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LIMITS AND C^*-ALGEBRAS OF LOW RANK OR DIMENSION

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Dedicated to the memory of Gert K. Pedersen

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ABSTRACT. We explore various limit constructions for C^*-algebras, such as composition series and inverse limits, in relation to the notions of real rank, stable rank, and extremal richness. We also consider extensions and pullbacks. We identify some conditions under which the constructions preserve low rank for the C^*-algebras or their multiplier algebras. We also discuss the version of topological dimension theory appropriate for primitive ideal spaces of C^*-algebras and provide an analogue for rank of the countable sum theorem of dimension theory. As an illustration of how the main results can be applied, we show that a CCR algebra has stable rank one if and only if it has topological dimension zero or one, and we characterize those σ-unital CCR algebras whose multiplier algebras have stable rank one or extremal richness. (The real rank zero case was already known.)

KEYWORDS: *Extensions of C^*-algebras, extremally rich C^*-algebras, inverse limits, pullbacks, real rank zero, stable rank one, subhomogeneous C^*-algebras.*

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1. INTRODUCTION

The concept of dimension for a topological space $\mathcal{X}$ originates in the basic fact that manifolds are locally homeomorphic to euclidean spaces, which have an obvious linear dimension. In the more abstract version given by Čech's covering dimension of a normal space $\mathcal{X}$, the dimension gives conditions under which certain functions extend and certain cohomology groups vanish.

Regarding a C^*-algebra A as the non-commutative analogue of $C(\mathcal{X})$ (or $C_0(\mathcal{X})$) for a compact (or just locally compact) Hausdorff space $\mathcal{X}$, it is natural to try to extend the notion of topological dimension of $\mathcal{X}$ to the analogous setting. The more so as the covering dimension of $\mathcal{X}$ is easily characterized in terms of elements in $C(\mathcal{X})$. In [38] Rieffel defined the *(topological) stable rank*, $\mathrm{tsr}(A)$, of an arbitrary C^*-algebra A, using concepts from dimension theory: If A is unital, $\mathrm{tsr}(A)$ is the smallest d in $\mathbb{N}$ (or ∞) such that unimodular d-tuples (namely, those which are left invertible as $d \times 1$ matrices) are dense in A^d. Shortly after, the stable rank was identified with the Bass stable rank of A, [15], which is a purely algebraic concept. In particular, by an earlier result of Vaserstein, [43], we have $\mathrm{tsr}(C_0(\mathcal{X})) = [\frac{1}{2}\dim(\mathcal{X} \cup \{\infty\})] + 1$, the factor $\frac{1}{2}$ arising from the use of complex scalars in $C_0(\mathcal{X})$.

The *real rank* of a C^*-algebra was introduced in [5] as an alternative to Rieffel's stable rank. Formally the only difference is that self-adjoint elements replace the general elements in Rieffel's definition, but this has unexpected consequences, especially for small values of the rank. In general one has $\mathrm{RR}(A) \leqslant 2\mathrm{tsr}(A) - 1$, and — pleasing for the eye — $\mathrm{RR}(C_0(\mathcal{X})) = \dim(\mathcal{X} \cup \{\infty\})$. However, in the lowest possible cases, $\mathrm{tsr}(A) = 1$ and $\mathrm{RR}(A) = 0$, the two notions are independent: one may be satisfied without the other.

One of the real surprises is the symmetry with which stable rank one and real rank zero sometimes interact with the two K-groups for a unital C^*-algebra A: If I is a closed ideal in A and $\mathrm{tsr}(A) = 1$, the natural map $K_0(I) \to K_0(A)$ is injective, whereas the map $K_1(I) \to K_1(A)$ is injective if $\mathrm{RR}(A) = 0$. Also, the natural map from Murray–von Neumann equivalence classes of projections in A to $K_0(A)$ is injective if A has stable rank one, whereas its image generates the whole group if A is of real rank zero.

Recall from [6] that a unital C^*-algebra A is *extremally rich* if the open set A_q^{-1} of *quasi-invertible* elements is dense in A. Here A_q^{-1} can be defined as $A^{-1}\mathcal{E}(A)$ A^{-1}, where $\mathcal{E}(A)$ denotes the set of extreme points in the closed unit ball, A_1, of A. Equivalently, cf. [7], A is extremally rich if $\mathrm{conv}(\mathcal{E}(A)) = A_1$, so that — as a Banach space — A has the λ-property, cf. [34]. If $A = C(\mathcal{X})$, extremal richness is equivalent to $\dim(\mathcal{X}) \leqslant 1$. In general, extremal richness is a generalization of Rieffel's notion of stable rank one suitable for not necessarily finite C^*-algebras. Thus every purely infinite, simple C^*-algebra is extremally rich, as is every von Neumann algebra.

The low ranks, i.e. stable rank one, real rank zero, and extremal richness, have different formal properties from the higher ranks. In particular, the low ranks are invariant under Rieffel–Morita equivalence, but $\mathrm{tsr}(A \otimes \mathbb{K}) = 2$ whenever $\mathrm{tsr}(A) > 1$ and $\mathrm{RR}(A \otimes \mathbb{K}) = 1$ whenever $\mathrm{RR}(A) > 0$. In this paper we primarily consider low ranks, though a few results include higher rank cases.

For general (non-commutative) C^*-algebras the relationship between rank and dimension is an analogy rather than a theorem. Nevertheless, topological dimension theory has some applications, and we provide a brief treatment in Section 2. This is largely, but not entirely, just a matter of using the appropriate results from topology, but, as we explain, it would be wrong simply to apply covering dimension to primitive ideal spaces. Our treatment includes all C^*-algebras with almost Hausdorff primitive ideal spaces, in particular all type I C^*-algebras. Section 2 also contains a result about rank, Theorem 2.10, which is analogous to and inspired by the countable sum theorem of dimension theory.

Section 3 treats C^*-inverse limits, mainly those where all maps are surjective. We provide a framework for representing certain multiplier algebras as such inverse limits. In Theorem 3.7 we prove that low rank is preserved by surjective C^*-inverse limits, and in Theorem 3.11 we prove that real rank zero and stable rank one are preserved in certain multiplier algebras which are non-surjective inverse limits.

Section 4 has results on low rank of pullbacks where at least one of the maps is surjective. Via the Busby construction this leads to results about low ranks of extensions and multiplier algebras of extensions. And Section 5 contains the applications to CCR algebras and concluding remarks and questions.

Determination of which extensions of low rank C^*-algebras have low rank has been a matter of continuing interest to many mathematicians. In all three cases A has low rank if and only if I and A/I do and an additional hypothesis is satisfied. For real rank zero and stable rank one the additional hypothesis is just the natural lifting condition, but for extremal richness it is the natural lifting condition plus a technical hypothesis. In all three cases it is desirable to identify circumstances in which the hypotheses can be simplified. In the extremal richness case this means more than merely eliminating the technical hypothesis. Results and remarks on this subject occur in Corollary 4.4, statements 4.6–4.8, and 5.11–5.14.

The authors previously announced a paper entitled, "Extremally rich ideals in C^*-algebras". The present paper and [9] constitute an expanded version of that paper.

2. TOPOLOGICAL DIMENSION AND LOW RANK

2.1. COMPOSITION SERIES. (i) Unless expressly mentioned, the word *ideal* will in this paper designate a closed (and therefore $*$-invariant) ideal in a C^*-algebra. We say that an increasing series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ of ideals of A, indexed by a segment of the ordinal numbers, is a *composition series of ideals* for A if $I_0 = 0$, $I_\beta = A$, and $I_\alpha = \Big(\bigcup\limits_{\gamma<\alpha} I_\gamma\Big)^=$ for limit ordinals α. Also $\widetilde{A}$ denotes the unitization of A, and "$=$" denotes norm closure.

(ii) Rørdam shows in 4.1–4.3 of [39] that in every C^*-algebra A there is a largest ideal $I_{\mathrm{tsr}\,1}(A)$ of stable rank one. If we define $\alpha(y) = \mathrm{dist}(y, \widetilde{A}^{-1})$, then the ideal is given by

$$I_{\mathrm{tsr}\,1}(A) = \{x \in A \mid \forall y \in \widetilde{A} :\ \alpha(x+y) = \alpha(y)\} = \{x \in A \mid x + \widetilde{A}^{-1} \subset (\widetilde{A}^{-1})^=\}.$$

Similar constructions are possible with respect to ideals of real rank zero and of extremal richness, cf. Theorems 2.3 and 2.16 of [9]. Thus if we let $\alpha_{\mathrm{sa}}(z) = \mathrm{dist}(z, \widetilde{A}^{-1}_{\mathrm{sa}})$ and define

$$R = \{x \in A_{\mathrm{sa}} \mid \forall y \in \widetilde{A}_{\mathrm{sa}} :\ \alpha_{\mathrm{sa}}(x+y) = \alpha_{\mathrm{sa}}(y)\} = \{x \in A_{\mathrm{sa}} \mid x + \widetilde{A}^{-1}_{\mathrm{sa}} \subset (\widetilde{A}^{-1}_{\mathrm{sa}})^=\},$$

then $I_{\mathrm{RR}\,0}(A) = R + \mathrm{i}R$ is an ideal of real rank zero in A, and the largest such.

(iii) It may happen, of course, that $A/I_{\mathrm{tsr}\,1}(A)$ has a non-zero ideal of stable rank one (consider e.g. the Toeplitz algebra, $\mathcal{T}$), or that $A/I_{\mathrm{RR}\,0}(A)$ has a non-zero ideal of real rank zero (consider e.g. a non-trivial extension (of real rank one) of a stabilized Bunce–Deddens algebra by $\mathbb{C}$, arising from a non-liftable projection in its corona algebra). In the general case we therefore obtain a strictly increasing series $\{I_\alpha | 0 \leqslant \alpha \leqslant \beta\}$ of ideals of A, which is a composition series for I_β, such that $I_{\alpha+1}/I_\alpha = I_{\mathrm{tsr}\,1}(A/I_\alpha)$ and $I_{\mathrm{tsr}\,1}(A/I_\beta) = 0$, and similarly for the real rank zero case.

(iv) If a C^*-algebra A has a composition series of ideals such that every subquotient $I_{\alpha+1}/I_\alpha$ has stable rank one (this implies $I_\beta = A$ above) we say that A has *generalized stable rank one*. Similarly we say that A has *generalized real rank zero* if it has a composition series such that $\mathrm{RR}(I_{\alpha+1}/I_\alpha) = 0$ for all α.

(v) If A has generalized stable rank one and if we choose $I_{\alpha+1}$ such that $I_{\alpha+1}/I_\alpha = I_{\mathrm{tsr}\,1}(A/I_\alpha)$, we obtain an *essential* composition series, i.e. $I_{\alpha+1}/I_\alpha$ is an essential ideal in A/I_α for all α. For if I were a non-zero ideal of A/I_α orthogonal to $I_{\alpha+1}/I_\alpha$, then there is a first ordinal μ such that $J = (I \cap I_\mu)/I_\alpha \neq 0$. Since μ cannot be a limit ordinal we see that J embeds as an ideal in $I_\mu/I_{\mu-1}$, and thus $\mathrm{tsr}(J) = 1$. As $J \cap (I_{\alpha+1}/I_\alpha) = 0$ this contradicts our choice of $I_{\alpha+1}$ as the largest ideal such that $\mathrm{tsr}(I_{\alpha+1}/I_\alpha) = 1$. Similar reasoning applies in the real rank zero case.

(vi) It follows from the observations made above that a C^*-algebra A has generalized stable rank one or generalized real rank zero precisely when $I_{\mathrm{tsr}\,1}(I/J) \neq 0$ or $I_{\mathrm{RR}\,0}(I/J) \neq 0$, respectively, for every non-zero quotient I/J of ideals of A.

This means that if A has another composition series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ (determined by other interesting subquotient properties), then we can find a composition series such that each of its subquotients has stable rank one or real rank zero, respectively, and is also a subquotient of one of the algebras $I_{\alpha+1}/I_\alpha$.

2.2. THE PRIMITIVE IDEAL SPACE. (i) Recall from pp. 233–241 in [16] or Section 3 of [13] that the set $A^\vee$ of primitive ideals in a C^*-algebra A is a locally quasi-compact, not necessarily Hausdorff topological space with the Jacobson topology, defined by the closure operation:

$$\overline{\mathcal{S}} = \mathrm{hull}(\ker(\mathcal{S}))\,, \quad \mathcal{S} \subset A^\vee\,.$$

Here $\mathrm{hull}(I) = \{P \in A^\vee \mid I \subset P\}$, and $\ker(\mathcal{S}) = \bigcap\limits_{P \in \mathcal{S}} P$ for any ideal I of A and any subset $\mathcal{S}$ of $A^\vee$. We obtain the formulae $(A/I)^\vee = \mathrm{hull}(I)$ and $I^\vee = A^\vee \setminus \mathrm{hull}(I)$, together with $\ker(\mathrm{hull}(I)) = I$. Furthermore, for each x in A the norm function $\check{x}$ on $A^\vee$ given by $\check{x}(P) = \|x - P\|$, $P \in A^\vee$, is lower semicontinuous, so that each set $\{P \in A^\vee \mid \check{x}(P) > \varepsilon\}$ is open; and $\check{x}$ vanishes at infinity, so that the set $\{P \in A^\vee \mid \check{x}(P) \geqslant \varepsilon\}$ is compact for $\varepsilon > 0$, cf. 4.4.4 of [31].

(ii) For some purposes, including some results in Section 5, it is helpful to make direct use of topological dimension theory. Since by definition $\mathrm{tsr}(A) =$

$\operatorname{tsr}(\widetilde{A})$ and $\mathrm{RR}(A) = \mathrm{RR}(\widetilde{A})$ when A is a non-unital C^*-algebra, we will use $\dim(\mathcal{X} \cup \{\infty\})$ as the basic dimension function for any locally compact Hausdorff space $\mathcal{X}$. Here dim is Čech's covering dimension and $\mathcal{X} \cup \{\infty\}$ is the one-point compactification of $\mathcal{X}$. It follows from 3.5.6 of [30] that

$$\dim(\mathcal{X} \cup \{\infty\}) = \sup_{\mathcal{K} \subset \mathcal{X}} \{\dim(\mathcal{K})\},$$

where $\mathcal{K}$ is compact, and we see from 3.5.3 of [30] that $\dim(\mathcal{X} \cup \{\infty\}) = \dim(\mathcal{X})$ when $\mathcal{X}$ is σ-compact. The concept of *local dimension* is treated in Chapter 5 of [30], and it follows from standard results that $\operatorname{loc}\dim(\mathcal{X}) = \dim(\mathcal{X} \cup \{\infty\})$ whenever $\mathcal{X}$ is locally compact and Hausdorff.

(iii) Recall that a subset $\mathcal{S}$ of a topological space $\mathcal{X}$ is called *locally closed* if $\mathcal{S} = \mathcal{F} \cap \mathcal{G}$ for some closed and open subsets $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{X}$, respectively. For a C^*-algebra A a locally closed subset $\mathcal{S}$ of $A^\vee$ corresponds to a subquotient of the form I/J, where I and J are ideals of A such that $J \subset I$ and $\mathcal{S} = (I/J)^\vee = I^\vee \cap \operatorname{hull} J$. Here I and J are not uniquely determined by $\mathcal{S}$, but I/J is determined up to canonical isomorphism.

(iv) Recall further that a topological space $\mathcal{X}$ is called *almost Hausdorff* if every non-empty closed subset $\mathcal{F}$ contains a non-empty relatively open subset $\mathcal{F} \cap \mathcal{G}$ (so $\mathcal{F} \cap \mathcal{G}$ is locally closed in $\mathcal{X}$) which is Hausdorff. If A is a C^*-algebra of type I then $A^\vee$ is almost Hausdorff since every non-zero quotient contains a non-zero ideal with continuous trace, cf. 6.2.11 of [31].

