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ALUTHGE TRANSFORMS AND n-CONTRACTIVITY OF WEIGHTED SHIFTS

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ABSTRACT. If W is a subnormal weighted shift, one might transform W in various ways: take the qth root of each weight, pick out a subsequence of weights, or take iterated Aluthge transforms of W, in each case producing another weighted shift. Via an approach to subnormality based on n-contractivity and completely monotone functions, we exhibit shifts whose transforms are again subnormal (or in related classes of interest), generalizing some recent results obtained by an approach through k-hyponormality.

KEYWORDS: *Weighted shifts, subnormal operators, completely hyperexpansive, n-contractive, k-hyponormal.*

MSC (2000): 47B37, 47B20.

INTRODUCTION

The subnormality of weighted shifts has been studied extensively; we concentrate here on some subnormal shifts and some transformations of those shifts under which subnormality is robust. In the next section we assemble some definitions and preliminary results; in the second section we assemble some further tools (in particular, some classes of useful functions interpolating the moment sequence for some shifts) and give some applications; in the third section we generalize considerably some results of [13] obtained by a different approach (and give proofs that are perhaps more straightforward); in the final section we give some remarks and pose some questions.

1. PREMIMINARIES

Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Recall that an operator N is normal if $N^*N = NN^*$, and an operator T acting on $\mathcal{H}$ is subnormal if there is a Hilbert space $\mathcal{K}$

containing $\mathcal{H}$ and a normal operator N on $\mathcal{K}$ so that $\mathcal{H}$ is invariant for N and T is (unitarily equivalent to) the restriction of N to $\mathcal{H}$. An operator T is hyponormal if $T^*T - TT^* \geqslant 0$. The subnormal operators have been extensively studied (see [7]); in the past fifteen years there has been considerable study of classes of operators between subnormal and hyponormal, namely the k-hyponormal operators. The operator T is k-hyponormal ($k = 1, 2, \ldots$) if the $(k+1)$ by $(k+1)$ operator matrix

$$\begin{pmatrix} I & T^* & \ldots & T^{*k} \\ T & T^*T & \ldots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \ldots & T^{*k}T^k \end{pmatrix} \geqslant 0.$$

An operator is hyponormal if and only if it is 1-hyponormal, and the study of these classes is motivated by the Bram–Halmos characterization of subnormality ([6], [18]): an operator is subnormal if and only if it is k-hyponormal for each $k = 1, 2, \ldots$. Some examples of the very extensive literature in this area are [8], [10], [11], and [12].

More recently interest has been revived in another characterization of subnormality due to Agler [1] based on work of Embry [16], and some classes more general than subnormal that then arise naturally. An operator T is n-contractive ($n = 1, 2, \ldots$) if

$$A_n(T) := \sum_{j=0}^{n} (-1)^j C(n, j) (T^*)^j T^j \geqslant 0, \tag{1.1}$$

where $C(n, j)$ denotes the usual binomial coefficient. The operator T is said to be n-hypercontractive if it is j-contractive for $j = 1, \ldots, n$. An alternative characterization of subnormality is that an operator is a contractive subnormal operator if and only if it is n-hypercontractive for all n [1]. These operators have been studied in [14] and [15]. The following recursion turns out to be very useful in the sequel.

$$A_{n+1}(T) = A_n(T) - T^* A_n(T) T, \quad n = 0, 1, \ldots. \tag{1.2}$$

The class of *completely hyperexpansive* operators is in some sense dual to the contractive subnormal operators and has been studied in a sequence of papers beginning with [4]. An operator is n-expansive if non-positivity (instead of non-negativity) holds in (1.1), and an operator is completely hyperexpansive if and only if it is n-expansive for each n.

The class of weighted shift operators has been a fruitful laboratory for study of the various classes weaker than subnormal, and we recall some notation. We let α denote a weight sequence, $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \ldots$. The weighted shift W_α acting on ℓ^2, with standard basis $e_0, e_1, \ldots$, is defined by $W_\alpha(e_j) = \sqrt{\alpha_j} e_{j+1}$ for all j.

The moments of the shift are defined by $\gamma_0 = 1$ and $\gamma_i = \prod_{j=0}^{i-1} \alpha_j$ for $i = 1, \ldots$. (Observe that some authors use "moment" for the products of the weights and not their squares.) It is well known [19] that any subnormal weighted shift with two weights equal other than the zeroth and first has all weights equal except possibly the zeroth so we usually assume that $\alpha_{j+1} > \alpha_j$ for $j \geqslant 1$ to avoid trivialities. A weighted shift with our chosen notation is hyponormal (that is, 1-hyponormal) if and only if the weights are weakly increasing. Finally, a simple calculation shows that to test positivity of $A_n(W_\alpha)$ for a weighted shift W_α, it is enough to test positivity of $A_n(W_\alpha)$ at the standard basis vectors, which considerably simplifies certain calculations. The k-hyponormality and n-contractivity conditions may be simplified considerably for weighted shifts: the weighted shift W_α is k-hyponormal if and only if

$$\begin{pmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k+1} \\ \cdots & \cdots & & \cdots \\ \gamma_{n+k} & \gamma_{n+k+1} & \cdots & \gamma_{n+2k} \end{pmatrix} \geqslant 0, \quad n = 0, 1, \ldots. \tag{1.3}$$

The test for m-contractivity at the nth basis vector is equivalent to

$$\gamma_n - C(n,1)\gamma_{n+1} + \cdots + (-1)^m C(n,n)\gamma_{n+m} \geqslant 0. \tag{1.4}$$

This may also be phrased in terms of forward differences: the forward differences operator ∇ is defined on $\phi : \mathbb{N} \cup \{0\} \to \mathbb{R}$ by $\nabla(\phi)(n) = \phi(n) - \phi(n+1)$. The iterated forward difference operators ∇^k are then defined by recursion: $\nabla^{k+1} = \nabla\nabla^k$. The condition in (1.4) may then be phrased as $\nabla^m(\gamma)(n) \geqslant 0$, where γ is a function yielding the moment sequence, and therefore m-contractivity for a shift is just positivity of the function $\nabla^m(\gamma)$.

A weighted shift is said to be *recursively generated* if there exists $r \geqslant 1$ and $\varphi_0, \ldots, \varphi_{r-1} \in \mathbb{R}$ such that

$$\gamma_{n+r} = \varphi_0\gamma_n + \cdots + \varphi_{r-1}\gamma_{n+r-1}, \quad n \geqslant 0.$$

We will have occasion to consider some weighted shifts appearing in [1] which we call the Agler model shifts; for each $n \geqslant 1$ let M_n denote the weighted shift with weight sequence $\sqrt{\frac{1}{n}}, \sqrt{\frac{2}{n+1}}, \sqrt{\frac{3}{n+2}}, \ldots$. Note that M_1 is just the unweighted shift and M_2 is just the familiar Bergman shift; it is known that the M_n are not recursively generated. Recall also that a subnormal weighted shift has an associated "representing measure" (Berger measure): if W_α is a weighted shift with moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ there exists a (probability) measure μ with closed support in $[0, \|W_\alpha\|^2]$ so that

$$\gamma_n = \int t^n \, \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots. \tag{1.5}$$

It is well-known (see [10]) that a weighted shift is recursively generated if and only if its Berger measure has finite support.

