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THE C^*-ALGEBRA OF SYMMETRIC WORDS IN TWO UNIVERSAL UNITARIES

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ABSTRACT. We compute the K-theory of the C^*-algebra of symmetric words in two universal unitaries. This algebra is the fixed point C^*-algebra for the order-two automorphism of the full C^*-algebra of the free group on two generators which switches the generators. Our calculations relate the K-theory of this C^*-algebra to the K-theory of the associated C^*-crossed-product by $\mathbb{Z}_2$.

KEYWORDS: *C^*-algebra, C^*-crossed-product, fixed point C^*-algebra, action of finite groups, free group C^*-algebras, symmetric algebras, K-theory of C^*-algebras.*

MSC (2000): Primary 46L55; Secondary 46L80.

1. INTRODUCTION

This paper investigates an example of a C^*-algebra of symmetric words in noncommutative variables. Our specific interest is in the C^*-algebra of symmetric words in the two universal unitaries generating the full C^*-algebra $C^*(\mathbb{F}_2)$ of the free group on two generators. Our main result is the computation of the K-theory [1] of this algebra.

The two canonical unitary generators of $C^*(\mathbb{F}_2)$ are denoted by U and V. The C^*-algebra of symmetric words in two universal unitaries U, V is precisely defined as the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$ for the order-2 automorphism σ which maps U to V and V to U. Our strategy to compute the K-theory of $C^*(\mathbb{F}_2)_1$ relies upon the work of Rieffel ([6], Proposition 3.4]) about Morita equivalence between fixed point C^*-algebras and C^*-crossed-products.

The first part of this paper describes the two algebras of interest: the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$ and the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ [8], [5]. We also exhibit an ideal $\mathcal{J}$ in $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ which is strongly Morita equivalent to $C^*(\mathbb{F}_2)_1$ and can be easily described as the kernel of an very simple $*$-morphism. We thus reduce the problem to the calculation of the K-theory of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$.

The second part of this paper starts with this calculation. We use a standard result from Cuntz [4] to calculate the K-theory of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$. We then deduce the K-theory of $C^*(\mathbb{F}_2)_1$, using the ideal $\mathcal{J}$ and the six-terms exact sequence in K-theory.

We conclude the paper by looking a little closer at the obstruction to the existence of unitaries of nontrivial K-theory in $C^*(\mathbb{F}_2)_1$. This amounts to comparing the structure of the ideal $\mathcal{J}$ and an ideal in $C^*(\mathbb{F}_2)_1$ related to the same representation as $\mathcal{J}$.

We also should mention that, in principle, using the results in [3], one could derive information on the representation theory of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ from the representations of $C^*(\mathbb{F}_2)$.

2. THE FIXED POINT C^*-ALGEBRA

Let $C^*(\mathbb{F}_2) = C^*(U, V)$ be the universal C^*-algebra generated by two unitaries U and V. We consider the order-2 automorphism σ of $C^*(\mathbb{F}_2)$ uniquely defined by $\sigma(U) = V$ and $\sigma(V) = U$. These relations indeed define an automorphism by universality of $C^*(\mathbb{F}_2)$. The automorphism σ will be called the *flip automorphism* in this paper.

Our main object of interest is the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$ defined by

$$C^*(\mathbb{F}_2)_1 = \{a \in C^*(\mathbb{F}_2) : \sigma(a) = a\}$$

which can be seen as the C^*-algebra of symmetric words in two universal unitaries. The C^*-algebra $C^*(\mathbb{F}_2)_1$ is related [6] to the C^*-crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$, which we will consider in our calculations. By definition, $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ is the universal C^*-algebra generated by three unitaries U, V and W subjects to the relations $W^2 = 1$ and $WUW^* = V$. Our objective is to gain some understanding of the structure of the unitaries and projections in $C^*(\mathbb{F}_2)_1$.

The first step in our work is to describe concretely the two C^*-algebras $C^*(\mathbb{F}_2)_1$ and $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$. Using Proposition 3.4 of [6], we also exhibit an ideal in the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ which is Morita equivalent to $C^*(\mathbb{F}_2)_1$.

The following easy lemma will be useful in our work:

LEMMA 2.1. *Let A be a unital C^*-algebra and σ an order-two automorphism of A. The fixed-point C^*-algebra of A for σ is the set $A_1 = \{a + \sigma(a) : a \in A\}$. Set $A_{-1} = \{a - \sigma(a) : a \in A\}$. Then $A = A_1 + A_{-1}$ and $A_1 \cap A_{-1} = \{0\}$.*

Proof. Let $\omega \in A$. Since $\sigma^2 = 1$ we have $\sigma(\omega + \sigma(\omega)) = \sigma(\omega) + \omega \in A_1$. On the other hand, if $a \in A$ then $a = \frac{1}{2}(a + \sigma(a)) + \frac{1}{2}(a - \sigma(a))$. Hence if $a \in A_1$ then $a - \sigma(a) = 0$ and $a \in \{\omega + \sigma(\omega) : \omega \in A\}$. Moreover this proves that $A = A_1 + A_{-1}$. Last, if $a \in A_1 \cap A_{-1}$ then $a = \sigma(a) = -\sigma(a) = 0$. ∎

