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# A HYPERCYCLIC OPERATOR WHOSE DIRECT SUM $T \oplus T$ IS NOT HYPERCYCLIC

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ABSTRACT. The present article answers in the negative the great and long-standing problem in hypercyclicity posed by D. Herrero: *Is $T \oplus T$ hypercyclic whenever $T$ is?* It also answers simultaneously the significant question asked by J. Bès, A. Peris, F. León-Saavedra and A. Montes-Rodríguez: *Does every hypercyclic operator satisfy the Hypercyclicity Criterion?*

KEYWORDS: *Hypercyclic operators, hypercyclicity criterion, cyclic vectors, direct sums of hypercyclic operators, Banach spaces.*

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## 1. INTRODUCTION

If $T$ is a linear bounded operator on a Banach space $X$, then the *orbit* of a vector $x \in X$ for $T$ is the set $\mathrm{Orb}(T, x) = \{x, T(x), T^2(x), T^3(x), \ldots\}$. A vector $x$ is called *hypercylic* for $T$ if $\mathrm{Orb}(T, x)$ is dense in $X$ or, in other words, there is no proper closed $T$-invariant subset of $X$ containing $x$. An example of a Banach space that supports an operator without non-trivial, closed, invariant subsets was found by C. Read [18]. $T$ is called *hypercyclic* if it has a hypercyclic vector. A vector $x \in X$ is said to be *cyclic* for an operator $T \in B(X)$ if the linear span of $\mathrm{Orb}(T, x)$ is dense in $X$. An operator $T \in B(X)$ is *cyclic* if it has a cyclic vector. It is evident that hypercyclicity implies cyclicity.

The first example of hypercyclicity appeared in the space of entire functions in 1929 by Birkhoff [6]. He showed essentially that the translation operator is a hypercyclic operator, while in 1952, MacLane [16] proved the hypercyclicity of the differentiation operator. Hypercyclicity on Banach spaces was discussed in 1969 by Rolewicz [20], who showed that whenever $|\lambda| > 1$, $\lambda T$ is hypercyclic where $T$ is the unilateral backward shift on $\ell^p$ $(1 \leqslant p < \infty)$ or $c_0$.

In 1982, C. Kitai determined in her Ph.D. Dissertation [13] conditions that ensure a continuous linear operator to be hypercyclic. This result, commonly referred to as the Hypercyclicity Criterion, was never published, and a few years later it was rediscovered in a broader form by R.M. Gethner and J.H. Shapiro [8], who used it to unify the previously mentioned results of Birkhoff, MacLane and Rolewicz, among others. We state here the criterion in the weak form due to J. Bès and A. Peris [5]. We say that an operator *T satisfies the Hypercyclicity Criterion* if it satisfies the hypothesis of the following criterion. Remember that a Fréchet space is a complete, linear, metric and locally convex space.

THE HYPERCYCLICITY CRITERION. *Let $X$ be a separable Fréchet space and let $T$ be a continuous linear operator on $X$. If there are dense subsets $X_0$ and $Y_0$ of $X$ and an increasing sequence of natural numbers $(n_k)_k$ and maps $S_k : Y_0 \to X$, $k \in \mathbb{N}$ such that:*

(i) $T^{n_k} S_k y \to y \ \forall y \in Y_0$;

(ii) $S_k y \to 0 \ \forall y \in Y_0$;

(iii) $T^{n_k} x \to 0 \ \forall x \in X_0$;

*then $T$ is hypercyclic.*

The direct sum of two hypercyclic operators is not in general a hypercyclic operator, indeed, H. Salas [21] and D. Herrero [11] showed that there exist $T_1$ and $T_2$ hypercyclic operators such that the direct sum $T_1 \oplus T_2$ is not hypercyclic.

On the other hand if $T$ satisfies the Hypercyclicity Criterion, then $T \oplus T$ is hypercyclic. Therefore D. Herrero [12] posed the following question:

PROBLEM 1. Is $T \oplus T$ hypercyclic whenever $T$ is?

H. Salas [21] and D. Herrero [11] showed in 1991 that there are hypercyclic operators on a Hilbert space that do not satisfy the Hypercyclicity Criterion with $n_k = k$, nonetheless they satisfy the criterion in its general form. In 1995, H. Salas [22] showed that every perturbation of the identity by a unilateral weighted backward shift with non-zero bounded weights is hypercyclic, and some years later, A. Montes-Rodríguez and F. León-Saavedra [14] showed that these hypercyclic operators also satisfy the Hypercyclicity Criterion. It seemed by then that every hypercyclic operator satisfied the criterion and for this reason J. Bès, A. Peris [5] and F. León-Saavedra, A. Montes-Rodríguez [15] asked the following:

PROBLEM 2. Does every hypercyclic operator satisfy the Hypercyclicity Criterion?

