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CLASSIFICATION OF CERTAIN INDUCTIVE LIMIT TYPE ACTIONS OF $\mathbb{R}$ ON C^*-ALGEBRAS

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ABSTRACT. In this paper we present a classification, up to equivariant isomorphism, of C^*-dynamical systems $(A, \mathbb{R}, \alpha)$ arising as inductive limits of directed systems $\{(A_n, \mathbb{R}, \alpha_n), \varphi_{nm}\}$ where each A_n is a finite direct sum of matrix algebras over graphs, the φ_{nm} are unital and injective, and the α_ns are generated by inner $*$-derivations coming from diagonalisable self-adjoint elements with distinct eigenvalues.

KEYWORDS: *Classification, C^*-dynamical systems, derivations.*

MSC (2000): 46L57, 46L35.

1. INTRODUCTION

There are now many classification results for C^*-dynamical systems of inductive limit type. For various cases involving compact groups, see [1],[8], [9],[10], and [12]. For non-compact groups, inductive limit type actions of the Euclidean motion group were studied in [4], actions of $SL(2, \mathbb{R})$ were considered in [5], and AF flows were classified in [2] and [3].

In this paper we shall consider C^*-dynamical systems $(A, \mathbb{R}, \alpha_n)$ arising as limits of inductive systems $\{(A_n, \mathbb{R}, \alpha_n), \varphi_{nm}\}$ where each A_n is a finite direct sum of matrix algebras over graphs, the α_ns are generated by inner derivations given by diagonalisable self-adjoint elements having distinct eigenvalues in all minimal quotients, and the φ_{nm} are unital, injective equivariant $*$-homomorphisms. In the case where the algebras are simple, these algebras were classified in [13] by showing that they are $A\mathbb{T}$, reducing the problem to that already solved in [7] and [15]. Our theorem does not include this result, as the trivial action does not satisfy our hypotheses. When attention is restricted to the algebra, our invariant closely resembles that used in [17] to classify pairs (A, B) with B what is called in [17] an approximately trivial homogemeous C^*-algebra and A a diagonal in B.

For the rest of this paper, the term C^*-dynamical system shall always imply that the group is $\mathbb{R}$, and we shall write A^α for the fixed point subalgebra of A under the action α. We refer the reader to [16] for general facts about $\mathbb{R}$ actions.

2. THE INVARIANT

DEFINITION 2.1. By the term *graph* we shall mean a 1-dimensional simplicial complex. Thus, each edge has exactly two vertices and any two edges have at most one intersection point, which must be a vertex. We shall assume that all graphs are connected unless stated otherwise.

DEFINITION 2.2. Let A be a C^*-algebra that is isomorphic to a matrix algebra over a graph X. A self-adjoint element $a \in A$ is called *diagonalisable* if, and only if, there exist mutually orthogonal projections $P_1, \dots, P_m \in A$ and functions $f_1, \dots, f_m$ from X to $\mathbb{R}$ such that $a(t) = f_1(t)P_1(t) + \cdots + f_m P_m(t)$ for all $t \in X$. If A is a finite direct sum of matrix algebras over graphs we say that a self adjoint element $a \in A$ is diagonalisable if its cut-down by each minimal central projection is.

In the special case where X is a tree and the eigenvalues of a are distinct in every fibre, it follows from Theorem 1.4 of [11] that diagonalisability of a is automatic. It is not in general (cf. the counterexamples in [11]).

DEFINITION 2.3. We shall say that a C^*-dynamical system (A, α) is of *special form* if, and only if, A is a finite direct sum of matrix algebras over graphs and α is generated by an inner derivation given by a diagonalisable self-adjoint element h of A having the property that the image of h has distinct eigenvalues in every simple quotient of A.

DEFINITION 2.4. If A is a sub-C^*-algebra of a C^*-algebra B, an element $b \in B$ is called an *A-normaliser* if $bAb^* \subseteq A$ and $b^*Ab \subseteq A$. If B is a unital C^*-algebra, a system of matrix units in B is a set $\{e_{ij}\}_{i,j=1}^n$ such that $e_{ij}^* = e_{ji}$, $e_{ij}e_{kl} = \delta_{jk}e_{il}$, and $\sum_{i=1}^n e_{ii} = 1$. If $B \cong C(X) \otimes M_n$, where X is a compact Hausdorff space, a *diagonal* in B was defined in [17] to be an Abelian subalgebra $A \subseteq B$ such that A contains the unit of B and there exists a system of matrix units in B consisting of normalisers of A that, together with A, generate B. If $B \cong \bigoplus_{m=1}^k C(X_m) \otimes M_{n_m}$, where each X_m is a compact Hausdorff space, we shall call an Abelian subalgebra $A \subseteq B$ a diagonal in B if its intersection with each $C(X_m) \otimes M_{n_m}$ is a diagonal in the above sense.

LEMMA 2.5. *Let (A, α) be a C^*-dynamical system of special form. Then the fixed point subalgebra, A^α, of A is a diagonal in A.*

Proof. Let (A, α) be a C^*-dynamical system of special form. It will be sufficient to consider the case where A has only one direct summand. Suppose $A \cong M_n(C(X))$, where X is a graph, and that α is generated by an inner $*$-derivation coming from a diagonalisable self-adjoint element h. Our hypotheses imply that we may write $h(t) = f_1(t)p_1(t) + \cdots + f_n(t)p_n(t)$, $t \in X$, where $p_1, \ldots, p_n$ are pairwise orthogonal minimal projections in $A \cong C(X, M_n)$, and $f_1, \ldots, f_n$ are real valued functions on X with the property that $f_i(t) \neq f_j(t)$ if $i \neq j$ for all ts. If $g \in A^\alpha$, we must have $g(t) = p_1(t)g(t)p_1(t) + \cdots + p_n(t)g(t)p_n(t)$ for every $t \in X$. It follows that $A^\alpha \cong p_1Ap_1 \oplus \cdots \oplus p_nAp_n \cong C(X) \oplus \cdots \oplus C(X)$ (n copies). The minimal projections $p_1, \ldots p_n$ are pairwise equivalent, so there exist partial isometries $v_{11}, \ldots, v_{1n}$ in A such that $v_{1i}v_{1i}^* = p_1$ and $v_{1i}^*v_{1i} = p_i$ for each i. Setting $e_{ij} = v_{1i}^*v_{1j}$ we see that $\{e_{ij}\}$ is a system of matrix units consisting of normalisers of A^α. Since A^α contains the centre of A, it follows that A^α is a diagonal. ∎

DEFINITION 2.6. Let X and Y be compact Hausdorff spaces. A unital $*$-homomorphism $\varphi : C(X) \otimes M_n \to C(Y) \otimes M_{nk}$ is called a *standard homomorphism* if there exist continuous functions $f_1, \ldots, f_k$ from Y to X such that $\varphi(h) = \mathrm{diag}(h \circ f_1, \ldots, h \circ f_k)$ for all $h \in C(X) \otimes M_n$, for some identifications of $C(X) \otimes M_n$ and $C(Y) \otimes M_{nk}$ with $C(X, M_n)$ and $C(Y, M_{nk})$ respectively. If $A \cong \bigoplus_{m=1}^{k} C(X_m) \otimes M_{n_m}$ and $B \cong \bigoplus_{j=1}^{l} C(X_m) \otimes M_{n_j}$, where the X_ms and Y_js are graphs, we shall call a unital $*$-homomorphism $\varphi : A \to B$ standard if, and only if, each of the partial maps from $C(X_m) \otimes M_{n_m}$ to $\varphi(1_m)(C(Y_j) \otimes M_{n_j})\varphi(1_m)$ for $1 \leqslant m \leqslant k$, $1 \leqslant j \leqslant l$ are standard in the above sense, where 1_m is the unit of the m-th direct summand of A. (Note that all of the projections in a matrix algebra over a graph are trivial, so $\varphi(1_m)(C(Y_j) \otimes M_{n_j})\varphi(1_m) \cong M_{\mathrm{rank}(\varphi(1_m))}(C(Y_j))$.)

It follows from Lemma 2.5 above and Corollary 1.13 of [17] that if (A, α) and (B, β) are two C^*-dynamical systems of special form, and $\varphi : A \to B$ is a unital equivariant $*$-homomorphism, then φ is unitarily equivalent to a standard homomorphism.

LEMMA 2.7. *Suppose $\{(A_n, \alpha_n), \varphi_{nm}\}$ is an inductive system of C^*-dynamical systems where each (A_n, α_n) is of special form and let (A, α) denote the inductive limit C^*-dynamical system. Then* $A^\alpha = \overline{\bigcup_{n=1}^{\infty} \varphi_{n\infty}(A_n^{\alpha_n})}$.