(v) We define the *topological dimension*, $\operatorname{top}\dim(A)$, of a C^*-algebra A for which $A^\vee$ is almost Hausdorff by

$$\operatorname{top}\dim(A) = \sup_{\mathcal{S} \subset A^\vee} \{\operatorname{loc}\dim(\mathcal{S})\} = \sup_{\mathcal{K} \subset A^\vee} \{\dim(\mathcal{K})\},$$

where $\mathcal{S}$ is any locally closed Hausdorff subset and $\mathcal{K}$ is any locally closed compact Hausdorff subset of $A^\vee$.

REMARK 2.1. In the simplest case where A is unital and $A^\vee$ is Hausdorff we have

$$\operatorname{top}\dim(A) = \dim(A^\vee) = \mathrm{RR}(Z(A))$$

by the Dauns–Hofmann Theorem, ([31], 4.4.8) where $Z(A)$ denotes the center of A. From [1] we then deduce, in the case where A is homogeneous of degree m and the corresponding Fell bundle is trivial, so that $A = Z(A) \otimes \mathbb{M}_m$, that we have $\mathrm{RR}(A) \leqslant r$ if and only if $\operatorname{top}\dim(A) \leqslant (2m-1)r$. In particular we learn that it is in general false that $\operatorname{top}\dim(A) \leqslant \mathrm{RR}(A)$ — unless $\mathrm{RR}(A) = 0$, cf. Proposition 2.7.

PROPOSITION 2.2. *If I is an ideal of a C^*-algebra A, then $A^\vee$ is almost Hausdorff if and only if $I^\vee$ and $(A/I)^\vee$ are both almost Hausdorff, and in that case*

$$\operatorname{top}\dim(A) = \max\{\operatorname{top}\dim(I), \operatorname{top}\dim(A/I)\}.$$

Proof. Every open or closed subset of an almost Hausdorff space is evidently almost Hausdorff. Moreover, any subset of an open or a closed set which is relatively locally closed and compact Hausdorff is also locally closed and compact Hausdorff in the global sense. This proves the first part of the the proposition and shows that $\operatorname{top\,dim}(A)$ majorizes the other two. The reverse inequality follows from 3.5.6. of [30]. ∎

PROPOSITION 2.3. *If $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ is a composition series for a C^*-algebra A then $A^\vee$ is almost Hausdorff if and only if $I_{\alpha+1}/I_\alpha$ is almost Hausdorff for each $\alpha < \beta$, and if this is so, then*

$$\operatorname{top\,dim}(A) = \sup_{\alpha<\beta}\{\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha)\}\,.$$

Proof. Assume that for some ordinal λ we have proved that

$$\operatorname{top\,dim}(I_\mu) = \sup_{\alpha<\mu}\{\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha)\}$$

for all $\mu < \lambda$. If λ is a limit ordinal then $I_\lambda = \Big(\bigcup\limits_{\mu<\lambda} I_\mu\Big)^{=}$. Since every compact subset of $I_\lambda^\vee$ is contained in some $I_\mu^\vee$ we conclude that

$$\operatorname{top\,dim}(I_\lambda) = \sup_{\mu<\lambda}\{\operatorname{top\,dim}(I_\mu)\} = \sup_{\alpha<\lambda}\{\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha)\}\,.$$

If λ is not a limit ordinal, i.e. $\lambda = \mu + 1$ for some $\mu < \lambda$, then again

$$\begin{aligned}\operatorname{top\,dim}(I_\lambda) &= \max\Big\{\operatorname{top\,dim}(I_\lambda/I_\mu), \sup_{\alpha<\mu}\{\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha)\}\Big\}\\ &= \sup_{\alpha<\lambda}\{\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha)\}\end{aligned}$$

by Proposition 2.2. The argument can now be completed by transfinite induction. ∎

PROPOSITION 2.4. *If A is a C^*-algebra such that $A^\vee$ is almost Hausdorff and if $A^\vee = \bigcup\limits_n \mathcal{S}_n$ where each $\mathcal{S}_n$ is locally closed, then* $\operatorname{top\,dim}(A) = \sup\limits_n\{\operatorname{top\,dim}(A_n)\}$, *where A_n is the subquotient of A with $A_n^\vee = \mathcal{S}_n$.*

Proof. It follows from Proposition 2.2 that $\operatorname{top\,dim}(A_n) \leqslant \operatorname{top\,dim}(A)$ for every n.

To prove the reverse inequality we shall use transfinite induction to construct a composition series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ for A such that each locally closed subset $(I_{\alpha+1}/I_\alpha)^\vee$ is contained in some $\mathcal{S}_n$.

Assume that for some ordinal λ we have defined these ideals for all $\mu < \lambda$. If λ is a limit ordinal we just put $I_\lambda = \Big(\bigcup\limits_{\mu<\lambda} I_\mu\Big)^{=}$. If λ is not a limit ordinal, i.e.

$\lambda = \mu + 1$ for some μ such that $I_\mu \neq A$, we observe that

$$\mathrm{hull}(I_\mu) = (A/I_\mu)^\vee = \bigcup_n \mathcal{S}_n \cap \mathrm{hull}(I_\mu)\,.$$

Since $(A/I_\mu)^\vee$ is a Baire space there is an n such that $(\mathcal{S}_n \cap \mathrm{hull}(I_\mu))^-$ has non-empty interior (relative to $\mathrm{hull}(I_\mu)$). As any locally closed subset $\mathcal{S}$ is a dense, relatively open subset of $\overline{\mathcal{S}}$, this implies that $\mathcal{S}_n \cap \mathrm{hull}(I_\mu)$ has a relative interior $\mathcal{G} \neq \emptyset$. Thus we may define I_λ such that $(I_\lambda/I_\mu)^\vee = \mathcal{G} \subset \mathcal{S}_n$.

Since I_λ is strictly larger than I_μ, the inductive process must eventually terminate with $I_\beta = A$ for some ordinal β, giving us the desired composition series. The result now follows from Proposition 2.3. ∎

REMARK 2.5. (i) We have not been able to locate a precise reference for the topological analogues of Propositions 2.3 and 2.4, but we note that Proposition 2.3 is a special case of Proposition 2.4 when the composition series is countable and that the topological analogue of Proposition 2.4 follows from standard results when $\mathcal{X}$ is second countable.

(ii) It follows from Proposition 2.3 that when $A^\vee$ is almost Hausdorff then $\dim(\mathcal{K}) \leqslant \operatorname{top\,dim}(A)$ for any compact Hausdorff subset of $A^\vee$, regardless of whether $\mathcal{K}$ is locally closed or not. We do not know whether such non-locally closed subsets can actually exist.

(iii) Note that $\operatorname{top\,dim}(A)$ depends only on $A^\vee$, but we are not showing this in the notation because it is not the same as $\dim(A^\vee)$. It would be a mistake if we had simply defined $\operatorname{top\,dim}(A)$ to be $\dim(A^\vee)$, ignoring the fact that $A^\vee$ may not be Hausdorff. To see this let $\mathcal{X}$ be a locally compact Hausdorff space and define $A = (C_0(\mathcal{X}) \otimes \mathbb{K})\tilde{}$. Then $A^\vee = \mathcal{X} \cup \{\infty\}$, where $\mathcal{X}$ has the given topology; but $\mathcal{X} \cup \{\infty\}$ is not the one-point-compactification. In fact, the only open set containing ∞ is the whole of $A^\vee$. Thus $\dim(A^\vee) = 0$, whereas we have correctly defined $\operatorname{top\,dim}(A)$ to be $\operatorname{loc\,dim}(\mathcal{X})$. Note also that $A^\vee$ is compact, so $\operatorname{loc\,dim}(A^\vee)$, as defined in [30], is zero. The space $A^\vee$, for $\mathcal{X} = [0,1]$, appears in Example 3.6.1 of [30].

The phenomenon above occurs annoyingly often, and means that we have to work with non-unital C^*-algebras in order not to destroy the Hausdorff properties of their primitive ideal spaces.

(iv) Kirchberg and Winter [17] have defined the decomposition rank, $\mathrm{dr}(A)$, for nuclear C^*-algebras A and have presented strong evidence that it is a non-commutative analogue of topological dimension. Unlike $\operatorname{top\,dim}(A)$, $\mathrm{dr}(A)$ is not a property of $A^\vee$, and therefore it can give much deeper information about A than can $\operatorname{top\,dim}(A)$. Winter [44] has shown that $\mathrm{dr}(A) = \operatorname{top\,dim}(A)$ when A is subhomogeneous, but this does not hold for all type I A, since by Example 4.8 of [17], $\mathrm{dr}(\mathcal{T}) = \infty$.

(v) It can be shown that for type I A, $\operatorname{top}\dim(A) \leqslant \sup\{\operatorname{top}\dim(A_\alpha)\}$ when $A = (\bigcup A_\alpha)^=$ for an upward directed family $\{A_\alpha\}$ of C^*-subalgebras, cf. Axiom X3 of [42].

(vi) Although we have defined the topological dimension of A only when $A^\vee$ is almost Hausdorff, the concept can be extended to arbitrary A in the special case of dimension zero. Thus we define $\operatorname{top}\dim(A) = 0$ to mean that $A^\vee$ has a basis consisting of compact-open subsets. This concept has already been used to good effect by Bratteli and Elliott [2] and recently by Pasnicu and Rørdam [29]. In Corollary 4.4 of [29] it is shown that the analogue of Proposition 2.2 holds for the extended concept in the separable case. However, the proof depends on the main theorem of [29]. We provide in the next proposition a simple direct proof that the analogue of Proposition 2.2 always holds. Then it is routine to show that the new definition for $\operatorname{top}\dim(A) = 0$ agrees with that given in Subsection 2.2 when $A^\vee$ is almost Hausdorff and that Proposition 2.3 and Proposition 2.4 still hold for the new definition. We are grateful to M. Rørdam for providing a copy of [29] and for helpful discussions.

PROPOSITION 2.6. *Let $\mathcal{X}$ be a locally quasi-compact topological space and $\mathcal{F}$ a closed subset. Then $\mathcal{X}$ has a basis of compact-open sets if and only if both $\mathcal{F}$ and $\mathcal{X} \setminus \mathcal{F}$ have bases of (relatively) compact-open sets.*

Proof. Since one direction is trivial, it is enough to assume $\mathcal{F}$ and $\mathcal{X} \setminus \mathcal{F}$ have the property and prove that $\mathcal{X}$ does. Thus we are given a point p in $\mathcal{X}$ and an open set $\mathcal{U}$ containing p, and we need to find a compact-open set $\mathcal{C}$ such that $p \in \mathcal{C}$ and $\mathcal{C} \subset \mathcal{U}$. We assume p is in $\mathcal{F}$, since otherwise the existence of $\mathcal{C}$ is obvious. Then there is a compact relatively open subset $\mathcal{K}$ of $\mathcal{F}$ such that $p \in \mathcal{K}$ and $\mathcal{K} \subset \mathcal{U}$. Let $\mathcal{V}$ be an open set such that $\mathcal{V} \subset \mathcal{U}$ and $\mathcal{V} \cap \mathcal{F} = \mathcal{K}$. By local quasi-compactness there is a compact set $\mathcal{L}$ such that $\mathcal{L} \subset \mathcal{V}$ and $\mathcal{K} \subset \mathcal{L}^\circ$, where $\mathcal{L}^\circ$ is the interior of $\mathcal{L}$. Then $\mathcal{L} \setminus \mathcal{L}^\circ$ is a compact subset of $\mathcal{V} \setminus \mathcal{F}$. Thus we can find a compact-open subset $\mathcal{C}_1$ (open relative to $\mathcal{X} \setminus \mathcal{F}$ and hence open in $\mathcal{X}$) such that $\mathcal{L} \setminus \mathcal{L}^\circ \subset \mathcal{C}_1 \subset \mathcal{V} \setminus \mathcal{F}$. Then let $\mathcal{C} = \mathcal{L}^\circ \cup \mathcal{C}_1 = \mathcal{L} \cup \mathcal{C}_1$. ∎

Recall that A is said to have the *ideal property* if every ideal of A is (ideally) generated by its projections. This property was defined by Stevens [41] and extensively studied by Pasnicu, cf. [27]. A weaker property is that $A \otimes \mathbb{K}$ have the ideal property.

PROPOSITION 2.7. *If A is a C^*-algebra of generalized real rank zero, or more generally if A has a composition series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ such that $(I_{\alpha+1}/I_\alpha) \otimes \mathbb{K}$ has the ideal property for each $\alpha < \beta$, then* $\operatorname{top}\dim(A) = 0$.

Proof. Since A and $A \otimes \mathbb{K}$ have the same primitive ideal spaces, and since topological dimension is compatible with composition series (Proposition 2.3 and Remark 2.5(vi)), it is sufficient to assume A has the ideal property and prove $A^\vee$ has a basis of compact-open sets. For this we use the fact that for every projection

p in A, $\{P \in A^\vee \mid p \notin P\}$ is a compact-open subset of $A^\vee$. Thus the ideal property yields directly the fact that every open subset of $A^\vee$ is a union of compact-open sets. ∎

REMARK 2.8. Since the study of ranks is our primary concern, we have only introduced the topological dimension for C^*-algbras as a tool. But it raises some natural questions. One of the main theorems in topological dimension theory is that if a normal space $\mathcal{X}$ is written as $\mathcal{X} = \bigcup \mathcal{F}_n$, where each $\mathcal{F}_n$ is closed, then $\dim(\mathcal{X}) = \sup_n \dim(\mathcal{F}_n)$. We proceed to establish an analogue of this countable sum theorem. Another will appear in Theorem 3.7, cf. Corollary 3.8 and Remark 3.9.

DEFINITION 2.9. Recall from [9] that a C^*-algebra A is *isometrically rich* if the union of the left and right invertible elements of $\widetilde{A}$ is dense in $\widetilde{A}$. Equivalently, cf. Proposition 4.2 of [9], A is isometrically rich if it is extremally rich and $\mathcal{E}(\widetilde{A})$ consists only of isometries and co-isometries.

THEOREM 2.10. *Let (I_n) be a sequence of ideals in a C^*-algebra A such that $\bigcup_{n=1}^{\infty} \mathrm{hull}(I_n) = A^\vee$. Then:*

(i) $\mathrm{RR}(A) = \sup_n \{\mathrm{RR}(A/I_n)\}$.

(i′) *If A is σ-unital and* $\mathrm{RR}(M(A/I_n)) = 0$ *for all n, then* $\mathrm{RR}(M(A)) = 0$.

(ii) $\mathrm{tsr}(A) = \sup_n \{\mathrm{tsr}(A/I_n)\}$.

(ii′) *If A is σ-unital and* $\mathrm{tsr}(M(A/I_n)) = 1$ *for all n, then* $\mathrm{tsr}(M(A)) = 1$.

(iii) *If each quotient A/I_n is isometrically rich and $I_{n+1} \subset I_n$ for all n, then A is isometrically rich.*

(iv) *If each quotient A/I_n is extremally rich and either $\{I_n\}$ is finite or $A^\vee$ is Hausdorff, then A is extremally rich.*

Proof. Without loss of generality we may assume in cases (i), (ii) and (iii) that A is unital. In case (i) we assume for some $d \geqslant 0$ that $\mathrm{RR}(A/I_n) \leqslant d$ for all n and take a tuple $\underline{x}$ in $(A_{\mathrm{sa}})^{d+1}$. We then wish to approximate $\underline{x}$ by a unimodular self-adjoint $(d+1)$-tuple. In case (ii) we assume for some $d \geqslant 1$ that $\mathrm{tsr}(A/I_n) \leqslant d$ for all n and take a tuple $\underline{x}$ in A^d. We then seek the same kind of approximation by a unimodular d-tuple. In cases (iii) and (iv) we take an arbitrary element x of A or $\widetilde{A}$ and wish to approximate it by a one-sided invertible element of A or by a general quasi-invertible element of $\widetilde{A}$. The basic construction is the same in all these cases, so we will write it out in case (iii), the most difficult, and then indicate the minor changes to be made for the others.