We can use Lemma 2.1 to obtain a more concrete description of the fixed-point C^*-algebra $C^*(\mathbb{F}_2)_1$ of symmetric words in two unitaries:

THEOREM 2.2. *Let σ be the automorphism of $C^*(\mathbb{F}_2) = C^*(U,V)$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Then the fixed point C^*-algebra of σ is $C^*(\mathbb{F}_2)_1 = C^*(U^n + V^n : n \in \mathbb{N})$.*

Proof. Obviously $\sigma(U^n + V^n) = U^n + V^n$ for all $n \in \mathbb{Z}$. Hence

$$C^*(\{U^n + V^n : n \in \mathbb{Z}\}) \subseteq C^*(\mathbb{F}_2)_1.$$

Conversely, $C^*(\mathbb{F}_2)_1 = \{\omega + \sigma(\omega) : \omega \in C^*(\mathbb{F}_2)\}$ by Lemma 2.1. So $C^*(\mathbb{F}_2)_1$ is generated by elements of the form $\omega + \sigma(\omega)$ where ω is a word in $C^*(\mathbb{F}_2)$, since $C^*(\mathbb{F}_2)$ is generated by words in U and V, i.e. by monomials of the form $U^{a_0}V^{a_1}\cdots U^{a_n}$ with $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{Z}$. It is thus enough to show that for any word $\omega \in C^*(\mathbb{F}_2)$ we have

$$\omega + \sigma(\omega) \in S \quad \text{where } S = C^*(U^n + V^n : n \in \mathbb{Z}).$$

Since, if ω starts with a power of V then $\sigma(\omega)$ starts with a power of U, we may as well assume that ω always starts with a power of U by symmetry. Since the result is trivial for $\omega = 1$ we assume that ω starts with a nontrivial power of U. Such a word is of the form

$$\omega = U^{a_0}V^{a_1}\cdots U^{a_{n-1}}V^{a_n} \quad \text{with } a_0, \ldots, a_{n-1} \in \mathbb{Z}\backslash\{0\} \text{ and } a_n \in \mathbb{Z}.$$

We define the order of such a word ω as the integer $o(\omega) = n$ if $a_n \neq 0$ and $o(\omega) = n - 1$ otherwise. In other words, $o(\omega)$ is the number of times we go from U to V or V to U in ω. The proof of our result follows from the following induction on $o(\omega)$.

By definition, if ω is a word such that $o(\omega) = 0$ then $\omega + \sigma(\omega) \in S$. Let us now assume that for some $m \geqslant 1$ we have shown that for all words ω starting in U such that $o(\omega) \leqslant m - 1$ we have $\omega + \sigma(\omega) \in S$. Let ω be a word of order m and let us write $\omega = U^{a_0}\omega_1$ with ω_1 a word starting in a power of V. By construction, ω_1 is of order $m - 1$. Let $\omega_2 = V^{a_0}\omega_1$. By construction, $o(\omega_2) = m - 1$ or $m - 2$. Either way, by our induction hypothesis, we have $\omega_1 + \sigma(\omega_1) \in S$ and $\omega_2 + \sigma(\omega_2) \in S$. Now:

$$\omega + \sigma(\omega) = U^{a_0}\omega_1 + V^{a_0}\sigma(\omega_1) = (U^{a_0} + V^{a_0})(\omega_1 + \sigma(\omega_1)) - (\omega_2 + \sigma(\omega_2))$$

hence $\omega + \sigma(\omega) \in S$ and our induction is complete. Hence $C^*(\mathbb{F}_2)_1 = S$ as desired. ∎

We wish to understand more of the structure of the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$. Using Morita equivalence, we can derive its K-theory. According to Proposition 3.4 of [6], $C^*(\mathbb{F}_2)_1$ is strongly Morita equivalent to the ideal $\mathcal{J}$ generated in the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ by the spectral projection $p = \frac{1}{2}(1 + W)$ of the canonical unitary W in $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ such that $WUW = V$. We first provide a simple yet useful description of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ in term of unitary generators, before providing a description of the ideal $\mathcal{J}$ which will ease the calculation of its K-theory.

LEMMA 2.3. *Let σ be the flip automorphism on the universal C^*-algebra $C^*(\mathbb{F}_2)$ generated by two universal unitaries U and V. The C^*-crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ is the universal C^*-algebra generated by two unitaries U and W with the relation $W^2 = 1$, or equivalently $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ is $*$-isomorphic to $C^*(\mathbb{Z} * \mathbb{Z}_2)$. In particular, the C^*-subalgebra of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ generated by U and the canonical unitary W implementing σ is actually equal to $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$.*

Proof. Let W be the canonical unitary $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ implementing σ. The C^*-subalgebra $C^*(U, W)$ of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ contains $WUW = V$ and thus equals $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$.