In considering "transformations" of weighted shifts we will use the (iterated) Aluthge transform. If T is any operator with $T = |T|U$ its polar decomposition, the Aluthge transform is defined by $AT(T) = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$; define the iterated Aluthge transform $AT^n(\cdot)$ by $AT^{n+1}(T) = AT(AT^n(T))$. This transform has been used in the study of p-hyponormality, where p is merely assumed positive: an operator T is p-hyponormal if $(T^*T)^p \geqslant (TT^*)^p$. In the list of transformations it will as well be useful to consider some restrictions of weighted shifts, and we will use $\bigvee\{e_{j+1}, e_{j+2}, \ldots\}$ to denote the obvious closed linear span and the restriction of W_α to some invariant subspace $\mathcal{N}$ by $W_\alpha|_{\mathcal{N}}$. As well we consider a shift with weights some p-th root of the weights of another shift W_α, and refer to such a shift briefly as the p-th root shift of W_α.

2. MOMENT INTERPOLATING FUNCTIONS, ROOT SHIFTS, AND ALUTHGE TRANSFORMS

The goal of this section is to show that the Aluthge transform of at least some subnormal operators is again subnormal. We begin with some results and tools of independent interest. If $\{c_n\}_{n=0}^\infty$ is a sequence, we say that it is interpolated by $f : \mathbb{R}_+ \to \mathbb{R}_+$ if $c_n = f(n)$, $n \geqslant 0$. The following result is analogous to some results in [3] and is essentially to be found in [5].

PROPOSITION 2.1. *Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$, f is infinitely differentiable on $\mathbb{R}_+$ (in the sense of right hand derivatives at 0), f' is non-negative, $f^{(2j)}$ is non-positive for $j \geqslant 1$, and $f^{(2j+1)}$ is non-negative for $j \geqslant 1$. Suppose further that W_α is a (bounded) weighted shift whose moment sequence $\{\gamma_n\}_{n=0}^\infty$ is interpolated by f. Then W_α is completely hyperexpansive.*

Proof. Let $g_1(x) = f(x) - f(x+1)$. Since $f' \geqslant 0$, f is (weakly) increasing, and thus $g_1(x) = f(x) - f(x+1) \leqslant 0$ for any x in $\mathbb{R}_+$. Since f'' is non-positive, f' is (weakly) decreasing, and thus $g_1'(x) = f'(x) - f'(x+1)$ is non-negative. By considering further derivatives of f, we find $g_1^{(n)}$ is non-positive for even n and non-negative for odd n. Now let $g_2(x) = g_1(x) - g_1(x+1)$. Since g_1' is non-negative, g_1 is increasing and thus g_2 is non-positive. Since g_1'' is non-positive, g_1' is decreasing, and therefore $g_2'(x) = g_1'(x) - g_1'(x+1) > 0$, and g_2' is non-negative. Continuing with the derivatives of g_1', we determine that $g_2^{(n)}$ is non-positive for even n and non-negative for odd n. Defining functions g_m iteratively in the obvious way, it is clear that $g_m^{(n)}$ is non-positive for even n and non-negative for odd n.

A trivial computation shows that

$$g_m(n) = \gamma_n - C(n,1)\gamma_{n+1} + \cdots + (-1)^m\gamma_{n+m} = \gamma_n\langle A_m(W_\alpha)e_n, e_n\rangle.$$

Since these are all non-positive, it follows that W_α is completely hyperexpansive. ∎

The proof of the following is much the same as that just given (indeed, it is practically the portion of that just given from g_1 onward) and is omitted; it relies on the result of [1] that an operator is a contractive subnormal operator if and only if it is n-hypercontractive for all n.

PROPOSITION 2.2. *Suppose $f : \mathbb{R}_+ \to \mathbb{R}_+$, f is infinitely differentiable on $\mathbb{R}_+$ (in the sense of right hand derivatives at 0), $f^{(2j)}$ is non-negative for $j \geqslant 1$, and $f^{(2j+1)}$ is non-positive for $j \geqslant 0$. Suppose further that W_α is a (bounded) weighted shift whose moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is interpolated by f. Then W_α is a subnormal contraction.*

The previous proposition motivates the study of the sort of functions in its hypothesis, so we set some notation. A function $f : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ is *completely monotone* if $f^{(2j)}$ is non-negative for $j \geqslant 1$, and $f^{(2j+1)}$ is non-positive for $j \geqslant 0$. Let CMF denote the set of completely monotone functions f such that $f(0) = 1$. It will be useful in what follows to adjoin some functions to this class: for $a \in [0,1)$ let $g_a : \mathbb{R}^+ \to \mathbb{R}^+$ denote the function $g_a(0) = 1$ and $g_a(x) = a$ for all $x > 0$. Let $CMFE$ denote the set of functions of the form $\alpha_1 f + \alpha_2 g_a$, where $f \in CMF$, $\alpha_1, \alpha_2 \geqslant 0$, and $\alpha_1 + \alpha_2 = 1$. It is easy to check that weighted shifts whose moments are interpolated by elements of $CMFE$ are contractive subnormal, as in Proposition 2.2. The following captures some properties of functions in CMF and $CMFE$ to be used in the sequel (note that comparison with a similar set in [3] related to the moments of completely hyperexpansive operators is enlightening). Let Γ denote the Euler Gamma function, defined by

$$\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t.$$

PROPOSITION 2.3. *The sets CMF and CMFE have the following closure properties:*

(i) *if f and g are in CMF so is their product;*

(ii) *if $g : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ has all its derivatives non-negative on : $\mathbb{R}_+$, $g(1) = 1$, and f is in CMF, then the composition $g \circ f$ is in CMF;*

(iii) *if $n, m \in \mathbb{N}$, and f is in CMF (respectively, CMFE), then so is g defined by $g(x) = \frac{f(nx+m)}{f(m)}$;*

(iv) *if $f_1, \ldots, f_n$ are in CMF (respectively, CMFE), then so is any convex combination $\sum_{j=1}^{n} a_j f_j$ with the a_j non-negative and so that $\sum_{j=1}^{n} a_j = 1$.*

Further, the following functions are in CMF:

(v) *$i \equiv 1$;*

(vi) *f defined by $f(x) = \frac{1}{(x+1)^p}$, $p > 0$;*

(vii) *g defined by $g(x) = a^{-x}$, $a > 1$;*

(viii) *h defined by $h(x) = \frac{\Gamma(t)}{\Gamma(s)} \left(\frac{\Gamma(x+s)}{\Gamma(x+t)} \right)^p$, where $0 \leqslant s \leqslant t$ and $0 < p < 1$.*

Proof. Claim (i) follows from considering the product rule terms arising from iterated derivatives of $f \cdot g$, and the other assertions, except the very last, are equally simple calculations. We defer the proof that h is in CMF to the next section. ∎

We next identify a useful function of the kind in Proposition 2.1, omitting the easy proof, and then give an operator theoretic consequence.

PROPOSITION 2.4. *Let* $0 < p < 1$ *and let* $K, J > 0$*. Then* $f : \mathbb{R}_+ \to \mathbb{R}_+ \setminus \{0\}$ *defined by*

$$f(x) = \Big(\frac{Kx + J}{J}\Big)^p$$

satisfies the hypotheses of Proposition 2.1*, and in addition* $f(0) = 1$.

