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STINESPRING'S THEOREM FOR HILBERT C^*-MODULES

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ABSTRACT. We provide an analogue of Stinespring's theorem for Hilbert C^*-modules.

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Let E and F be Hilbert C^*-modules over C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\varphi : \mathcal{A} \to \mathcal{B}$ be a linear map. A map $\Phi : E \to F$ is said to be a φ-map if $\langle \Phi(x), \Phi(y) \rangle = \varphi(\langle x, y \rangle)$ is satisfied for all $x, y \in V$. If φ is a morphism of C^*-algebras, then Φ is called a φ-morphism of Hilbert C^*-modules. A φ-morphism $\Phi : E \to B(H_1, H_2)$, where H_1, H_2 are Hilbert spaces and $\varphi : \mathcal{A} \to B(H_1)$ is a representation of $\mathcal{A}$, is called a *representation* of E. The representation Φ is said to be a *faithful representation* of E if Φ is injective.

It is well known that for every Hilbert C^*-module there is a (faithful) representation to $B(H_1, H_2)$ for some Hilbert spaces H_1, H_2. It is easy to check that each φ-morphism Φ is necessarily a linear operator and a module map in the sense $\Phi(xa) = \Phi(x)\varphi(a)$ for all $a \in \mathcal{A}, x \in E$.

We recall that the Hilbert C^*-module E, together with norm $\|\cdot\|_n$ on $M_n(E)$ given by $\|[x_{ij}]\|_n = \Big\| \Big[\sum\limits_{k=1}^{n} \langle x_{ki}, x_{kj} \rangle \Big] \Big\|^{1/2}$, is an operator space. By a direct calculation, we have:

PROPOSITION. *Let E, F be Hilbert C^*-modules over unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and $\varphi : \mathcal{A} \to \mathcal{B}$ be a completely bounded map. Then every φ-map $\Phi : E \to F$ is completely bounded. If E is full and φ is completely positive, then*

$$\|\Phi\|_{\mathrm{cb}} = \|\Phi\| = \|\varphi\|^{1/2} = \|\varphi(1)\|^{1/2}.$$

Complete positivity and its related concepts form an interesting and useful part of the theory of operator algebras. Stinespring's theorem is an essential result in complete positivity which is a natural generalization of the Gelfand–Naimark–Segal theorem to operator-valued mappings.

Now, we are going to provide an analogue of Stinespring's theorem for Hilbert C^*-modules.

MAIN THEOREM. *If E is a Hilbert C^*-module over the unital C^*-algebra $\mathcal{A}$, and $\varphi : \mathcal{A} \to B(H_1)$ is a completely positive map with $\varphi(1) = 1$ and $\Phi : E \to B(H_1, H_2)$ is a φ-map with the additional property $\Phi(x_0)\Phi(x_0)^* = 1_{B(H_2)}$ for some $x_0 \in E$, where H_1, H_2 are Hilbert spaces, then there exist Hilbert spaces K_1, K_2 and isometries $V : H_1 \to K_1$ and $W : H_2 \to K_2$ and a $*$-homomorphism $\rho : \mathcal{A} \to B(K_1)$ and a ρ-representation $\Psi : E \to B(K_1, K_2)$ such that*

$$\varphi(a) = V^*\rho(a)V \quad \Phi(x) = W^*\Psi(x)V$$

for all $x \in E, a \in \mathcal{A}$.

Proof. As in the proof of Stinespring's theorem, we consider the algebraic tensor product $\mathcal{A} \otimes H_1$ consisting of all formal sums $\sum_j a_j \otimes h_j$. Define a form $\langle \cdot, \cdot \rangle$ on $\mathcal{A} \otimes H_1$ by

$$\Big\langle \sum_j a_j \otimes h_j, \sum_j b_j \otimes g_j \Big\rangle = \sum_{ij} \langle \varphi(b_i^* a_j) h_j, g_i \rangle.$$

It can be checked that this is a sesquilinear form on $\mathcal{A} \otimes H_1$. The positivity of the sesquilinear form is a consequence of the complete positivity of φ.

Let $\mathcal{N} = \{v \in \mathcal{A} \otimes H_1 : \langle v, v \rangle = 0\}$, and consider the quotient space $K_0 = (\mathcal{A} \otimes H_1)/\mathcal{N}$. As usual, $\langle x + \mathcal{N}, y + \mathcal{N} \rangle = \langle x, y \rangle$ is a well-defined inner product on K_0. Let K_1 be the completion of K_0 with respect to the norm defined by this inner product and define $\rho : \mathcal{A} \to B(K_1)$ by

$$\rho(a)\Big(\sum_j a_j \otimes h_j + \mathcal{N}\Big) = \sum_j a a_j \otimes h_j + \mathcal{N}.$$

It can be shown that $\rho : \mathcal{A} \to B(K_1)$ is a unital $*$-homomorphism. Now define an operator $V : H_1 \to K_1$ by $V(h_1) = 1 \otimes h_1 + \mathcal{N}$. A direct calculation shows that V is an isometry and $\varphi(a) = V^*\rho(a)V$.