Let us now prove that the C^*-subalgebra $C^*(U, W)$ is universal for the given relations. Let u, w be two arbitrary unitaries in some arbitrary C^*-algebra such that $w^2 = 1$. Let $v = wuw \in C^*(u, w)$. By universality of the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ there exists a unique $*$-morphism $\varphi : C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2 \longrightarrow C^*(u, w)$ such that $\varphi(U) = u$, $\varphi(V) = v$ and $\varphi(W) = w$. Thus $C^*(U, W)$ is universal for the proposed relations. In particular, it is $*$-isomorphic (by uniqueness of the universal C^*-algebra for the given relations) to $C^*(\mathbb{Z} * \mathbb{Z}_2)$. ∎

Now, the ideal $\mathcal{J}$ can be described as the kernel of a particularly explicit $*$-morphism of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ to $C(\mathbb{T})$.

PROPOSITION 2.4. *Let σ be the flip automorphism of the universal C^*-algebra $C^*(\mathbb{F}_2)$ generated by two universal unitaries U and V. Let W be the canonical unitary of the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ such that $WUW = V$. The fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$ is strongly Morita equivalent to the kernel $\mathcal{J}$ in $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ of the $*$-morphism $\varphi : C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2 \longrightarrow C(\mathbb{T})$ defined by $\varphi(W) = -1$ and $\varphi(U)(z) = \varphi(V)(z) = z$ for all $z \in \mathbb{T}$.*

Proof. Let φ be the unique $*$-morphism from $C^*(U, W)$ into $C(\mathbb{T})$ defined using the universal property of Lemma 2.3 by: $\varphi(W)(z) = -1$ and $\varphi(U)(z) = z$ for all $z \in \mathbb{T}$. Since $\varphi(p) = 0$ by construction, $\mathcal{J} \subseteq \ker \varphi$.

On the other hand, if π is the canonical surjection $C^*(U, W) \twoheadrightarrow C^*(U, W)/\mathcal{J}$ then $\pi(W) = -1$, so

$$\pi(V) = \pi(WUW) = \pi(W)\pi(U)\pi(W) = \pi(U).$$

Hence any representation ψ of $C^*(U, W)/\mathcal{J}$ lifts to $C^*(U, V)$ as a representation of the form $\psi = \eta \circ \varphi$ for some representation η of $C(\mathbb{T})$. Hence $\ker \varphi \subseteq \mathcal{J} = \ker \pi$. ∎

Our goal now is to compute the K-theory of the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$. Since $C^*(\mathbb{F}_2)_1$ and the ideal $\mathcal{J}$ are Morita equivalent, they have the same K-theory. By Proposition 2.4, we have the short exact sequence $0 \to \mathcal{J} \to C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2 \xrightarrow{\varphi} C(\mathbb{T}) \to 0$, and it seems quite reasonable to use the six-term exact sequence of K-theory to deduce the K-groups of $\mathcal{J}$ from the K-groups of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$, as long as the later can be computed. The next section precisely follows this path, starting by computing the K-theory of the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$.

3. K-THEORY OF THE C^*-CROSSED PRODUCT AND THE FIXED POINT C^*-ALGEBRA

We use the homotopy-based result in [4] to compute the K-theory of the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$.

PROPOSITION 3.1. *Let $C^*(\mathbb{F}_2) = C^*(U, V)$ be the universal C^*-algebra generated by two unitaries U and V and let σ be the order-2 automorphism of $C^*(\mathbb{F}_2)$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Then $K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}^2$ is generated by the classes of the spectral projections of W and $K_1(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}$ is generated by the class of U.*

Proof. By Lemma 2.3, the crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ is $*$-isomorphic to

$$C^*(\mathbb{Z} * \mathbb{Z}_2) = C^*(\mathbb{Z}) *_{\mathbb{C}} C^*(\mathbb{Z}^2) = C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2$$

where the free product is amalgamated over the C^*-algebra generated by the respective units in each C^*-algebra. More precisely, we embed $\mathbb{C}$ via, respectively, $i_1 : \lambda \in \mathbb{C} \mapsto \lambda 1 \in C(\mathbb{T})$ and $i_2 : \lambda \in \mathbb{C} \mapsto (\lambda, \lambda) \in \mathbb{C}^2$. There are natural $*$-morphisms from $C(\mathbb{T})$ and from $\mathbb{C}^2$ onto $\mathbb{C}$ defined respectively by $r_1 : f \in C(\mathbb{T}) \mapsto f(1)$ and $r_2 : \lambda \oplus \mu \mapsto \lambda$. Now, $r_1 \circ i_1 = r_2 \circ i_2$ is the identity on $\mathbb{C}$. By [4], we conclude that the following sequences for $\varepsilon = 0, 1$ are exact:

$$0 \to K_\varepsilon(\mathbb{C}) \xrightarrow{j_\varepsilon} K_\varepsilon(C(\mathbb{T}) \oplus \mathbb{C}^2) \xrightarrow{k_\varepsilon} K_\varepsilon(C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2) \to 0$$

where $j_\varepsilon = K_\varepsilon(i_1) \oplus K_\varepsilon(-i_2)$ and $k_\varepsilon = K_\varepsilon(k_1 + k_2)$ where k_1 is the canonical embedding of $C(\mathbb{T})$ into $C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2$ and k_2 is the canonical embedding of $\mathbb{C}^2$ into $C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2$.