THEOREM 2.5. *Let* $0 < p < 1$ *and* $K, J > 0$ *be arbitrary, and consider the weight sequence* α *defined by*

$$\alpha_i = \Big(\frac{K(i+1) + J}{Ki + J}\Big)^{\frac{p}{2}}, \quad i = 0, 1, \dots.$$

The associated W_α *is completely hyperexpansive. In particular, the weighted shift with weight sequence* α : $\sqrt[4]{2}, \sqrt[4]{\frac{3}{2}}, \sqrt[4]{\frac{4}{3}}, \dots$ *(the square roots of those of the Dirichlet shift) is completely hyperexpansive.*

Proof. For the first claim, it suffices to compute that the moments are $\gamma_n = \left(\frac{Kn+J}{J}\right)^p$ $(n \geqslant 0)$, and cite Propositions 2.1 and 2.4. The second is the special case $K = J = 1$ and $p = \frac{1}{2}$ to be used in the sequel. ∎

To obtain the desired results concerning Aluthge transforms, we shall use the mechanism of Schur products. Recall that if $\alpha = \{\alpha_n\}_{n=0}^\infty$ and $\beta = \{\beta_n\}_{n=0}^\infty$ are sequences, their Schur product $\alpha \circ \beta$ is the sequence $= \{\alpha_n\beta_n\}_{n=0}^\infty$. If W_α and W_β are the associated weighted shifts, it is a trivial computation to note that the moment sequence for $W_{\alpha\circ\beta}$ is the Schur product of the relevant moment sequences. With a slight abuse of language, we call $W_{\alpha\circ\beta}$ the Schur product of W_α and W_β. The next lemma is from [3]; recall that $\nabla^{(k)}$ denotes the kth iterated forward difference.

LEMMA 2.6. *If* $\alpha = \{\alpha_n\}_{n=0}^\infty$ *and* $\beta = \{\beta_n\}_{n=0}^\infty$ *are sequences, for any nonnegative integer* k*,*

$$\nabla^{(k)}(\alpha \circ \beta)(m) = \sum_{p=0}^{k} C(k,p)\nabla^{(k-p)}(\alpha)(m+p)\nabla^{(p)}(\beta)(m).$$

The subnormality portion of next theorem is from [12] and is proved there from the point of view of k-hyponormality; we provide the easy proof based on an n-contractivity approach.

THEOREM 2.7. *Suppose W_α and W_β are n-hypercontractive weighted shifts. Then $W_{\alpha\circ\beta}$ is n-hypercontractive. Further, if W_α and W_β are contractive subnormal, then $W_{\alpha\circ\beta}$ is contractive subnormal.*

Proof. Via the lemma and the remarks before it, the kth iterated difference of the moment sequence for $W_{\alpha\circ\beta}$ is a positive sum of products of lower order iterated differences for W_α and W_β, and is therefore positive since they are. The second statement follows since a weighted shift is contractive subnormal if and only if it is n-hypercontractive for all $n = 1, 2, \ldots$. ∎

We finally have things in place for a first result about the Aluthge transform.

THEOREM 2.8. *The Aluthge transform $AT(B)$ of the Bergman shift B is subnormal.*

Proof. Consider first the weight sequence $\alpha : \sqrt[4]{\frac{1}{2}}, \sqrt[4]{\frac{2}{3}}, \sqrt[4]{\frac{3}{4}}, \ldots$ and the associated W_α. This shift has moment sequence $\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{4}}, \ldots$, and is therefore subnormal citing Proposition 2.2 with $f(x) = \frac{1}{\sqrt{x}}$. (Alternatively, one may note this shift has weights the reciprocals of $\sqrt[4]{\frac{2}{1}}, \sqrt[4]{\frac{3}{2}}, \sqrt[4]{\frac{4}{3}}, \ldots$, invoke Proposition 2.1 to deduce that shift is completely hyperexpansive, and cite Remark 4 of [4] to deduce W_α is subnormal.) Observe that $AT(B)$ has weight sequence $\sqrt[4]{\frac{1}{2} \cdot \frac{2}{3}}, \sqrt[4]{\frac{2}{3} \cdot \frac{3}{4}}, \sqrt[4]{\frac{3}{4} \cdot \frac{4}{5}}, \ldots$, and is therefore the Schur product of W_α and the restriction of W_α to $\bigvee_{i=1}^{\infty} e_i$. Since each of these is subnormal, so is $AT(B)$ by Theorem 2.7. ∎

COROLLARY 2.9. *Let M_3 be the third of the Agler model operators (defined in the introduction), having weight sequence $\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{4}}, \sqrt{\frac{3}{5}}, \ldots$. The shift with weights the square roots of these is subnormal.*

Proof. This shift is exactly $AT(B)$. ∎

This result motivates a look at the Aluthge transforms of the other Agler model operators M_n. The desired result follows from an approach considering shifts with weights the square roots of those of M_n, but in fact a better result may be obtained. Recall that $AT^m(\cdot)$ denotes the mth iterated Aluthge transform.

THEOREM 2.10. *Let M_n denote the nth Agler model operator, let $M_{n,p}$ denote the weighted shift with weight sequence $(\frac{1}{n})^p, (\frac{2}{n+1})^p, (\frac{3}{n+2})^p, \ldots$ and let $n, m \in \mathbb{N}$ and $p > 0$ be arbitrary. Then $M_{n,p}$ is subnormal, the pth root shift of M_n is subnormal, and $AT^m(M_n)$ is subnormal for each m.*

Proof. It is an easy computation that the moment sequence for $M_{n,p}$ is

$$\Big(\frac{(n-1)!}{1 \cdot 2 \cdots (n-1)}\Big)^{(2p)}, \Big(\frac{(n-1)!}{2 \cdot 3 \cdots (n)}\Big)^{(2p)}, \Big(\frac{(n-1)!}{3 \cdot 4 \cdots (n+1)}\Big)^{(2p)}, \ldots.$$

This moment sequence is interpolated by f_n defined by

$$f_n(x) = \Big(\frac{(n-1)!}{(x+1)(x+2)\cdots(x+(n-1))}\Big)^{(2p)}.$$

But f_n is the product of the $n-1$ functions of the form

$$\Big(\frac{j}{(x+j)}\Big)^{(2p)}, \quad j = 1, \ldots, n-1. \tag{2.1}$$

And the jth of the functions in (2.1) is the composition $\frac{f(x+j-1)}{f(j-1)}$ where f is defined by $f(x) = \frac{1}{(x+1)^{(2p)}}$. By Proposition 2.3, f is in CMF, these compositions are in CMF, and the resulting product is in CMF as well. Therefore $M_{n,p}$ is subnormal by Proposition 2.2. It follows readily that for $p > 0$, the pth root shift of M_n is subnormal, since it is $M_{n,2p}$.

The iterated Aluthge transform of a weighted shift is straightforward to compute: if $\alpha : \alpha_0, \alpha_1, \alpha_2, \ldots$ is the weight sequence for W_α, then $AT^m(W_\alpha)$ is a weighted shift with weight sequence $\beta^{(m)}$ with

$$\beta_j^{(m)} = \Big(\prod_{i=0}^{m}(\alpha_{j+i})^{C(m,i)}\Big)^{\frac{1}{2^m}}, \quad j = 0, 1, \ldots.$$