J. Bès and A. Peris [5] have shown that $T \oplus T$ is hypercyclic if and only if $T$ satisfies the Hypercyclicity Criterion or equivalent if $T$ is hereditarily hypercyclic with respect to some increasing sequence of natural numbers $(n_k)_k$, this is, for all subsequences $(n_{k_j})_j$ of $(n_k)_k$ there exists $x \in X$ such that $\{T^{n_{k_j}} x : j \in \mathbb{N}\}$ is dense.

THEOREM 1.1 (Bès and Peris, 1999). *Let $X$ be a separable Fréchet space and let $T$ be a continuous linear operator on $X$. The following assertions are equivalent:*

(i) *$T$ satisfies the Hypercyclicity Criterion;*

(ii) *$T \oplus T$ is hypercyclic;*

(iii) *$T$ is hereditarily hypercyclic with respect to some sequence $(n_k)_k$.*

In this paper we show that there exists a continuous linear operator $T$ which is hypercyclic but its direct sum $T \oplus T$ is not. Thus answering in the negative Problem 1 and Problem 2. The counterexample is given in the following way: we shall find some "universal objects" for the Hypercyclicity Criterion consisting of hypercyclic operators where the norm on the underlying space is in a specific sense maximal; we shall show that if there is any counterexample for Problem 1, then one of these specific operators must be a counterexample; and then we shall show that some of these operators are indeed counterexamples.

## 2. A SQUARE-NORM HYPERCYCLICITY CRITERION

Let $(X, \|\cdot\|)$ be a Banach space, let $T$ be a bounded linear operator on $X$, and let $e_0$ be a cyclic vector for $T$. If $e_i = T^i e_0$ for every $i \in \mathbb{N}$, and $c_{00}$ is the collection of finite linear combinations of the vectors $e_i$, then $X$ will be the completion of $c_{00}$ under the given norm $\|\cdot\|$. We define a new seminorm on $X$ (we shall call it the "square norm" — it is in fact a norm in the cases of interest to us):

$$\|x\|_{[2]} = \inf\Big\{\sum_{i=0}^{n} \|x_i\| : x = x_0 + \sum_{i=1}^{n} p_i(T)x_i \text{ with } \|p_i(T)e_0\| \leqslant 1\Big\}. \tag{2.1}$$

Obviously $\|x\|_{[2]} \leqslant \|x\|$ for all $x \in X$; note that the two would be equal if the norm $\|p\| = \|p(T)e_0\|$ were an algebra norm on the algebra $\mathbb{C}[X]$.

THEOREM 2.1. *Let $T$ be a bounded operator on the Banach space $X$, such that $T \oplus T$ is hypercyclic. If $e_0$ is a cyclic vector in $X$ for $T$, then $\|e_0\|_{[2]} = 0$.*

*Proof.* We may assume that $\|e_0\| = 1$. If $T \oplus T$ is hypercyclic, then $T \oplus T$ has a dense set of hypercyclic vectors, so for every $\delta > 0$ there exist $y, z \in X$ such that $\|y\| < \delta$, $\|z\| < \delta$ and $(e_0 + y, e_0 + z)$ is a hypercyclic vector. So, there exists $N \in \mathbb{N}$ such that $(T \oplus T)^N(e_0 + y, e_0 + z)$ is very close to $(e_0, 0)$, specifically we can find an $N$ such that

$$\|T^N e_0 + T^N y - e_0\| < \delta \quad \text{and} \quad \|T^N e_0 + T^N z\| < \delta.$$

Since $e_0$ is a cyclic vector, we can approximate $y, z$ by $p(T)e_0$, $q(T)e_0$ where $p$ and $q$ are complex polynomials, in such a way that this is still true:

$$\|p(T)e_0\| < \delta,\ \|q(T)e_0\| < \delta,\ \|T^N e_0 + T^N p(T)e_0 - e_0\| < \delta,$$
$$\|T^N e_0 + T^N q(T)e_0\| < \delta, \quad \text{and hence} \quad \|T^N(p(T) - q(T))e_0 - e_0\| < 2\delta.$$

Let us consider the vector

$$x = T^N(p(T) - q(T))q(T)e_0$$

in two different ways:

$$\begin{aligned} x &= q(T)(T^N(p(T)-q(T))e_0) = q(T)(e_0+\gamma), \quad \|\gamma\| < 2\delta; \quad \text{and} \\ x &= (p(T)-q(T))T^N q(T)e_0 = (p(T)-q(T))(-T^N e_0+\gamma_2) \\ &= (p(T)-q(T))\gamma_2 - (e_0+\gamma), \quad \|\gamma_2\| < \delta; \end{aligned}$$

therefore

$$e_0 = (p(T)-q(T))\gamma_2 - \gamma - x = (p(T)-q(T))\gamma_2 - \gamma - q(T)(e_0+\gamma). \tag{2.2}$$