Now, $K_1(C(\mathbb{T}) \oplus \mathbb{C}^2) = \mathbb{Z}$ generated by the identity map $\mathrm{Id}_{\mathbb{T}} \in C(\mathbb{T})$. Since $K_1(\mathbb{C}) = 0$ we conclude that $K_1(C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2) = \mathbb{Z}$ generated by the canonical unitary generator of $C(\mathbb{T})$. On the other hand, $K_0(C(\mathbb{T}) \oplus \mathbb{C}^2) = \mathbb{Z}^3$ (where the first copy of $\mathbb{Z}$ is generated by the unit $1_{C(\mathbb{T})}$ of $C(\mathbb{T})$ and the two other copies are generated by each of the projections $(1, 0)$ and $(0, 1)$ in $\mathbb{C}^2$). Now, the range of j_0 is the subgroup of $\mathbb{Z}^3$ generated by the class of $1_{C(\mathbb{T})} \oplus -1_{\mathbb{C}^2}$ where $1_{\mathbb{C}^2}$ is the unit of $\mathbb{C}^2$. This class is $(1, -1, -1)$, so we conclude easily that $K_0(C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2) = \mathbb{Z}^2$ is generated by the two projections in $\mathbb{C}^2$ (whose classes are $(0, 1, 0)$ and $(0, 0, 1)$).

Using the $*$-isomorphism between $C(\mathbb{T}) *_{\mathbb{C}} \mathbb{C}^2$ and $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ we conclude that $K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}^2$ is generated by the spectral projections of W while $K_1(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}$ is generated by the class of U (or V as these are equal by construction). ∎

REMARK 3.2. It is interesting to compare our results with the K-theory of the C^*-crossed-product by $\mathbb{Z}$ instead of $\mathbb{Z}_2$. One could proceed with the standard six-terms exact sequence ([1], Theorem 10.2.1), but it is even simpler to observe the following similar result to Lemma 2.3: the C^*-crossed-product $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}$ is $*$-isomorphic to $C^*(\mathbb{F}_2)$. If $C^*(\mathbb{F}_2)$ is generated by the two universal unitaries U, V and W is the canonical unitary in $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}$ such that $WUW = V$ then once again $C^*(U, V, W) = C^*(U, W)$ and, following a similar argument as for Lemma 2.3 we observe that $C^*(U, W)$ is the C^*-algebra universal for two arbitrary unitaries.

Hence, $K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}) = \mathbb{Z}$ is generated by the identity, while the classes of U and W generate $K_1(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}) = \mathbb{Z}^2$.

We can now deduce the K-theory of $C^*(\mathbb{F}_2)_1$ from Proposition 2.4 and Theorem 3.1.

THEOREM 3.3. *Let $C^*(\mathbb{F}_2) = C^*(U, V)$ be the universal C^*-algebra generated by two unitaries U and V and let σ be the order-2 automorphism of $C^*(\mathbb{F}_2)$ defined by $\sigma(U) = V$ and $\sigma(V) = U$. Let $C^*(\mathbb{F}_2)_1 = \{a \in C^*(\mathbb{F}_2) : \sigma(a) = a\}$ be the fixed point C^*-algebra for σ. Then $K_0(C^*(\mathbb{F}_2)_1) = \mathbb{Z}$ is generated by the identity in $C^*(\mathbb{F}_2)_1$ and $K_1(C^*(\mathbb{F}_2)_1) = 0$.*

Proof. By Proposition 3.4 of [6], $C^*(\mathbb{F}_2)_1$ is strongly Morita equivalent to the ideal $\mathcal{J}$ generated in $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ by the projection $p = \frac{1}{2}(1 + W)$. Using Proposition 2.4, we can apply the six-terms exact sequence to the short exact sequence

$$0 \longrightarrow \mathcal{J} \xrightarrow{i} C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2 \xrightarrow{\varphi} C(\mathbb{T}) \longrightarrow 0 \tag{3.1}$$

where i is the canonical injection and φ is the $*$-morphism of Proposition 2.4. Since we know the K-theory of $C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2$ by Theorem 3.1, including a set of generators of the K-groups, and the K-theory of $C(\mathbb{T})$, we can easily deduce the K-theory of $\mathcal{J}$. Indeed, using the six-term exact sequence Theorem 9.3 of [1] applied to (3.1), the following six-terms cyclic sequence is exact:

$$\begin{array}{ccccc} K_0(\mathcal{J}) & \xrightarrow{K_0(i)} & K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z}^2 & \xrightarrow{K_0(\varphi)} & K_0(C(\mathbb{T})) \\ \delta \uparrow & & & & \downarrow \beta \\ K_1(C(\mathbb{T})) & \xleftarrow{K_1(\varphi)} & K_1(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2) = \mathbb{Z} & \xleftarrow{K_1(i)} & K_1(\mathcal{J}) \end{array} \tag{3.2}$$