Proof. Let (A, α) be a C^*-dynamical system. For each natural number n, let $g_n : \mathbb{R} \to \mathbb{R}$ be the function defined by $g_n(t) = 1/(2n+1)$ if $t \in [-n, n]$, $g_n(t) = 0$ if $t \notin [-n-1, n+1]$, and being linear on $[-n-1, -n]$ and $[n, n+1]$. Then $\int_{\mathbb{R}} g_n(t)\,\mathrm{d}t = 1$. Consider the mapping $\Gamma_n : A \to A$ given by $\Gamma_n(a) =$

$\int_{\mathbb{R}} g_n(t)\alpha_t(a)\,\mathrm{d}t$. If $a, b \in A$, we have

$$\|\Gamma_n(a) - \Gamma_n(b)\| = \Big\| \int_{\mathbb{R}} g_n(t)\alpha_t(a)\,\mathrm{d}t - \int_{\mathbb{R}} g_n(t)\alpha_t(b)\,\mathrm{d}t \Big\|$$
$$\leqslant \int_{\mathbb{R}} g_n(t)\|\alpha_t(a) - \alpha_t(b)\|\,\mathrm{d}t = \|a - b\|.$$

Furthermore, if (B, β) is another C^*-dynamical system, and $\varphi : A \to B$ is an equivariant $*$-homomorphism, then $\Gamma_n \circ \varphi = \varphi \circ \Gamma_n$, where we define Γ_n on B the same way.

Let $\{(A_n, \alpha_n), \varphi_{nm}\}$ be an inductive system of C^*-dynamical systems where each (A_n, α_n) is of special form and let (A, α) denote the inductive limit C^*-dynamical system. Let $a \in A^\alpha$ and let $\varepsilon > 0$ be given. For some natural number m, there exists a $b \in A_m$ such that $\|\varphi_{m\infty}(b) - a\| < \varepsilon$. Suppose that $A_m \cong C(X) \otimes M_l$ for a graph X (the case of several direct summands is similar) and that α_m is generated by a self-adjoint element $h(x) = f_1(x)p_1(x) + \cdots + f_l(x)p_l(x)$, $x \in X$, where $p_1, \ldots, p_l$ are orthogonal minimal projections with sum 1 and the f_js are real valued functions with distinct values for every $x \in X$. It is then easy to see that $\alpha_{m,t}(p_k(x)b(x)p_j(x)) = \exp(\mathrm{i}t(f_k(x) - f_j(x)))(p_k(x)b(x)p_j(x))$ for all $x \in X$ and $t \in \mathbb{R}$. Since the values of the f_js are distinct for every $x \in X$, and X is compact, it follows that $f_k(x) - f_j(x)$ is bounded and bounded away from zero, $M \geqslant |f_k(x) - f_j(x)| \geqslant \delta > 0$, say. If r is a non-zero real number, $\int_I \mathrm{e}^{\mathrm{i}rt}\,\mathrm{d}t$ will be zero on any interval of length $2\pi/|r|$. Noticing that

$$\Big(\int_{\mathbb{R}} g_n(t)\alpha_t(p_k b p_j)\,\mathrm{d}t\Big)(x) = \Big(\int_{\mathbb{R}} g_n(t)\exp(\mathrm{i}t(f_k(x) - f_j(x)))\,\mathrm{d}t\Big)(p_k(x)b(x)p_j(x)),$$

it is easy to see that $\Gamma_n(p_k b p_j) \to 0$ as $n \to \infty$ for $j \neq k$, so $\Gamma_n(b) \to p_1 b p_1 + \cdots + p_l b p_l \in A_m^{\alpha_m}$ as $n \to \infty$. It follows that $\|a - \varphi_{m\infty}(p_1 b p_1 + \cdots + p_l b p_l)\| \leqslant \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $a \in \overline{\bigcup_{n=1}^{\infty} \varphi_{n\infty}(A_n^{\alpha_n})}$. The other inclusion is obvious, so the lemma follows. ∎

It is immediate from this lemma, and important for what follows, that A^α is commutative.

DEFINITION 2.8. Let (A, α) be a C^*-dynamical system. A non-zero partial isometry $v \in A$ will be called a *generalised eigenisometry* if, and only if, vv^* and v^*v are in A^α; and there exists a self-adjoint element $a \in A^\alpha$ such that $vv^*avv^* = a$, and, for all $t \in \mathbb{R}$, $\alpha_t(v) = \mathrm{e}^{\mathrm{i}at}v$. If v is a generalised eigenisometry, the element a satisfying the above conditions is easily seen to be unique, and we shall refer to it as the generalised eigenvalue of v. We shall consider the zero partial isometry to be a generalised eigenisometry with generalised eigenvalue zero.

LEMMA 2.9. *Let (A, α) be a C^*-dynamical system of special form, and let v be a partial isometry with $vv^* \in A^\alpha$ and $v^*v \in A^\alpha$. Then v is a generalised eigenisometry . Furthermore, if u is another partial isometry with $uu^* \in A^\alpha$, $u^*u \in A^\alpha$, and $\|u - v\| < 1/10$, then $uu^* = vv^*$, $u^*u = v^*v$, and u has the same generalised eigenvalue as v.*

Proof. Let (A, α) be a C^*-dynamical system with A^α commutative, and suppose that u and v are generalised eigenisometries with generalised eigenvalues a and b respectively, and that $v^*v \perp u^*u$ and $uu^* \perp vv^*$. Then $u + v$ is a generalised eigenisometry with generalised eigenvalue $a + b$. Furthermore, if p is a projection in A^α and $p \leqslant u^*u$, then up is a generalised eigenisometry with generalised eigenvalue $a(upu^*)$.

Suppose now that (A, α) is of special form. From the above observations, it will be sufficient to consider the case of one direct summand. Suppose that $A_m \cong C(X) \otimes M_l$ for a graph X and that α_m is generated by a self-adjoint element $h(x) = f_1(x)p_1(x) + \cdots + f_l(x)p_l(x)$, $x \in X$, where $p_1, \ldots, p_l$ are orthogonal minimal projections with sum 1 and the f_js are real valued functions with distinct values for every $x \in X$. If v is a partial isometry with $v^*v \in A^\alpha$, then v^*v is a sum of certain of the ps. If $p_j \leqslant v^*v$, then vp_jv^* is another minimal projection in A^α, and is therefore another of the ps, p_k say. If $b \in p_kAp_j$, we have $\alpha_t(p_k(x)b(x)p_j(x)) = \exp(\mathrm{i}t(f_k(x) - f_j(x)))(p_k(x)b(x)p_j(x))$ for all $x \in X$ and $t \in \mathbb{R}$, so we see that vp_j is a generalised eigenisometry, and it follows that v is.

Let u and v be two partial isometries with $vv^* \in A^\alpha$, $v^*v \in A^\alpha$, $uu^* \in A^\alpha$, $u^*u \in A^\alpha$, and $\|u - v\| < 1/10$. Then $\|u^*u - v^*v\| < 1/5$ and $\|uu^* - vv^*\| < 1/5$. Since A^α is commutative, two projections in A^α that differ by less than 1 in norm are equal, so $uu^* = vv^*$ and $u^*u = v^*v$. Similarly, we see that for each j, $\|up_ju^* - vp_jv^*\| \leqslant \|up_j(u^* - v^*)\| + \|(u - v)p_jv^*\| \leqslant 1/5$, so $up_ju^* = vp_jv^*$. It follows that up_j and vp_j are generalised eigenisometries with the same generalised eigenvalue. Summing over the js, we see that u and v have the same generalised eigenvalue. ∎

DEFINITION 2.10. Let (A, α) be a C^*-dynamical system. Let $E(A, \alpha)$ denote the subset of $D(A^\alpha) \times D(A^\alpha) \times A^\alpha$ consisting of those triples (x, y, a) such that there exists a generalised eigenisometry $v \in A$ with $x = [vv^*]$, $y = [v^*v]$, and generalised eigenvalue a. If (B, β) is another C^*-dynamical system and $\varphi : A \to B$ is an equivariant $*$-homomorphism, then the image of under φ of a generalised eigenisometry with generalised eigenvalue a is a generalised eigenisometry in B with generalised eigenvalue $\varphi(a)$, so we have that the map $D(\varphi) \times D(\varphi) \times \varphi$ takes $E(A, \alpha)$ into $E(B, \beta)$. We shall write $E(\varphi)$ for the restriction of $D(\varphi) \times D(\varphi) \times \varphi$ to $E(A, \alpha)$.

DEFINITION 2.11. Let (A, α) and (B, β) be two C^*-dynamical systems of special form, and suppose that $\varphi : A \to B$ is a $*$-homomorphism such that $\varphi(A^\alpha) \subseteq B^\beta$. If v is a generalised eigenisometry in A, with generalised eigenvalue a say, then, since $\varphi(vv^*)$ and $\varphi(v^*v)$ are both in B^β, it follows by Lemma 2.9

that $\varphi(v)$ is a generalised eigenisometry in B, with generalised eigenvalue b say. If w is another generalised eigenisometry in A with $w^*w = v^*v$, $ww^* = vv^*$, and generalised eigenvalue a, then $u = wv^*$ is a partial unitary in A^α with $uu^* = u^*u = vv^*$ and $w = uv$. We have $\varphi(w) = \varphi(u)\varphi(v)$ and $\varphi(u) \in B^\beta$, so $\beta_t(\varphi(w)) = \varphi(u)\mathrm{e}^{\mathrm{i}tb}\varphi(v) = \mathrm{e}^{\mathrm{i}tb}\varphi(u)\varphi(v) = \mathrm{e}^{\mathrm{i}tb}\varphi(w)$, where we have used that B^β is commutative. Thus $\varphi(w)$ has the same generalised eigenvalue as $\varphi(v)$. It follows that we may define a map $\widetilde{E} : E(A,\alpha) \to E(B,\beta)$ by $\widetilde{E}([vv^*],[v^*v],a) = ([\varphi(vv^*)],[\varphi(v^*v)],b)$, where v is a generalised eigenisometry in A with generalised eigenvalue a, and b is the generalised eigenvalue of $\varphi(v)$ in B. Note that if φ is equivariant, $\widetilde{E}(\varphi) = E(\varphi)$.