Recall that for a quasi-invertible element y in A we have defined $m_q(y) = \mathrm{dist}(y, A \setminus A_q^{-1})$, cf. 1.4 and 1.5 of [6], and that $m_q(y)$ also measures the distance from 0 to the rest of the spectrum of $|y|$. Now take $\varepsilon > 0$ and let $x_0 = x$. Also, let $\pi_n \colon A \to A/I_n$ denote the quotient morphism. By induction we will construct a sequence (x_n) in A, such that $\pi_n(x_n) \in (A/I_n)_q^{-1}$ and $\|x_n - x_{n-1}\| < \delta_n$, where

$$\delta_n = \min\left\{2^{-n}\varepsilon\,, \frac{1}{2}m_q(\pi_{n-1}(x_{n-1}))\,, \frac{1}{2}\delta_{n-1}\right\}$$

for all n (with the convention that $m_q(\pi_0(x_0)) = \delta_0 = 1$). Assume that we have defined x_k for all natural numbers $k < n$. Since A/I_n is extremally rich we can find y in $(A/I_n)_q^{-1}$ such that $\|y - \pi_n(x_{n-1})\| < \delta_n$, and we may then choose x_n in A such that $\pi_n(x_n) = y$ and $\|x_n - x_{n-1}\| < \delta_n$, completing the induction step.

The sequence (x_n) is evidently convergent, so we can define $x_\infty = \lim x_n$. The inequalities in the construction imply that $\|x_\infty - x\| < \varepsilon$. Moreover,

$$\|x_\infty - x_n\| < \sum_{k>n} \delta_k \leqslant 2\delta_{n+1} \leqslant m_q(\pi_n(x_n))\,,$$

so $\pi_n(x_\infty) \in (A/I_n)_q^{-1}$ for all n. By assumption A/I_n is isometrically rich, so $\pi_n(x_\infty)$ is either left or right invertible. However, since $I_{n+1} \subset I_n$, if $\pi_n(x_\infty)$ is not right invertible then $\pi_m(x_\infty)$ cannot be right invertible for any $m \geqslant n$. We may therefore assume that $\pi_n(x_\infty)$ is, say, left invertible for all n. Equivalently, $\pi_n(x_\infty^* x_\infty)$ is invertible for all n. Since $A^\vee = \bigcup \mathrm{hull}(I_n)$ this implies that $\rho(x_\infty^* x_\infty)$ is invertible for every irreducible representation $(\rho, \mathcal{H})$ of A. Therefore $x_\infty^* x_\infty$ is invertible in A, so that x_∞ is left invertible in A, as desired.

In case (iv), when $A^\vee$ is Hausdorff, the same construction will produce an approximant x_∞ in $\mathbf{1} + A \subset \widetilde{A}$ such that $\rho(x_\infty)$ is either left or right invertible for every irreducible representation $(\rho, \mathcal{H})$ of A. Thus

$$m_q(\rho(x_\infty)) = \max\{m(\rho(x_\infty)), m(\rho(x_\infty^*))\} > 0\,,$$

where, as usual,

$$m(\rho(x_\infty)) = \max\{\varepsilon > 0 \mid [0, \varepsilon[\cap \mathrm{sp}(\rho(|x_\infty|)) = \emptyset\}.$$

Thus $\min(m(\rho(x_\infty)), 1) = 1 - \|(\mathbf{1} - \rho(|x_\infty|))_+\|$. Since this function is continuous on $A^\vee$ and approaches 1 at ∞, there is an $\varepsilon > 0$ such that $m_q(\rho(x_\infty)) \geqslant \varepsilon$ for all $(\rho, \mathcal{H})$, whence $m_q(x_\infty) \geqslant \varepsilon$ and $x_\infty \in \widetilde{A}_q^{-1}$, cf. Proposition 1.2 of [6]. And of course the completion of case (iv) when $\{I_n\}$ is finite is obvious.

In cases (i) and (ii) a key fact is that the set of (self-adjoint) unimodular tuples in A^d is open. The distance of a unimodular tuple to the set of non-unimodular tuples will then replace the function m_q in the previous argument. We therefore obtain a tuple $\underline{x}_\infty = (y_1, \ldots, y_d)$ such that $\rho(\sum y_k^* y_k)$ is invertible for every irreducible representation of A, which means that $\sum y_k^* y_k$ is invertible in A and $\underline{x}_\infty$ is a unimodular tuple.

The proofs of cases (i′) and (ii′) are almost identical, so we write it out only for case (ii′), the more difficult. Thus we are given x in $M(A)$ and seek to approximate x by an invertible element y of $M(A)$. By the same basic argument as above, we can approximate x by x_∞ in $M(A)$ such that for all n $\overline{\pi}_n(x_\infty)$ is invertible in $M(A/I_n)$. Here $\overline{\pi}_n \colon M(A) \to M(A/I_n)$ is the natural extension of π_n, and we use the non-commutative Tietze extension theorem ([32], Theorem 10), to deduce that $\overline{\pi}_n$ is surjective. We shall show that x_∞ has a polar decomposition, $x_\infty = u|x_\infty|$, where u is unitary in $M(A)$. Then let $y = u(|x_\infty| + \varepsilon\mathbf{1})$.

For each n let $(\rho_n, \mathcal{H}_n)$ be a non-degenerate representation of A with kernel I_n. Let $(\rho, \mathcal{H})$ be the direct sum representation, and let $\overline{\rho}_n$ and $\overline{\rho}$ be the extensions of ρ_n and ρ to $M(A)$. Then $\overline{\rho}$ is faithful and $\overline{\rho}(M(A))$ is the idealizer of $\rho(A)$ in $B(\mathcal{H})$. Since $\overline{\rho}_n(x_\infty)$ is invertible, we can write $\overline{\rho}_n(x_\infty) = U_n\overline{\rho}_n(|x_\infty|)$, where U_n is unitary in $B(\mathcal{H}_n)$. Let $U = \bigoplus_n U_n$ and note that $U\overline{\rho}(|x_\infty|) = \overline{\rho}(x_\infty) = \overline{\rho}(|x_\infty^*|)U$. It is sufficient to show that U idealizes $\rho(A)$. Clearly $U\rho(R) \subset \rho(A)$ and $\rho(L)U \subset \rho(A)$, where $L = (A|x_\infty^*|)^=$ and $R = (|x_\infty|A)^=$, one-sided ideals of A. We claim that $L = A = R$. To see this, use the corresponding hereditary C^*-subalgebras, $B = (|x_\infty^*|A|x_\infty^*|)^=$ and $C = (|x_\infty|A|x_\infty|)^=$. If, for example, $R \neq A$, then $C \neq A$; and hence $\varphi_{|C} = 0$ for some pure state φ. But since $\bigcup_{n=1}^{\infty} \operatorname{hull}(I_n) = A^\vee$, φ factors through A/I_n for some n. This contradicts the invertibility of $\overline{\pi}_n(|x_\infty|)$. ∎

REMARK 2.11. (i) It follows from Theorem 2.10 that we have:

(1) $\mathrm{RR}(A/(I\cap J)) = \max\{\mathrm{RR}(A/I), \mathrm{RR}(A/J)\}$;
(2) $\mathrm{tsr}(A/(I\cap J)) = \max\{\mathrm{tsr}(A/I), \mathrm{tsr}(A/J)\}$; and
(3) $A/(I\cap J)$ is extremally rich$\Leftrightarrow$ A/I and A/J are for any pair I, J of ideals in A. Since $A/(I\cap J)$ is a surjective pullback of A/I and A/J, (1) and (2) are not new, cf. Theorem 4.1 and Remark 4.2 below.

(ii) Since the extra conditions in cases (iii) and (iv) are disappointing, we mention a couple of complements.

(4) if A/I_1 is isometrically rich and $\mathrm{tsr}(A/I_n) = 1$ for $n > 1$, then A is isometrically rich;
(5) if $A/I_1, \ldots, A/I_m$ are isometrically rich, $\mathrm{tsr}(A/I_n) = 1$ for $n > m$, and if $\operatorname{hull}(I_1), \ldots, \operatorname{hull}(I_m)$ are mutually disjoint, then A is extremally rich.

If we replace the sequence (I_n) by $\left(\bigcap_{k=1}^{n} I_k\right)$, it is easy to deduce (4) from case (iii). The deduction of (5) is also elementary, though not so immediate.

(iii) We show in Example 4.10 that if A/I_1 is only extremally rich in (4), then A need not be extremally rich, and that the disjointness hypothesis cannot be omitted from (5). Thus the extra conditions in cases (iii) and (iv) of Theorem 2.10 cannot be omitted. We also show in Example 4.10 that (iii′) and (iv′), the analogues of (iii) and (iv) for multiplier algebras, are false. It can also be shown that (i′) and (ii′) would be false for real ranks > 0 or stable ranks > 1.

(iv) Part (i) of Theorem 2.10 gives in principle a new proof of the topological countable sum theorem for compact Hausdorff spaces. However, the standard proof as found in Theorem 2.5 of [30] is also of a function-theoretic nature. But it uses the fact that $\dim \mathcal{X} \leqslant n$ if and only if each unimodular $(n+1)$-tuple in any quotient of $C(\mathcal{X})_{\mathrm{sa}}$ lifts to a unimodular tuple in $C(\mathcal{X})_{\mathrm{sa}}$. Since general C^*-algebras may not have very many ideals this definition does not generalize, and the commutative proof cannot be used as it stands.

3. LOW RANK OF INVERSE LIMITS

3.1. INVERSE LIMITS. (i) If $\{A_i\}$ is a family of C^*-algebras indexed by a directed set $\mathbb{I}$, and if $\{\varphi_{ij} \mid i, j \in \mathbb{I},\ i > j\}$ is a family of morphisms $\varphi_{ij}\colon A_i \to A_j$ satisfying the coherence relations $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all $i > j > k$ in $\mathbb{I}$, we define the C^*-*inverse limit* as the C^*-algebra $\varprojlim A_i$ of bounded strings $x = (x_i)$ in $\prod A_i$ such that $\varphi_{ij}(x_i) = x_j$ for all $i > j$. If $\rho_i\colon A \to A_i$, $i \in \mathbb{I}$, is a family of morphisms which is coherent with respect to (φ_{ij}), i.e. $\varphi_{ij} \circ \rho_i = \rho_j$ for all $i > j$, there is a unique morphism $\rho\colon A \to \varprojlim A_i$ given by $\rho(x) = (\rho_i(x))$. This universal property provides an alternative definition of $\varprojlim A_i$.

We shall here be exclusively interested in the case where $\mathbb{I} = \mathbb{N}$. If each morphism $\varphi_n = \varphi_{n,n-1}$ is surjective, we shall refer to $\varprojlim A_n$ as the *surjective C^*-inverse limit* of the A_n's. For the rest of this section we shall assume $\varprojlim A_n$ denotes a surjective C^*-inverse limit unless we explicitly say otherwise.

In stark contrast to the direct limit, the inverse limit of C^*-algebras is practically absent from the theory. The reason is that it tends to be unmanageably large. To circumvent this difficulty Phillips considered in [36] and [37] the category of pro-C^*-algebras in which full inverse limits (containing unbounded strings in $\prod A_i$) are allowed, but a much weaker topology is used. Roughly speaking, this is the non-commutative analogue of passing from the category of compact spaces (where an infinite topological union need not be in the category) to the category of normal spaces (where this process is allowed, cf. 1.4.3 of [30]). In fact, if $\mathcal{X}$ is the direct limit (in the category of topological spaces) of a directed family $(\mathcal{X}_i)$ of compact Hausdorff spaces then $\varprojlim C(\mathcal{X}_i) = C(\beta\mathcal{X})$, where $\beta\mathcal{X}$ denotes the Stone–Čech compactification of $\mathcal{X}$ (so that $\beta\mathcal{X}$ is the direct limit of $(\mathcal{X}_i)$ in the category of compact topological spaces). We shall here turn the usual disadvantage of inverse limits into an advantage, describing a rather general method of writing the multiplier algebra $M(A)$ of some C^*-algebras A as a surjective C^*-inverse limit of quotients of A.

(ii) For every m we define the surjective morphism $\pi_m\colon \varprojlim A_n \to A_m$ by evaluating an element $x = (x_n)$ of $\varprojlim A_n$ at m. Note that $\varphi_n \circ \pi_n = \pi_{n-1}$ for $n > 1$. For each C^*-subalgebra B of $\varprojlim A_n$ we therefore obtain a sequence of C^*-subalgebras $B_n = \pi_n(B)$ such that $\varphi_n(B_n) = B_{n-1}$ for $n > 1$. Conversely, and more to the point, given such a coherent sequence (B_n), there is a natural embedding of $\varprojlim B_n$ as a C^*-subalgebra $\overline{B}$ of $\varprojlim A_n$. Evidently $B \subset \overline{B}$, and in general the inclusion is strict. It is straightforward to verify that if B is an ideal or a hereditary C^*-subalgebra of $\varprojlim A_n$, then this is also the case for every B_n

in A_n. Conversely, if a coherent sequence (B_n) consists of ideals or hereditary C^*-subalgebras, then $\overline{B}$ will have the same property in $\varprojlim A_n$. With the obvious modifications this holds even for one-sided ideals.

THEOREM 3.1. *If $\varprojlim A_n$ is the surjective C^*-inverse limit of a sequence (A_n) of σ-unital C^*-algebras, and if A_o is an ideal of $\varprojlim A_n$ such that $\pi_m(A_o) = A_m$ for every m, then $M(A_o) = \varprojlim M(A_n)$, a surjective C^*-inverse limit.*

Proof. Put $A = \varprojlim A_n$ and $M = \varprojlim M(A_n)$. We then claim that there is a commutative diagram

$$\begin{tikzcd}
A_o \arrow[r, "\iota_o"] \arrow[d, "\iota"] & A \arrow[r, "\pi_n"] \arrow[d, "\rho"] & A_n \arrow[r, "\varphi_n"] \arrow[d, "\iota_n"] & A_{n-1} \arrow[d, "\iota_{n-1}"] \\
M(A_o) & M \arrow[r, "\overline{\pi}_n"] & M(A_n) \arrow[r, "\overline{\varphi}_n"] & M(A_{n-1})
\end{tikzcd}$$

Here ι and ι_k for $k \geqslant 0$ are the natural embeddings, and $\overline{\varphi}_n$ is the surjective morphism obtained from Theorem 10 of [32]. It follows that M is a surjective inverse limit, and the coordinate evaluations $\overline{\pi}_n$ are therefore also surjective.

Since the rightmost square of the diagram is commutative, we can define the morphism ρ by $(x_n) \mapsto (\iota_n(x_n))$ for every string $x = (x_n)$ in A, and we note that $\overline{\pi}_n \circ \rho = \iota_n \circ \pi_n$ by this definition. Evidently ρ is injective and $\rho(A)$ is an essential ideal of M.

We claim that A_o is essential in A. For if $xA_o = 0$ for some x in A, then $\pi_n(x)A_n = 0$ for every n by our assumption on A_o, whence $\pi_n(x) = 0$, and therefore $x = 0$. It follows that $\rho(\iota_o(A_o))$ is an essential ideal in M. By the universal property of multiplier algebras there is therefore an injective morphism $\varphi\colon M \to M(A_o)$ such that $\varphi \circ \rho \circ \iota_o = \iota$.

Each surjective morphism $\pi_n \circ \iota_o$ extends uniquely to a (not yet claimed surjective) morphism $\psi_n\colon M(A_o) \to M(A_n)$. Since $\varphi_n \circ \pi_n \circ \iota_o = \pi_{n-1} \circ \iota_o$, then also $\overline{\varphi}_n \circ \psi_n = \psi_{n-1}$ for all n. By the universal property of inverse limits this means that we have a unique morphism $\psi\colon M(A_o) \to M$ such that $\overline{\pi}_n \circ \psi = \psi_n$ for all n. It follows that

$$\overline{\pi}_n \circ \rho \circ \iota_o = \iota_n \circ \pi_n \circ \iota_o = \psi_n \circ \iota = \overline{\pi}_n \circ \psi \circ \iota$$

for all n, which implies that $\psi \circ \iota = \rho \circ \iota_o$.