Each statement in the following argument follows from the exactness of (3.2). Trivially, $K_0(\varphi)$ is a surjection, so $\beta = 0$. Hence $K_1(i)$ is injective. Yet, as $\varphi(U) : z \in \mathbb{T} \mapsto z$, we conclude that $K_1(\varphi)$ is an isomorphism (since it maps a generator to a generator), and thus $K_1(i) = 0$. Now $K_1(i) = 0$ and $\beta = 0$ implies that $K_1(\mathcal{J}) = 0$. Since $K_1(\varphi)$ is surjective, $\delta = 0$ and thus $K_0(i)$ is injective. Its image is thus isomorphic to $K_0(\mathcal{J})$ and coincide with $\ker K_0(\varphi)$. Now, $K_0(\varphi)(p) = 0$ and $K_0(\varphi)(1 - p) = 1$ (by Theorem 3.1, p and $1 - p$ generate $K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2)$). Hence the image of $K_0(i)$ is isomorphic to the copy of $\mathbb{Z}$ generated by p in $K_0(C^*(\mathbb{F}_2) \rtimes_\sigma \mathbb{Z}_2)$. ∎

We go a little deeper in the structure of the fixed point C^*-algebra $C^*(\mathbb{F}_2)_1$. Of interest is to compare the ideal $\mathcal{J}$ and its natural restrictions to $C^*(\mathbb{F}_2)$ and

$C^*(\mathbb{F}_2)_1$. The motivation for this comparison is to understand the obstruction to the existence of any nontrivial unitary in $C^*(\mathbb{F}_2)_1$ in the sense of K-theory.

THEOREM 3.4. *Let θ be the $*$-epimorphism $C^*(U,V) \twoheadrightarrow C(\mathbb{T})$ defined by $\theta(U) = \theta(V) : z \in \mathbb{T} \mapsto z$. Let $\mathcal{I} = \ker\theta$. Then $K_0(\mathcal{I}) = 0$ and $K_1(\mathcal{I}) = \mathbb{Z}$ where the generating unitary in the smallest unitization $\mathcal{I}^+$ of $\mathcal{I}$ of the K_1 group is UV^*.*

Let $\mathcal{I}_1 = C^*(\mathbb{F}_2)_1 \cap \mathcal{I}$. We then have $C^*(\mathbb{F}_2)_1/\mathcal{I}_1 = C(\mathbb{T})$. Then $K_1(\mathcal{I}_1) = 0$ while $K_0(\mathcal{I}_1) = \mathbb{Z}$.

Proof. The K-theory of the ideal $\mathcal{I}$ is based upon the simple six-term exact sequence:

$$\begin{array}{ccccc} K_0(\mathcal{I}) = 0 & \xrightarrow{K_0(i)} & K_0(C^*(\mathbb{F}_2)) = \mathbb{Z} & \xrightarrow{K_0(\theta)} & K_0(C(\mathbb{T})) = \mathbb{Z} \\ \delta = 0 \uparrow & & & & \downarrow \beta = 0 \\ K_1(C(\mathbb{T})) = \mathbb{Z} & \xleftarrow{K_1(\theta)} & K_1(C^*(\mathbb{F}_2)) = \mathbb{Z}^2 & \xleftarrow{K_1(i)} & K_1(\mathcal{I}) = \mathbb{Z} \end{array}$$

corresponding to the defining exact sequence $\mathcal{I} \overset{i}{\hookrightarrow} C^*(\mathbb{F}_2) \overset{\theta}{\twoheadrightarrow} C(\mathbb{T})$ with i the canonical injection. Now, $K_0(C^*(\mathbb{F}_2))$ is generated by 1, and as $\theta(1) = 1$ we see that $K_0(\theta)$ is the identity. Hence $\beta = 0$ and $K_0(i) = 0$ so $K_0(\mathcal{I}) = \delta(\mathbb{Z})$. On the other hand, $K_1(C^*(\mathbb{F}_2))$ is generated by U and V, respectively identified with $(1,0)$ and $(0,1)$ in $\mathbb{Z}^2$. We have $\theta(U) = \theta(V) : z \mapsto z$ which is the generator of $K_1(C(\mathbb{T}))$, so $K_1(\theta)$ is surjective and thus $\delta = 0$. Hence $K_0(\mathcal{I}) = 0$. On the other hand, $\ker K_1(\theta)$ is the group generated by $(1,-1)$, the class of UV^*. Thus $K_1(i)$, which is an injection, is in fact a bijection from $K_1(\mathcal{I})$ onto its range $\ker K_1(\theta)$ and thus $K_1(\mathcal{I}) = \mathbb{Z}$ generated by UV^*, as indeed $\theta(UV^* - 1) = 0$ and thus $UV^* - 1 \in \mathcal{I}$.

Now, we turn to the ideal $\mathcal{I}_1$. We recall from Lemma 2.1 the notation $C^*(\mathbb{F}_2)_{-1} = \{a - \sigma(a) : a \in C^*(\mathbb{F}_2)\}$. Our first observation is that $C^*(\mathbb{F}_2)_{-1} \subseteq \mathcal{I}$. Indeed, since $\theta(U) = \theta(V)$ we have $\theta \circ \sigma = \theta$ and thus $\theta(a - \sigma(a)) = 0$ for all $a \in C^*(\mathbb{F}_2)$. Therefore, $C^*(\mathbb{F}_2)/\mathcal{I} = C^*(\mathbb{F}_2)_1/(\mathcal{I} \cap C^*(\mathbb{F}_2)_1)$ since $C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2)_1 \oplus C^*(\mathbb{F}_2)_{-1}$ as vector spaces by Lemma 2.1.