A further computation then shows that $AT^m(W_\alpha)$ is a 2^m-fold Schur product of shifts with weights the 2^mth roots of those of W_α: let $W_{\alpha,m}$ denote the weighted shift with weight sequence $\alpha_0^{\frac{1}{2^m}}, \alpha_1^{\frac{1}{2^m}}, \ldots$. Then $AT^m(W_\alpha)$ turns out to be the product of one copy of $W_{\alpha,m}$, $C(m,1)$ copies of $W_{\alpha,m}|\bigvee\{e_1, e_2, \ldots\}$, $C(m,2)$ copies of $W_{\alpha,m}|\bigvee\{e_2, e_3, \ldots\}$, and so on. (For example, the zeroth and first weight of $AT^3(W_\alpha)$ are $\alpha_0^{\frac{1}{8}}\alpha_1^{\frac{3}{8}}\alpha_2^{\frac{3}{8}}\alpha_3^{\frac{1}{8}}$ and $\alpha_1^{\frac{1}{8}}\alpha_2^{\frac{3}{8}}\alpha_3^{\frac{3}{8}}\alpha_4^{\frac{1}{8}}$ respectively. As well, the shift $AT^3(W_\alpha)$ is the Schur product of one copy of the shift with weight sequence $\alpha_0^{\frac{1}{8}}, \alpha_1^{\frac{1}{8}}, \alpha_2^{\frac{1}{8}}, \ldots$, three copies of the shift with weights $\alpha_1^{\frac{1}{8}}, \alpha_2^{\frac{1}{8}}, \alpha_3^{\frac{1}{8}}, \ldots$, three copies of the shift with weights $\alpha_2^{\frac{1}{8}}, \alpha_3^{\frac{1}{8}}, \alpha_4^{\frac{1}{8}}, \ldots$, and one copy of the shift with weights $\alpha_3^{\frac{1}{8}}, \alpha_4^{\frac{1}{8}}, \alpha_5^{\frac{1}{8}}, \ldots$.)

Since the restriction of a subnormal operator is subnormal, it suffices for the result on iterated Aluthge transforms to show that for each n and m, the shift with weights $\frac{1}{2^m}$th roots of the weights of M_n is subnormal. But this is the first part of the theorem, and we are done. ∎

EXAMPLE 2.11. We turn to an example to illustrate some of the preceding results. In Example 1.2 of [13] the authors consider the shift with weight sequence δ defined by

$$\delta_n = \begin{cases} \sqrt{\frac{1}{2}} & n = 0, \\ \sqrt{\frac{2^n + \frac{1}{2}}{2^n + 1}} & n \geqslant 1. \end{cases}$$

This shift is shown to be subnormal by computation of its moments:

$$\gamma_n = \frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{1}{3}, \quad n \geqslant 1,$$

and by then showing that $\chi := \frac{1}{3}\delta_0 + \frac{1}{3}\delta_{1/2} + \frac{1}{3}\delta_1$ is a satisfactory candidate for a Berger measure. An alternative argument is to use Propositions 2.2 and 2.3: observe that the moments are interpolated by f defined by $f(x) = \frac{1}{3} \cdot 2^{-x} + \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot g_0$. This is a convex combination of three functions in $CMFE$ (citing Proposition 2.3 (i) and (iii)), and the resulting shift is therefore subnormal.

Neither the square root shift of W_δ nor the Aluthge transform of W_δ is subnormal: indeed, neither is even 2-hyponormal. One may check that the matrix

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{\frac{5}{3}}}{2} & \frac{\sqrt{\frac{3}{2}}}{2} \\ \frac{\sqrt{\frac{5}{3}}}{2} & \frac{\sqrt{\frac{3}{2}}}{2} & \frac{\sqrt{\frac{17}{3}}}{4} \\ \frac{\sqrt{\frac{3}{2}}}{2} & \frac{\sqrt{\frac{17}{3}}}{4} & \frac{\sqrt{\frac{11}{2}}}{4} \end{pmatrix}$$

for the square root shift of W_δ (as in (1.3) for $m = 1$) is not positive and the matrix

$$\begin{pmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{\frac{5}{3}}}{2} & \frac{\sqrt{5}}{4} & \frac{\sqrt{17}}{8} \\ \frac{\sqrt{5}}{4} & \frac{\sqrt{17}}{8} & \frac{\sqrt{\frac{187}{3}}}{16} \\ \frac{\sqrt{17}}{8} & \frac{\sqrt{\frac{187}{3}}}{16} & \frac{\sqrt{\frac{715}{3}}}{32} \end{pmatrix}$$

for $AT(W_\delta)$ is not positive either. (A similar computation, which we omit, shows that $AT^2(W_\delta)$ is not 2-hyponormal either.) It is somewhat interesting to note also that W_ε defined to be the square root shift of W_δ is in fact 6-hypercontractive (but not 7-contractive) while not even 2-hyponormal, and thus the contractivity conditions fare better under the square root operation than the hyponormality conditions. To obtain the contractivity claims, we may compute individually the various tests for n-contractivity ($1 \leqslant n \leqslant 6$) at the basis vector e_0 and see that they are positive. For $n \geqslant 1$, the moments γ_n of W_ε are interpolated by g defined by $g(x) = \sqrt{\frac{1}{3}2^{-x} + \frac{1}{3}}$. One computes easily that for $x > 0$, $g' < 0$, $g'' > 0$, $g''' < 0$, and with a little more work that $g^{(4)} > 0$, and this guarantees positivity up through 4th iterated differences for W_ε (4-hypercontractivity) as in the proof of Proposition 2.2. Now $g^{(5)}$ is not non-positive, but is for $x \geqslant 1.5$, and as in the proof of Proposition 2.2 this guarantees positivity of 5th iterated differences except for those at e_0 and e_1, which turn out to be positive. Similarly, $g^{(6)}$ is non-negative for $x > 3.5$, and the needed tests at e_j for low values of j are positive. However, the test at e_1 for 7-contractivity fails.

It is well known that the Aluthge transform of an operator "improves" p-hyponormality of an operator for $p < 1$, as shown by the following theorem.

THEOREM 2.12 ([2]). *If T is p-hyponormal for $\frac{1}{2} \leqslant p < 1$, then $AT(T)$ is hyponormal. If T is p-hyponormal for $0 < p < \frac{1}{2}$, then $AT(T)$ is $(p + \frac{1}{2})$-hyponormal.*

It is then reasonable to ask what, if anything, the Aluthge transform does for k-hyponormality or the other properties under consideration. The previous example shows that preservation of subnormality by the Aluthge transform is not guaranteed. The next example shows that neither k-hyponormality nor n-contractivity is improved, or even necessarily preserved, by the Aluthge transform.

EXAMPLE 2.13. Let $\alpha(x)\ :\ \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ be a perturbation in the first weight of the Bergman shift, and $W_{\alpha(x)}$ be the associated shift. Let $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$ be the first and second iterated Aluthge transform. For each of these three operators, the only weight involving x is the zeroth, so all the tests for the various n-contractivity conditions are met automatically except that for e_0. As a result, the upper bound in x for n-contractivity can be readily computed, and some representative (approximate) data using [20] for the cutoffs for n-contractivity for $W_{\alpha(x)}$, $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$ are below:

n	$W_{\alpha(x)}$	$AT(W_{\alpha(x)})$	$AT^2(W_{\alpha(x)})$
1	1	1.5	3
2	.75	.897	1.19
3	.666	.737	.855
4	.625	.6657	.728
5	.60	.6259	.6629
6	.5833	.6008	.6246
7	.5714	.5836	.5996
8	.5625	.5711	.5823
9	.5556	.5617	.5696
10	.55	.5544	.5600
$\vdots$	$\vdots$	$\vdots$	$\vdots$
20	.525	.5240	.5233

For (some) low values of n, at least n-contractivity of $W_{\alpha(x)}$ guarantees both n-contractivity and $(n+1)$-contractivity of $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$. For larger values of n, the n-contractivity but not the $(n+1)$-contractivity of $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$ are guaranteed by n-contractivity of $W_{\alpha(x)}$. For $n = 20$, n-contractivity of $W_{\alpha(x)}$ does not even guarantee n-contractivity of $AT(W_{\alpha(x)})$ or $AT^2(W_{\alpha(x)})$; the Aluthge transform has actually made things worse. (While the above data are approximate, exact rational number calculation using [20] shows that the $n = 20$ contractivity tests at e_0 for $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$ are negative at $x = \frac{5245}{10000}$ and $x = \frac{5235}{10000}$, respectively.)