Combining these results we find that

$$(\varphi \circ \psi) \circ \iota = \varphi \circ \rho \circ \iota_o = \iota \quad \text{and} \quad (\psi \circ \varphi) \circ \rho \circ \iota_o = \psi \circ \iota = \rho \circ \iota_o\,.$$

Since $\iota(A_o)$ is an essential ideal in $M(A_o)$ and $\rho(\iota_o(A_o))$ is an essential ideal in M these equations imply that φ and ψ are the inverses of one another, and we have our natural isomorphism. ∎

COROLLARY 3.2. *If (A_n) is a sequence of σ-unital C^*-algebras with surjective morphisms $\varphi_n\colon A_n \to A_{n-1}$, and if $\overline{\varphi}_n\colon M(A_n) \to M(A_{n-1})$ denote the unique (surjective) extensions of the φ_n's then*

$$M(\varprojlim A_n) = \varprojlim M(A_n)\,.$$

3.2. CONSTANT IDEALS. We say that an ideal I in $\varprojlim A_n$ is *m-constant* if $I \cap \ker \pi_m = 0$. Equivalently, $I \subset (\ker \pi_m)^\perp$. Since $(\ker \pi_n)$ is a decreasing sequence, $I \cap \ker \pi_n = 0$ for all $n \geqslant m$. Since $\varphi_n \circ \pi_n = \pi_{n-1}$ we see that $\ker \varphi_n \subset \pi_n(\ker \pi_m)$ for $n > m$. If therefore $I_n = \pi_n(I)$ denotes the associated sequence of ideals in A_n, then $I_n \cap \ker \varphi_n = 0$ for $n > m$. Thus I is isomorphic to I_m and I_n is isomorphic to I_m for all $n \geqslant m$. In particular, $\varprojlim I_n = I$. Conversely, if (I_n) is a sequence of ideals in (A_n) such that $\varphi_n(I_n) = I_{n-1}$ for $n > 1$ and $I_n \cap \ker \varphi_n = 0$ for all $n > m$ for some m then $I = \varprojlim I_n$ will be an m-constant ideal in $\varprojlim A_n$.

If I is an n-constant and J an m-constant ideal with $n \leqslant m$, then $I + J \subset (\ker \pi_m)^\perp$ since $\ker \pi_m \subset \ker \pi_n$, so $I + J$ is an m-constant ideal. Since $(\ker \pi_m)^\perp$ is the largest m-constant ideal, it follows that $A_c = (\bigcup (\ker \pi_m)^\perp)^=$ is equal to the sum of all constant ideals, and we shall refer to it as the *quasi-constant ideal* of $\varprojlim A_n$.

The motivating example for considering constant and quasi-constant ideals arises from the Stone–Čech compactification. If $\mathcal{X}$ is a locally compact Hausdorff space then $C_b(\mathcal{X})$ is always a C^*-inverse limit. In the case where $\mathcal{X}$ is also σ-compact we can write $\mathcal{X} = \bigcup \mathcal{X}_n$, where each $\mathcal{X}_n$ is compact and $\mathcal{X}_n \subset (\mathcal{X}_{n+1})^\circ$. Put $A_n = C(\mathcal{X}_n)$ and let $\varphi_n(f) = f|\mathcal{X}_n$ for each f in $C(\mathcal{X}_{n+1})$. Then $C_b(\mathcal{X}) = \varprojlim A_n$. The large constant ideals will be of the form $(\ker \pi_m)^\perp = C_0((\mathcal{X}_m)^\circ)$, so the quasi-constant ideal of $C_b(\mathcal{X})$ can be identified with $C_0(\mathcal{X})$.

Theorem 3.1 provides an immediate generalization of this construction:

COROLLARY 3.3. *If $A = \varprojlim A_n$ is the surjective C^*-inverse limit of a sequence of σ-unital C^*-algebras (A_n), such that the quasi-constant ideal A_c of A satisfies $\pi_m(A_c)$ $= A_m$ for every m, then $M(A_c) = \varprojlim M(A_n)$.*

THEOREM 3.4. *Let (J_n) and (I_n) be two sequences of ideals in a C^*-algebra A, one increasing, the other decreasing, but such that $I_n \cap J_n = 0$ for all n. If $A_o = (\bigcup J_n)^=$ is essential in A and $A_o + I_n = A$ for every n, then with $A_n = A/I_n$ and $\varphi_n\colon A_n \to A_{n-1}$ the natural morphisms we have an embedding $\rho\colon A \to \varprojlim A_n$ such that $\rho(A_o)$ is an ideal. If also each A/I_n is σ-unital, then $M(A_o) = \varprojlim M(A_n)$.*

Proof. If $x \in \bigcap I_n$ then it annihilates J_n for every n, whence $x \in A_o^\perp$. But then $x = 0$ since A_o is essential. Thus our assumptions imply that $\bigcap I_n = 0$. The quotient morphisms $\rho_n\colon A \to A_n$ satisfy $\varphi_n \circ \rho_n = \rho_{n-1}$ for all n, and therefore define a morphism $\rho\colon A \to \varprojlim A_n$. Since $\ker \rho = \bigcap I_n = 0$, this is an embedding. Observe that $\rho_n(J_m) = (J_m + I_n)/I_n$ is an ideal in A_n for every n and m. Since $\ker \varphi_n = I_{n-1}/I_n$ we see moreover that $\rho_n(J_m) \cap \ker \varphi_n = 0$ for $n > m$. Consequently $(\rho_n(J_m))$ is a coherent sequence of ideals in (A_n), all isomorphic for $n \geqslant m$, thus giving rise to the m-constant ideal $\rho(J_m)$ in $\varprojlim A_n$. It follows from this that $\rho(A_o)$ is an ideal in $\varprojlim A_n$, isomorphic to A_o (and contained in the quasi-constant ideal of $\varprojlim A_n$). Since by assumption

$$\pi_n(\rho(A_o)) = \rho_n(A_o) = (A_o + I_n)/I_n = A_n\,,$$

it follows from Theorem 3.1 that $M(A_o) = \varprojlim M(A_n)$. ∎

REMARK 3.5. (i) If $A = \varprojlim A_n$ where each A_n is separable, it can be shown that A_c is the largest separable ideal of A, a result analogous to the result from [3] that every separable C^*-algebra B is the largest separable ideal of $M(B)$. With the help of these facts it can be shown that if $M(A_0) = \varprojlim M(A_n)$ for a separable ideal A_0 of A, then $A_0 = A_c$ and $\pi_m(A_0) = A_m$, as in Corollary 3.3. Also if (I_n) is a decreasing sequence of ideals of a separable C^*-algebra A such that $\bigcap I_n = 0$, and if A_0 is an ideal of A such that $M(A_0)$ is identified as above with $\varprojlim M(A/I_n)$, then $A_0 = (\bigcup I_n^\perp)^=$ and $A_0 + I_n = A$, as in Theorem 3.4. These facts provide some justification for our approach in Corollary 3.3 and Theorem 3.4.

(ii) In Theorem 3.4 we could enlarge I_n to $J_n^\perp$ or J_n to $I_n^\perp$ and still have the hypotheses. In the first case we see that $\varprojlim M(A_n)$ doesn't change. In the second, $(\bigcup J_n)^=$ doesn't change, at least in the separable case.

COROLLARY 3.6. *Let (I_n) and (J_n) be two sequences of ideals in a C^*-algebra A, one decreasing, the other increasing, such that $I_n \perp J_n$ for every n. If $\bigcup J_n$ is dense in A and each quotient $A_n = A/I_n$ is unital, then with $\varphi_n\colon A_{n+1} \to A_n$ the natural morphisms we have an embedding of A into $\varprojlim A_n$, such that $\varprojlim A_n = M(A)$.*

THEOREM 3.7. *Let $A = \varprojlim A_n$ be the surjective C^*-inverse limit of a sequence of C^*-algebras. Then:*

(i) *If* $\mathrm{RR}(A_n) = 0$ *for all n, then* $\mathrm{RR}(A) = 0$.

(ii) $\mathrm{tsr}(A) = \sup_n\{\mathrm{tsr}(A_n)\}$.

(iii) *If each A_n is isometrically rich, then A is isometrically rich.*

(iv) *If each A_n is extremally rich, then A is extremally rich.*

Proof. We shall use the same basic construction as in the proof of Theorem 2.10, relative to the surjective morphisms $\pi_n\colon A \to A_n$. Assuming, as we may, that A is unital we find in cases (i), (iii) or (iv) for each x in A_{sa} or in A and each $\varepsilon > 0$ an x_∞ in A_{sa} or in A, such that $\|x_\infty - x\| < \varepsilon$ and $\pi_n(x_\infty) \in (A_n)^{-1}_{\mathrm{sa}}$ (in case (i)) and $\pi_n(x_\infty) \in (A_n)^{-1}_q$ (in cases (iii) and (iv)). Moreover, in case (iii) where each A_n is isometrically rich, we see from the connecting morphisms $\varphi_n\colon A_n \to A_{n-1}$ that either all $\pi_n(x_\infty)$ are left invertible or all are right invertible.

Let $\pi_n(x_\infty) = w_n|\pi_n(x_\infty)|$ be the polar decomposition, so that w_n is in $\mathcal{E}(A_n)$. Since w_n is unique and $|\pi_n(x_\infty)| = \pi_n(|x_\infty|)$ it follows that $\varphi_n(w_n) = w_{n-1}$ for $n > 1$, so that $w = (w_n)$ is in $\mathcal{E}(A)$. Moreover, if all $\pi_n(x_\infty)$ are self-adjoint invertibles, then every w_n is a symmetry, so w is a symmetry, and if all $\pi_n(x_\infty)$ are, say, left invertible, then each w_n is an isometry, so w is an isometry.

Now put $y = w(|x_\infty| + \varepsilon\mathbf{1})$. Then $y \in A_q^{-1}$ (and y is self-adjoint invertible if w is a symmetry, whereas y is left invertible if w is an isometry) with $\|y - x\| \leqslant 2\varepsilon$.

In the remaining case (ii) we are given a tuple $\underline{x}$, and the approximant $\underline{x}_\infty$ is also a tuple. Thus $\underline{x}_\infty = (y_1, \dots, y_d)$ is in A^d, and $h_n = \pi_n(\sum y_k^* y_k)$ is invertible in A_n for all n. Put $\underline{w}_n = \pi_n(\underline{x}_\infty) h_n^{-1/2}$. Then as above $\varphi_n(\underline{w}_n) = \underline{w}_{n-1}$ and we approximate $\underline{x}$ with $\underline{w}((\sum y_k^* y_k)^{1/2} + \varepsilon\mathbf{1})$. ∎

COROLLARY 3.8. *Let (J_n) be an increasing sequence of ideals in a C^*-algebra A such that $\bigcup J_n$ is dense in A. Assume furthermore that each annihilator quotient $A/(J_n)^\perp$ is unital. Then:*

(i) *If* $\mathrm{RR}(A) = 0$, *then* $\mathrm{RR}(M(A)) = 0$.

(ii) $\mathrm{tsr}(M(A)) = \mathrm{tsr}(A)$.

(iii) *If A is isometrically rich, then $M(A)$ is isometrically rich.*

(iv) *A is extremally rich if and only if $M(A)$ is extremally rich if and only if each $A/(J_n)^\perp$ is extremally rich.*

Proof. Combine Corollary 3.6 and Theorem 3.7. ∎

REMARK 3.9. (i) The idea in Corollary 3.8 of combining properties of ideals and their annihilators is found in Proposition 3.15 of [40], which is labeled a technical proposition. Viewed as a generalization of writing $C_\mathrm{b}(\mathcal{X})$ as a C^*-inverse limit $\varprojlim C(\mathcal{X}_n)$, cf. Subsection 3.2, the condition seems more intuitive. Sheu's result calculates $\mathrm{tsr}(A)$ using weaker hypotheses on the ideals than those in Corollary 3.8. It helped inspire some of our results and in turn could be deduced from Theorem 2.10(ii).

(ii) It is instructive to realize that these formulae are non-commutative analogues of the well-known identities $\dim(\beta\mathcal{X}) = \dim(\mathcal{X})$, valid for any normal space $\mathcal{X}$ ([30], 6.4.3). By contrast, the identities $\mathrm{tsr}(M(A)) = \mathrm{tsr}(A)$ and $\mathrm{RR}(M(A)) = \mathrm{RR}(A)$ are often false for non-commutative C^*-algebras. A partial "explanation" might be that $M(A)$ is not always obtainable as a C^*-inverse limit in the non-commutative case.

(iii) The hypotheses of Corollary 3.8 imply that $A^\vee = \bigcup \mathrm{hull}(J_n^\perp)^\circ$, a considerably stronger condition than the one used in Theorem 2.10, but cases (iii) and (iv) and case (ii) for higher ranks don't follow from Theorem 2.10.

(iv) The reader may have wondered at the asymmetry in the treatment of stable rank and real rank in Theorem 3.7 parts (i) and (ii). The truth is that we have — at the moment — no argument to prove that if $\mathrm{RR}(A) \leqslant n$ for some unital C^*-algebra A and $n > 0$, then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for every tuple $(x_0, \dots, x_n)$ in A_sa there is a tuple $(y_0, \dots, y_n)$ in A_sa with $\sum y_k^2 \geqslant \delta$ and $\|x_k - y_k\| \leqslant \varepsilon$ for all k. In the similar situation for stable ranks we can take δ to be any number less that ε^2, as we saw in the proof of Theorem 3.7. This missing information means that the higher real ranks of inverse limits and even direct products cannot be estimated, a fact that seems not to be widely known.

(v) We show that the surjectivity hypothesis in Theorem 3.7 cannot be omitted, even in the separable, commutative, unital case. Note that if (A_n) is a decreasing sequence of C^*-subalgebras of B, then $\bigcap A_n$ is the C^*-inverse limit of the A_n's. Let $\mathcal{X}$ be an arbitrary compact metric space and $f\colon \mathcal{C} \to \mathcal{X}$ a surjective continuous map, where $\mathcal{C}$ is the Cantor set. Let $\mathcal{G} = \{(s,t) \in \mathcal{C} \times \mathcal{C} \mid f(s) = f(t)\}$ and $\mathcal{D} = \{(s_n, t_n)\}$, a countable dense subset of $\mathcal{G}$. Then if $B = C(\mathcal{C})$ and $A_n = \{g \in B \mid g(s_k) = g(t_k), k = 1, \dots, n\}$, then $\operatorname{top\,dim}(A_n) = 0$ and $\bigcap A_n \cong C(\mathcal{X})$.

Despite this example, we have some positive results about one class of non-surjective inverse limits: Let (I_n) be an increasing sequence of ideals such that $A = (\bigcup I_n)^=$. Then $M(A)$ is the C^*-inverse limit of the $M(I_n)$'s relative to the restriction maps $\rho_n\colon M(A) \to M(I_n)$ and $\rho_{n,n-1}\colon M(I_n) \to M(I_{n-1})$. (But A is the direct limit of the I_n's.)

LEMMA 3.10. *Let $A = (\bigcup I_n)^=$, where (I_n) is an increasing sequence of ideals, and let $g \in A_+$. Then there is an increasing sequence (g_n) such that $g_n \in I_{n+}$ and $\|g - g_n\| \to 0$.*

Proof. Let $P_n = \{x \in I_{n+}\colon \|x\| < 1\}$ and $P = \bigcup P_n$. By 1.4.3 of [31] P_n is directed upward and forms an approximate identity of I_n. Hence P is directed upward and forms an approximate identity of A. Thus we can recursively construct an increasing sequence (r_j) in P such that $\|g^{1/2}(\mathbf{1} - r_j)g^{1/2}\| < \frac{1}{j}$ and a strictly increasing sequence (n_j) with $r_j \in P_{n_j}$. Finally, let $g_n = g^{1/2} r_j g^{1/2}$ if $n_j \leqslant n < n_{j+1}$ ($g_n = 0$ if $n < n_1$). ∎

THEOREM 3.11. *Assume A is a σ-unital C^*-algebra and (I_n) an increasing sequence of ideals such that $A = (\bigcup_n I_n)^=$. Let $\rho_n\colon M(A) \to M(I_n)$ be the restriction maps.*

(i) *If* $\operatorname{tsr}(A) = 1$ *and if p and q are projections in $M(A)$ such that $\rho_n(p) \sim \rho_n(q)$ for all n, then $p \sim q$, where $\sim$ denotes Murray–von Neumann equivalence.*

(ii) *If* $\operatorname{tsr}(M(I_n)) = 1$ *for all n, then* $\operatorname{tsr}(M(A)) = 1$.