Now, we consider the algebraic tensor product $E \otimes H_2$ consisting of all formal sums $\sum_j x_j \otimes h_j$. Define a form $\langle \cdot, \cdot \rangle$ on $E \otimes H_2$ by

$$\Big\langle \sum_j x_j \otimes h_j, \sum_j y_j \otimes g_j \Big\rangle = \sum_{ij} \langle \Phi(x_i)\Phi(y_j)^* h_i, g_j \rangle.$$

for all $i, j = 1, \ldots, n$ and $x_i \in E, h_i \in H_2$.

It can be checked that this is a sesquilinear form on $E \otimes H$. The positivity of the sesquilinear form is a consequence of the fact that the matrix $[\Phi(x_i)\Phi(x_j)^*]$ is positive in $B(H_2^{(n)})$.

In a similar way, we can let $\mathcal{N}' = \{x \in E \otimes H_2 : \langle x, x \rangle = 0\}$, and consider the quotient space $K_0' = (E \otimes H_2)/\mathcal{N}'$. As usual, $\langle x + \mathcal{N}', y + \mathcal{N}' \rangle = \langle x, y \rangle$, for $x, y \in E \otimes H_2$, is a well-defined inner product on K_0'. Let K_2 be the completion of

K_0' with respect to the norm defined by this inner product. Now define $\Psi : E \to B(K_1, K_2)$ by

$$\Psi(x)\Big(\sum_j a_j \otimes h_j + \mathcal{N}\Big) = \sum_j x_0 \otimes \Phi(xa_j)h_j + \mathcal{N}'.$$

Also, we define an operator $W : H_2 \to K_2$ by $W(h_2) = x_0 \otimes h_2 + \mathcal{N}'$ for $h_2 \in H_2$. Note that

$$\|W(h_2)\|^2 = \langle x_0 \otimes h_2, x_0 \otimes h_2 \rangle = \langle \Phi(x_0)\Phi(x_0)^* h_2, h_2 \rangle = \langle h_2, h_2 \rangle = \|h_2\|^2.$$

Thus W is an isometry. Now we show that $\Phi(x) = W^*\Psi(x)V$, for all $x \in E$. For this, let $h_1 \in H_1$, $h_2 \in H_2$ and $x \in E$. Then we have

$$\begin{aligned}\langle W^*\Psi(x)V(h_1), h_2\rangle &= \langle \Psi(x)(1 \otimes h_1 + \mathcal{N}), x_0 \otimes h_2 + \mathcal{N}'\rangle \\ &= \langle x_0 \otimes \Phi(x)h_1 + \mathcal{N}', x_0 \otimes h_2 + \mathcal{N}'\rangle \\ &= \langle \Phi(x_0)\Phi(x_0)^*\Phi(x)h_1, h_2\rangle = \langle \Phi(x)h_1, h_2\rangle.\end{aligned}$$

Finally, it must be shown that Ψ is a ρ-morphism. In fact, it must be checked that $\Psi(x)^*\Psi(y) = \rho(\langle x, y\rangle)$ is satisfied for all $x, y \in E$. For this, let $a, b \in \mathcal{A}$, $x, y \in E$ and $h_1, h_1' \in H_1$, then we have

$$\begin{aligned}\langle \Psi(x)^*\Psi(y)(a\otimes h_1 + \mathcal{N}), b\otimes h_1' + \mathcal{N}\rangle &= \langle \Psi(y)(a \otimes h_1 + \mathcal{N}), \Psi(x)(b \otimes h_1' + \mathcal{N})\rangle \\ &= \langle x_0 \otimes \Phi(ya)h_1, x_0 \otimes \Phi(xb)h_1'\rangle \\ &= \langle \Phi(x_0)\Phi(x_0)^*\Phi(ya)h_1, \Phi(xb)h_1'\rangle \\ &= \langle \Phi(xb)^*\Phi(ya)h_1, h_1'\rangle \\ &= \langle \varphi(\langle xb, ya\rangle)h_1, h_1'\rangle \\ &= \langle V^*\rho(b^*\langle x, y\rangle a)Vh_1, h_1'\rangle \\ &= \langle \rho(\langle x, y\rangle)\rho(a)(1\otimes h_1 + \mathcal{N}), \rho(b)(1\otimes h_1' + \mathcal{N})\rangle \\ &= \langle \rho(\langle x, y\rangle)(a \otimes h_1 + \mathcal{N}), (b \otimes h_1' + \mathcal{N})\rangle.\end{aligned}$$

Since $\Psi(x)^*\Psi(y)$ and $\rho(\langle x, y\rangle)$ are bounded linear operators, then we have $\Psi(x)^*\Psi(y) = \rho(\langle x, y\rangle)$ on the whole K_1. ∎

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