LEMMA 2.12. *Let (A,α) be a C^*-dynamical system of special form. Then $E(A,\alpha)$ is a finite set.*

Proof. It is easy to see that each non-zero generalised eigenisometry can be uniquely expressed as a sum of generalised eigenisometries between minimal projections in A^α, and that any two partial isometries with the same minimal projections in A^α for both range and support are generalised eigenisometries with the same generalised eigenvalue. Since there are only finitely many projections in A^α, the lemma follows. ∎

LEMMA 2.13. *Suppose $\{(A_n,\alpha_n),\varphi_{nm}\}$ is an inductive system of C^*-dynamical systems where each (A_n,α_n) is of special form, and let (A,α) denote the inductive limit C^*-dynamical system. Then we have $E(A,\alpha) = \bigcup_{n=1}^{\infty} E(\varphi_{n\infty})(E(A_n,\alpha_n))$.*

Proof. Let v be a generalised eigenisometry in A with generalised eigenvalue a. Then for some large enough m we have that there exist projections $p,q \in A_m^{\alpha_m}$ such that $vv^* = \varphi_{m\infty}(p)$ and $v^*v = \varphi_{m\infty}(q)$. Passing to a subsequence, we may assume that there exist such projections in $A_1^{\alpha_1}$, and we fix a pair p,q. Since A has cancelation of projections, there exists a partial isometry $x \in A$ such that $x^*x = 1 - v^*v$ and $xx^* = 1 - vv^*$. Thus $(x+v)$ is a unitary. There exist unitaries $u_n \in A_n$ such that $\varphi_{n\infty}(u_n) \to x+v$. Set $s_n = u_n\varphi_{1n}(q)$. Then s_n is a partial isometry in A_n with $\varphi_{n\infty}(s_n) \to v$. For large enough n, there exist partial isometries $r_n \in A_n$ with $r_n^*r_n = s_n^*s_n$ and $r_nr_n^* = \varphi_{1n}(p)$. Furthermore, r_n may be chosen so that $\|r_ns_n - s_n\| \to 0$ as $n \to \infty$. Setting $w_n = r_ns_n$, we have $w_nw_n^* = \varphi_{1n}(p)$, $w_n^*w_n = \varphi_{1n}(q)$, and $\varphi_{n\infty}(w_n) \to v$. It follows from Lemma 2.9 that each w_n is a generalised eigenisometry with generalised eigenvalue b_n say. Since $\{\varphi_{n\infty}(w_n)\}$ is convergent it is Cauchy, so for some M we have $\|\varphi_{k\infty}(w_k) - \varphi_{j\infty}(w_j)\| < 1/20$ for all $k > j > M$. It follows that for any b_k and b_j with $k > j > M$, there exists an $l > k$ such that $\|\varphi_{kl}(w_k) - \varphi_{jl}(w_j)\| < 1/10$. It follows from Lemma 2.9 that the generalised eigenvalues of $\varphi_{kl}(w_k)$ and $\varphi_{jl}(w_j)$ in A_l are the same, so $\varphi_{kl}(b_k) = \varphi_{jl}(b_j)$. This implies that the sequence $\{\varphi_{k\infty}(b_k)\}$ is eventually constant, equal to some element b say. Since $\varphi_{n\infty}(w_n) \to v$, we have, for all large enough n, $\alpha_t(\varphi_{n\infty}(w_n)) = \mathrm{e}^{\mathrm{i}t\varphi_{n\infty}(b_n)}\varphi_{n\infty}(w_n) = \mathrm{e}^{\mathrm{i}tb}\varphi_{n\infty}(w_n) \to \alpha_t(v) = \mathrm{e}^{\mathrm{i}ta}v$ for all $t \in \mathbb{R}$. It follows that $b = a$ and $([vv^*], [v^*v], a) = E(\varphi_{n\infty})([w_n w_n^*], [w_n^* w_n], b_n)$ for all large enough n. This shows one inclusion, and the other is obvious, so the lemma follows. ∎

DEFINITION 2.14. Suppose that (A, α) is a C^*-dynamical system with A^α commutative, and suppose that u and v are two generalised eigenisometries with $uu^* = vv^*$, $u^*u = v^*v$, and having the same generalised eigenvalue. Then u^*v is a partial unitary in A^α, so $uau^* = vav^*$ for all $a \in (v^*v)A$. Thus we get a map, which we shall denote R, from $E(A, \alpha)$ into the $*$-homomorphisms from A^α to A^α defined by $R(([vv^*], [v^*v], a))(b) = vbv^*$ for $b \in A^\alpha$, $([vv^*], [v^*v], a) \in E(A, \alpha)$.

LEMMA 2.15. *Suppose that (A, α) and (B, β) are C^*-dynamical systems with A^α and B^β commutative, and that $\varphi : A \to B$ is an equivariant $*$-homomorphism. Then $R((E(\varphi))(x))(\varphi(b)) = \varphi(R(x)(b))$ for every $x \in E(A, \alpha)$ and $b \in A^\alpha$. Furthermore, if (A, α) and (B, β) are of special form and $\varphi : A \to B$ is any $*$-homomorphism with $\varphi(A^\alpha) \subseteq B^\beta$, then $R((\widetilde{E}(\varphi))(x))(\varphi(b)) = \varphi(R(x)(b))$ for every $x \in E(A, \alpha)$ and $b \in A^\alpha$.*

Proof. Both sides are equal to $\varphi(v)\varphi(b)\varphi(v^*)$, where $x = ([vv^*], [v^*v], a)$. ∎

DEFINITION 2.16. Define a category C as follows. An object in C is a triple (A, E, R), where A is a C^*-algebra, E is a subset of $D(A) \times D(A) \times A$, and R is a map from E into the endomorphisms on A^α. A morphism in C from an object (A, E, R) to an object (B, F, V) is a $*$-homomorphism $\varphi : A \to B$ such that $(D(\varphi) \times D(\varphi) \times \varphi)(E) \subset F$ and $V((D(\varphi) \times D(\varphi) \times \varphi)(x))(\varphi(b)) = \varphi(R(x)(b))$ for $x \in E$ and $b \in A^\alpha$. Composition in C is just the usual composition of $*$-homomorphisms. For a C^*-dynamical system (A, α) with A^α commutative, write $\mathrm{Inv}(A, \alpha)$ for the triple $(A^\alpha, E(A, \alpha), R)$. If (B, β) is another C^*-dynamical system with B^β commutative, and $\psi : A \to B$ is an equivariant $*$-homomorphism, write $\mathrm{Inv}(\psi)$ for $\psi|_{A^\alpha}$. Then Inv is a functor from the category of C^*-dynamical systems with commutative fixed point subalgebras with equivariant $*$-homomorphisms to the category C.

3. CLASSIFICATION

Suppose that (A, α) is a C^*-dynamical system of special form. It follows from the theory of stable relations (cf. [14]) that the commutative C^*-algebra A^α has the following stability property: The algebra A^α can be finitely presented, $A^\alpha \cong C^*\langle x_1, \dots, x_n : R\rangle$, in terms of elements $\{x_1, \dots, x_n\}$ such that for any $\varepsilon > 0$ and finite subset $F \subseteq A$, there exists a $\delta > 0$ such that if B and C are C^*-algebras with $C \subseteq B$ and $\varphi : A^\alpha \to B$ is a $*$-homomorphism with $\{x_1, \dots, x_n\} \subseteq_\delta (C)$, then there exists a $*$-homomorphism $\psi : A^\alpha \to C$ such that $\|\varphi(f) - \psi(f)\| < \varepsilon$ for all $f \in F$.

DEFINITION 3.1. Let (A,α) and (B,β) be two C^*-dynamical systems. We shall call a sequence $\{\varphi_n\}$ of $*$-homomorphisms from A to B *approximately equivariant* if, and only if, for every fixed $x \in A$ and $t \in \mathbb{R}$ we have that $\|\varphi_n(\alpha_t(x)) - \beta_t(\varphi_n(x))\| \to 0$ as $n \to \infty$.