(iii) *If* $\operatorname{RR}(M(I_n)) = 0$ *for all n, then* $\operatorname{RR}(M(A)) = 0$.

Proof. (ii) In the proof we make frequent use of a C^*-algebraic operation which has already occurred in connection with C^*-algebras of low rank, but which has no standard notation. If x is an element of a C^*-algebra B which is faithfully represented on a Hilbert space, then x has a canonical polar decomposition, $x = v|x|$, where $|x|$ is in B but v need not be. If $f\colon [0,\infty[\to \mathbb{C}$ is a continuous function such that $f(0) = 0$, let $x_{[f]}$ denote $vf(|x|) = f(|x^*|)v$. Then, as is well known, $x_{[f]}$ is in B and is independent of the representation of B. Also this operation is compatible with morphisms. For $\delta \geqslant 0$ let E_δ and F_δ be the spectral projections of $|x|$ and $|x^*|$ for the interval $]\delta, \infty[$, which again need not be in B. Then, when B is unital, Rørdam [39] showed that $\operatorname{tsr}(B) = 1$ if and only if for each x in B and each $\delta > 0$ there is a unitary u in B with $uE_\delta = vE_\delta$ (equivalently

$F_\delta u = F_\delta v$). In other words u "extends" the partial isometry vE_δ. This property of u is independent of the representation of B and can equivalently be stated:

(6) $x_{[f]} = uf(|x|)$ whenever $f_{|[0,\delta]} = 0$, or, still equivalently;

(7) $xg(|x|) = u|x|g(|x|)$ whenever $g_{|[0,\delta]} = 0$.

Let x be in $M(A)$ and $0 < \delta < 1$. Fix a strictly positive element g of A, and let (g_n) be as in Lemma 3.10. Define functions h_0, h, k, and θ_n on $[0, \infty[$ by:

(8) $h_{0|[0,\frac{\delta}{4}]} = 0$, $h_{0|[\delta,\infty[} = 1$, h_0 is linear on $[\frac{\delta}{4}, \delta]$;

(9) $h_{|[0,\frac{1}{2}]} = 0$, $h_{|[1,\infty[} = 1$, h is linear on $[\frac{1}{2}, 1]$;

(10) $k_{|[0,\frac{1}{4}]} = 0$, $k_{|[\frac{1}{2},\infty[} = 1$, k is linear on $[\frac{1}{4}, \frac{1}{2}]$;

(11) $\theta_1 = k \circ h_0$, and $\theta_{n+1} = k \circ h \circ \theta_n$.

Also we shall use the same symbol ρ_n to denote the restriction maps from $M(I_m)$ to $M(I_n)$ for $m \geqslant n$.

We shall recursively define w_n, e_n, and f_n in $M(I_n)$ and y_n in $M(A)$ such that:

(12) w_n is unitary, $0 \leqslant e_n, f_n \leqslant \mathbf{1}$;

(13) w_n extends $v_n E_\delta^{(n)}$, referring to a polar decomposition of $\rho_n(x)$;

(14) $(\rho_{n-1}(w_n) - w_{n-1})e_{n-1} = 0 = f_{n-1}(\rho_{n-1}(w_n) - w_{n-1})$ for $n > 1$;

(15) $w_n e_n = f_n w_n$;

(16) $\|(\mathbf{1} - e_n)g_n\|$, $\|g_n(\mathbf{1} - f_n)\| < \frac{1}{n}$;

(17) $\rho_{n-1}(e_n)e_{n-1} = e_{n-1}$ and $\rho_{n-1}(f_n)f_{n-1} = f_{n-1}$ for $n > 1$;

(18) $y_n \in x_{[\theta_n]} + I_n$;

(19) $\rho_n(y_n) = w_n|\rho_n(y_n)|$;

(20) $(\mathbf{1} - |y_n|)E_\delta = 0 = F_\delta(\mathbf{1} - |y_n^*|)$; and

(21) $(\mathbf{1} - \rho_n(|y_n|))e_n = 0 = f_n(\mathbf{1} - \rho_n(|y_n^*|))$.

To start the construction, choose a unitary w_1 which extends $v_1 E_{\delta/4}^{(1)}$, establishing (13). Then let (r_i) be an approximate identity for I_1 and let

$$e_i' = h(\rho_1(h_0(|x|) + (\mathbf{1} - h_0(|x|))^{1/2} r_i (\mathbf{1} - h_0(|x|))^{1/2}))\,.$$

Since (e_i') converges strictly to $\mathbf{1}$ in $M(I_1)$, we may define $e_1 = e_{i_0}'$, where i_0 is chosen large enough that $\|(\mathbf{1} - e_1)g_1\|$, $\|g_1(\mathbf{1} - w_1 e_1 w_1^*)\| < 1$. Using (15) as the definition of f_1, we have (16). Now let

$$c_1 = k(\rho_1(h_0(|x|) + (\mathbf{1} - h_0(|x|))^{1/2} r_{i_0} (\mathbf{1} - h_0(|x|))^{1/2}))\,,$$

and $m_1 = w_1 c_1$. From the choice of w_1 and the fact that $c_1 \in \rho_1(\theta_1(|x|)) + I_1$, it follows that $m_1 = \rho_1(x_{[\theta_1]}) + z_1$ for some z_1 in I_1. Then let $y_1 = x_{[\theta_1]} + z_1$, so that (18) and (19) are clear. To prove (20) it is enough to show that the relations hold both modulo I_1 and after application of ρ_1. Modulo I_1 the relations are obvious from (18) and the fact that $\theta_{1_{|[\delta,\infty]}} = 1$. Since $\rho_1(y_1) = m_1$, it is clear that $\rho_1(|y_1|)$ unitizes anything unitized by $\rho_1(h_0(|x|))$. This shows the first part of (20), and the second part follows by conjugation with w_1 (using (13)). The first part of (21) follows from the fact that $kh = h$, and the second again follows by conjugation with w_1.

Now let $n > 1$ and assume the first $n-1$ steps. Choose a unitary w_n in $M(I_n)$ which extends $v'E'_{1/4}$, referring to a polar decomposition of $\rho_n(y_{n-1})$. To verify (13) it suffices to verify it both modulo I_{n-1} and after application of ρ_{n-1}. Then (18) for $n-1$ easily implies the relation modulo I_{n-1}. From (19) for $n-1$ we see:

(22) $\rho_{n-1}(w_n)t = w_{n-1}t$ whenever $\rho_{n-1}(|y_{n-1}|)t = t$, and $t\rho_{n-1}(w_n) = tw_{n-1}$ whenever $t\rho_{n-1}(|y^*_{n-1}|) = t$.

In view of (20) and (13) for $n-1$, the proof of (13) is complete. Next let (r_i) be an approximate identity for I_n and let

$$e'_i = h(\rho_n(h(|y_{n-1}|) + (\mathbf{1} - h(|y_{n-1}|))^{1/2} r_i (\mathbf{1} - h(|y_{n-1}|))^{1/2}))\,.$$

Since (e'_i) converges strictly to $\mathbf{1}$ in $M(I_n)$, we may define $e_n = e'_{i_0}$ where i_0 is chosen large enough that $\|(\mathbf{1} - e_n)g_n\|$, $\|g_n(\mathbf{1} - w_n e_n w_n^*)\| < \frac{1}{n}$. As in step one, we define f_n by (15) and have (16). Now we can deduce (14) from (21) for $n-1$, using (22). Since $\rho_{n-1}(e_n)$ unitizes anything unitized by $\rho_{n-1}(|y_{n-1}|)$, we have the first part of (17). The second part follows by conjugation, using (14). Next let

$$c_n = k(\rho_n(h(|y_{n-1}|) + (\mathbf{1} - h(|y_{n-1}|))^{1/2} r_{i_0} (\mathbf{1} - h(|y_{n-1}|))^{1/2}))\,,$$

and $m_n = w_n c_n$. Since $c_n \in \rho_n(k(h(|y_{n-1}|))) + I_n$, $m_n \in w_n k(h(\rho_n(|y_{n-1}|))) + I_n$. Also, the choice of w_n and the fact that $k \circ h$ is supported on $[\frac{1}{4}, \infty)$ imply that $w_n k(h(\rho_n(|y_{n-1}|))) = \rho_n(y_{n-1})_{[k\circ h]}$. It follows, using (18) for $n-1$, that $m_n = \rho_n(x_{[\theta_n]}) + z_n$ for some z_n in I_n. Then let $y_n = x_{[\theta_n]} + z_n$, so that (18) and (19) are clear. Finally (20) and (21) are proved as in step one.

Now given the recursion, we construct w essentially as the strict limit of (w_n). Of course w_n is only in $M(I_n)$, but for each a in $\bigcup I_n$, $w_n a$ and $a w_n$ are defined for n sufficiently large; and we claim that these sequences converge. In fact from (14) and (17), $f_n \rho_n(w_m)$ and $\rho_n(w_m)e_n$ are constant for $m \geqslant n$. In particular $(w_m a)$ is convergent for a in $e_n g_n I_n$. Since $e_n g_n \to g$ by (16) and the choice of (g_n), we conclude that $(w_m a)$ is convergent for a in gI_n; and since g is strictly positive, $(gI_n)^- = I_n$. The convergence of $(a w_m)$ is proved similarly. Finally, it is clear that w is unitary and the fact that w extends vE_δ follows from (13).

(iii) It was shown in both [4] and [10] that a C^*-algebra B has real rank zero if and only if it satisfies an interpolation by projections property. When B is unital and is faithfully represented on a Hilbert space this property can be stated as follows: Let x be in B_{sa}, and for $\delta > 0$ let E^+_δ and E^-_δ be the spectral projections of x for the intervals $]\delta, \infty[$ and $]-\infty, -\delta[$. Then there is a projection p in B such that $E^+_\delta \leqslant p \leqslant \mathbf{1} - E^-_\delta$. This property can be equivalently stated:

(7′) $pg(x) = g(x)$ whenever $g_{|]-\infty,\delta]} = 0$, and $pg(x) = 0$ whenever $g_{|[-\delta,\infty[} = 0$.

Thus the interpolation property is independent of the representation of B. Note that for the canonical polar decomposition, $x = v|x|$, we have that $v^* = v$,

$E_\delta = F_\delta = E_\delta^+ + E_\delta^-$, $vE_\delta = E_\delta^+ - E_\delta^-$, and $v|x| = |x|v$ in the notation of case (ii). It was further observed in [10] that B has real rank zero if and only if for each such δ and x, vE_δ can be extended to a self-adjoint unitary u in B. The proof of case (iii) can now proceed analogously to that of case (ii).

Thus we start with $0 < \delta < 1$ and x in $M(A)_{\mathrm{sa}}$ and recursively define w_n and e_n in $M(I_n)_{\mathrm{sa}}$ and y_n in $M(A)_{\mathrm{sa}}$ such that, with the same notations as in case (ii):

(12′) w_n is unitary and $0 \leqslant e_n \leqslant 1$;
(13′) w_n extends $v_n E_\delta^{(n)}$, referring to a polar decomposition of $\rho_n(x)$;
(14′) $(\rho_{n-1}(w_n) - w_{n-1})e_{n-1} = 0$ for $n > 1$;
(15′) $w_n e_n = e_n w_n$;
(16′) $\|(\mathbf{1} - e_n)g_n\| < \frac{1}{n}$;
(17′) $\rho_{n-1}(e_n)e_{n-1} = e_{n-1}$ for $n > 1$;
(18′) $y_n \in x_{[\theta_n]} + I_n$;
(19′) $\rho_n(y_n) = w_n|\rho_n(y_n)| = |\rho_n(y_n)|w_n$;
(20′) $(\mathbf{1} - |y_n|)E_\delta = 0$; and
(21′) $(\mathbf{1} - \rho_n(|y_n|))e_n = 0$.

To start the construction, we choose two projections p and q in $M(I_1)$ such that $E_{\delta/4}^{(1)+} \leqslant p \leqslant E_{\delta/8}^{(1)+}$ and $E_{\delta/4}^{(1)-} \leqslant q \leqslant E_{\delta/8}^{(1)-}$. (The existence of p and q follows from the stated interpolation property and functional calculus.) Note that $pq = 0$, $\rho_1(h_0(x_+)) \in pM(I_1)p$, and $\rho_1(h_0(x_-)) \in qM(I_1)q$. Let (r_i) be an approximate identity for pI_1p, (s_j) an approximate identity for qI_1q, (t_k) an approximate identity consisting of projections for $(\mathbf{1} - p - q)I_1(\mathbf{1} - p - q)$, and

$$e'_{i,j,k} = h(\rho_1(h_0(|x|) + (\mathbf{1} - h_0(|x|))^{1/2}(r_i + s_j)(\mathbf{1} - h_0(|x|))^{1/2})) + t_k\,.$$

Then $(e'_{i,j,k})$ converges strictly to $\mathbf{1}$ in $M(I_1)$. Hence we may define $e_1 = e'_{i_0,j_0,k_0}$, where i_0, j_0, k_0 are chosen large enough that $\|(\mathbf{1} - e_1)g_1\| < 1$. Now let

$$c_1 = k(\rho_1(h_0(|x|) + (\mathbf{1} - h_0(|x|))^{1/2}(r_{i_0} + s_{j_0})(\mathbf{1} - h_0(|x|))^{1/2})) + t_{k_0}\,,$$

$w_1 = 2(p + t_{k_0}) - \mathbf{1}$, and $m_1 = w_1 c_1 = c_1 w_1$. As above, $m_1 = \rho_1(x_{[\theta_1]}) + z_1$ for some z_1 in I_1, and we define $y_1 = x_{[\theta_1]} + z_1$. All the conditions follow as in the proof of case (ii), or more easily.

As in the proof of case (ii), the recursive step is essentially the same as the initial step, using y_{n-1} instead of x, h instead of h_0, $\frac{1}{n}$ instead of 1, and $\frac{1}{4}$ instead of $\frac{\delta}{4}$. All the conditions are verified as in case (ii) or more easily. And the completion of the proof after the recursive construction is also the same as in case (ii).

(i) Let $p_n = \rho_n(p)$, $q_n = \rho_n(q)$, $B_n = p_n I_n p_n$, $C_n = q_n I_n q_n$, and $X_n = p_n I_n q_n$. Fix strictly positive elements g' of pAp and g'' of qAq and choose g'_n in B_n and g''_n in C_n as in Lemma 3.10. Choose u_n in $M(I_n)$ such that $u_n u_n^* = p_n$ and $u_n^* u_n = q_n$, and let $\widetilde{B}_n = B_n + \mathbb{C}p_n$, $\widetilde{C}_n = C_n + \mathbb{C}q_n$, and $\widetilde{X}_n = X_n + \mathbb{C}u_n$. Thus $\widetilde{B}_n, \widetilde{C}_n, \widetilde{X}_n \subset M(I_n)$, and $\widetilde{X}_n$ is a $\widetilde{B}_n$-$\widetilde{C}_n$ Hilbert C^*-bimodule.

An element u of $\widetilde{X}_n$ will be called unitary if $uu^* = p_n$ and $u^*u = q_n$. Since $\widetilde{B}_n$ is unital of stable rank one, and since the map $x \mapsto xu_n^*$ is an isomorphism of $\widetilde{X}_n$ with $\widetilde{B}_n$, the result of Rørdam [39] applies: If $x \in \widetilde{X}_n$ with polar decomposition $x = v|x|$, and if $\delta > 0$, then there is a unitary u in $\widetilde{X}_n$ such that $uE_\delta = vE_\delta$.

