, where $\varphi_J(z) = a_J z^J$ is an arbitrary nonzero term in the Fourier expansion of φ. The desired result then follows from the first corollary to the proposition below. ∎

PROPOSITION 3.4. *Suppose $n \geqslant 1$ and $\varphi(z) = z^m$ is a monomial (not necessarily holomorphic) on $\mathbb{T}^n$. Then the operator $T = H_\varphi^* H_\varphi$ is diagonal with respect to the natural basis $\{z^J / \|z^J\| : J \geqslant 0\}$ of $H^{2,\alpha}(\mathbb{T}^n)$. Moreover, if the eigenvalue of T corresponding to the eigenvector z^J is denoted by λ_J, then $\lambda_J = 0$ whenever $m + J \geqslant 0$, and we have the following whenever some component of $m + J$ is negative:*

$$\lambda_J = \prod_{k=1}^{n} \Big(\frac{|m_k + j_k| + 1}{j_k + 1} \Big)^\alpha .$$

Proof. We first show that the operator

$$T = H_\varphi^* H_\varphi : H^{2,\alpha}(\mathbb{T}^n) \to H^{2,\alpha}(\mathbb{T}^n)$$

is diagonal with respect to the natural basis of $H^{2,\alpha}(\mathbb{T}^n)$. For any holomorphic monomial z^J on $\mathbb{T}^n$, we have

$$H_\varphi(z^J) = z^{m+J} - P(z^{m+J}).$$

There are two cases to consider: if $m + J \geqslant 0$ (meaning that each of the components of the n-tuple $m + J$ is nonnegative), then

$$H_\varphi(z^J) = z^{m+J} - z^{m+J} = 0;$$

if at least one component of $m + J$ is negative, then

$$P(z^{m+J}) = 0, \quad H_\varphi(z^J) = z^{m+J}.$$

In particular, for any two different holomorphic monomials z^J and $z^{J'}$ we always have $\langle H_\varphi(z^J), H_\varphi(z^{J'})\rangle = 0$. Equivalently, $\langle T(z^J), z^{J'}\rangle = 0$ whenever $J \neq J'$. Therefore, T is a diagonal operator with respect to the natural basis of $H^{2,\alpha}(\mathbb{T}^n)$.

For any holomorphic z^J we write $T(z^J) = \lambda_J z^J$. If we take the inner product with the vector z^J on both sides of this equation, the result is

$$\|H_\varphi(z^J)\|^2 = \lambda_J \|z^J\|^2.$$

It follows that $\lambda_J = 0$ whenever $m + J \geqslant 0$, and we have the following whenever some component of $m + J$ is negative:

$$\lambda_J = \frac{\|z^{m+J}\|^2}{\|z^J\|^2} = \prod_{k=1}^{n} \Big(\frac{|m_k + j_k| + 1}{j_k + 1}\Big)^\alpha. \quad ∎$$

COROLLARY 3.5. *If $n > 1$ and $\varphi(z) = z^m$ is not holomorphic on $\mathbb{T}^n$, then H_φ is not compact on $H^{2,\alpha}(\mathbb{T}^n)$.*

Proof. Without loss of generality we may assume that $m_1 < 0$. We consider monomials of the form z_2^j. By Proposition 3.4 above, the eigenvalues of $T = H_\varphi^* H_\varphi$ corresponding to the eigenvectors z_2^j are given by

$$\lambda_j = \Big[\frac{|m_2 + j| + 1}{j + 1}\Big]^\alpha \prod_{k\neq 2} (|m_k| + 1)^\alpha.$$

Regardless of the values of α and m, we always have

$$\lim_{j\to\infty} \lambda_j = 1.$$

This shows that the operator T cannot be compact, because the eigenvalues of a compact operator must converge to zero. ∎

COROLLARY 3.6. *Suppose $n = 1$ and $\varphi(z) = z^m$ is a monomial on $\mathbb{T}$. If $m \geqslant 0$, then $H_\varphi = 0$; if $m < 0$, then H_φ is a finite rank operator whose rank equals $|m|$.*

Once again, we mention that for general α and $n > 1$, we do not have a characterization of symbols $f \in L^{2,\alpha}(\mathbb{T}^n)$ such that H_f is bounded on $H^{2,\alpha}(\mathbb{T}^n)$.

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