THEOREM 3.2. *Let $\{(A_n,\alpha_n),\varphi_{nm}\}$ and $\{(B_n,\beta_n),\psi_{nm}\}$ be two inductive systems of C^*-dynamical systems of special form with unital injective connecting maps, let (A,α) and (B,β) denote the respective inductive limit C^*-dynamical systems, and suppose $\gamma : \mathrm{Inv}(A,\alpha) :\to \mathrm{Inv}(B,\beta)$ and $\eta : \mathrm{Inv}(B,\beta) \to \mathrm{Inv}(A,\alpha)$ are a pair of inverse isomorphisms in the category C. Then there exist sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers and $*$-homomorphisms $\gamma_k : A_{n_k} \to B_{m_k}$ and $\eta_k : B_{m_k} \to A_{n_{k+1}}$ such that $\gamma_k : A_{n_k}^{\alpha_{n_k}} \to B_{m_k}^{\beta_{m_k}}$, $\eta_k : B_{m_k}^{\beta_{m_k}} \to A_{n_{k+1}}^{\alpha_{n_{k+1}}}$,*

$$\begin{tikzcd}
A_{n_1}^{\alpha_{n_1}} \arrow[r] \arrow[d, "\gamma_1"] & A_{n_2}^{\alpha_{n_2}} \arrow[r] \arrow[d, "\gamma_2"] & A_{n_3}^{\alpha_{n_3}} \arrow[r] \arrow[d, "\gamma_3"] & \cdots \arrow[r] & A^{\alpha} \arrow[d, shift left, "\gamma"] \\
B_{m_1}^{\beta_{m_1}} \arrow[r] \arrow[ur, "\eta_1"'] & B_{m_2}^{\beta_{m_2}} \arrow[r] \arrow[ur, "\eta_2"'] & B_{m_3}^{\beta_{m_3}} \arrow[r] \arrow[ur] & \cdots \arrow[r] & B^{\beta} \arrow[u, shift left, "\eta"]
\end{tikzcd}$$

is approximately commutative in the sense of [6], *the diagram*

$$\begin{tikzcd}
E(A_{n_1},\alpha_{n_1}) \arrow[r] \arrow[d, "\widetilde{E}(\gamma_1)"] & E(A_{n_2},\alpha_{n_2}) \arrow[r] \arrow[d, "\widetilde{E}(\gamma_2)"] & E(A_{n_3},\alpha_{n_3}) \arrow[r] \arrow[d, "\widetilde{E}(\gamma_3)"] & \cdots \arrow[r] & E(A,\alpha) \arrow[d, shift left, "E(\gamma)"] \\
E(B_{m_1},\beta_{m_1}) \arrow[r] \arrow[ur, "\widetilde{E}(\eta_1)"'] & E(B_{m_2},\beta_{m_2}) \arrow[r] \arrow[ur, "\widetilde{E}(\eta_2)"'] & E(B_{m_3},\beta_{m_3}) \arrow[r] \arrow[ur, "\widetilde{E}(\eta_3)"'] & \cdots \arrow[r] & E(B,\beta) \arrow[u, shift left, "E(\eta)"]
\end{tikzcd}$$

is commutative, and the sequences $\{\psi_{m_l\infty} \circ \gamma_l \circ \varphi_{n_k n_l}\}$ and $\{\varphi_{n_{l+1}\infty} \circ \eta_l \circ \psi_{m_k m_l}\}$ are approximately equivariant.

Proof. Let the inductive systems $\{(A_n,\alpha_n),\varphi_{nm}\}$ and $\{(B_n,\beta_n),\psi_{nm}\}$, the C^*-dynamical systems (A,α) and (B,β), and the isomorphisms $\gamma : \mathrm{Inv}(A,\alpha) \to \mathrm{Inv}(B,\beta)$ and $\eta : \mathrm{Inv}(B,\beta) \to \mathrm{Inv}(A,\alpha)$ be as in the statement of the theorem.

By Lemma 2.12, $E(A_1,\alpha_1)$ is a finite set. By Lemma 2.13, we have $E(B,\beta) = \bigcup_{n=1}^{\infty} E(\psi_{n\infty})\ (E(B_n,\beta_n))$, so there exists a natural number l such that $E(\gamma\circ\varphi_{1\infty})$ $E(A_1,\alpha_1) \subseteq E(\psi_{l\infty})E(B_l,\beta_l)$. Since $E(\psi_{l\infty})$ is injective, we have a map from $E(A_1,\alpha_1)$ to $E(B_l,\beta_l)$ that makes the diagram

$$\begin{tikzcd}
E(A_1,\alpha_1) \arrow[r] \arrow[dr] & E(A_2,\alpha_2) \arrow[r] & \cdots \arrow[r] & E(A,\alpha) \arrow[d, shift left, "E(\gamma)"] \\
\cdots \arrow[r] & E(B_l,\beta_l) \arrow[r] & \cdots \arrow[r] & E(B,\beta) \arrow[u, shift left, "E(\eta)"]
\end{tikzcd}$$

commute. Repeating the process with $E(B_l,\beta_l)$, we get an $m > 1$ such that $E(\eta \circ \psi_{l\infty})E(B_l,\beta_l) \subseteq E(\varphi_{m\infty})E(A_m,\alpha_m)$ and a map from $E(B_l,\beta_l)$ to $E(A_m,\alpha_m)$. It is easy to see that the composition of the map from $E(B_l,\beta_l)$ to $E(A_m,\alpha_m)$ with that from $E(A_1,\alpha_1)$ to $E(B_l\beta_l)$ is equal to $E(\varphi_{1m})$. Repeating the above, passing to subsequences, and renumbering, we get a commutative diagram:

$$\begin{tikzcd}
E(A_1,\alpha_1) \arrow[r] \arrow[d] & E(A_2,\alpha_2) \arrow[r] \arrow[d] & E(A_3,\alpha_3) \arrow[r] \arrow[d] & \cdots \arrow[r] & E(A,\alpha) \arrow[d, shift left, "E(\gamma)"] \\
E(B_1,\beta_1) \arrow[r] \arrow[ur] & E(B_2,\beta_2) \arrow[r] \arrow[ur] & E(B_3,\beta_3) \arrow[r] \arrow[ur] & \cdots \arrow[r] & E(B,\beta) \arrow[u, shift left, "E(\eta)"]
\end{tikzcd} \tag{3.1}$$

Since each (A_n,α_n) is of special form, we may choose, for the l-th minimal direct summand of A_n, a system of matrix units $\{v_{ij}^{n,l}\}_{ij}$ consisting of generalised eigenisometries. We fix such a system for each A_n, and another, $\{w_{ij}^{m,k}\}_{ij}$, for the k-th minimal direct summand of B_m, for each B_m. Let $a_{ij}^{n,l}$ denote the generalised eigenvalue for $v_{ij}^{n,l}$, and let $b_{ij}^{m,k}$ denote that of $w_{ij}^{m,k}$. Consider $(v_{11}^{n,l})A_n^{\alpha_n}(v_{11}^{n,l}) \cong C(X_l)$, for some graph X_l. Let $F_{n,l}$ be a finite set of generators for $(v_{11}^{n,l})A_n^{\alpha_n}(v_{11}^{n,l})$. Then $(v_{j1}^{n,l})F_{n,l}(v_{1j}^{n,l})$ is a finite set of generators for $(v_{jj}^{n,l})A_n^{\alpha_n}(v_{jj}^{n,l})$. Let $F_n = \Big(\bigcup_{l,j}(v_{j1}^{n,l})F_{n,l}(v_{1j}^{n,l})\Big) \cup \Big(\bigcup_{l,i,j} a_{ij}^{n,l}\Big)$. Then F_n is a finite set of generators for A_n^{α} that contains the generalised eigenvalue for every generalised eigenisometry with a minimal support projection. Finally, let $\widetilde{F}_n = \bigcup_{m\leqslant n} \varphi_{mn}(F_m)$, and let $H_{m,k}$, H_m and $\widetilde{H}_m$ be defined similarly for the other inductive system.

Let $\{\varepsilon_n\}$ be summable sequence of positive real numbers such that the sequence $\{\delta_n\}$ defined by $\delta_n = \sum_{k=n}^{\infty} \varepsilon_k$ is also summable. It follows from the semiprojectivity of the $A_n^{\alpha_n}$s that for some natural number l, there exists a $*$-homomorphism $\pi : A_1^{\alpha_1} \to B_l^{\beta_l}$ such that the diagram

$$\begin{tikzcd}
 & A_1^{\alpha_1} \arrow[r] \arrow[dr, "\pi"] & A_2^{\alpha_2} \arrow[r] & \cdots \arrow[r] & A^{\alpha} \arrow[d, shift left, "\gamma"] \\
 & \cdots \arrow[r] & B_l^{\beta_l} \arrow[r] & \cdots \arrow[r] & B^{\beta} \arrow[u, shift left, "\eta"]
\end{tikzcd}$$

commutes to within ε_1 on $\widetilde{F}_1$. Similarly, there exists an $m > 1$ and a $*$-homomorphism $\zeta : B_l^{\beta_l} \to A_m^{\alpha_m}$ such that the analogous diagram commutes to within ε_1 on $\widetilde{H}_l \cup \pi_1(\widetilde{F}_1)$. It follows that the triangle

$$\begin{tikzcd}
A_1^{\alpha_1} \arrow[rr] \arrow[dr, "\pi"'] & & A_m^{\alpha_m} \\
 & B_l^{\beta_l} \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