Using (2.2) and applying (2.1) we have:

$$\begin{aligned} \|e_0\|_{[2]} &= \|(p(T)-q(T))\gamma_2 - \gamma - q(T)(e_0+\gamma)\|_{[2]} \\ &\leqslant \|(p(T)-q(T))\gamma_2\|_{[2]} + \|\gamma\|_{[2]} + \|q(T)e_0\|_{[2]} + \|q(T)\gamma\|_{[2]} \\ &\leqslant \|(p(T)-q(T))e_0\|\|\gamma_2\| + \|\gamma\| + \|q(T)e_0\| + \|q(T)e_0\|\|\gamma\| \\ &\leqslant 2\delta\cdot\delta + 2\delta + \delta + 2\delta\cdot\delta = 3\delta + 4\delta^2. \end{aligned}$$

This is true for every $\delta > 0$, so $\|e_0\|_{[2]} = 0$. ∎

## 3. UNIVERAL OBJETS FOR THE HYPERCYCLICITY PROBLEM

If $p$ is a complex polynomial, let deg$p$ denote the degree of $p$, and let $|p|$ denote the sum of the absolute values of the coefficients of $p$.

Let $(p_i)_{i=1}^\infty \subset \mathbb{C}[z]$ be a fixed sequence of complex polynomials such that:

(i) $\deg p_i \leqslant i$ for all $i \in \mathbb{N}$;

(ii) $|p_i| \leqslant i$ for all $i \in \mathbb{N}$; and

(iii) for all $n \in \mathbb{N}$, the set $\{p_i : \deg p_i < n\}$ is dense in the finite dimensional space of all complex polynomials of degree less than $n$.

Now let $\Lambda \subset \mathbb{N}^\mathbb{N}$ be an "upper interval" in the sense of Read [19], i.e. a collection of sequences of natural numbers $\mathbf{d} = (d_i)_{i=1}^\infty$ which increase at least at a rate specified by a fixed sequence of growth conditions. We have the following lemma:

LEMMA 3.1. *Let $T$ be a bounded operator on the Banach space $X$ and let $e_0$ be a hypercyclic vector for $T$. Write $e_i = T^i e_0$ for all $i \in \mathbb{N}$. Then for some $\mathbf{d} \in \Lambda$, writing $a_i = d_{2i-1}$ and $b_i = d_{2i}$, we have*

$$\|e_{b_i} - p_i(T)e_0\| < \frac{1}{a_i} \quad \text{for every } i \in \mathbb{N}. \tag{3.1}$$

*Proof.* Let us say the "growth conditions" which define $\Lambda$ are:

$$d_1 > c \quad \text{and} \quad d_{i+1} > F_i(d_1, d_2, \dots, d_i) \quad \text{for given } c \in \mathbb{N} \text{ and } F_i : \mathbb{N}^i \to \mathbb{N}.$$

We choose $a_i$ and $b_i$ recursively as follows:

(i) Choose $a_1 = c + 1$ and find $b_1 > \max(a_1, F_1(a_1))$ such that

$$\|e_{b_1} - p_1(T)e_0\| \leqslant \frac{1}{a_1}.$$

This is possible because $p_1(T)e_0$ lies in the closed orbit $\overline{\{e_i : i > 0\}}$, in fact, it lies in $\overline{\{e_i : i > \alpha\}}$ for any fixed $\alpha$.

(ii) Given $a_1, b_1, a_2, \ldots, a_n, b_n$ we define

$$a_{n+1} = 1 + \max(b_n, F_{2n}(a_1, b_1, \ldots, b_n))$$

and find $b_{n+1} > \max(a_{n+1}, F_{2n+1}(a_1, b_1, \ldots, a_{n+1}))$ such that

$$\|e_{b_{n+1}} - p_{n+1}(T)e_0\| \leqslant \frac{1}{a_{n+1}}.$$

This is again possible because for any polynomial $p$ and constant $\alpha$, we have $p(T)e_0$ belongs to $\overline{\{e_i : i > \alpha\}}$.

Thus we find a sequence $a_1, b_1, a_2, b_2, \ldots$ which grows fast enough to be in $\Lambda$ and satisfies (3.1) for all $i \in \mathbb{N}$. ∎

LEMMA 3.2. *Let $c_{00}$ denote the vector space of all terminating sequences $\sum_{i=0}^{n} \lambda_i e_i$ where $n \in \mathbb{N}$ and $\lambda_i \in \mathbb{C}$. For every $C \geqslant 1$ and all sequences $(a_i), (b_i), (p_i)$ as above, there is a unique largest seminorm $\|\cdot\|$ on $c_{00}$ with the following properties:*

(i) $\|e_0\| \leqslant 1$;

(ii) $\|Tx\| \leqslant C\|x\|$ *for all* $x \in c_{00}$;

(iii) $\|e_{b_n} - p_n(T)e_0\| \leqslant \frac{1}{a_n}$ *for all* $n \in \mathbb{N}$.