We shall recursively construct x_n in X_n and w_n in $\widetilde{X}_n$ such that:

(12″) w_n is unitary and $\|x_n\| \leqslant 1$;
(14″) $(w_n - w_{n-1})|x_{n-1}| = 0 = |x_{n-1}^*|(w_n - w_{n-1})$ for $n > 1$;
(15″) $x_n = w_n|x_n|$;
(16″) $\|(\mathbf{1} - x_n^*x_n)g_n''\|, \|g_n'(\mathbf{1} - x_nx_n^*)\| < \frac{1}{n}$;
(17″) $(\mathbf{1} - |x_n|)|x_{n-1}| = 0 = |x_{n-1}^*|(\mathbf{1} - |x_n^*|)$ for $n > 1$; and
(21″) there is c_n in C_n such that $0 \leqslant c_n \leqslant \mathbf{1}$ and $(\mathbf{1} - c_n)|x_n| = 0$.

To start the construction, let $w_1 = u_1$ and choose an approximate identity (r_i) for C_1. Since $(h(r_i))$ is also an approximate identity, we may define $|x_1| = h(r_{i_0})^{1/2}$, where i_0 is chosen large enough that $\|(\mathbf{1} - h(r_{i_0}))g_1''\|, \|g_1'(\mathbf{1} - w_1h(r_{i_0})w_1^*)\| < 1$. Then using (15″) as the definition of x_1 and taking $c_1 = k(r_{i_0})$, we have all the conditions.

For the recursive step, we take w_n in $\widetilde{X}_n$ to be a unitary extension of vE_δ for some $\delta < 1$, referring to a polar decomposition of $w_{n-1}c_{n-1}$. If (r_i) is an approximate identity for C_n, then $(h(c_{n-1} + (\mathbf{1} - c_{n-1})^{1/2}r_i(\mathbf{1} - c_{n-1})^{1/2}))$ is also an approximate identity. Thus we may define $|x_n| = (h(c_{n-1}+(\mathbf{1}-c_{n-1})^{1/2}r_{i_0}(\mathbf{1}-c_{n-1})^{1/2}))^{1/2}$, where i_0 is chosen large enough that $\|(\mathbf{1}-|x_n|^2)g_n''\|, \|g_n'(\mathbf{1}-w_n|x_n|^2 w_n^*)\| < 1$. The rest is similar to the above or clear.

Then we can see, as in the proof of case (ii), that (x_n) converges strictly to an element w of $M(A)$ such that $ww^* = p$ and $w^*w = q$ (also $w_n \to w$). ∎

REMARK 3.12. (i) We don't know whether $M(I_n)$ extremally rich for all n implies $M(A)$ extremally rich. Nevertheless Theorem 3.11 can be used in conjunction with Theorem 4.8 below to help prove extremal richness for some multiplier algebras, as in Corollary 4.9.

(ii) Case (i) of Theorem 3.11 is relevant in the context of low rank because there are known relationships, and interest in investigating possible further relationships, between low rank and various cancellation properties for equivalence classes of projections.

4. PULLBACKS, EXTENSIONS AND LOW RANK

Recall that the meaning of *pullback diagram*, in the notation below, is that (η, ρ) gives an isomorphism of A with $B \oplus_D C = \{(b, c) \in B \oplus C \mid \tau(b) = \pi(c)\}$.

THEOREM 4.1. *Consider a pullback diagram of C^*-algebras*

$$\begin{array}{ccc} A & \xrightarrow[\eta]{} & B \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\tau} \\ C & \xrightarrow{\pi} & D \end{array}$$

in which π (hence also η) is surjective. Then:

(i) $\operatorname{tsr}(A) \leqslant \max(\operatorname{tsr}(B), \operatorname{tsr}(C)))$.

(ii) *If B and C have real rank zero, then A has real rank zero.*

(iii) *If B and C (hence also D) are extremally rich, then: A is extremally rich and ρ extreme-point-preserving* (e.p.p.) $\Leftrightarrow$ *τ is* e.p.p.

Proof. By forced unitization we may assume that all C^*-algebras and all morphisms are unital.

Note first that $\rho|\ker\eta$ is an isomorphism onto $\ker\pi$, so that we may put $I = \ker\eta = \ker\pi$ to obtain the commutative diagram of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow[\eta]{} & B & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle\tau} & & \\ 0 & \longrightarrow & I & \longrightarrow & C & \xrightarrow{\pi} & D & \longrightarrow & 0 \end{array}$$

cf. Remark 3.2 of [35].

(i) Let $d = \max(\operatorname{tsr}(B), \operatorname{tsr}(C))$. If $\underline{x}$ is a tuple in A^d, then we first approximate $\eta(\underline{x})$ by a unimodular tuple in B^d. By the definition of quotient norm, we approximate $\underline{x}$ by a tuple $\underline{y}$ such that $\eta(\underline{y})$ is unimodular. Then $\pi(\rho(\underline{y})) = \tau(\eta(\underline{y}))$, which is unimodular. By Lemma 2.1 of [23] we can approximate $\rho(\underline{y})$ by a unimodular tuple $\underline{z}$ such that $\pi(\underline{z}) = \pi(\rho(\underline{y}))$. Then the pair $(\eta(\underline{y}), \underline{z})$ satisfies the pullback condition and gives a unimodular approximant to $\underline{x}$.

(iii) If τ is e.p.p. and $u \in \mathcal{E}(B)$, then since C is extremally rich there is v in $\mathcal{E}(C)$ such that $\pi(v) = \tau(u)$. Then the pair (u, v) represents a lifting of u to $\mathcal{E}(A)$.

To show that ρ is e.p.p., consider $w = (u, v)$ in $\mathcal{E}(A)$. Then v is a partial isometry in C such that $(\mathbf{1} - v^*v)C(\mathbf{1} - vv^*) \subset I$, since $\pi(v) = \tau(u) \in \mathcal{E}(D)$. However, $(\mathbf{1} - v^*v)I(\mathbf{1} - vv^*) = 0$ because $I \subset A$ and $w \in \mathcal{E}(A)$. Taken together this means that v is in $\mathcal{E}(C)$ as desired.

According to Theorem 6.1 of [6], to finish the proof that A is extremally rich, we must check that

$$I + \mathcal{E}(A) \subset (A_q^{-1})^= .$$

For this, consider $w = (u, v)$ in $\mathcal{E}(A)$ and x in I. Since C is extremally rich $v + x \in (C_q^{-1})^=$, so $v + x$ is the limit of a sequence (a_n) from C_q^{-1}. By 2.13 of [9] we may assume that $v - a_n \in I$. In the standard decomposition $a_n = v_n e_n$, with v_n in $\mathcal{E}(C)$ and $e_n = (|a_n| + \mathbf{1} - v_n^* v_n) \in C_+^{-1}$ (cf. Theorem 1.1 of [6],), we then have $\pi(e_n) = \mathbf{1}$. It follows that $w_n = (u, v_n) \in \mathcal{E}(A)$ and $x_n = (\mathbf{1}, e_n) \in A_+^{-1}$; and since $w_n x_n \to (u, v + x)$ we have shown that $(u, v + x) \in (A_q^{-1})^=$, as desired.

Finally, to show the reverse implication, assume $u \in \mathcal{E}(B)$. Then $u = \eta(w)$, $w \in \mathcal{E}(A)$, since A is extremally rich. Then by hypothesis, $\rho(w) \in \mathcal{E}(C)$, and hence $\tau(u) = \pi(\rho(w)) \in \mathcal{E}(D)$.

(ii) The proof is similar to, and slightly easier than, the first part of the proof of (iii). By Theorem 3.14 of [5] we need only prove that projections lift from B to A. ∎

REMARK 4.2. A number of papers contain results similar to ours. Thus Corollary 3.16 of [40] and Corollary 2.7 of [23] cover Theorem 4.1 for stable rank, and Lemma 1.3 of [25] covers it for real rank, in the special case where both π and τ (hence also η and ρ) are surjective. Sheu's, Nistor's, and Osaka's results cover arbitrary values of the rank, not just low ranks.

More recently, independently of our result (and after it was first obtained in 1998), Nagisa, Osaka, and Phillips ([21], Proposition 1.6) proved Theorem 4.1(ii) for arbitrary values of the real rank. After Corollary 1.12 of [21] they remark that their proof (which is considerably longer than ours) also works for stable rank. The idea, used in Corollary 4.4 below, of combining Theorem 4.1 with Busby's analysis of extensions, is also found in Osaka's survey article ([26], Proposition 3.4).

The surjectivity condition cannot be entirely dropped from Theorem 4.1. To show that surjectivity of π cannot be omitted for the case of real rank zero take $A = C([0,1])$, embedded in the algebra D of all bounded functions on [0,1]. Let C be the subalgebra of D consisting of functions that are continuous on [0,1], except for possible jump discontinuities at points of the form $n2^{-m}$, where $0 < n < 2^m$, $m \in \mathbb{N}$. Let B be defined as C, except that the jump discontinuities are now allowed at points of the form $n3^{-m}$, where $0 < n < 3^m$, $m \in \mathbb{N}$. Realizing B and C as inductive limits of algebras of step-functions with only finitely many jumps, we see easily that they both have real rank zero. In fact, B and C are both isomorphic to $C(\mathcal{C})$, where $\mathcal{C}$ denotes the Cantor set. Since $B \subset D$ and $C \subset D$ it is easy to verify that we have $B \oplus_D C = B \cap C = A$. But A has real rank one, not zero.

Tensoring A, B, C and D with $C([0,1])$, we obtain an example for functions on the unit square, which shows that surjectivity of π cannot be left out in the stable rank one or extremal richness cases either.

If τ in Theorem 4.1(iii) is not e.p.p., it can actually happen that ρ is e.p.p. and A not extremally rich (cf. Corollary 4.4) or that A is extremally rich and ρ not e.p.p.

COROLLARY 4.3. *Let I be an ideal in an extremally rich C^*-algebra A and denote by $\pi\colon A \to A/I$ the quotient morphism. Then for each extremally rich C^*-subalgebra B which is* e.p.p. *embedded in A/I, the C^*-subalgebra $\pi^{-1}(B)$ is extremally rich and* e.p.p. *embedded in A.*

Proof. We have the commutative diagram

$$\begin{array}{ccccc} I & \longrightarrow & \pi^{-1}(B) & \xrightarrow[\pi]{} & B \\ \| & & \downarrow & & \downarrow \\ I & \longrightarrow & A & \xrightarrow{\pi} & A/I \end{array}$$

which shows that $\pi^{-1}(B)$ is one of the pullbacks covered by Theorem 4.1. ∎

Every extension of C^*-algebras is associated with a Busby diagram, cf. [12] or [14],

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & A & \xrightarrow[\eta]{} & B & \longrightarrow & 0 \\ & & \| & & \downarrow\rho & & \downarrow\tau & & \\ 0 & \longrightarrow & I & \longrightarrow & M(I) & \xrightarrow{\pi} & Q(I) & \longrightarrow & 0 \end{array}$$

where $Q(I) = M(I)/I$ denotes the corona algebra of I. Here the right hand square is a pullback, and A is completely determined by the Busby invariant τ. Either A is unital, which implies that also τ is unital; or A is non-unital, in which case we obtain a new pullback diagram replacing η and τ by the forced unitized morphisms $\widetilde{\eta}\colon \widetilde{A} \to \widetilde{B}$ and $\widetilde{\tau}\colon \widetilde{B} \to Q(I)$. Since this will not effect the rank of any of the algebras involved, we may as well assume that the extension is unital.

Applying Theorem 4.1 to the pullback diagrams described above, we obtain a simple but powerful tool for producing examples of extremally rich C^*-algebras.

COROLLARY 4.4. *Consider an extension of C^*-algebras*

$$0 \to I \to A \to B \to 0$$

determined by the Busby invariant $\tau\colon B \to Q(I)$. Then

(a) $\mathrm{tsr}(A) \leqslant \max(\mathrm{tsr}(B), \mathrm{tsr}(M(I)))$.

(b) *If both B and $M(I)$ have real rank zero, then A has real rank zero.*

(c) *If both B and $M(I)$ are extremally rich, then the following are equivalent:*

(i) τ *is* e.p.p;

(ii) $\eta(\mathcal{E}(A)) = \mathcal{E}(B)$, *where $\eta\colon A \to B$ is the quotient map;*

(iii) *A is extremally rich.*

Proof. In this situation the map ρ is always e.p.p. In fact if $u \in \mathcal{E}(A)$, then $(\mathbf{1} - u^*u)I(\mathbf{1} - uu^*) = 0$. Since $\rho(I)$ is strictly dense in $M(I)$, this implies that $(\mathbf{1} - \rho(u)^*\rho(u))M(I)(\mathbf{1} - \rho(u)\rho(u)^*) = 0$. The rest follows from Theorem 4.1 and its proof. ∎

Sometimes it is useful to know whether $\eta(\mathcal{E}(A)) = \mathcal{E}(B)$ for purposes other than determining whether A is extremally rich. For example, we may want to know whether η induces a surjective map on the extremal K-sets, cf. [8]. Also if I is the largest ideal of A which is a dual C^*-algebra, then the description of elements of A with persistently closed range, cf. Section 7 of [8], can be simplified if extremals lift. Thus it is worthwhile to state the following corollary, which follows from the arguments above.

COROLLARY 4.5. *If I is an ideal in a unital C^*-algebra A, $\eta\colon A \to A/I$ is the quotient map, and if $M(I)$ is extremally rich, then $\eta(\mathcal{E}(A)) = \mathcal{E}(A/I)$ if and only if the Busby invariant $\tau\colon A/I \to Q(I)$ is extreme-point-preserving.*

REMARK 4.6. (i) For stable rank one or real rank zero the hypotheses of Corollary 4.4 can be weakened: If $\mathrm{tsr}(B) = \mathrm{tsr}(I) = 1$ and the map $\iota_0\colon K_0(I) \to K_0(M(I))$ is injective then $\mathrm{tsr}(A) = 1$. It is well-known by now that the vanishing of the map $\partial_1\colon K_1(B) \to K_0(I)$ suffices for an extension of stable rank one algebras to have stable rank one, and this fact follows from the injectivity of ι_0 by an easy diagram chase. According to [24] the statement about ∂_1 is an unpublished result of G. Nagy, who later published a different proof in Corollary 2 of [22]. Similarly, if $\mathrm{RR}(B) = 0 = \mathrm{RR}(I)$ and the map $\iota_1\colon K_1(I) \to K_1(M(I))$ is injective then $\mathrm{RR}(A) = 0$. Here we need the injectivity of ι_1 to show that $\partial_0\colon K_0(B) \to K_1(I)$ vanishes. This statement is due to Zhang, cf. Propositions 3.14 and 3.15 of [5].

(ii) Since any morphism from an isometrically rich C^*-algebra is extreme-point-preserving, Corollary 4.4(c) yields extremal richness for A whenever B is isometrically rich, and this is not a severe restraint. By contrast, the condition that not only I but also $M(I)$ should be extremally rich or of low rank is quite restrictive.

EXAMPLE 4.7. To apply Corollary 4.4(c) we need a supply of C^*-algebras (necessarily extremally rich) whose multiplier algebras are extremally rich. As shown in Corollary 3.8 of [18] this happens for every σ-unital purely infinite simple C^*-algebra. However, if A is σ-unital, simple (but not elementary) and has a finite trace then neither $M(A)$ nor $M(A \otimes \mathbb{K})$ are extremally rich by Theorems 3.1 and 3.2 of [18]. Secondly, we immediately observe that if A is a *dual* C^*-algebra, then $M(A)$ is extremally rich. Indeed, A is the direct sum $\bigoplus A_i$ of elementary C^*-algebras (full matrix algebras or algebras of compact operators on some Hilbert space), so $M(A)$ is the direct product $\prod M(A_i)$, where each $M(A_i) = \mathbb{B}(\mathcal{H}_i)$ for some Hilbert space $\mathcal{H}_i$. But if $A = c \otimes \mathbb{K}$, the algebra of norm convergent sequences of compact operators on ℓ^2, then $M(A)$ is not extremally rich. This will be shown in Example 4.10, below. Finally, Theorem 5.9 below provides additional examples.