commutes to within $2\varepsilon_1$ on $\widetilde{F}_1$. Proceeding in this fashion, and again passing to subsequences and renumbering, we get a diagram

$$
\begin{tikzcd}
A_1^{\alpha_1} \arrow[r] \arrow[d, "\pi_1"] & A_2^{\alpha_2} \arrow[r] \arrow[d, "\pi_2"] & A_3^{\alpha_3} \arrow[r] \arrow[d, "\pi_3"] & \cdots \arrow[r] & A^{\alpha} \arrow[d, shift left, "\gamma"] \\
B_1^{\beta_1} \arrow[r] \arrow[ur, "\zeta_1"'] & B_2^{\beta_2} \arrow[r] \arrow[ur, "\zeta_2"'] & B_3^{\beta_3} \arrow[r] \arrow[ur] & \cdots \arrow[r] & B^{\beta} \arrow[u, shift left, "\eta"]
\end{tikzcd}
\tag{3.2}
$$

such that for each n, the triangle with $A_n^{\alpha_n}$, $A_{n+1}^{\alpha_{n+1}}$, and $B_n^{\beta_n}$ commutes within $2\varepsilon_n$ on the union of $\widetilde{F}_n$ and all of the images of $\widetilde{F}_k$, $\widetilde{H}_k$ for $k < n$ under all paths in the diagram ending at $A_n^{\alpha_n}$, the triangle with $B_n^{\beta_n}$, $A_{n+1}^{\alpha_{n+1}}$, and $B_{n+1}^{\beta_{n+1}}$ commutes to within $2\varepsilon_n$ on the union of $\widetilde{H}_n$ and all of the images of $\widetilde{F}_k$, $\widetilde{H}_k$ for $k < n$ under all paths in the diagram ending at $B_n^{\beta_n}$, and the whole diagram approximately commutes in the sense of [6].

Now we define our maps γ_n and η_n. Consider $A_n \cong M_{k_1}(C(X_1)) \oplus \cdots \oplus M_k(C(X_q))$ with the systems of matrix units $\{v_{ij}^{n,l}\}_{ij}$ already chosen. We define γ_n to be equal to the restriction of π_n to $(v_{11}^{n,1} A_n^{\alpha_n} v_{11}^{n,1}) \oplus \cdots \oplus (v_{11}^{n,q} A_n^{\alpha_n} v_{11}^{n,q})$. Next, we consider the partial isometries $\{v_{ij}^{n,l}\}_{ij}$, $l = 1, \ldots, q$, $j = 1, \ldots, k_l$ for each l. From diagram (3.1) we have that there exist generalised eigenisometries $\{u_{ij}^{n,l}\}_{ij}$ in B_n such that $(u_{ij}^{n,l})^*(u_{ij}^{n,l}) = \gamma((v_{ij}^{n,l})^*(v_{ij}^{n,l})) = \gamma(v_{jj}^{n,l})$, $(u_{ij}^{n,l})(u_{ij}^{n,l})^* = \gamma((v_{ij}^{n,l})(v_{ij}^{n,l})^*) = \gamma(v_{ii}^{n,l})$, and $u_{ij}^{n,l}$ has generalised eigenvalue $\gamma(a_{ij}^{n,l})$. We may choose the $u_{ij}^{n,l}$s to be finite sums of the $w_{ij}^{n,l}$s. We define $\gamma_n(v_{ij}^{n,l})$ to be $u_{ij}^{n,l}$. This assignment then extends to a map from A_n to B_n which we call γ_n. The map η_n is defined similarly.

Since any element of $A_n^{\alpha_n}$ can be written as a finite sum of elements of the form $v_{j1}^{n,l} f v_{1j}^{n,l}$, where $f \in v_{11}^{n,l} A_n v_{11}^{n,l}$, it follows that $\gamma_n(A_n^{\alpha_n}) \subseteq B_n^{\beta_n}$, and similarly for the η_ns. If (A, α) and (B, β) are two C^*-dynamical systems of special form and $\gamma : A \to B$ is a $*$-homomorphism that satisfies $\gamma(A^\alpha) \subseteq B^\beta$, it follows from Lemma 2.15 that $R(\widetilde{E}(\gamma)(x))(\gamma(b)) = \gamma(R(x)(b))$ for every $x \in E(A, \alpha)$ and $b \in A^\alpha$. It is easy to see that the $\widetilde{E}(\gamma_n)$s and $\widetilde{E}(\eta_n)$s are exactly the maps in our diagram (3.1).

Next we show that the restrictions of the γ_ns and η_ns to the fixed point subalgebras make the diagram

$$
\begin{tikzcd}
A_1^{\alpha_1} \arrow[r] \arrow[d, "\gamma_1"] & A_2^{\alpha_2} \arrow[r] \arrow[d, "\gamma_2"] & A_3^{\alpha_3} \arrow[r] \arrow[d, "\gamma_3"] & \cdots \arrow[r] & A^{\alpha} \arrow[d, shift left, "\gamma"] \\
B_1^{\beta_1} \arrow[r] \arrow[ur, "\eta_1"'] & B_2^{\beta_2} \arrow[r] \arrow[ur, "\eta_2"'] & B_3^{\beta_3} \arrow[r] \arrow[ur] & \cdots \arrow[r] & B^{\beta} \arrow[u, shift left, "\eta"]
\end{tikzcd}
$$

approximately commutative in the sense of [6]. We do this by showing that the maps γ_n differ from the corresponding π_ns by at most $4\delta_n$ on the generators, and similarly for the η_ns and ζ_ns, at which point it will follow from the approximate commutativity of diagram (3.2).

Consider $A_n^{\alpha_n}$. We have that $\pi_n = \gamma_n$ on $(v_{11}^{n,l} A_n^{\alpha_n} v_{11}^{n,l})$. Let $f \in F_{n,l} \subseteq (v_{11}^{n,l} A_n^{\alpha_n} v_{11}^{n,l})$ and consider $g = (v_{j1}^{n,l}) f (v_{1j}^{n,l}) \in (v_{jj}^{n,l} A_n^{\alpha_n} v_{jj}^{n,l})$. Let x denote the element $([v_{j1}^{n,l} v_{1j}^{n,l}], [v_{1j}^{n,l} v_{j1}^{n,l}], a_{j1}^{n,l}) \in E(A_n, \alpha_n)$. We then have $g = R(x)(f)$.

Now $\|\pi_n(g) - (\gamma \circ \varphi_{n\infty})(g)\|_B \leqslant \sum_{k=n}^{\infty} 2\varepsilon_k = 2\delta_n$ from the approximate commutativity of (3.2), and

$$(\gamma \circ \varphi_{n\infty})(g) = (\gamma \circ \varphi_{n\infty})(R(x)(f)) = R(E(\gamma \circ \varphi_{n\infty})(x))((\gamma \circ \varphi_{n\infty})(f)).$$

By definition, $\gamma_n(g) = R(E(\gamma \circ \varphi_{n\infty})(x))(\gamma_n(f)) = R(E(\gamma \circ \varphi_{n\infty})(x))(\pi_n(f))$. As $\|\pi_n(f) - (\gamma \circ \varphi_{n\infty})(f)\|_B \leqslant \sum_{k=n}^{\infty} 2\varepsilon_k = 2\delta_n$ from the approximate commutativity of (3.2), we have $\|\pi_n(g) - \gamma_n(g)\|_B \leqslant 4\delta_n$. The calculation for the ηs and ζs is similar.

Finally, we show that the sequences $\{\psi_{l\infty} \circ \gamma_l \circ \varphi_{kl}\}$ and $\{\varphi_{l+1\infty} \circ \eta_l \circ \psi_{kl}\}$ are approximately equivariant. Consider A_n and the element $v_{ij}^{n,l}$ with generalised eigenvalue $a_{ij}^{n,l}$. We have that $\gamma_n(v_{ij}^{n,l})$ is a generalised eigenisometry in $B_n \subseteq B$ with generalised eigenvalue $b_{ij}^{n,l} = \gamma \circ \varphi_{n\infty}(a_{ij}^{n,l})$. From above, we have

$$\|\gamma_n(a_{ij}^{n,l}) - b_{ij}^{n,l}\|_B \leqslant \|\gamma_n(a_{ij}^{n,l}) - \pi_n(a_{ij}^{n,l})\|_B + \|\pi_n(a_{ij}^{n,l}) - b_{ij}^{n,l}\|_B \leqslant 4\delta_n.$$

Since

$$\|\gamma_n(\alpha_{n,t}(v_{ij}^{n,l})) - \beta_{n,t}(\gamma_n(v_{ij}^{n,l}))\|_B = \|(\exp(t\gamma_n(a_{ij}^{n,l})) - \exp(tb_{ij}^{n,l}))\gamma_n(v_{ij}^{n,l})\|_B,$$

the result follows easily. ∎

LEMMA 3.3. *Let (A, α) and (B, β) be two C^*-dynamical systems of special form and let $\varphi : A \to B$ be a unital, equivariant, $*$-homomorphism from A to B. Let $\{v_{ij}^l\}_{ij}$ be a system of matrix units for the l-th minimal direct summand of A consisting of generalised eigenisometries. Let $\psi : A \to B$ be a $*$-homomorphism such that $\psi(A^\alpha) \subseteq B^\beta$ and $\widetilde{E}(\psi) = E(\varphi)$. Then there exists a unitary $u \in B^\beta$ such that $\mathrm{Ad}\, u \circ \psi(v_{ij}^l) = \varphi(v_{ij}^l)$ for each l, i, j.*