*Specifically, this seminorm is*

$$\|x\|_{\max} = \inf\Big\{ \sum_{i,j=0}^{N} C^j |\lambda_{ij}| : x = \sum_{j=0}^{N} \lambda_{0j} e_j + \sum_{i=1}^{N}\sum_{j=0}^{N} \lambda_{ij} a_i (e_{j+b_i} - p_i(T)e_j) \Big\}. \tag{3.2}$$

*Proof.* The $\|x\|_{\max}$ is clearly a seminorm and clearly satisfies

$$\|e_0\|_{\max} \leqslant 1, \quad \|Tx\|_{\max} \leqslant C\|x\|_{\max} \quad \text{and} \quad \|e_{b_i} - p_i(T)e_0\|_{\max} \leqslant \frac{1}{a_i} \quad \text{for all } i.$$

Thus it satisfies all conditions of the lemma. However any other seminorm $\|\cdot\|_0$ satisfying these conditions satisfies:

$$\|e_j\|_0 \leqslant C^j \quad \text{and} \quad \|e_{j+b_i} - p_i(T)e_j\|_0 \leqslant \frac{C^j}{a_i}$$

for all $i, j \in \mathbb{N}$; hence $\|x\|_0 \leqslant \|x\|_{\max}$ for all $x \in c_{00}$. Then $\|x\|_{\max}$ is the largest such seminorm. ∎

DEFINITION 3.3. Let $C \geqslant 1$ be given, and an upper interval $\Lambda \subset \mathbb{N}^{\mathbb{N}}$. Let $T \in B(X)$, let $e_0 \in X$, and write $e_i = T^i e_0$. We say $T$ is a $(C, \Lambda)$*-maximal hypercyclic operator* if $X$ is the completion of $(c_{00}, \|\cdot\|_{\max})$ where the norm $\|\cdot\|_{\max}$ is one of the family of seminorms defined in Lemma 3.2.

In the view of equation (3.1) it is plain that $T$ will have a hypercyclic vector $e_0$; and if $T$ is any hypercyclic operator satisfying the conditions of Lemma 3.2, then $\|p(T)e_0\| \leqslant \|p(T)e_0\|_{\max}$ as in (3.2).

THEOREM 3.4. *Let $C \geqslant 1$ be given, and an upper interval $\Lambda$. If $T \oplus T$ is hypercyclic for every $(C, \Lambda)$-maximal operator $T$, then $T \oplus T$ is hypercyclic for every hypercyclic operator $T$ of norm less than or equal to $C$. In particular, for each hypercyclic operator $T$ of norm at most $C$, with hypercyclic vector $e_0$, we can find a $(C, \Lambda)$-maximal $T'$ with hypercyclic vector $e'_0$, such that $\|p(T)e_0\| \leqslant \|p(T')e'_0\|$ for all complex polynomials $p$.*

*Proof.* Let $T \in B(X)$ be hypercyclic with $\|T\| \leqslant C$, and pick a hypercyclic vector $e_0$ for $T$ with norm 1. By Lemma 3.1, we can find $a_1 < b_1 < a_2 < b_2 < \cdots$ in $\Lambda$ such that writing $e_i = T^i e_0$, we have

$$\|e_{b_i} - p_i(T)e_0\| < \frac{1}{a_i}.$$

We can identify $X$ with the completion of $c_{00}$ under its norm, and we have

$$\|p(T)e_0\| \leqslant \|p(T)e_0\|_{\max}$$

for every complex polynomial $p$. Let $Y$ describe the completion of $(c_{00}, \|\cdot\|_{\max})$ and let $(y_1, y_2) \in Y \oplus Y$ be hypercyclic for $T \oplus T$. We can write

$$y_i = \sum_{j=0}^{\infty} q_{ij}(T)e_0 \quad (i = 1, 2)$$

where

$$\sum_{j=0}^{\infty} \|q_{ij}(T)e_0\|_{\max} < \infty.$$

In the original space $X$ the sums $y'_i = \sum\limits_{j=0}^{\infty} q_{ij}(T)e_0$ converge just as fast. We claim $(y'_1, y'_2)$ is hypercyclic for $T \oplus T \in B(X \oplus X)$. For given $i$ and $j$ in $\mathbb{N}$ and $\delta > 0$ we can find an $N$ such that

$$\|T^N y_1 - p_i(T)e_0\|_{\max} < \delta \quad \text{and} \quad \|T^N y_2 - p_j(T)e_0\|_{\max} < \delta;$$

hence

$$\|T^N y'_1 - p_i(T)e_0\|_X < \delta \quad \text{and} \quad \|T^N y'_2 - p_j(T)e_0\|_X < \delta;$$

so $(y'_1, y'_2) \in X \oplus X$ is indeed hypercyclic for $T \oplus T$. ∎

## 4. $\Lambda$-MAXIMAL HYPERCYCLIC OPERATORS $T$ WHERE $T \oplus T$ IS NOT HYPERCYCLIC

The previous theorem shows that the maximal hypercyclic operators are in some sense the most likely to fail to have $T \oplus T$ hypercyclic. That this does indeed happen is shown by our main theorem:

THEOREM 4.1. *If $\Lambda$ is a sufficiently fast increasing upper interval, then for no $(2, \Lambda)$-maximal hypercyclic operator $T$ is $T \oplus T$ hypercyclic. (The growth conditions on $\Lambda$ that we require for this to be so will be explained below.)*

The reason why $T \oplus T$ is not hypercyclic is that, contrary to Theorem 2.1, we have $\|e_0\|_{[2]} > 0$, where $\|\cdot\|_{[2]}$ is the square norm (2.1) derived from the original maximal norm $\|\cdot\|$ as in (3.2). To show this, we shall exhibit a $\|\cdot\|_{[2]}$-continuous linear funtional $\varphi$ on $c_{00}$ such that $\varphi(e_0) = 1$. We make a fairly common-sense recursive definition of the key numbers $\varphi(e_i)$ as follows:

For $\mathbf{d} = (d_i)_{i=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ we will always write

$$a_i = d_{2i-1} \quad \text{and} \quad b_i = d_{2i}.$$

We impose the growth conditions

$$b_1 > 2 \quad \text{and} \quad b_n > 3b_{n-1} + n + 1 \quad (n \geqslant 2).$$

We define $\varphi(e_0) = 1$, $\varphi(e_i) = 0$ for $i \in (0, b_1)$ and $\varphi(e_{b_1}) = \varphi(p_1(T)e_0)$ which is known because $\deg p_1 \leqslant 1$ and $\varphi(e_i) = 0$ is known for $i < b_1$; in fact $\varphi(p_1(T)e_0)$ is precisely the constant coefficient of $p_1$. We define $\varphi(e_i) = 0$ for $i \in (b_1, 2b_1)$ and $\varphi(e_{2b_1}) = \varphi(p_1^2(T)e_0)$ (which is the square of the constant coefficient of $p_1$, since $\varphi(e_1) = \varphi(e_2) = 0$ and $\deg p_1 \leqslant 1$). We define $\varphi(e_i) = 0$ for $i \in (2b_1, b_2)$. For each $n > 1$, we define $\varphi(e_i)$ as follows for $i$ in the interval $[b_n, b_{n+1})$:

$$\varphi(e_i) = \begin{cases} \varphi(p_n(T)e_{i-b_n}) & \text{if } b_n \leqslant i \leqslant b_n + 3b_{n-1}, \\ & \text{or } 2b_n \leqslant i \leqslant 2b_n + 3b_{n-1}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that since $b_n > 3b_{n-1}$, the intervals $[b_n, b_n + 3b_{n-1}]$ and $[2b_n, 2b_n + 3b_{n-1}]$ do not overlap with each other, nor with the corresponding intervals for different $n$; also since $\deg p_n \leqslant n$ and $b_n > n$ the vector $p_n(T)e_{i-b_n}$ is supported on $[0, i-1]$ so the recursive definition $\varphi(e_i) = \varphi(p_n(T)e_{i-b_n})$ makes sense. We will have $\varphi(e_i) = 0$ whenever

$$i \notin \{0, b_1, 2b_1\} \cup \Big( \bigcup_{n=2}^{\infty} ([b_n, b_n + 3b_{n-1}] \cup [2b_n, 2b_n + 3b_{n-1}]) \Big).$$

We will need the following lemma:

LEMMA 4.2. *Let $n \in \mathbb{N}$ with $n \geqslant 2$. Thus we have $\varphi(e_i) = \varphi(p_n(T)e_{i-b_n})$ not only for*

$$i \in [b_n, b_n + 3b_{n-1}] \cup [2b_{n_1}, 2b_n + 3b_{n-1}]$$

*but also for*

$$i \in [b_n, 2b_n - n) \cup [2b_n, 3b_n - n).$$

*Proof.* If $i \in (b_n + 3b_{n-1}, 2b_n - n)$ certainly $\varphi(e_i) = 0$ by (4.1) but also $p_n(T)e_{i-b_n}$ is supported on $(3b_{n-1}, b_n - n + \deg p_n) \subset (3b_{n-1}, b_n)$ so

$$\varphi(p_n(T)e_{i-b_n}) = 0.$$

Hence

$$\varphi(e_i) = \varphi(p_n(T)e_{i-b_n}) = 0.$$

Similarly if $i \in (2b_n + 3b_{n-1}, 3b_n - n)$ we have

$$\varphi(e_i) = \varphi(p_n(T)e_{i-b_n}) = 0. \quad \blacksquare$$

THEOREM 4.3. *Given suitable growth conditions on the upper interval $\Lambda$, we will have $\|\varphi\|^*_{[2]} = 1$, where this denotes the norm of $\varphi$ as an element of $(c_{00}, \|\cdot\|_{[2]})^*$.*

Given what we already know, this is enough to prove that $\|e_0\|_{[2]} = 1$ and so complete the proof of Theorem 4.1. Let us therefore prove Theorem 4.3.