THEOREM 4.8. *Let I be an ideal in a σ-unital C^*-algebra A. Then:*

(i) *If* $\mathrm{RR}(M(I)) = 0$ *and* $\mathrm{RR}(M(A/I)) = 0$, *then* $\mathrm{RR}(M(A)) = 0$.

(ii) *If* $\mathrm{tsr}(M(I)) = 1$ *and* $\mathrm{tsr}(M(A/I)) = 1$, *then* $\mathrm{tsr}(M(A)) = 1$.

(iii) *If both $M(I)$ and $M(A/I)$ are extremally rich and* $\mathrm{tsr}(M(A/(I + I^{\perp}))) = 1$, *then $M(A)$ is extremally rich and the natural morphism $\overline{\rho}\colon M(A) \to M(I)$ is extreme-point-preserving.*

Proof. Setting $B = A/I$ we have a commutative diagram in which each row contains an extension and ι denotes an unspecified embedding:

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, equal] & A \arrow[r, "\eta"] \arrow[d, "\rho"] & B \arrow[r] \arrow[d, "\tau"] & 0 & \\
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, equal] & \rho(A) \arrow[r, "\pi"] \arrow[d, "\iota"] & \tau(B) \arrow[r] \arrow[d, "\iota"] & 0 & \\
0 \arrow[r] & I \arrow[r, "\iota"] \arrow[d, "\iota"] & M(I) \arrow[r, "\pi"] & Q(I) \arrow[r] & 0 & \\
0 \arrow[r] & M(\rho(A), I) \arrow[r, "\iota"] & I(\rho(A)) \arrow[r, "\pi"] \arrow[u, "\iota"] & I(\tau(B)) \arrow[r, "\varphi"] \arrow[u, "\iota"] & M(\tau(B)) \arrow[r] & 0 \\
0 \arrow[r] & M(A, I) \arrow[r, "\iota"] \arrow[u, "\iota"] & M(A) \arrow[rr, "\overline{\eta}"] \arrow[u, "\overline{\rho}"] & & M(B) \arrow[r] \arrow[u, "\overline{\tau}"] & 0
\end{tikzcd}$$

Here $M(A, I) = \{x \in M(A) \mid xA + Ax \subset I\}$ and $M(\rho(A), I) = \{x \in M(I) \mid x\rho(A) + \rho(A)x \subset I\}$, whereas $I(\rho(A))$ and $I(\tau(B))$ denote the idealizers of the two algebras inside $M(I)$ and $Q(I)$, respectively. The two quotient morphisms η and π are central to the picture and describe the extensions in the first and third row. The morphism $\rho\colon A \to M(I)$ is the natural map arising from the embedding of I as an ideal in A; and $\ker \rho = I^\perp$. The Busby invariant τ is derived from ρ to make the upper right square commutative. The morphisms $\overline{\eta}, \overline{\tau}$ and $\overline{\rho}$ are the canonical extensions to the multiplier algebras of η, τ and ρ, respectively. Since A, hence also B and $\tau(B)$ are σ-unital, the overlined morphisms are surjective by Theorem 10 of [32]. Finally, the natural morphism φ is surjective by Corollary 3.2 of [14].

The kernel of the morphism $\overline{\rho}$ is

$$\ker \overline{\rho} = \{x \in M(A) \mid xA + Ax \subset I^\perp\} = M(A, I^\perp)\,,$$

which intersects $M(A, I)$ in 0. Thus $\overline{\rho}$ gives an isomorphism of $M(A, I)$ onto the hereditary C^*-subalgebra $M(\rho(A), I)$ of $M(I)$, which is an ideal of $I(\rho(A))$. In the diagram we may therefore identify the two isomorphic ideals in $M(A)$ and $I(\rho(A))$.

We claim that the lower right rectangle is a pullback, so that

$$M(A) = I(\rho(A)) \oplus_{M(\tau(B))} M(B)\,.$$

For this it suffices to show that $M(\rho(A), I) = \ker(\varphi \circ \pi)$ in $I(\rho(A))$, cf. Proposition 3.1 of [35]. But this is evident, since the kernel of φ is the (two-sided) annihilator $\tau(B)^\perp$ in $I(\tau(B))$

(i) Since $M(\rho(A), I)$ is hereditary in $M(I)$, it follows that $\mathrm{RR}(M(A, I)) = 0$. Given that also $\mathrm{RR}(M(B)) = 0$ we need only show that projections lift from $M(B)$

to $M(A)$, cf. 3.14 of [5]. Given a projection p in $M(B)$ let $\overline{h}$ be a self-adjoint lift in $M(A)$ of the symmetry $2p-1$, and put $h=\overline{\rho}(\overline{h})$. By [4], see also the explanatory version in [10], there is an interpolating projection q in $M(I)$ such that if $f_\pm$ are (any) two continuous functions vanishing on $[-1,\frac{1}{2}]$ and $[-\frac{1}{2},1]$, respectively, with $f_+(1)=1$ and $f_-(-1)=1$ for future use, then

$$qf_+(h)=f_+(h) \quad \text{and} \quad (\mathbf{1}-q)f_-(h)=f_-(h).$$

For every b in B we have

$$\pi(f_+(h))\tau(pb)=\tau(pb) \quad \text{and} \quad \pi(f_-(h))\tau((\mathbf{1}-p)b)=\tau((\mathbf{1}-p)b)\,.$$

Consequently,

$$\pi(q)\tau(pb)=\pi(qf_+(h))\tau(pb)=\tau(pb)\,.$$

Similarly $\pi(\mathbf{1}-q)\tau((\mathbf{1}-p)b)=\tau((\mathbf{1}-p)b)$. Taken together, this means that $\pi(q)\tau(b)=\tau(pb)$. Similarly $\tau(b)\pi(q)=\tau(bp)$. Thus $\pi(q)\in I(\tau(B))$ and $\varphi(\pi(q))=\overline{\tau}(p)$. It follows that $\overline{p}=(q,p)\in M(A)$ and is a projection lift of p, as desired.

(ii) We now know that both $M(B)$ and $M(A,I)$ (being isomorphic to a hereditary C^*-subalgebra of $M(I)$) have stable rank one, so in order to prove that $\mathrm{tsr}(M(A))=1$ we need only show that unitaries lift from $M(B)$ to $M(A)$, cf. 6.4 of [6]. Given a unitary u in $M(B)$ let $\overline{h}$ be a lift of u to $M(A)$ and put $h=\overline{\rho}(\overline{h})$. Since $\mathrm{tsr}(M(I))=1$ there is, by Theorem 2.2 of [39] or Corollary 8 of [33], for any continuous function f vanishing on $[0,\frac{1}{2}]$, with $f(1)=1$, a unitary w in $M(I)$ such that if $h=v|h|$ is the polar decomposition in some $B(H)$ (cf. the proof of Theorem 3.11) then

$$vf(|h|)=wf(|h|) \quad \text{and} \quad f(|h^*|)v=f(|h^*|)w\,.$$

For each b in B we compute

$$\pi(f(|h|))\tau(b)=\tau(f(|u|)b)=\tau(b)\,.$$

Similarly $\tau(b)\pi(f(|h^*|))=\tau(b)$. Consequently,

$$\pi(w)\tau(b)=\pi(wf(|h|))\tau(b)=\pi(vf(|h|))\tau(b)=\tau(ub)\,,$$

and similarly $\tau(b)\pi(w)=\tau(bu)$. Thus $\varphi(\pi(w))=\overline{\tau}(u)$ and $\overline{u}=(w,u)\in M(A)$ and is a unitary lift of u, as desired.

(iii) Now $M(A,I)$ and $M(B)$ are both extremally rich, so to prove that $M(A)$ is extremally rich we must show that extreme partial isometries in $\mathcal{E}(M(B))$ lift in a "good" way, cf. Theorem 6.1 of [6]. Given u in $\mathcal{E}(M(B))$ let $\overline{h}$ be a lift in $M(A)$ and put $h=\overline{\rho}(\overline{h})$ as in case (ii). Since $M(I)$ is extremally rich there is for any continuous function f vanishing on $[0,\frac{1}{2}]$, with $f(1)=1$, an extreme partial isometry w in $\mathcal{E}(M(I))$ such that if $h=v|h|$ is the polar decomposition, then

$$vf(|h|)=wf(|h|) \quad \text{and} \quad f(|h^*|)v=f(|h^*|)w\,,$$

cf. Theorem 2.2 of [6]. We have assumed that $\mathrm{tsr}(M(\tau(B))) = 1$, so $\overline{\tau}(u)$ is unitary. Consequently

$$\pi(f(|h|))\tau(b) = \tau(f(|u|)b) = \tau(b)$$

for every b in B, and similarly $\tau(b)\pi(f(|h^*|)) = \tau(b)$. As in case (ii) we therefore obtain the equations

$$\pi(w)\tau(b) = \tau(ub) \quad \text{and} \quad \tau(b)\pi(w) = \tau(bu)\,,$$

so that $\pi(w) \in I(\tau(B))$ with $\varphi(\pi(w)) = \overline{\tau}(u)$. Thus $\overline{u} = (w, u) \in M(A)$ and is an extremal lift of u as we wanted.

Let $p_\pm$ denote the defect projections of $\overline{u}$. The extra technical condition needed in Theorem 6.1 of [6] is that the two bimodules $p_\pm M(A)r$ are extremally rich for any defect projection r arising from an element s in $\mathcal{E}(M(A,I)^{\sim})$. Since $\overline{\tau}(\overline{\eta}(\overline{u})) = \overline{\tau}(u)$ is unitary it follows that $\pi(\overline{\rho}(p_\pm)) \in \tau(B)^\perp$, i.e. $\overline{\rho}(p_\pm) \in M(\rho(A), I)$. Since moreover $r \in M(A, I)$, also $\overline{\rho}(r) \in M(\rho(A), I)$. But this is a hereditary C^*-subalgebra of $M(I)$, so

$$\overline{\rho}(p_\pm M(A)r) = \overline{\rho}(p_\pm)M(\rho(A), I)\overline{\rho}(r) = \overline{\rho}(p_\pm)M(I)\overline{\rho}(r)\,.$$

Note now that by construction $\overline{\rho}(\overline{u}) = w \in \mathcal{E}(M(I))$, so $\overline{\rho}(p_\pm)$ are both extreme defect projections of $M(I)$. Moreover, since $\overline{\rho}(M(A,I))$ is hereditary in $M(I)$, $\overline{\rho}(s)$ is in $\mathcal{E}(M(I))$ and $\overline{\rho}(r)$ is also an extreme defect projection of $M(I)$. It now follows from Proposition 4.4 of [6] that $\overline{\rho}(p_\pm)M(I)\overline{\rho}(r)$ is extremally rich.

Finally, the fact that $\overline{\rho}$ is e.p.p. follows by the same argument as in Corollary 4.4. ∎

COROLLARY 4.9. *Let $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ be a composition series for a separable C^*-algebra A. Then*

(i) *If* $\mathrm{RR}(M(I_{\alpha+1}/I_\alpha)) = 0$ *for all $\alpha < \beta$, then* $\mathrm{RR}(M(A)) = 0$.

(ii) *If* $\mathrm{tsr}(M(I_{\alpha+1}/I_\alpha)) = 1$ *for all $\alpha < \beta$, then* $\mathrm{tsr}(M(A)) = 1$.

(iii) *If $M(I_1)$ is extremally rich and* $\mathrm{tsr}(M(I_{\alpha+1}/I_\alpha)) = 1$ *for $1 \leqslant \alpha < \beta$, then $M(A)$ is extremally rich.*

Proof. If we assume, as we may, that (I_α) is strictly increasing, then β is countable, since $A^\vee$ is second countable. Then the result follows by a routine transfinite induction from Theorems 3.11 and 4.8. ∎

EXAMPLE 4.10. In Example 7.9 of [8] we explored a non-extremally rich C^*-algebra A that illustrates a number of points in the present paper. (There is a typographical error in the second-to-last paragraph of Example 7.9 of [8]: Please change (vii) to (viii).) Let $I = \bigoplus_{n=1}^{\infty} \mathbb{M}_n$ and let B be the C^*-subalgebra of $M(I)\ (= \prod_{n=1}^{\infty} \mathbb{M}_n)$ consisting of norm convergent sequences (relative to the standard embedding of $\mathbb{M}_n$ in $\mathbb{B}(\ell^2)$). Then $M(B)$ consists of the strong* convergent sequences. If s_n denotes the forward truncated shift in $\mathbb{M}_n$ and s the forward shift on ℓ^2, we put $\widetilde{s} = (s_n)$ in $M(B)$ and now $A = C^*(B, \widetilde{s}, \mathbf{1})$ in $M(B)$. The quotient A/I is isomorphic to $\mathcal{T}_e$, the extended Toeplitz algebra generated by the element $s \oplus s^*$ in $\mathbb{B}(\ell^2 \oplus \ell^2)$. The primitive ideal space of A (equal to the spectrum $\widehat{A}$) is the disjoint union

$$\{\pi_1, \pi_2, \dots\} \cup \{\sigma_+\} \cup \{\sigma_-\} \cup \mathbb{T},$$

where each π_n corresponds to an n-dimensional representation, $\sigma_\pm$ correspond to the infinite dimensional representations of $\mathcal{T}_e$, and the circle $\mathbb{T}$ (with the usual topology) consists of one-dimensional representations. The set $\mathcal{F} = \{\sigma_+, \sigma_-\} \cup \mathbb{T}$ is the hull of I, and (π_n) converges to all points in $\mathcal{F}$ simultaneously.

Each set $\{\pi_n\}$ is closed in $A^\vee$ and corresponds to a stable rank one quotient of A. Thus the hypothesis in (4) of Remark 2.11(ii) that A/I_1 be isometrically rich cannot be weakened to extremal richness. Also, each of the sets $\{\sigma_+\} \cup \mathbb{T}$ and $\{\sigma_-\} \cup \mathbb{T}$ is closed in $A^\vee$ and corresponds to an isometrically rich quotient of A. Thus the disjointness condition in (5) of Remark 2.11(ii) cannot be omitted. Since $\widehat{A}$ is an almost Hausdorff space, the Hausdorff demand in part (iv) of Theorem 2.10 cannot be replaced by almost Hausdorff.

Since A/B is isometrically rich and A is not extremally rich, it follows from Corollary 4.4 that $M(B)$ is not extremally rich. This shows that the non-existent parts (iii′) and (iv′) of Theorem 2.10 are false if we consider $B^\vee = \Big(\bigcup_n \{\pi_n\}\Big) \bigcup \{\sigma_+\}$. Note that $B^\vee$ is Hausdorff and all but one of the quotients of B are unital and of stable rank one. This also shows that the hypothesis that $\operatorname{tsr}(M(A/(I+I^\perp))) = 1$ cannot be dropped from Theorem 4.8, since $\operatorname{tsr}(M(I)) = 1$ and $M(B/I)$ is isometrically rich, thus suggesting some sharpness in the hypotheses of Theorem 4.8. Finally, since B is a corner of $c \otimes \mathbb{K}$, the assertion in Example 4.7 that $M(c \otimes \mathbb{K})$ is not extremally rich has been verified.

5. APPLICATIONS, REMARKS, AND QUESTIONS

The main purpose of this section is to provide some sample applications of the basic results of the paper by determining when CCR algebras or their multiplier algebras have low rank. In deciding how much material to present and how to present it, we have tried to walk a fine line. On the one hand we don't want to obscure the main purpose with too many technical proofs, and on the other hand we don't want to complicate the statements of the results with unnecessary technical hypotheses. But we begin with two light contributions concerning type I algebras. These together with Proposition 2.7 constitute our best effort to give a somewhat general characterization of low topological dimension. (The "lightness" of Proposition 5.1 lies in the fact that the most important parts were already known.)