Using the six-terms exact sequence and Theorem 3.3, we can compute the K-theory of the ideal $\mathcal{I}_1$:

$$\begin{array}{ccccc} K_0(\mathcal{I}_1) & \xrightarrow{K_0(i)} & K_0(C^*(\mathbb{F}_2)_1) = \mathbb{Z} & \xrightarrow{K_0(\theta)} & K_0(C(\mathbb{T})) = \mathbb{Z} \\ \delta \uparrow & & & & \downarrow \beta \\ K_1(C(\mathbb{T})) = \mathbb{Z} & \xleftarrow{K_1(\theta)} & K_1(C^*(\mathbb{F}_2)_1) = 0 & \xleftarrow{K_1(i)} & K_1(\mathcal{I}_1) \end{array}$$

where i is again the canonical injection and we denote the restriction of θ to $C^*(\mathbb{F}_2)_1$ by θ again. Note that θ thus restricted is still an epimorphism. Each subsequent argument follows from the exactness of the six-terms sequence. Since $K_0(C^*(\mathbb{F}_2)_1)$ is generated by the class of the unit in $C^*(\mathbb{F}_2)_1$ and $\theta(1) = 1$ generated $K_0(C(\mathbb{T}))$, we conclude that $K_0(\theta)$ is the identity, so $K_0(i) = 0$ and $\beta = 0$. Hence $K_0(\mathcal{I}_1) = \delta(\mathbb{Z})$ and $K_1(i)$ is injective. Since $K_1(C^*(\mathbb{F}_2)_1) = 0$ and $\beta = 0$

we conclude that $K_1(\mathcal{I}_1) = 0$. Therefore $K_1(\theta) = 0$, so δ is injective and we get $K_0(\mathcal{I}_1) = \mathbb{Z}$. ∎

It is not too surprising that $K_1(\mathcal{I}_1) = 0$ since $K_1(\mathcal{I}) = \mathbb{Z}$ is generated by $UV^*(-1)$ which is an element in $C^*(\mathbb{F}_2)_{-1}$ of class $(1,-1)$ in $K_1(C^*(\mathbb{F}_2))$, so it is not connected to any unitary in $C^*(\mathbb{F}_2)_1$. Of course, this is not a direct proof of this fact, as homotopy in $M_n(\mathcal{I}_1) + 1_n$ ($n \in \mathbb{N}$) is a more restrictive notion than in $M_n(C^*(\mathbb{F}_2)_1)$ ($n \in \mathbb{N}$). But more remarkable is the fact that $K_0(\mathcal{I}_1)$ contains some nontrivial element. Of course, $\mathcal{I}_1$ is projectionless since $C^*(\mathbb{F}_2)$ is by [2], so the projection generating $K_0(\mathcal{I}_1)$ is at least (and in fact, exactly in) $M_2(\mathcal{I}_1) + 1_2$. We now turn to an explicit description of the generator of $K_0(\mathcal{I}_1)$ and we investigate why this projection is trivial in both $K_0(C^*(\mathbb{F}_2)_1)$ and $K_0(\mathcal{I})$ but not in $K_0(\mathcal{I}_1)$. By exactness of the six-terms exact sequence, this projection is exactly the obstruction to the nontriviality of $K_1(C^*(\mathbb{F}_2)_1)$.

For any C^*-algebra A, we denote by $[q]_A$ the K_0-class of any projection $q \in M_n(A) + 1_n$ for any $n \in \mathbb{N}$.

THEOREM 3.5. *Let $\theta : C^*(\mathbb{F}_2) \longrightarrow C(\mathbb{T})$ be the $*$-homomorphism defined by $\theta(U) = \theta(V) : z \in \mathbb{T} \mapsto z$ and let $\mathcal{I} = \ker\theta$. Let $\mathcal{I}_1 = \mathcal{I} \cap C^*(\mathbb{F}_2)_1$. Let $Z = \frac{1}{2}(U+V) \in C^*(\mathbb{F}_2)_1$. Let β be the projection in $M_2(\mathcal{I}_1) + 1_2$ defined by:*

$$\beta = \begin{bmatrix} Z^*Z & Z^*(\sqrt[2]{1-ZZ^*}) \\ (\sqrt[2]{1-ZZ^*})Z & 1-ZZ^* \end{bmatrix}.$$

Then the generator of $K_0(\mathcal{I}_1)$ is $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$ where $p_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. On the other hand, the projection β is homotopic to p_2 in $M_2(\mathcal{I}) + 1_2$ and in $M_2(C^*(\mathbb{F}_2)_1)$. Thus $[\beta]_{\mathcal{I}} = 0$ in $K_0(\mathcal{I})$ and $[\beta]_{C^*(\mathbb{F}_2)_1} = [p_2]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}$ in $K_0(C^*(\mathbb{F}_2)_1)$.