*Proof.* $\varphi(e_0) = 1$ so obviously $\|\varphi\|^*_{[2]} \geqslant 1$. In view of the definition of the square norm (2.1), it is sufficient to show that $|\varphi(y)| \leqslant 1$ whenever $\|y\| \leqslant 1$ and whenever $y = p(T)z$ with $\|y\| \leqslant 1$ and $\|p(T)e_0\| \leqslant 1$. If we use the convolution multiplication on $c_{00}$ (cautiously, for it isn't continuous with respect to either $\|\cdot\|$ or $\|\cdot\|_{[2]}$) such that $e_i \cdot e_j = e_{i+j}$, this is equivalent to showing that $|\varphi(y)| \leqslant 1$ and $|\varphi(y \cdot z)| \leqslant 1$ whenever $\|y\|, \|z\| \leqslant 1$. Now our norm is one of the "max" norms defined in (3.2), with $C = 2$. In these norms the unit ball is the absolutely convex hull of the vectors $2^{-j}e_j$ ($j \geqslant 0$), and $2^{-j}a_i(e_{j+b_i} - p_i(T)e_j)$ ($j \geqslant 0, i \geqslant 1$). So it is enough to check that $|\varphi(y)| \leqslant 1$ when $y$ is a vector of form $2^{-j}e_j$ ($j \geqslant 0$), or $2^{-j}a_k(e_{j+b_k} - p_k(T)e_j)$ ($j \geqslant 0, k \geqslant 1$), or $2^{-j}a_k a_l(T^{b_k} - p_k(T))(T^{b_l} - p_l(T))e_j$ ($j \geqslant 0, l \geqslant k \geqslant 1$). We proceed to check all these cases.

*Case 1:* Suppose $y = 2^{-j}e_j$ for some $j \geqslant 0$.

From our recursive definition (4.1), it is evident that since $|p_1| \leqslant 1$, we have

$$\max_{i<b_2} |\varphi(e_i)| \leqslant 1, \tag{4.2}$$

so certainly $|\varphi(y)| \leqslant 1$ if $j < b_2$; and for $n \geqslant 2$, we have

$$\max_{i\in[b_n,2b_n)} |\varphi(e_i)| = \max_{i\in[b_n,b_n+3b_{n-1}]} |\varphi(p_n(T)e_{i-b_n})| \leqslant |p_n| \cdot \max_{i\in[0,b_n)} |\varphi(e_i)|$$

since $3b_{n-1} + \deg p_n < b_n$ ($n \geqslant 2$); and

$$\max_{i\in[2b_n,b_{n+1})} |\varphi(e_i)| = \max_{i\in[b_n,b_n+3b_{n-1}]} |\varphi(p_n(T)e_i)| \leqslant |p_n|^2 \cdot \max_{i\in[0,b_n)} |\varphi(e_i)|.$$

Hence for all $n \geqslant 3$,

$$\max_{i\in[0,b_n)} |\varphi(e_i)| \leqslant \max(1, |p_n|^2) \cdot \max_{i\in[0,b_{n-1})} |\varphi(e_i)|.$$

Using (4.2) and the fact that $|p_n| \leqslant n$, we find that for all $n$,

$$\max_{i\in[0,b_n)} |\varphi(e_i)| \leqslant (n!)^2. \tag{4.3}$$

If we impose the growth conditions $b_n \geqslant 2^{(n+1)^2}$ ($n \geqslant 2$), we find that for all $n \geqslant 3$, $j \in [b_{n-1}, b_n)$ and $y = 2^{-j}e_j$ we have

$$|\varphi(y)| \leqslant 2^{-b_{n-1}}(n!)^2 \leqslant 2^{-n^2}(n!)^2 \leqslant 1$$

as required.

*Case 2:* Suppose $y = 2^{-j} a_k (e_{j+b_k} - p_k(T)e_j)$ $(j \geqslant 0, k \geqslant 1)$. We split this case into subcases depending on how large $j$ is.

*Subcase 2a:* Suppose $j < 2\log_2 a_k$. Then

$$j + b_k \in [b_k, b_k + 2\log_2 a_k) \subset [b_k, 2b_k - k)$$

(given the very mild growth condition

$$b_k > b_{k-1} + 2k + 3\log_2 a_k. \tag{4.4}$$

— this could be even milder for the present need but we will use it again below). So by Lemma 4.2, $\varphi(e_{j+b_k}) = \varphi(p_k(T)e_j)$ hence

$$\varphi(y) = 0.$$

*Subcase 2b:* Suppose $j \in [2\log_2 a_k, b_{k+1} - b_k)$. Let $|\cdot|_1$ denote the $l_1$ norm on $c_{00}$, that is

$$\Big| \sum_{i=0}^{n} \lambda_i e_i \Big|_1 = \sum_{i=0}^{n} |\lambda_i|.$$

Then since $2^{-j} a_k \leqslant a_k^{-1}$, we have

$$|y|_1 \leqslant a_k^{-1}(1 + |p_k|) \leqslant a_k^{-1}(1+k),$$

and since $y$ is supported on $[0, b_{k+1})$ we have

$$|\varphi(y)| \leqslant |y|_1 \cdot \max_{r < b_{k+1}} |\varphi(e_r)| \leqslant a_k^{-1}(1+k)(k!)^2 \leqslant 1,$$

given the growth condition $a_k \geqslant (1+k)(k!)^2$ $(k > 0)$.