The situation for k-hyponormality is quite similar; only the positivity of the various matrices

$$\begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_k \\ \gamma_1 & \gamma_2 & \cdots & \gamma_{k+1} \\ \gamma_2 & \gamma_3 & \cdots & \gamma_{k+2} \\ \vdots & & \ddots & \vdots \\ \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k} \end{pmatrix}$$

(as in (1.3) for $n = 0$, for each of $W_{\alpha(x)}$, $AT(W_{\alpha(x)})$, and $AT^2(W_{\alpha(x)})$) depend on x. For low values of k if $W_{\alpha(x)}$ is k-hyponormal then so are $AT(W_{\alpha(x)})$ and $AT^2(W_{\alpha(x)})$, but they need not be $(k+1)$-hyponormal. Also, in the case $k = 6$, k-hyponormality of $W_{\alpha(x)}$ is not even sufficient for k-hyponormality of $AT(W_{\alpha(x)})$ or $AT^2(W_{\alpha(x)})$. We leave the computations to the interested reader.

We may use the sort of first weight perturbation above to obtain a situation in which the perturbed shift is subnormal if and only if its square root shift is. Given a weight sequence $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \ldots$, let $\alpha(x)$ denote the weight sequence $\sqrt{x}, \sqrt{\alpha_1}, \ldots$, and $\sqrt{\alpha(x)}$ denote the weight sequence $\sqrt[4]{x}, \sqrt[4]{\alpha_1}, \ldots$. Clearly via Schur products if $W_{\sqrt{\alpha(x)}}$ is subnormal so is $W_{\alpha(x)}$. It turns out that the reverse is true for the various M_n; let $M_{n,x}$ denote the perturbation of M_n in the zeroth weight by $\sqrt{x}$ and $M_{n,x,1/2}$ its square root shift. A lemma is required.

LEMMA 2.14. *For each $n = 1, 2, \ldots$ the zeroth weight of M_n is maximal for subnormality; that is, $x = \frac{1}{n}$ is the maximal x such that $M_{n,x}$ is subnormal.*

Proof. We omit the case $n = 1$ (the unweighted shift) as trivial. It is a computation to verify that for $n = 2, 3, \ldots$ the Berger measure for M_n is $\mathrm{d}\mu_n = (n-1)(1-t)^{n-2}\mathrm{d}t$, and then a further computation to show that the Berger measure for the restriction of M_n to $\bigvee_{j=1}^{\infty} e_j$ is $\mathrm{d}\nu_n = nt(n-1)(1-t)^{n-2}\mathrm{d}t$. It follows from Proposition 8 of [9] that a backstep extension of this restriction is subnormal if and only if $\frac{1}{t}$ is in $L^1(\nu_n)$ (obvious in this case from the form of $\mathrm{d}\nu_n$) and

$$x \leqslant \left(\left\|\frac{1}{t}\right\|_{L^1(\nu_n)}\right) = \frac{1}{n}. \quad \blacksquare$$

PROPOSITION 2.15. *For each $n = 1, 2, \ldots$ $M_{n,x}$ is subnormal if and only if $M_{n,x,1/2}$ is subnormal.*

Proof. The weight sequence for $M_{n,x}$ is $\sqrt{x}, \sqrt{\frac{2}{n+1}}, \sqrt{\frac{3}{n+2}}, \ldots$ and that for $M_{n,x,1/2}$ is $\sqrt[4]{x}, \sqrt[4]{\frac{2}{n+1}}, \sqrt[4]{\frac{3}{n+2}}, \ldots$. If the second yields a subnormal so does the first via Schur products, as noted before the lemma. If the first yields a subnormal, then $x \leqslant \frac{1}{n}$, so $\sqrt[4]{x} \leqslant \sqrt[4]{\frac{1}{n}}$. Since the square root of M_n is subnormal, any downward perturbation in its zeroth weight is again subnormal by Proposition 8 of [9], as required. ∎

The relationship between subnormality of the shift and subnormality of the Aluthge transform is less clear, but the following holds.

PROPOSITION 2.16. *Let $\alpha(x) : \sqrt{x}, \sqrt{\frac{2}{3}}, \sqrt{\frac{3}{4}}, \ldots$ denote the perturbation of the Bergman shift in the first weight, with $B(x)$ the associated weighted shift. Then the shift $AT(B(x))$ is subnormal if and only if $B(x)$ is subnormal.*

Proof. As observed above in the proof of Corollary 2.9, the Aluthge transform of B is just M_3. It follows that $AT(B(x))$ is just a perturbation in the zeroth weight of the square root shift M_3 (the perturbed zeroth weight has the form $\sqrt[4]{x \cdot \frac{1}{3}}$). Since we know the largest weight for which such a perturbation is subnormal by Lemmas 2.14 and Proposition 2.15, the proof is completed by an argument similar to that in the proof of Proposition 2.15. ∎

Since we lack the Berger measure for the operators $AT(M_n)$ for $n = 3, 4, \ldots$, it is hard to make further progress. Via another approach, there is numerical evidence that the expected result — $M_{n,x}$ is subnormal if and only if $AT(M_{n,x})$ is subnormal — indeed holds. In [15] there is the following result.

PROPOSITION 2.17 (Theorem 3.3, [15]). *Suppose $\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \ldots$ is the weight sequence corresponding to a contractive subnormal weighted shift not the unweighted unilateral shift. Then the sequence whose nth term is*

$$C(n,1) - C(n,2)\alpha_1 + C(n,3)\alpha_1\alpha_2 - \cdots + (-1)^{n-1}\alpha_1\alpha_2\cdots\alpha_{n-1}$$

is increasing and its terms are positive. Further, if $\alpha : \sqrt{x}, \sqrt{\alpha_1}, \ldots$ is a perturbation in the first weight, $W_{\alpha(x)}$ is subnormal if and only if

$$x \leqslant \lim_{n\to\infty} \frac{1}{C(n,1) - C(n,2)\alpha_1 + C(n,3)\alpha_1\alpha_2 - \cdots + (-1)^{n-1}\alpha_1\alpha_2\cdots\alpha_{n-1}}.$$

Exact rational calculations using [20] show that $AT(M_{3,x})$ is subnormal for $x \leqslant \frac{1}{3}$ and not subnormal if $x \geqslant \frac{1}{3} + .001$, while $M_{3,x}$ is subnormal if and only if $x \leqslant \frac{1}{3}$. Similarly, the cutoffs for subnormality of $AT(M_{4,x})$ and $AT(M_{5,x})$ are within one-thousandth of the cutoffs for subnormality of $M_{4,x}$ and $M_{5,x}$, respectively. There is thus modest evidence that for these operators subnormality of the operator and of its Aluthge transform coincide.

Implicit in the proof of Proposition 2.2, and used in the discussion of Example 2.13, is the following result.

PROPOSITION 2.18. *Let W_α be a weighted shift whose moment sequence $\{\gamma_n\}_{n=0}^{\infty}$ is interpolated by $f : \mathbb{R}_+ \to \mathbb{R}_+$, where $f^{(j)}$ is non-positive, $1 \leqslant j \leqslant m$ and j odd, and $f^{(j)}$ is non-negative, $1 \leqslant j \leqslant m$ and j even. Then W_α is m-hypercontractive. The same* *conclusion holds if the moment sequence is interpolated by a convex combination of such an f and some g_a.*

An application of this proposition is as follows. Suppose $\alpha : \alpha_0, \alpha_1, \ldots$ is a weight sequence such that the moments of W_α are interpolated by some f in *CMFE*. Let $\alpha(x)$ denote the perturbation in the first weight in which α_0 is replaced by x. We have used already (from Proposition 8 of [9]) that if $0 \leqslant x \leqslant \alpha_0$ then $W_{\alpha(x)}$ is subnormal; after a computational lemma, we reprove this result in our special case.