A simple calculation shows that β is a projection. We organize the proof of Theorem 3.5 in several lemmas. We start with the two quick observations that β is homotopic to p_2 in $M_2(C^*(\mathbb{F}_2)_1)$ and in $M_2(\mathcal{I}) + 1_2$, and then we prove that $[\beta]_{\mathcal{I}_1} \neq [p_2]_{\mathcal{I}_1}$ in $K_0(\mathcal{I}_1)$.

LEMMA 3.6. *The projections β and $1 - p_2$ are homotopic in $M_2(C^*(\mathbb{F}_2)_1)$. Thus $[\beta]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}$.*

Proof. For all $t \in [0,1]$ we set:

$$\beta_t = \begin{bmatrix} t^2Z^*Z & tZ^*\sqrt[2]{1-tZZ^*} \\ (\sqrt[2]{1-tZZ^*})tZ & 1-tZZ^* \end{bmatrix}.$$

Then $(\beta_t)_{t\in[0,1]}$ is by construction an homotopy in $M_2(C^*(\mathbb{F}_2)_1)$ between $\beta_1 = \beta$ and $\beta_0 = 1 - p_2$. Trivially $1 - p_2$ and p_2 are homotopic, and $[p_2]_{C^*(\mathbb{F}_2)_1} = [1]_{C^*(\mathbb{F}_2)_1}$, hence our result. ∎

The important observation in the proof of Lemma 3.6 is that although $\beta_t \in M_2(C^*(\mathbb{F}_2)_1)$ for all $t \in [0,1]$, we have

$$\theta(tZ\sqrt[2]{1-tZZ^*})(z) = tz\sqrt[2]{1-t} \neq 0$$

for $t \in (0,1)$ and $z \in \mathbb{T}$, so β_t does not belong to the ideals $\mathcal{I}$ and $\mathcal{I}_1$.

Since $K_0(\mathcal{I}) = 0$, it is trivial that the class of β in $K_0(\mathcal{I})$ is null. It is however interesting to look for a concrete homotopy between β and the projection p_2 in $M_2(\mathcal{I}) + 1_2$, in parallel to the construction of Lemma 3.6.

LEMMA 3.7. *In $M_2(\mathcal{I}) + 1_2$ the projection β is homotopic to p_2. Hence in $K_0(\mathcal{I})$ we verify that we have indeed $[\beta]_{\mathcal{I}} = 0$.*

Proof. The unitary equivalence in Lemma 3.8 does not carry over to the unitization of the ideal $\mathcal{I}$, but we can check that β is homotopic to p_2 in $M_2(\mathcal{I}) + 1_2$. Set $Z_t = tU + (1-t)V$ and set:

$$\beta_t = \begin{bmatrix} Z_t^* Z_t & Z_t^*(\sqrt[2]{1-Z_tZ_t^*}) \\ (\sqrt[2]{1-Z_tZ_t^*})Z_t & 1 - Z_tZ_t^* \end{bmatrix}$$

for all $t \in [0,1]$. As before, β_t is a projection for all $t \in [0,1]$ since $\|Z_t\| = 1$ for all $t \in [0,1]$. Now, $\beta_0 = \beta_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ while $\beta_{1/2} = \beta$. Of course, $t \in [0, \frac{1}{2}] \mapsto \beta_t$ is continuous. Moreover:

$$(tU + (1-t)V)(tU + (1-t)V)^* = 1 - 2(t-t^2) + (t-t^2)(UV^* + VU^*)$$

so $\theta(1 - Z_tZ_t^*) = 0$. Hence, $\beta_t \in M_2(\mathcal{I}) + 1_2$ for all $t \in [0,1]$. Hence β is homotopic to p_2 in $M_2(\mathcal{I}) + 1_2$. ∎

Unlike in the case of Lemma 3.6, the homotopy used in the proof of Lemma 3.8 is in $M_2(\mathcal{I})$, but it is not in $M_2(C^*(\mathbb{F}_2)_1)$ and hence not in $M_2(\mathcal{I}_1)$.

The crux of this matter is that β is the obstruction to the existence of a nontrivial element in $K_1(C^*(\mathbb{F}_2)_1)$. In view of Lemmas 3.6 and 3.7, we wish to see a concrete reason why β can not have the same class as p_2 in $K_0(\mathcal{I}_1)$. We start with a useful calculation: since β and p_2 are homotopic in $C^*(\mathbb{F}_2)_1$, they are unitarily equivalent as well, and we now make explicit a unitary implementing this equivalence:

LEMMA 3.8. *Let $Y = \begin{bmatrix} Z^* & \sqrt[2]{1-Z^*Z} \\ \sqrt[2]{1-ZZ^*} & -Z \end{bmatrix}$. Then Y is a unitary in $M_2(C^*(\mathbb{F}_2)_1)$ such that $Yp_2Y^* = \beta$.*