Proof. The unitary $u = \sum_l \sum_k \varphi(v_{k1}^l)\psi(v_{1k}^l)$ is easily seen to do the job. ∎

THEOREM 3.4. *Let $\{(A_n, \alpha_n), \varphi_{nm}\}$ and $\{(B_n, \beta_n), \psi_{nm}\}$ be two inductive systems of C^*-dynamical systems of special form with unital injective connecting maps and let (A, α) and (B, β) denote the respective inductive limit C^*-dynamical systems, and suppose $\gamma : \mathrm{Inv}(A, \alpha) :\to \mathrm{Inv}(B, \beta)$ and $\eta : \mathrm{Inv}(B, \beta) \to \mathrm{Inv}(A, \alpha)$ are a pair of* *inverse isomorphisms in the category* $\mathcal{C}$. *Then there exists a pair of inverse equivariant* $*$*-isomorphisms* $\widetilde{\eta} : B \to A$ *and* $\widetilde{\gamma} : A \to B$ *such that* $\eta = \mathrm{Inv}(\widetilde{\eta})$ *and* $\gamma = \mathrm{Inv}(\widetilde{\gamma})$.

Proof. Consider the $*$-homomorphisms γ_k and η_k provided by Theorem 3.2. We have a diagram:

$$
\begin{tikzcd}
A_{n_1} \arrow[r] \arrow[d, "\gamma_1"] & A_{n_2} \arrow[r] \arrow[d, "\gamma_2"] & A_{n_3} \arrow[r] \arrow[d, "\gamma_3"] & \cdots \arrow[r] & A \\
B_{m_1} \arrow[r] \arrow[ur, "\eta_1"'] & B_{m_2} \arrow[r] \arrow[ur, "\eta_2"'] & B_{m_3} \arrow[r] \arrow[ur] & \cdots \arrow[r] & B
\end{tikzcd}
\tag{3.3}
$$

If (A, α) and (B, β) are two C^*-dynamical systems of special form, and $\psi : A \to B$ is a $*$-homomorphism with $\psi(A^\alpha) \subseteq B^\beta$, then composing with $\mathrm{Ad}\, u$ for a unitary $u \in B^\beta$ does not change either $\widetilde{E}(\psi)$ or the restriction of ψ to A^α, so applying Lemma 3.3 to alter each of the vertical maps in turn, from left to right through the diagram, we may assume that (3.3) is approximately commutative in the sense of [6]. By [6], there exists a pair of inverse $*$-isomorphisms $\widetilde{\eta} : B \to A$ and $\widetilde{\gamma} : A \to B$ that make the diagram

$$
\begin{tikzcd}
A_{n_1} \arrow[r] \arrow[d, "\gamma_1"] & A_{n_2} \arrow[r] \arrow[d, "\gamma_2"] & A_{n_3} \arrow[r] \arrow[d, "\gamma_3"] & \cdots \arrow[r] & A \arrow[d, shift left, "\widetilde{\gamma}"] \\
B_{m_1} \arrow[r] \arrow[ur, "\eta_1"'] & B_{m_2} \arrow[r] \arrow[ur, "\eta_2"'] & B_{m_3} \arrow[r] \arrow[ur] & \cdots \arrow[r] & B \arrow[u, shift left, "\widetilde{\eta}"]
\end{tikzcd}
$$

approximately commutative. Furthermore, we have $\eta = \widetilde{\eta}|_{B^\beta}$ and $\gamma = \widetilde{\gamma}|_{A^\alpha}$. Since composing with $\mathrm{Ad}\, u$ for a unitary $u \in B^\beta$ does not affect approximate equivariance, we see that $\widetilde{\eta}$ and $\widetilde{\gamma}$ are equivariant and fulfil our requirements. ∎

4. EXAMPLES AND CLOSING REMARKS

We begin this section with a theorem indicating how, given a C^*-dynamical system (A, α) in our classified class, one can determine the Elliott invariant of the algebra A. Recall that this consists of the K_0 group, its positive cone K_0^+, along with the class of the unit in K_0, or, equivalently, the dimension range, $D(A)$; the K_1 group; the tracial state space $T(A)$, or, equivalently, the order unit Banach space of continuous affine functions on the tracial state space, $\mathrm{Aff}\, T(A)$, with the distinguished element determined by the unit of the algebra; and the natural map $\Gamma : K_0(A) \to \mathrm{Aff}\, T(A)$.

THEOREM 4.1. *Let* $\{(A_n, \alpha_n), \varphi_{nm}\}$ *be an inductive system where each* (A_n, α_n) *is of special form and the connecting maps are unital and injective. Let* (A, α) *denote the limit* C^**-dynamical system. Then the Elliott invariant of* A *is computed from* $\mathrm{Inv}(A, \alpha)$ *as follows:*

(K_0) *The map from* $D(A^\alpha)$ *to* $D(A)$ *induced by the inclusion of* A^α *into* A

is surjective. The projection of $E(A,\alpha)$ onto its first two coordinates is an equivalence relation on $D(A^\alpha)$. Two elements in $D(A^\alpha)$ are sent to the same element of $D(A)$ if, and only if, they are equivalent under this relation.

(K_1) *Each element γ of $E(A,\alpha)$ defines a $*$-homomorphism $R(\gamma) : A^\alpha \to A^\alpha$, and hence a group homomorphism $K_1(R(\gamma)) : K_1(A^\alpha) \to K_1(A^\alpha)$. Let N denote the subgroup of $K_1(A^\alpha)$ generated by $\{x - K_1(R(\gamma))(x) : x \in K_1(A^\alpha), \gamma \in E(A,\alpha)\}$. The homomorphism $i_* : K_1(A^\alpha) \to K_1(A)$ induced by the inclusion of A^α into A is surjective, and has kernel N, so $K_1(A) \cong K_1(A^\alpha)/N$.*

(Aff $T(A)$) *As above, for each γ in $E(A,\alpha)$ we have a $*$-homomorphism $R(\gamma) : A^\alpha \to A^\alpha$. Let V denote the closed linear subspace of $\operatorname{Aff} T(A)$ generated by $\{w - \operatorname{Aff} T(R(\gamma))(w) : w \in \operatorname{Aff} T(A^\alpha), \gamma \in E(A,\alpha)\}$. The map $i_* : \operatorname{Aff} T(A^\alpha) \to \operatorname{Aff} T(A)$ induced by the inclusion of A^α into A is surjective and its kernel is V. Furthermore, the map it induces from $\operatorname{Aff} T(A^\alpha)/V$ to $\operatorname{Aff} T(A)$ is positive and isometric, so $\operatorname{Aff} T(A) \cong \operatorname{Aff} T(A^\alpha)/V$ as order unit Banach spaces.*

(Pairing) *The pairing is determined by demanding that the diagram*

$$\begin{array}{ccc} D(A^\alpha) & \longrightarrow & \operatorname{Aff} T(A^\alpha) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ D(A^\alpha)/\mathfrak{E} & \longrightarrow & \operatorname{Aff} T(A^\alpha)/V, \end{array}$$

where $\mathfrak{E}$ denotes the equivalence relation coming from $E(A,\alpha)$ as above, commutes.

Proof. We begin with (K_0). If (A,α) is of special form, all three assertions are obvious. In general, we have $A^\alpha = \varinjlim A_n^{\alpha_n}$ by Lemma 2.7, so $D(A^\alpha) = \varinjlim D(A_n^{\alpha_n})$ and $D(A) = \varinjlim D(A_n)$. Surjectivity of the map from $D(A^\alpha)$ to $D(A)$ induced by inclusion now follows from the surjectivity of the maps from $D(A^{\alpha_n})$ to $D(A_n)$ for all n and the commutativity of the diagrams

$$\begin{array}{ccc} D(A_n^{\alpha_n}) & \longrightarrow & D(A^\alpha) \\ \downarrow & & \downarrow \\ D(A_n) & \longrightarrow & D(A) \end{array}$$

induced by the inclusions. That $\pi_1 \times \pi_2(E(A,\alpha))$ is an equivalence relation whenever A^α is commutative is easily checked (commutativity of A^α is required when showing that the relation is transitive). Two elements x, y of $D(A^\alpha)$ are mapped to the same element of $D(A)$ if, and only if, there exist a number n and elements $w, z \in D(A_n^{\alpha_n})$ such that $x = D(\varphi_{n\infty})(w)$, $y = D(\varphi_{n\infty})(z)$, and $D(i_n)(w) = D(i_n)(z)$, where $i_n : A_n^{\alpha_n} \to A_n$ is the inclusion. By Lemma 2.9, in this case we have $(w,z) \in \pi_1 \times \pi_2(E(A_n,\alpha_n))$, and it follows that $(x,y) \in \pi_1 \times \pi_2(E(A,\alpha))$. That $(x,y) \in \pi_1 \times \pi_2(E(A,\alpha))$ implies that x and y are mapped to the same element of $D(A)$ is obvious, so this finishes the case for (K_0).