LEMMA 2.19. *Consider f defined by*

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t < 0, \\ 0 & t \geqslant 0. \end{cases}$$

Then f is non-negative and infinitely differentiable; further, for any $m = 1, 2, \ldots$ there is an interval $[k_m, 0]$ with $k_m < 0$ such that $f^{(j)}$ is non-positive, $1 \leqslant j \leqslant m$ and j odd, and $f^{(j)}$ is non-negative, $1 \leqslant j \leqslant m$ and j even.

Proof. The first assertions are essentially contained in Remark 9.5 of [17]. The facts about derivatives follow from a computation showing that the dominant term in $f^{(j)}$ for t negative and close to zero is

$$e^{-\frac{1}{t^2}} \frac{2^j}{t^{3j}}. \quad \blacksquare$$

PROPOSITION 2.20. *Suppose $\alpha : \alpha_0, \alpha_1, \ldots$ is a weight sequence such that the moments of W_α are interpolated by some f in CMFE. Let $\alpha(x)$ denote the perturbation in the first weight in which α_0 is replaced by x. If $0 \leqslant x \leqslant \alpha_0$ then $W_{\alpha(x)}$ is subnormal.*

Proof. It suffices to show that $W_{\alpha(x)}$ is m-hypercontractive for each $m = 1, 2, \ldots$, so let m be arbitrary. The result is trivial if $x = \alpha_0$ or $x = 0$. Suppose $0 < x < \alpha_0$, so $\frac{x}{\alpha_0} < 1$. Let k_m be as in the conclusion of the previous lemma. A computation then shows that f_m defined by

$$f_m(t) = \begin{cases} e^{\frac{1}{k_m^2}} (1 - (\frac{x}{\alpha_0})^2) e^{-\frac{1}{(k_m(t-1))^2}} + (\frac{x}{\alpha_0})^2 & t < 1, \\ (\frac{x}{\alpha_0})^2 & t \geqslant 1, \end{cases}$$

yields f_m so that f_m is non-negative on $[0, \infty)$, $f_m(0) = 1$, $f_m(t) = \left(\frac{x}{\alpha_0}\right)^2$ $(t \geqslant 1)$, and $f_m^{(j)}$ is non-positive, $1 \leqslant j \leqslant m$ and j odd, and $f_m^{(j)}$ is non-negative, $1 \leqslant j \leqslant m$ and j even. The (non-negative) product $f \cdot f_m$ interpolates the moments of $W_{\alpha(x)}$, and its derivatives alternate in sign up through order m, and $W_{\alpha(x)}$ is m-hypercontractive by the proposition. $\blacksquare$

3. SUBSHIFTS AND CERTAIN SUBNORMAL SHIFTS

This section takes up the matter of "j-subshifts" of a weighted shift, and provides the promised proof of Proposition 2.3 part (vii). It is motivated by the following: in [13] we find the definitions and results below.

DEFINITION 3.1 (Definition 2.9, [13]). Suppose $\alpha : \alpha_0, \alpha_1, \ldots$ is a sequence and j a positive integer. A sequence $\beta : \beta_0, \beta_1, \ldots$ is a j-subsequence of α if there exists an integer r, $0 \leqslant r < j$, so that $\beta_n = \alpha_{jn+r}$ for all n. If α is a weight sequence, the shift W_β associated with β is called a *j-subshift* of W_α.

DEFINITION 3.2 (Definition 2.5, [13]). Let $a, b, c, d \geqslant 0$ and satisfy $ad - bc > 0$. Define the weight sequence $\alpha : \alpha_0, \ldots$ by $\alpha_n = \sqrt{\frac{an+b}{cn+d}}$, $n = 0, 1, 2, \ldots$, and denote the associated weighted shift by $S(a, b, c, d)$.

THEOREM 3.3 (Theorems 2.7 and 2.13, [13]). *Let $a, b, c, d \geqslant 0$ and satisfy $ad - bc > 0$. Then $S(a, b, c, d)$ is subnormal and so are all its j-subshifts, $j \geqslant 1$.*

(Remark that we consider only bounded shifts in what follows, and so require $c > 0$. Note then that $a = 0$ forces $b = 0$, a triviality we ignore henceforth.) The second conclusion of the theorem follows since a j-subshift of $S(a, b, c, d)$ turns out to be $S(aj, ar + b, cj, cr + d)$, and with the necessary inequality $(aj)(cr + d) - (ar + b)(cj) = j(ad - bc) > 0$. The authors prove the first assertion of the theorem from the point of view of k-hyponormality, using Schur products, and via an insightful matrix calculation. We first give a more direct proof of the first part of the theorem using a n-contractivity approach. Proposition 2.3 part (vii) then yields a generalization of the theorem.

Proof. First observe that is it enough, given our assumption $c > 0$, to check that each $S(1, \frac{b}{a}, 1, \frac{d}{c})$ is subnormal, since this is a multiple of the given shift and they are subnormal, or not, together. To ease the notation slightly, denote the shift by $S(1, s, 1, t)$, and note that $t > s$ from the assumptions. The test for 1-contractivity of $S(1, s, 1, t)$ at the basis vector e_n is to check that $1 - \frac{n+s}{n+t} > 0$, but this is $\frac{t-s}{n+t} > 0$, which is obvious. The test for 2-contractivity, using the usual recursion (1.2), is

$$1 - \frac{n+s}{n+t} - \frac{n+s}{n+t}\Big(1 - \frac{n+1+s}{n+1+t}\Big) > 0,$$

and a little algebra reduces this to

$$\frac{(t-s)(1+t-s)}{(n+t)(n+t+1)} > 0,$$

which is obvious. An induction using the recursion again shows that the test at e_n for m-contractivity of $S(1, s, 1, t)$ reduces to the positivity of

$$\frac{(t-s)(t+1-s)\cdots(t+m-1-s)}{(n+t)(n+t+1)\cdots(n+t+m-1)},$$

immediate, which shows the shift m-contractive for all m and thus subnormal. ∎

The function in Proposition 2.3 part (vii) arises more naturally than it may appear: the Pochhammer "rising factorial" $P(r,n) = r(r+1)(r+2)\cdots(r+n-1)$ (n an integer) is exactly the term appearing in the numerator or denominator (inside the square root) of the moments of a shift $S(1,s,1,t)$ with r appropriately s or t. It is known that $P(r,n) = \frac{\Gamma(r+n)}{\Gamma(r)}$, and therefore the correct root of $\frac{\Gamma(x+n)}{\Gamma(x)}$ is a natural candidate for the interpolating function.

Recall that the derivative of the Gamma function is the Digamma function ψ, which is known to be increasing. The successive derivatives of ψ are denoted $\psi^{(n)}$, $n = 1, 2, \ldots$, and these satisfy the equations

$$\psi^{(n)}(x) = (-1)^{n+1} n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}}.$$

In particular, if n is odd then $\psi^{(n)}$ is decreasing, and if n is even $\psi^{(n)}$ is increasing.