Proof. Observe that $Z^*(1-ZZ^*) = Z^* - Z^*ZZ^* = (1-Z^*Z)Z^*$. Thus, for any $n \in \mathbb{N}$ we get by a trivial induction that $Z^*(1-ZZ^*)^n = (1-Z^*Z)^nZ^*$. Hence, for any polynomial p, we have $Z^*(p(1-ZZ^*)) = (p(1-Z^*Z))Z^*$. By Stone–Weierstrass, we deduce that $Z^*f(1-ZZ^*) = f(1-Z^*Z)Z^*$ for any continuous function f on the spectrum of $1-ZZ^*$ and $1-Z^*Z$ which is the compact space $[0,1]$, and in particular for the square root. Therefore:

$$Z^*\sqrt[2]{1-ZZ^*} = (\sqrt[2]{1-Z^*Z})Z^*. \tag{3.3}$$

Now, we have:

$$\begin{aligned} YY^* &= \begin{bmatrix} Z^* & \sqrt[2]{1-Z^*Z} \\ \sqrt[2]{1-ZZ^*} & -Z \end{bmatrix} \begin{bmatrix} Z & \sqrt[2]{1-ZZ^*} \\ \sqrt[2]{1-Z^*Z} & -Z^* \end{bmatrix} \\ &= \begin{bmatrix} Z^*Z+1-Z^*Z & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

using (3.3) since $Z^*\sqrt[2]{1-ZZ^*} - (\sqrt[2]{1-Z^*Z})Z^* = 0$. Similarly, we get $Y^*Y = 1_2$.

Now, we compute Yp_2Y^*:

$$\begin{aligned} &\begin{bmatrix} Z^* & \sqrt[2]{1-Z^*Z} \\ \sqrt[2]{1-ZZ^*} & -Z \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z & \sqrt[2]{1-ZZ^*} \\ \sqrt[2]{1-Z^*Z} & -Z^* \end{bmatrix} \\ &\qquad = \begin{bmatrix} Z^* & \sqrt[2]{1-Z^*Z} \\ \sqrt[2]{1-ZZ^*} & -Z \end{bmatrix} \begin{bmatrix} Z & \sqrt[2]{1-ZZ^*} \\ 0 & 0 \end{bmatrix} \\ &\qquad = \begin{bmatrix} Z^*Z & Z^*\sqrt[2]{1-ZZ^*} \\ (\sqrt[2]{1-ZZ^*})Z & 1-ZZ^* \end{bmatrix} = \beta. \end{aligned}$$

Last, we observe that $\sigma(Z) = Z$ by construction and thus $\sigma(Y) = Y$ as well: in other words, $Y \in M_2(C^*(\mathbb{F}_2)_1)$ (and we recover that β is unitarily equivalent in $M_2(C^*(\mathbb{F}_2)_1)$ to p_2). ∎

Note that $Z - \lambda 1 \notin \mathcal{I}$ for all $\lambda \in \mathbb{C}$ and so Y does not belong to $M_2(\mathcal{I}) + 1_2$. Indeed, the following lemma shows that β and p_2 do not have the same K_0-class in $\mathcal{I}_1$, precisely because the conjunction of the conditions of symmetry and being in the kernel of θ make it impossible to deform one into the other, even though each condition alone does not create any obstruction.

LEMMA 3.9. *We have* $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1} \neq 0$ *in* $K_0(\mathcal{I}_1)$.

Proof. We shall prove that in fact $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$ is a generator for $K_0(\mathcal{I}_1)$. Let $\delta : K_1(C(\mathbb{T})) \longrightarrow K_0(\mathcal{I}_1)$ be the exponential map in the six-term exact sequence in K-theory induced by the exact sequence $0 \to \mathcal{I}_1 \to C^*(\mathbb{F}_2)_1 \xrightarrow{\theta} C(\mathbb{T}) \to 0$. Let us denote by z the canonical unitary $z : \omega \in \mathbb{T} \mapsto \omega$ in $C(\mathbb{T})$. Let us also denote by θ_2 the map induced by θ on $M_2(C^*(\mathbb{F}_2))$. By Proposition 9.2.3 of [7], if u is any unitary in $M_2(C^*(\mathbb{F}_2)_1)$ such that $\theta_2(u) = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}$, then $\delta([z]_{C(\mathbb{T})}) = [up_2u^*]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$. In particular, $\theta_2(Y) = \begin{bmatrix} z & 0 \\ 0 & z^* \end{bmatrix}$ so $\delta([z]_{C(\mathbb{T})}) = [Yp_2Y^*]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1} = [\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$.

On the other hand, by Theorem 3.4, δ is an isomorphism of group. Since $[z]_{C(\mathbb{T})}$ is a generator of $K_1(C(\mathbb{T}))$ we conclude that $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$ is a generator of $K_0(\mathcal{I}_1)$. ∎

We have thus proven Lemma 3.9 and completed our proof of Theorem 3.5 by identifying $[\beta]_{\mathcal{I}_1} - [p_2]_{\mathcal{I}_1}$ as the generator of $K_0(\mathcal{I}_1)$ and verifying that without the conjoint conditions of symmetry via σ and θ, the difference of the classes of β and p_2 is null in both $K_0(C^*(\mathbb{F}_2)_1)$ and in $K_0(\mathcal{I})$.

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