Next, we consider (K_1). Again, if (A,α) is of special form, the assertions in the theorem are obvious. In general, the surjectivity of the map from $K_1(A^\alpha)$ to $K_1(A)$ induced by the inclusion follows as for the dimension range. Applying K_1 to the commutative diagram

$$\begin{tikzcd}
A_1^{\alpha_1} \arrow[r] \arrow[d] & A_2^{\alpha_2} \arrow[r] \arrow[d] & \cdots \arrow[r] & A^\alpha \arrow[d] \\
A_1 \arrow[r] & A_2 \arrow[r] & \cdots \arrow[r] & A,
\end{tikzcd}$$

we see that the kernel of the map from $K_1(A^\alpha)$ to $K_1(A)$ is generated by the images of the N_ns in $K_1(A^\alpha)$. These are all contained in N, and that their union is equal to N follows from Lemma 2.13. The assertion for (K_1) follows.

Consider now $\operatorname{Aff} T(A)$. For (A,α) of special form the statements are easy to see. As in the case of K_1, the images of the V_ns are all contained in V. Let $w \in \operatorname{Aff} T(A^\alpha)$ be such that $i_*(w) = 0$. We may choose a sequence w_n such that $w_n \in \operatorname{Aff} T(A_n^{\alpha_n})$ and $\varphi_{nm*}(w_n) \to w$. Since $\operatorname{Aff} T(A_n^{\alpha_n}) \cong (A_n^{\alpha_n})_{\mathrm{sa}}$ isometrically, and the maps between the affine function spaces correspond to the restrictions of $*$-homomorphisms under the natural isomorphisms, the maps between the $\operatorname{Aff} T(A_n^{\alpha_n})$s are all isometric. For all n,m with $m > n$ the diagram

$$\begin{tikzcd}
\operatorname{Aff} T(A_n^{\alpha_n}) \arrow[r] \arrow[d, "i_*"] & \operatorname{Aff} T(A_m^{\alpha_m}) \arrow[d, "i_*"] \\
\operatorname{Aff} T(A_n) \arrow[r] & \operatorname{Aff} T(A_m)
\end{tikzcd}$$

commutes, so $\|\varphi_{nm*}(i_*(w_n))\| = d(\varphi_{nm*}(w_n), V_m)$. It follows that we have $d\Big(\varphi_{n\infty}(w_n), \bigcup_{m=1}^{\infty} \varphi_{m\infty}(V_m)\Big) \to 0$ as $n \to \infty$, and $w \in \overline{\bigcup_{m=1}^{\infty} \varphi_{m\infty*}(V_m)}$. We have that the maps $i_n^* : T(A_n) \to T(A_n^{\alpha_n})$ are all injective, from which it follows that $i^* : T(A) \to T(A^\alpha)$ is injective. It follows from the Hahn–Banach theorem that continuous affine functions on $i^*(T(A))$ extend to continuous affine functions on all of $T(A^\alpha)$, so the map $i_* : \operatorname{Aff} T(A^\alpha) \to \operatorname{Aff} T(A)$ is surjective. Next, we show that $\overline{\bigcup_{m=1}^{\infty} \varphi_{m\infty*}(V_m)} = V$. Suppose that $v \in \operatorname{Aff} T(A^\alpha)$ and $\gamma \in E(A,\alpha)$. Then there exists a number n and a $\gamma_n \in E(A_n,\alpha_n)$ such that $\gamma = E(\varphi_{n\infty})(\gamma_n)$ and a sequence v_m, $m > n$, such that $v_m \in \operatorname{Aff} T(A_m^{\alpha_m})$ and $\varphi_{m\infty*}(v_m) \to v$. We then have $\varphi_{m\infty*}(v_m) - \varphi_{m\infty*}(R(E(\varphi_{nm})(\gamma_n))(v_m)) \to v - \operatorname{Aff} T(R(\gamma))(v)$ as $m \to \infty$, so V is the closure of the images of the V_ns. From above we see that $\|w + V\| = \|i_*(w)\|_{\operatorname{Aff} T(A)}$ for any $w \in \operatorname{Aff} T(A^\alpha)$. That $\operatorname{Aff} T(A) \cong \operatorname{Aff} T(A^\alpha)/V$ now follows.

The final assertion about the pairing now follows easily from above, so the proof is complete. ∎

Our next theorem gives a characterisation in terms of the invariant of those C^*-dynamical systems in our classified class that are AF flows.

THEOREM 4.2. *Let $\{(A_n, \alpha_n), \varphi_{nm}\}$ be an inductive system where each (A_n, α_n) is of special form and the connecting maps are unital and injective. Let (A, α) denote the limit C^*-dynamical system. Then (A, α) is an* AF *flow if, and only if, the fixed point subalgebra, A^α, is* AF *and all of the generalised eigenvalues have finite spectrum.*

Proof. Assume (A, α) is an inductive limit as in the statement of the theorem and that it is also an AF flow. It is easy to see that the fixed point subalgebra, A^α, is an AF algebra. Suppose that v is a generalised eigenisometry with generalised eigenvalue a. Since (A, α) is an AF flow, there exists an invariant finite dimensional subalgebra B of A such that v is approximately contained in B. Assuming the approximate containment is close enough, there exists a partial isometry $w \in B$ such that $ww^* \in A^\alpha$, $w^*w \in A^\alpha$, and w is norm close to v. By Lemma 2.9, w is a generalised eigenisometry with the same generalised eigenvalue as v. Since B is finite dimensional, we may write $w = w_1 + \cdots + w_k$ where the w_js are mutually orthogonal eigenisometries, with eigenvalues $a_1, \ldots, a_k$ say. It is then easy to see that $a = a_1(w_1 w_1^*) + \cdots + a_k(w_k w_k^*)$. This proves the necessity of the conditions.

Now suppose that (A, α) is a limit C^*-dynamical system as in the statement of the theorem, that A^α is an AF-algebra, and that all the generalised eigenvalues have finite spectrum. We shall construct an increasing sequence $\{m_k\}$ of integers and a sequence $\{B_k\}$ of C^*-algebras such that B_k is an invariant finite dimensional subalgebra of A_{m_k}, $\varphi_{m_k m_{k+1}}(B_k) \subseteq B_{k+1}$ for each k, and $\overline{\bigcup_k \varphi_{m_k \infty}(B_k)} = A$. Since the connecting maps are injective, we shall view them as inclusions.

First, we construct $B_1 \subseteq A_1$. We may choose, for the l-th minimal direct summand of A_1, a system $\{v_{ij}^{1,l}\}$ of matrix units consisting of generalised eigenisometries. Since the range projection of each $v_{ij}^{1,l}$ is minimal, and the generalised eigenvalue of $v_{ij}^{1,l}$ has finite spectrum, we see that the $v_{ij}^{1,l}$s are eigenisometries in the usual sense. Set B_1 equal to the sub-C^*-algebra of A_1 generated by the $v_{ij}^{1,l}$s for each l. Clearly B_1 is invariant.

Let F_1 be a finite subset of A. We may choose an n, a finite subset $G_n \subseteq A_n^{\alpha_n}$ and a system of matrix units $\{v_{ij}^{n,l}\}_l$ for each minimal direct summand of A_n consisting of generalised eigenisometies such that B_1 is contained in the sub-C^*-algebra of A_n generated by the $v_{ij}^{n,l}$s and F_1 is approximately contained in the sub-C^*-algebra generated by G_n and the $v_{ij}^{n,l}$s. Since A^α is AF, there exists a finite dimensional subalgebra, W_n of A^α such that G_n is approximately contained in W_n. Since W_n is finite dimensional, it is generated by its minimal projections, there are finitely many of these, and they lift, so for some $s > n$ we have $W_n \subseteq A_s^{\alpha_s}$. We may choose a system of matrix units $\{v_{ij}^{s,l}\}$ for each minimal direct summand of

A_s consisting of generalised eigenisometries in such a way that the $v_{ij}^{n,l}$s are contained in the finite dimensional subalgebra of A_s generated by the $v_{ij}^{s,l}$s. As in the case of B_1, the $v_{ij}^{s,l}$s are all eigenisometries in the usual sense. Set $m_2 = s$ and set B_2 equal to the sub-C^*-algebra of A_s generated by the $v_{ij}^{s,l}$s. Since the elements of $A_s^{\alpha_s}$ having finite spectrum are all in B_2, it follows that $W_n \subseteq B_2$. It follows that $B_1 \subseteq B_2$ and that F_1 is approximately contained in B_2. Proceeding in this fashion, it is easy to see that with appropriate choices of subsets F_n and tolerances for approximate containments we get sequences $\{m_k\}$ and $\{B_k\}$ meeting our requirements. Sufficiency of the conditions follows. ∎

Next, we describe a one parameter family of pairwise non-isomorphic actions on the CAR-algebra all having the same fixed point subalgebra, and, by the theorem above, exactly one of which is an AF flow.