Proof of Proposition 2.3 *part* (vii). Assume $0 < p$ and $0 \leqslant s < t$ (we ignore the trivial case $s = t$). Define h by

$$h(x) = \Big(\frac{\Gamma(t)}{\Gamma(s)} \frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^p,$$

and f by

$$f(x) = \Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^p.$$

(We compute with f for convenience, as the signs of its derivatives are obviously the same as those of h, while clearly $h(0) = 1$ as required.)

A computation shows that

$$\begin{aligned} f'(x) &= p\Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^{p-1} \Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)(\psi(x+s) - \psi(x+t)) \\ &= p\Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^{p}(\psi(x+s) - \psi(x+t)), \end{aligned}$$

and this is negative since $s < t$ and ψ is increasing. Another computation shows that

$$\begin{aligned} f''(x) = p^2&\Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^{p}(\psi(x+s) - \psi(x+t))^2 \\ &+ p\Big(\frac{\Gamma(x+s)}{\Gamma(x+t)}\Big)^{p}(\psi^{(1)}(x+s) - \psi^{(1)}(x+t)), \end{aligned}$$

which is positive as required, since each of the two summands is in fact positive.

One may then found an induction argument on the observation that $f^{(m)}$, for $m \geqslant 2$, is a sum of terms each of the form

$$p^j \left(\frac{\Gamma(x+s)}{\Gamma(x+t)}\right)^p (\psi(x+s)-\psi(x+t))^{m_1} \\ \cdot (\psi^{(1)}(x+s)-\psi^{(1)}(x+t))^{m_2} \cdots (\psi^{(m-1)}(x+s)-\psi^{(m-1)}(x+t))^{m_{m-1}}.$$

Assume for the induction that each of these terms is of the correct sign for the appropriate parity of the derivative (negative if m is odd and positive if m is even). This term in $f^{(m)}$ is the "ancestor" to a number of terms in $f^{(m+1)}$ arising from the product rule. If such a term arises from the derivative of $\left(\frac{\Gamma(x+s)}{\Gamma(x+t)}\right)^p$, we know that there is a sign change by the computation for f' above. If such a term arises from the derivative of $(\psi(x+s)-\psi(x+t))^{m_1}$ with $m_1 \geqslant 1$, the change in power of $(\psi(x+s)-\psi(x+t))$ induces a sign change, but the chain rule derivative term $(\psi^{(1)}(x+s)-\psi^{(1)}(x+t))$ does not. A similar argument applies as well to terms arising from the derivative of some $(\psi^{(j)}(x+s)-\psi^{(j)}(x+t))^{m_j}$ with j even and $m_j \geqslant 1$, using the fact that $\psi^{(j)}$ is increasing but $\psi^{(j+1)}$ is decreasing. On the other hand, terms arising from the derivative of some term $(\psi^{(j)}(x+s)-\psi^{(j)}(x+t))^{m_j}$ with j odd and $m_j \geqslant 1$ gain no sign change from the change in power of $(\psi^{(j)}(x+s)-\psi^{(j)}(x+t))$, but acquire a sign change from the chain rule term $(\psi^{(j+1)}(x+s)-\psi^{(j+1)}(x+t))$. Thus all terms of $f^{(m+1)}$ are of the appropriate sign, and the result follows by induction. ∎

What follows is the promised generalization of Theorem 3.3.

THEOREM 3.4. *Let $a,b,c,d \geqslant 0$ and satisfy $ad-bc>0$, and let $p>0$. Define the weight sequence $\alpha : \alpha_1, \dots$ by $\alpha_n = \left(\frac{an+b}{cn+d}\right)^p$, $n = 0,1,2,\dots$, and denote the associated weighted shift by $S_p(a,b,c,d)$. Then $S_p(a,b,c,d)$ is subnormal and so are all its j-subshifts, $j \geqslant 1$. Further, all qth root shifts of $S_p(a,b,c,d)$ are subnormal ($1 \leqslant q$), and the iterated Aluthge transforms $AT^m(S_p(a,b,c,d))$ for $m = 1,2,\dots$ are subnormal as well.*

Proof. As in the proof of Theorem 3.3 it suffices to consider $S_p(1,s,1,t)$, whose moment sequence is then exactly $\left\{\left(\frac{\Gamma(t)}{\Gamma(s)}\frac{\Gamma(n+s)}{\Gamma(n+t)}\right)^{2p}\right\}_{n=0}^{\infty}$. The function of Proposition 2.3 part (vii) then provides the needed interpolation, yielding subnormality of $S_p(1,s,1,t)$. The result for subshifts follows as in the remark after Theorem 3.3, and the result on qth root shifts is immediate since p is arbitrary. Finally, the subnormality of $AT^m(S_p(a,b,c,d))$ follows from qth root shifts and Schur products as in the proof of Theorem 2.10, and we are done. ∎

There are certain results available about j-subshifts of qth-roots, qth roots of Aluthge transforms, and similar "compositions" of transformations of shifts, which we leave to the interested reader.

4. REMARKS AND OPEN QUESTIONS

4.1. As seen above subnormal qth root shifts guarantee subnormal Aluthge transforms, but is certainly possible to produce a weighted shift whose Aluthge transform is subnormal but none of whose qth roots are. Indeed, one may construct a weight sequence with first element $\sqrt{\frac{2}{5}}$ and whose Aluthge transform is M_3, but which is not even hyponormal, and therefore none of its roots shifts is even hyponormal. However, we do not know of a subnormal weighted shift whose Aluthge transform is subnormal but whose root shifts are not.

4.2. We do not know of a recursively generated subnormal weighted shift (with the trivial exceptions of scalar multiples of the unweighted shift, and, for qth roots, shifts formed as qth Schur powers) for which any of the Aluthge transform, qth root shifts, or j-subshifts are subnormal.

4.3. In considering subnormality of an operator there are always the two routes of k-hyponormality and n-contractivity (among others), and arriving at subnormality via the n-contractivity approach one has then obtained k-hyponormality for all k. For weighted shifts, in particular, positivity for each m and n of the expression in (1.4) involving the moments thus yields the positivity of the moment matrix in (1.3) for each n and k. In the results above about roots and Aluthge transforms of weighted shifts the present paper has consistently taken the n-contractivity approach, and has therefore obtained the positivity of various moment matrices "for free." It is reasonable to ask whether the approach to subnormality via k-hyponormality (that is, via positivity of these moment matrices) is also feasible. The following result is a step in this direction yielding in particular the subnormality of the roots of the Bergman shift; it is due to James Rovnyak, and we are grateful for his permission to include it here.

PROPOSITION 4.1. *Let $r > 0$ and $q > 0$. Then the matrix M defined by*

$$M = \Big[\frac{1}{(r+j+k)}\Big]_{j,k=0}^{n}$$

is positive semidefinite and invertible.