*Subcase 2c:* Suppose $j \in [b_r - b_k, b_{r+1} - b_k)$ for some $r > k$. Then

$$|y|_1 \leqslant 2^{b_k - b_r} a_k (1 + |p_k|) \leqslant 2^{b_k - b_r} a_k (1+k).$$

Since $y$ is supported on $[0, b_{r+1})$ we have

$$\begin{aligned} |\varphi(y)| &\leqslant 2^{b_k - b_r} a_k (1+k) \cdot \max_{s < b_{r+1}} |\varphi(e_s)| \leqslant 2^{b_k - b_r} a_k (1+k)(r!)^2 \\ &\leqslant 2^{b_{r-1} - b_r} a_{r-1} \cdot r \cdot (r!)^2 \leqslant 1, \end{aligned}$$

given the growth conditions $b_r > b_{r-1} + \log_2(a_{r-1} \cdot r \cdot (r!)^2)$ $(r \geqslant 2)$.

*Case* 3: Suppose $y = 2^{-j} a_k a_l (T^{b_k} - p_k(T))(T^{b_l} - p_l(T))e_j$ $(j \geqslant 0, l \geqslant k \geqslant 1)$. We split this case into subcases depending on the value of $j$, and also according to whether $l = k$ or $l > k$.

*Subcase 3a:* Suppose $l = k$ and $0 \leqslant j \leqslant 3\log_2 a_l$. Then

$$j + 2b_l \in [2b_l, 3b_l - l)$$

by (4.4), so Lemma 4.2 tells us

$$\varphi(e_{j+2b_l}) = \varphi(p_l(T)e_{j+b_l}),$$

i.e.

$$\varphi(T^{b_l}(T^{b_l} - p_l(T))e_j) = 0.$$

The vector $p_l(T)e_j$ is supported on

$$[0, l + 3\log_2 a_l) \subset [0, b_l - l) \quad \text{by (4.4),}$$

so Lemma 4.2 again tells us

$$\varphi(T^{b_l} p_l(T)e_j) = \varphi(p_l^2(T)e_j).$$

Hence

$$\varphi((T^{b_l} - p_l(T))^2 e_j) = 0,$$

and so $\varphi(y) = 0$.

*Subcase 3b:* Suppose $l > k$ and $0 \leqslant j \leqslant 3\log_2 a_l$. Then the vector

$$u = (T^{b_k} - p_k(T))e_j$$

is supported on

$$[0, b_k + 3\log_2 a_l) \subset [0, b_l - l) \quad \text{by (4.4).}$$

Lemma 4.2 then tells us that

$$\varphi(T^{b_l}u) = \varphi(p_l(T)u)$$

hence

$$\varphi((T^{b_l} - p_l(T))(T^{b_k} - p_k(T))e_j) = 0$$

and so $\varphi(y) = 0$.

*Subcase 3c:* Suppose $j \in [3\log_2 a_l, b_{l+1} - 2b_l)$. Then $2^{-j} < a_l^{-3}$ hence

$$|y|_1 \leqslant a_k a_l^{-2}(1 + |p_l|)(1 + |p_k|) \leqslant a_l^{-1}(1 + l)^2;$$

$y$ is supported on $[0, b_{l+1})$, so

$$|\varphi(y)| \leqslant |y| \cdot \max_{i < b_{l+1}} |\varphi(e_i)| \leqslant a_l^{-1}((l+2)!)^2$$

because $|\varphi(e_i)| \leqslant ((l+1)!)^2$ for $i < b_{l+1}$ by (4.3). We can be sure $|\varphi(y)| \leqslant 1$ provided we include the growth conditions

$$a_l > ((l+2)!)^2 \quad (l > 0).$$

*Subcase 3d:* Suppose $j \in [b_r - 2b_l, b_{r+1} - 2b_l)$ for some $r \geqslant l + 2$. Then once again $y$ is supported on $[0, b_{r+1})$ and therefore (4.3) tells us

$$|\varphi(y)| \leqslant ((r+1)!)^2 \cdot |y|.$$

But in this final case $j$ is large enough that $|y|$ is very small indeed, in fact

$$|y| \leqslant 2^{2b_l - b_r} a_k a_l (1 + |p_k|)(1 + |p_l|) \leqslant 2^{2b_{r-1} - b_r} a_{r-1}^2 \cdot r^2,$$

and so we conclude that $|\varphi(y)| \leqslant 1$ provided we have the growth conditions $b_r > 2b_{r-1} + \log_2(((r+1)!)^2 \cdot a_{r-1}^2 \cdot r^2)$ $(r \geqslant 2)$.