EXAMPLE 4.3. Define an inductive system $\{(A_n, \alpha_n^r), \varphi_{nm}\}$ of C^*-dynamical systems as follows. For $n \geqslant 0$, let $A_n = C([0,1], M_{2^n})$, and $[\varphi_{n,n+1}(f)](t) = \mathrm{diag}(f(t/2), f((t+1)/2))$. For each $r \geqslant 1$, define $h_0^r \in A_0$ by $h_0^r(t) = t^r$ and $h_n^r \in A_n$ by $h_n^r = \varphi_{0n}(h_0^r)$. Then it is easy to check that h_n^r has distinct eigenvalues in every fibre of A_n, so setting α_n^r to be the action on A_n generated by h_n^r we have that (A_n, α_n^r) is of special form and the connecting maps are equivariant. Let (A, α^r) denote the limit C^*-dynamical system. Then α^r is an inner action generated by a self adjoint element with spectrum $[0,1]$. The φ_{nm} are approximately constant in the sense of [6], and it follows from the classification theorem in [6] that $A \cong M_{2^\infty}$. Considering the maps between the fixed point subalgebras, we see that these are also approximately constant, so from [6] A^{α^r} is an AF-algebra (its K_1 group is trivial). If $r = 1$, then all of the generalised eigenvalues have finite spectrum, so by Theorem 4.2, (A, α^1) is an AF flow. If $r > 1$, then there are generalised eigenvalues with infinite spectrum, so (A, α^r) is not an AF flow. That $(A, \alpha^s) \not\cong (A, \alpha^r)$ for $r, s > 1$, $r \neq s$ is not obvious, but may be seen as follows. We show that the projection e_{11} in $A_1 \subseteq A$ has a characterisation in terms of the invariant that is independent of the inductive system, and also of the particular value of r. This distinguished projection, along with the invariant, will then allow us to recover the value of r. The property that characterises e_{11} is the following: $e_{11} \in A^{\alpha^r}$, $2[e_{11}] = [1]$ in $K_0(A)$, and there exists a number $v \in (0,1)$ with the property that if $0 < \gamma < v$, and $\varepsilon > 0$ is given, then there exist projections $e, f, q \in A^{\alpha^r}$ with $e, f \leqslant e_{11}$, $q \perp e_{11}$ and self adjoint elements $a, b \in A^{\alpha^r}$ such that $([e],[q],b) \in E(A,\alpha^r)$, $\mathrm{sp}(b) \subseteq (1-\varepsilon, 1]$, $([e],[f],a) \in E(A,\alpha^r)$ and $\mathrm{sp}(a) \subseteq (\gamma - \varepsilon, \gamma + \varepsilon)$; and if $\nu > v$, then there exists a $\delta > 0$ such that there do not exist projections $x, y \in A^{\alpha^r}$ and a self adjoint element $c \in A^{\alpha^r}$ with $x, y \leqslant e_{11}$, $([x],[y],c) \in E(A,\alpha^r)$, and $\mathrm{sp}(c) \subseteq (\nu - \delta, \nu + \delta)$. To see this, consider $A_m^{\alpha_m^r}$ for some $m > 1$. The spectrum of $A_m^{\alpha_m^r}$ consists of 2^m disjoint compact intervals. Identify these with their images under the canonical map from $\mathrm{sp}(A_m^{\alpha_m^r})$

to $\mathrm{sp}(A_0) = [0,1]$ induced by φ_{0m}. They then correspond to the intervals of the partition $\{k/2^m : k = 0,\ldots,2^m\}$ of $[0,1]$. For each pair of minimal projections in $A_m^{\alpha_m^r}$, any partial isometry from the second to the first is a generalised eigenisometry, and its generalised eigenvalue may be described as follows. Suppose the two projections correspond to the intervals $[j/2^m,(j+1)/2^m]$ and $[k/2^m,(k+1)/2^m]$ respectively under the identification above. Then the generalised eigenvalue, as a function on $[j/2^m,(j+1)/2^m]$ is $a(t) = t^r - (t - j/2^m + k/2^m)^r$. Now if P is a projection having the property above, then $2[P] = [1]$ in $K_0(A)$ implies that the collection of subintervals corresponding to P contains half the total number. The next condition implies that the image of the subset corresponding to P contains the interval $[0, v^{1/r})$, and the last part implies that it is disjoint from $(v^{1/r}, 1]$ (note that going further out in the inductive systems results in refining the partition). It is now easy to see that e_{11}, which corresponds to $[1,1/2]$, is the unique projection with these properties, and that $v = (1/2)^r$ is the unique number that meets the conditions.

Our next example shows that we may not drop the third part of our invariant.

EXAMPLE 4.4. For $n \geqslant 0$, let $A_n = C(\mathbb{T}, M_{2^n}) \cong M_{2^n} \otimes C(\mathbb{T})$. Let $\varphi_{0,1} : A_0 \to A_1$ be defined by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}$ and let $\psi_{0,1} : A_0 \to A_1$ be defined by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$, where z is the canonical generator of $C(\mathbb{T})$. For $n \geqslant 1$, define $\varphi_{n,n+1}, \psi_{n,n+1} : A_n \to A_{n+1}$ by $\varphi_{n,n+1} = \mathrm{id} \otimes \varphi_{0,1}$ and $\psi_{n,n+1} = \mathrm{id} \otimes \psi_{0,1}$. Let h_1 be the constant matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1/2 \end{pmatrix}$, in A_1, and for $n \geqslant 1$, let $h_{n+1} = 1 \otimes h_n + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes (1/2^n) \in A_{n+1} \cong M_2 \otimes A_n$. Then for all n, h_n is a constant matrix with distinct eigenvalues. Let α_n be the real action on A_n defined by h_n. Then (A_n, α_n) is of special form, and the maps ψ_{nm} and φ_{nm} are all equivariant. Consider the inductive systems $\{(A_n,\alpha_n),\varphi_{nm}\}$ and $\{(A_n,\alpha_n),\psi_{nm}\}$, with limits (A,α) and (B,β) respectively. These systems fall within our classified class. It is easy to see that $A_n^{\alpha_n} \cong \mathbb{C}^{2^n} \otimes C(\mathbb{T})$. In the case of $\varphi_{n,n+1}$, each partial map from a minimal direct summand of $A_n^{\alpha_n}$ to a minimal direct summand of $A_{n,n+1}^{\alpha_{n+1}}$ is just the identity map, whereas for $\psi_{n,n+1}$ some are the identity while others are the canonical orientation reversing map on the circle, where we have chosen an orientation on each minimal direct summand of $A_n^{\alpha_n}$ that is consistent with a choice on the whole algebra A_n. Thus we may define a sequence of automorphisms $\gamma_n : A_n^{\alpha_n} \to A_n^{\alpha_n}$

making the diagram

$$\begin{tikzcd}
A_n^{\alpha_n} \arrow[r, "\varphi"] \arrow[d, "\gamma_n"] & A_{n+1}^{\alpha_{n+1}} \arrow[r, "\varphi"] \arrow[d, "\gamma_{n+1}"] & A_{n+2}^{\alpha_{n+2}} \arrow[r, "\varphi"] \arrow[d, "\gamma_{n+2}"] & \cdots \arrow[r] & A^\alpha \\
A_n^{\alpha_n} \arrow[r, "\psi"] & A_{n+1}^{\alpha_{n+1}} \arrow[r, "\psi"] & A_{n+2}^{\alpha_{n+2}} \arrow[r, "\psi"] & \cdots \arrow[r] & B^\beta
\end{tikzcd}$$

commute and such that $E(\gamma_n)$ is the identity on $E(A_n, \alpha_n)$ (each partial map of γ_n is either the identity or the canonical orientation reversing map). We get an isomorphism $\Gamma : A^\alpha \to B^\beta$ that makes the whole diagram commute. Furthermore, the map $E(\Gamma)$ is an isomorphism from $E(A, \alpha)$ to $E(B, \beta)$. It is easy to see that $A \cong M_{2^\infty} \otimes C(\mathbb{T})$, but that B has trivial K_1 group, so $A \not\cong B$. Thus we may not drop the action of E on the fixed point subalgebra from our invariant. Notice also that the isomorphism Γ, together with $E(\Gamma)$, gives rise to an isomorphism of the $K\mathfrak{R}$ modules that were shown to classify AF flows in [3].

REMARK 4.5. We close this section by describing a natural groupoid structure that $E(A, \alpha)$ carries. Suppose that w and v are generalised eigenisometries with generalised eigenvalues a and b respectively, such that $w^*w = vv^*$. Then wv is a generalised eigenisometry with generalised eigenvalue $a + wbw^*$. This gives rise to a composition law on $E(A, \alpha)$ by

$$([ww^*], [w^*w], a) \circ ([vv^*], [v^*v], b) = ([ww^*], [v^*v], a + wbw^*).$$

It is easy to see that this composition is preserved by morphisms of the invariant. This extra structure may be useful in describing the range of the invariant in some special cases, but it is not sufficient to replace the third part of the invariant, as the example above shows.

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