Proof. Let $v \in \mathbb{C}^{n+1}$ with $v^{\mathrm{T}} = (c_0, \dots, c_n)^{\mathrm{T}}$. Then

$$\langle Mv, v\rangle_{\mathbb{C}^{n+1}} = \sum_{j,k=0}^{n} \frac{\overline{c}_k c_j}{(r+j+k)^q}.$$

Via a computation, one has

$$\frac{1}{y^q} = \frac{1}{\Gamma(q)} \int_0^\infty \mathrm{e}^{-yt} t^{q-1}\, \mathrm{d}t.$$

Then

$$\langle Mv, v\rangle_{\mathbb{C}^{n+1}} = \sum_{j,k=0}^{n} \frac{\bar{c}_k c_j}{\Gamma(q)} \int_0^\infty \mathrm{e}^{-(r+j+k)t} t^{q-1}\,\mathrm{d}t = \frac{1}{\Gamma(q)} \int_0^\infty \sum_{j,k=0}^{n} \bar{c}_k c_j \mathrm{e}^{-jt} \mathrm{e}^{-kt} \mathrm{e}^{-rt} t^{q-1}\,\mathrm{d}t$$
$$= \frac{1}{\Gamma(q)} \int_0^\infty \Big| \sum_{j=0}^{n} c_j \mathrm{e}^{-jt} \Big|^2 \mathrm{e}^{-rt} t^{q-1}\,\mathrm{d}t \geqslant 0.$$

Thus M is positive semidefinite, and further M is invertible since the quadratic form can vanish only if $\sum_{j=0}^{n} c_j \mathrm{e}^{-jt} \equiv 0$ and thus $v = 0$. ∎

4.4. The usefulness of the functions in Propositions 2.2 and 2.3 for subnormality and the appearance of analogous functions in Proposition 2.1 for complete hyperexpansivity would seem to motivate the study of these latter functions. We limit ourselves to basic remarks here; a more sophisticated study of these — the Bernstein functions — is to be found in [5]. It turns out that these functions are considerably less tractable. The analogues of (iii), (iv), and (v) of Proposition 2.3 do in fact hold. But easy computations show that the proofs of (i) and (viii) do not go through, and the results are in fact false. Consider the shift with weight sequence $\alpha : \sqrt{\frac{3}{1}}, \sqrt{\frac{4}{2}}, \sqrt{\frac{5}{3}}, \ldots$, which we refer to henceforth as the reciprocal shift of M_3. Its moment sequence is $\gamma_n = \frac{(n+1)(n+2)}{2}$, and as has been pointed out in [4] the resulting shift is not completely hyperexpansive. Further, the "natural" interpolating function f defined by $f(x) = \frac{(x+1)(x+2)}{2}$ does not have the requisite derivative properties — f'' is positive — even though it is the product of $\frac{(x+1)}{1}$ and $\frac{(x+2)}{2}$, each of which is in the class of functions under consideration. Therefore the analogue of (i) cannot hold, and further, $f(x) = \left(\frac{\Gamma(0)}{\Gamma(2)} \frac{\Gamma(x+2)}{\Gamma(x+0)}\right)$, showing that the analogue of (viii) fails.

The following do hold. (Note that D, the Dirichlet shift, is the reciprocal shift for M_2.)

PROPOSITION 4.2. *Let $0 < p < \frac{1}{2}$ be arbitrary and let α be the weight sequence $\alpha : \sqrt[p]{\frac{2}{1}}, \sqrt[p]{\frac{3}{2}}, \sqrt[p]{\frac{4}{3}}, \ldots$. Then W_α is completely hyperexpansive. It follows that any qth root shift of the Dirichlet shift D, and hence any iterated Aluthge transform $AT^n(D)$, is completely hyperexpansive.*

Proof. The interpolating function for the moments is f defined by $f(x) = (x+1)^{2p}$, and the needed derivative properties are immediate. ∎

PROPOSITION 4.3. *The shift with weights $\sqrt[4]{\frac{3}{1}}, \sqrt[4]{\frac{4}{2}}, \sqrt[4]{\frac{5}{3}}, \ldots$ (the "square root shift of the reciprocal of M_3") is completely hyperexpansive.*

Proof. The natural interpolating function for the moments is f defined by $f(x) = \sqrt{\frac{(x+1)(x+2)}{2}}$. Then $f'(x) = \frac{3x+2}{2\sqrt{(x+1)(x+2)}}$ is appropriately positive, and $f''(x) = \frac{-1}{4(x+1)(x+2)^{3/2}}$ is both appropriately negative and the product of functions whose derivatives alternate in sign, and whose further derivatives therefore themselves alternate in sign as needed. ∎

Since the reciprocal shift of M_3 is not completely hyperexpansive, the previous proposition shows that the Schur product of completely hyperexpansive shifts need not be completely hyperexpansive, as might have been expected by an examination of Lemma 2.6.

4.5. Finally, there is considerable numerical/computational evidence (with the aid of [20]) that the natural interpolating functions for the moments of the "cube root shift of the reciprocal of M_4" and the "fourth root shift of the reciprocal of M_5" are functions in the class yielding complete hyperexpansivity. We have at present no explanation for these phenomena, if true, and no approach for the proof of the obvious natural conjecture if they are.

REFERENCES

[1] J. AGLER, Hypercontractions and subnormality, *J. Operator Theory* **13**(1985), 203–217.

[2] A. ALUTHGE, On p-hyponormal operators for $0 < p < 1$, *Integral Equations Operator Theory* **13**(1990), 307–315.

[3] A. ATHAVALE, Some operator theoretic calculus for positive definite kernels, *Proc. Amer. Math. Soc.* **112**(1991), 701–708.

[4] A. ATHAVALE, On completely hyperexpansive operators, *Proc. Amer. Math. Soc.* **124**(1996), 3745–3752.

[5] A. ATHAVALE, A. RANJEKAR, Bernstein functions, complete hyperexpansivity, and subnormality. I, *Integral Equations Operator Theory* **43**(2002), 253–263.

[6] J. BRAM, Subnormal operators, *Duke Math. J.* **22**(1965), 75–94.

[7] J. CONWAY, *The Theory of Subnormal Operators*, Math. Surveys Monographs, vol. 36, Amer. Math. Soc., Providence, RI 1980.

[8] R. CURTO, Joint hyponormality: A bridge between hyponormality and subnormality, in *Operator Theory: Operator Algebras and Applications, Part 2 (Durham, NH, 1988)*, Proc. Sympos. Pure Math., vol. 51, Amer. Math. Soc., Providence, RI 1990, pp. 69–91.

[9] R. CURTO, Quadratically hyponormal weighted shifts, *Integral Equations Operator Theory* **13**(1990), 49–66.

[10] R. CURTO, L. FIALKOW, Recursively generated weighted shifts and the subnormal completion problem. I, *Integral Equations Operator Theory* **17**(1993), 202–246.

[11] R. CURTO, I. JUNG, W. LEE, Extension and extremality of recursively generated weighted shifts, *Proc. Amer. Math. Soc.* **130**(2001), 565–576.

[12] R. CURTO, S. PARK, k-hyponormality of powers of weighted shifts via Schur products, *Proc. Amer. Math. Soc.* **131**(2002), 2761–2769.

[13] R. CURTO, Y. POON, J. YOON, Subnormality of Bergman-like weighted shifts, *Integral Equations Operator Theory* **308**(2005), 334–342.

[14] G. EXNER, On n-contractive and n-hypercontractive operators, *Integral Equations Operator Theory* **56**(2006), 451–468.

[15] G. EXNER, I. JUNG, S. PARK, On n-contractive and n-hypercontractive operators. II, *Integral Equations Operator Theory* **60**(2008), 451–467.

[16] M. EMBRY, A generalizaton of the Halmos–Bram criterion for subnormality, *Acta Sci. Math. (Szeged)* **35**(1973), 61–64.

[17] E. FISCHER, *Intermediate Real Analysis*, Undergrad. Texts Math., Springer-Verlag, New York 1983.

[18] P. HALMOS, Normal dilations and extensions of operators, *Summa Bras. Math.* **2**(1950), 51–65.

[19] J. STAMPFLI, Which weighted shifts are subnormal?, *Pacific J. Math.* **17**(1966), 367–379.

[20] WOLFRAM RESEARCH, INC., *Mathematica, Version 5.0*, Wolfram Research Inc., Champaign, IL, 1996.

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