So given appropriate growth conditions in the definition of $\Lambda$, we can be sure that $\|\varphi\|_{[2]}^* \leqslant 1$ whenever $\varphi$ is as in (4.1) and the norm $\|\cdot\|$ is a $(2, \Lambda)$-maximal norm as defined in (3.2). This brings the proof of Theorem 4.3 to a close, and so also the proof of our main Theorem 4.1. ∎

## 5. CONCLUSION

We have shown not only that there exist hypercyclic operators such that $T \oplus T$ is not hypercyclic, but that in some sense all the "maximal" ones also have this property. For given an upper interval $\Lambda$ and a pair $(T', e'_0)$ consisting of an operator of norm at most 2 and a hypercyclic vector for it, we can use Theorem 3.4 to find a $(2, \Lambda)$-maximal operator $T$ on an appropriate completion of $c_{00}$, such that for every complex polynomial $p$, $\|p(T')e'_0\| \leqslant \|p(T)e_0\|$. If $\Lambda$ has been chosen appropriately, $T \oplus T$ will definitely not be hypercyclic.

## REFERENCES


[1] S.I. ANSARI, Hypercyclic and cyclic vectors, *J. Funct. Anal.* **128**(1995), 374–383.

[2] S.I. ANSARI, Existence of hypercyclic operators on topological vector spaces, *J. Funct. Anal.* **148**(1997), 384–390.

[3] B. BEAUZAMY, An operator on a separable Hilbert space with all polynomials hypercyclic, *Studia Math.* **96**(1990), 81–90.

[4] L. BERNAL-GONZÁLEZ, Hypercyclic sequences of differential and antidifferential operators, *J. Approx. Theory* **96**(1999), 323–337.

[5] J. BÈS, A. PERIS, Hereditarily hypercyclic operators, *J. Funct. Anal.* **167**(1999), 94–112.

[6] G.D. BIRKHOFF, Démonstration d'un théorème élémentaire sur les fonctions entières, *C. R. Acad. Sci. Paris* **189**(1929), 473–475.

[7] P.S. BOURDON, Invariant manifolds of hypercyclic vectors, *Proc. Amer. Math. Soc.* **118**(1993), 845–847.

[8] R.M. GETHNER, J.H. SHAPIRO, Universal vectors for operators on spaces of holomorphic functions, *Proc. Amer. Math. Soc.* **100**(1987), 281–288.

[9] K.G. GROSSE-ERDMANN, Universal families and hypercyclic operators, *Bull. Amer. Math. Soc.* **36**(1999), 345–381.

[10] K.G. GROSSE-ERDMANN, Recent developments in hypercyclicity, *RACSAM Rev. R. Acad. CienC. Exactas Fis. Nat. Serie A. Mat.* **97**(2003), 273–286.

[11] D.A. HERRERO, Limits of hypercyclic and supercyclic operators, *J. Funct. Anal.* **99**(1991), 179–190.

[12] D.A. HERRERO, Hypercyclic operator and chaos, *J. Operator Theory* **28**(1992), 93–103.

[13] C. KITAI, Invariant closed sets for linear operators, Ph.D. Dissertation, Univ. of Toronto, Toronto 1982.

[14] F. LEÓN-SAAVEDRA, A. MONTES-RODRÍGUEZ, Linear structure of hypercyclic vectors, *J. Funct. Anal.* **148**(1997), 524–545.

[15] F. LEÓN-SAAVEDRA, A. MONTES-RODRÍGUEZ, Spectral theory and hypercyclic subspaces, *Trans. Amer. Math. Soc.* **353**(2000), 247–267.

[16] G.R. MACLANE, Sequences of derivatives and normal families, *J. Anal. Math.* **2**(1952), 72–87.

[17] A. PERIS, Chaotic polynomials on Fréchet spaces, *Proc. Amer. Math. Soc.* **127**(1999), 3601–3603.

[18] C.J. READ, The invariant subspace problem for a class of Banach spaces. II. Hypercyclic operators, *Israel J. Math.* **63**(1988), 1–40.

[19] C.J. READ, Normed "upper interval" algebras without nontrivial closed subalgebras, *Studia Math.* **171**(2005), 295–303.

[20] S. ROLEWICZ, On orbits of elements, *Studia Math.* **32**(1969), 17–22.

[21] H. SALAS, A hypercyclic operator whose adjoint is also hypercyclic, *Proc. Amer. Math. Soc.* **112**(1991), 765–770.

[22] H. SALAS, Hypercyclic weighted shifts, *Trans. Amer. Math. Soc.* **347**(1995), 993–1004.

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