We will use the local definition of AF-algebra in order to cover non-separable algebras. Thus A is an *AF-algebra* if for every finite subset $\mathcal{F}$ of A and every $\varepsilon > 0$, there is a finite dimensional C^*-subalgebra B such that $\operatorname{dist}(a, B) < \varepsilon$ for each a in $\mathcal{F}$. In Proposition 5.1 below the equivalence of (i) and (iv) in the separable case was proved by Bratteli and Elliott [2], the fact that (i) implies (v) is a special case of results of Lin [19], [20], and Pasnicu ([28], Remark 2.12) shows that (i) is equivalent to the ideal property in the separable case.

PROPOSITION 5.1. *Let A be a type* I *C^*-algebra. Then* (v) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). *If also A is σ-unital, then all the conditions are equivalent:*

(i) *A is an* AF-*algebra.*

(ii) *A has real rank zero.*

(iii) *A has generalized real rank zero.*

(iv) $\operatorname{top\,dim}(A) = 0$.

(v) *$M(A)$ has real rank zero.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (ii) are obvious, and (iii) $\Rightarrow$ (iv) follows from Proposition 2.7.

(iv) $\Rightarrow$ (i) A has a composition series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ such that for each $\alpha < \beta$ $I_{\alpha+1}/I_\alpha$ is Rieffel–Morita equivalent to a commutative algebra $C_0(\mathcal{X}_\alpha)$. Each $\mathcal{X}_\alpha$ is totally disconnected, and hence $C_0(\mathcal{X}_\alpha)$ is AF. Therefore $I_{\alpha+1}/I_\alpha$ is AF. Since direct limits of AF-algebras are AF, and since an extension of one AF-algebra by another is AF (a fact which was not known when [2] was written), a routine transfinite induction shows that A is AF.

Finally, if A is σ-unital, the implication (i) $\Rightarrow$ (v) follows from Corollary 3.7 of [19]. ∎

PROPOSITION 5.2. *If A is a type* I *C^*-algebra, then the following conditions are equivalent:*

(i) *A has generalized stable rank one;*

(ii) *A has a composition series $\{I_\alpha \mid 0 \leqslant \alpha \leqslant \beta\}$ such that $I_{\alpha+1}/I_\alpha$ is extremally rich for each $\alpha < \beta$;*

(iii) $\operatorname{top\,dim}(A) \leqslant 1$.

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, and for (ii) $\Rightarrow$ (iii) we use a composition series such that $I_{\alpha+1}/I_\alpha$ is both extremally rich and Rieffel–Morita equivalent to a commutative C^*-algebra $C_0(\mathcal{X}_\alpha)$, cf. Subsection 2.1(vi). Then $I_{\alpha+1}/I_\alpha$ extremally rich implies $C_0(\mathcal{X}_\alpha)$ extremally rich which implies $\operatorname{tsr}(C_0(\mathcal{X}_\alpha)) = 1$ which implies $\operatorname{top\,dim}(C_0(\mathcal{X}_\alpha)) \leqslant 1$ which implies $\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha) \leqslant 1$. Then by Proposition 2.3 $\operatorname{top\,dim}(A) \leqslant 1$. The proof that (iii) implies (i) is similar. Now $\operatorname{top\,dim}(I_{\alpha+1}/I_\alpha) \leqslant 1$ implies $\operatorname{top\,dim}(C_0(\mathcal{X}_\alpha)) \leqslant 1$ which implies $\operatorname{tsr}(C_0(\mathcal{X}_\alpha)) = 1$ which implies $\operatorname{tsr}(I_{\alpha+1}/I_\alpha) = 1$. ∎

If A is type I and $\operatorname{top\,dim}(A) = 0$, then A has stable rank one since it is AF. But in the one-dimensional case there are numerous examples where A is extremally rich but not of stable rank one and numerous examples where A is not even extremally rich. But, as we proceed to show, such examples cannot be

CCR algebras, despite the fact that the (non-canonical) composition series for a CCR algebra with each $I_{\alpha+1}/I_\alpha$ of continuous trace can be very complicated.

LEMMA 5.3. *Let A be an n-homogeneous C^*-algebra such that* $\operatorname{top\,dim}(A) \leqslant 1$.
(i) *If A is σ-unital, then* $\operatorname{tsr}(M(A)) = 1$.
(ii) *In any case,* $\operatorname{tsr}(A) = 1$.

Proof. (i) By the structure theory for n-homogeneous algebras and the fact that $A^\vee$ is σ-compact, we can find a sequence (I_m) of ideals such that $\bigcup$hull $(I_m) = A^\vee$ and each A/I_m is isomorphic to $C(\mathcal{X}_m) \otimes \mathbb{M}_n$ where $\mathcal{X}_m$ is compact. Since $\dim(\mathcal{X}_m) \leqslant 1$, A/I_m is unital and of stable rank one. The conclusion follows from Theorem 2.10(ii′).

(ii) If A is σ-unital (ii) follows from (i). But A is the direct limit of an upward directed family of σ-unital ideals. ∎

LEMMA 5.4. *Assume each irreducible representation of A has dimension at most n and* $\operatorname{top\,dim}(A) \leqslant 1$.
(i) *Then* $\operatorname{tsr}(A) = 1$.
(ii) *If also A is separable, then* $\operatorname{tsr}(M(A)) = 1$.

Proof. (i) We use induction on n. The case $n = 1$ is known. For $n > 1$ A has an ideal I which is n-homogeneous such that all irreducibles of A/I have dimension at most $n-1$. Thus $\operatorname{tsr}(I) = \operatorname{tsr}(A/I) = 1$ by Lemma 5.3(ii) and induction. To complete the proof we show that $\iota_0 \colon K_0(I) \to K_0(A)$ is injective, cf. Remark 4.6. Write $I = (\bigcup I_\alpha)^=$ for an upward directed family of σ-unital ideals. Consider

$$K_0(I_\alpha) \overset{(\iota_\alpha)_*}{\longrightarrow} K_0(I) \overset{\iota_0}{\longrightarrow} K_0(A) \overset{(\rho_\alpha)_*}{\longrightarrow} K_0(M(I_\alpha)),$$

where $\rho_\alpha \colon A \to M(I_\alpha)$ is the natural map used above, cf. Corollary 4.4. The composite map is injective by Lemma 5.3(i). Since every element of $K_0(I)$ is in the image of $(\iota_\alpha)_*$ for α sufficiently large, this implies ι_0 is injective.

(ii) We use the same induction and the same I, but here we use Lemma 5.3(i) and Theorem 4.8. ∎

LEMMA 5.5. *Let A be a C^*-algebra, all of whose irreducible representations are finite dimensional, such that* $\operatorname{top\,dim}(A) \leqslant 1$.
(i) *Then* $\operatorname{tsr}(A) = 1$.
(ii) *If also A is separable, then* $\operatorname{tsr}(M(A)) = 1$.

Proof. There is a sequence (I_n) of ideals such that all irreducibles of A/I_n have dimension at most n and $\bigcup \operatorname{hull}(I_n) = A^\vee$. Thus (i) follows from Theorem 2.10(ii) and Lemma 5.4(i), and (ii) follows from Theorem 2.10(ii′) and Lemma 5.4(ii). ∎

THEOREM 5.6. *If A is a* CCR *C^*-algebra, then the following conditions are equivalent:*

(i) *A has stable rank one;*
(ii) *A is extremally rich;*
(iii) top dim$(A) \leqslant 1$.

Proof. We need only prove that (iii) implies (i). We claim that $A = (\bigcup B_\alpha)^=$, for an upward directed family of hereditary C^*-subalgebras, such that each B_α has only finite dimensional irreducibles. One way to see this is to use the theory of the minimal dense ideal, $K(A)$, found in Section 5.6 of [31]. By 5.6.2 of [31] the hereditary C^*-subalgebra generated by any finite subset of $K(A)$ is contained in $K(A)$. And for each a in $K(A)$ and each irreducible π, $\pi(a)$ has finite rank. Since $B_\alpha^\vee$ is an open subset of $A^\vee$, top dim$(B_\alpha) \leqslant 1$. Then the conclusion follows from Lemma 5.5(i) and the preservation of low rank by direct limits. ∎

THEOREM 5.7. *If A is a σ-unital* CCR *C^*-algebra, then $M(A)$ has stable rank one if and only if* top dim$(A) \leqslant 1$ *and all irreducible representations of A are finite dimensional.*

Proof. If A has an infinite dimensional irreducible representation, then A has a quotient algebra isomorphic to $\mathbb{K}$. Therefore $M(A)$ has a quotient isomorphic to $B(\mathcal{H})$. But $\mathrm{tsr}(B(\mathcal{H})) = \infty$.

For the other direction we write $A = (\bigcup A_\alpha)^=$ for a suitable upward directed family of separable C^*-subalgebras. Since A is σ-unital, we can choose the A_α's so that $M(A) = \bigcup M(A_\alpha)$. (Note that $M(A_\alpha) \subset M(A)$ if A_α contains an approximate identity of A.) Since $\mathrm{tsr}(A) = 1$ (by Lemma 5.5(i) or Theorem 5.6), we can also arrange that $\mathrm{tsr}(A_\alpha) = 1$ and hence top dim$(A_\alpha) \leqslant 1$. Then Lemma 5.5(ii) applies to each A_α, and the conclusion follows. ∎

We are not applying the inverse limit theory here, but we could have used Corollary 4.9 (which is based on Theorem 3.11) instead of Theorem 2.10 in the proof of Lemma 5.5(ii), and we could have used an easier inverse limit argument similar to the construction in Subsection 3.2 instead of Theorem 2.10 in the proof of Lemma 5.3(i). A negative answer to the first question below would allow the possibility of more crucial applications of Theorem 3.11.

QUESTIONS 5.8. (i) If A is a σ-unital (or separable) C^*-algebra (or type I C^*-algebra) such that $M(A)$ has stable rank one, does there necessarily exist a sequence (I_n) of ideals such that $\bigcup \mathrm{hull}(I_n) = A^\vee$ and each A/I_n is unital?
(ii) Does there exist a separable C^*-algebra A such that $\mathrm{tsr}(A) = 1$ and $1 < \mathrm{tsr}(M(A)) < \infty$?

NOTE: These questions are not conjectures, and nothing in this paper should be construed as a conjecture.

THEOREM 5.9. *Let I be an ideal of a σ-unital C^*-algebra A such that I is a dual C^*-algebra, A/I has only finite dimensional irreducible representations, and* top dim$(A/I) \leqslant 1$. *Then $M(A)$ is extremally rich.*

Proof. By Theorem 5.7 $M(A/I)$ has stable rank one. Hence the result follows from Theorem 4.8(iii). ∎

REMARK 5.10. It can be shown that if $M(A)$ is extremally rich for a σ-unital CCR algebra A, then A satisfies the hypotheses of Theorem 5.9. In other words:

If A is a σ-unital CCR C^*-algebra, then $M(A)$ is extremally rich if and only if $\operatorname{top}\dim(A) \leqslant 1$ and each infinite dimensional irreducible representation of A gives an isolated point of $A^{\vee}$.

We are omitting the proof of the converse but point out that the algebra called B in Example 4.10 is an instructive example. Note that not every C^*-algebra satisfying the hypotheses of Theorem 5.9 is CCR.

COROLLARY 5.11. *Let I be an ideal of a C^*-algebra A such that I has only finite dimensional irreducible representations and* $\operatorname{top}\dim(I) \leqslant 1$. *Then* $\operatorname{tsr}(A) = \operatorname{tsr}(A/I)$.

Proof. If I is σ-unital, this follows directly from Theorem 5.7 and Corollary 4.4. The general case follows from this via standard techniques for reducing to the separable case. In representing A as the direct limit of separable algebras A_α, we arrange directly that $\operatorname{tsr}(A_\alpha \cap I) = 1$ rather than trying to arrange directly that $\operatorname{top}\dim(A_\alpha \cap I) \leqslant 1$. (We don't know whether it is possible to control topological dimension in general constructions of this sort.) ∎

REMARK 5.12. As mentioned above in Remark 4.6, one can use K-theory instead of multiplier algebras to study the stable rank of an extension of one stable rank one C^*-algebra by another; and in this way one can prove the stable rank one case of Corollary 5.11 without using multiplier algebras. The following result, whose proof is omitted, can be used:

Let A be a CCR algebra such that $\operatorname{top}\dim(A) \leqslant 1$. If $\alpha \in K_0(A)$ and $\pi_*(\alpha) = 0$ in $K_0(\pi(A))$ for every irreducible representation π, then $\alpha = 0$.

THEOREM 5.13. *Let I be an ideal of a C^*-algebra A such that I has only finite dimensional irreducible representations,* $\operatorname{top}\dim(I) \leqslant 1$, *and A/I is extremally rich. Then* (i), (ii), *and* (iii) *below are equivalent and* (v) *implies* (i), (ii), *and* (iii). *If also I is σ-unital then* (iv) *is equivalent to* (i), (ii), *and* (iii).

(i) *A is extremally rich.*

(ii) $\pi(\mathcal{E}(\widetilde{A})) = \mathcal{E}((A/I)^{\sim})$, *where π is the quotient map.*

(iii) $\operatorname{tsr}(A/(I + I^{\perp})) = 1$.

(iv) *The Busby invariant $\tau\colon A/I \to M(I)/I$ is extreme-point-preserving.*

(v) *A/I is isometrically rich.*

Proof. We may assume A unital.

It is obvious that (i) implies (ii). Assume (ii) and let u be in $\mathcal{E}(A/(I + I^{\perp}))$. Lift u first to $\mathcal{E}(A/I)$ and then to v in $\mathcal{E}(A)$. Let ρ be a representation of A which is faithful, non-degenerate, and atomic on I. Hence $\ker(\rho) = I^{\perp}$. Clearly $\rho(v)$ is unitary. Therefore $v + I^{\perp}$ is unitary in $A/I^{\perp}$ and u is unitary. It now follows from 2.7 of [9] that $\mathrm{tsr}(A/(I+I^{\perp})) = 1$. If (iii) is true, then $\mathrm{tsr}(A/I^{\perp}) = 1$ by Corollary 5.11. Since A is a pullback of A/I and $A/I^{\perp}$, with both maps surjective, Theorem 4.1 implies that A is extremally rich. Thus (i), (ii), and (iii) are equivalent.

Now if I is σ-unital, the facts that (i), (ii), and (iv) are equivalent and (v) implies (i) follow from Theorem 5.7, Corollary 4.4(c), and Remark 4.6(ii). The fact that (v) still implies (i), (ii), and (iii) in general follows via standard techniques for reducing to the separable case. ∎

REMARK 5.14. (i) One should not view Theorem 5.13 as the prototype for results stating that certain C^*-algebras I are universally good ideals from the point of view of extremal richness of extensions. The reason is that the equivalence of (ii) and (iii) is an artifact of the particular class of ideals being considered. Instead, Corollary 4.4 and Remark 4.6 suggest two reasonable prototypes:

$$(\mathrm{v}) \Rightarrow (\mathrm{i}) \Leftrightarrow (\mathrm{ii}), \text{ or}$$
$$(\mathrm{v}) \Rightarrow (\mathrm{i}) \Leftrightarrow (\mathrm{ii}) \Leftrightarrow (\mathrm{iv}).$$

One may want technical hypotheses, such as the σ-unitality used above, in order to prove the stronger version. It can be shown that arbitrary purely infinite simple C^*-algebras satisfy the weaker version, cf. Example 4.7.

(ii) The fact that (v) implies (iii) in Theorem 5.13 amounts to the statement that $M(I)/I$ contains no proper isometries. Our proof of this uses $\mathrm{top\,dim}(I) \leqslant 1$. Could this hypothesis be omitted? In other words:

If I is a C^*-algebra with only finite dimensional irreducible representations, is $M(I)/I$ necessarily stably finite?

REMARK 5.15. Theorems on higher real rank and stable rank of CCR algebras can also be proved using our main results. This will be done in a future paper. In particular, a generalization, which doesn't involve topological dimension, of Corollary 5.11 to arbitrary ranks will be given.

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