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A CONDITION OF POSITIVITY FOR TOEPLITZ OPERATORS ON THE SPHERE

DONARA NGUON

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ABSTRACT. We extend the covariant symbolic calculus for Toeplitz operators on the unit sphere of $\mathbb{C}^n$ to all orders and we give a computable sufficient condition for them to be positive.

KEYWORDS: *Toeplitz operators, Berezin-quantization, symbolic calculus, positivity of operators.*

MSC (2000): 47B35, 47B32, 47B65, 32W50.

INTRODUCTION

In the theory of pseudodifferential operators, the sharp Gårding inequality plays an important role in the study of propagation of singularities as well as in local solvability [6], [3]; we may notice that if P is a pseudodifferential operator, then the fact that P satisfies the sharp Gårding inequality is equivalent to the positivity of the operator $\big(\frac{P+P^*}{2} + C \cdot \mathrm{Id}\big)$ for some constant C. Therefore, it is interesting to get criteria of positivity for pseudodifferential operators. However, if the symbol of a pseudodifferential operator is nonnegative, it does not imply that the operator is positive, not even bounded from below, and there are positive pseudodifferential operators whose symbols are not nonnegative [4]. It is thus advisable to understand better the inter-relations between lower bounds for pseudodifferential operators and lower bounds for their symbols. One deep result was given by C. Fefferman and D.H. Phong by the use of microlocal analysis, cf. [3]. We would like to study this problem in a different way. In [5], V. Guillemin recalls an isomorphism between operators in the Weyl algebra defined on $\mathbb{R}^n$ and Toeplitz operators defined on the complex n-ball ; this isomorphism is called the *Boutet de Monvel - Howe correspondence* [2], [5]. We can use it to obtain sufficient conditions of positivity for Weyl operators, but we need first obtain sufficient conditions of positivity for Toeplitz operators: this is the purpose of this paper. We describe now the setting of the theory of Toeplitz operators we need, cf. [1], [7], [8].

Let X be the unit sphere of $\mathbb{C}^n$ and $\mathrm{d}\sigma$ the Borel measure on X, invariant under rotation. The space $L^2(X)$ is endowed with the inner product $(f,g) = \int_X f(x)\overline{g(x)}\mathrm{d}\sigma(x)$. We denote by $H^s(X)$ the Sobolev space on X of order $s \in \mathbb{R}$ and $\mathcal{O}^s := H^s(X) \cap \mathrm{Ker}\overline{\partial}_{\mathrm{b}}$, the subspace of functions of $H^s(X)$ which have an analytic continuation in a neighborhood of X. Let $m \in \mathbb{R}$, a Toeplitz operator of order m is a linear operator $T : \mathcal{O}^s \to \mathcal{O}^{s-m}$ for all $s \in \mathbb{R}$, such that it is of the form $T = \mathbf{S}P$ where $\mathbf{S}$ is the Szegö orthogonal projection $H^s(X) \to \mathcal{O}^s$ and P is a pseudodifferential operator on X of order m. If P is a differential operator with polynomial coefficients, the associated Toeplitz operator will be called a *differential Toepliz operator*.

In [8] a symbolic calculus for Toeplitz operators on X of order zero was given. This symbolic calculus needs to be extended to Toeplitz operators of any order: This is the matter of the second section of this paper. The computable model for these operators lives in the spaces of entire functions defined on $\mathbb{C}^n$ of exponential type, so that we will need specific techniques to compute them: for this purpose, formulas are given in the first section. In the last section, a simple and computable sufficient condition on the covariant symbol is given for a Toeplitz operator to be positive.

Throughout this paper $\hbar$ is a positive constant and $n \geqslant 1$ is an integer.

1. REPRODUCING KERNELS AND DIVISION FORMULAS

The space of entire functions of exponential type will be denoted by $\mathcal{E}^0 := L^2(\mathbb{C}^n; \varpi_0 \mathrm{d}\nu) \cap \mathrm{Ker}\overline{\partial}$ with $\varpi_0(z) := (\pi\hbar)^{-2n} K_0\left(\frac{2}{\hbar}|z|\right)$, where, for every $x > 0$, $K_0(x) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-t-x^2/4t}\frac{\mathrm{d}t}{t}$ is the Macdonald function of order zero and $\mathrm{d}\nu$ is the Lebesgue measure on $\mathbb{C}^n$.

In [8] we gave a reproducing kernel for entire functions of exponential type in $\mathcal{E}^0$. Indeed, if we apply (15) in [8] to $A = \mathrm{Id}$, we get:

$$F(z) = (\tfrac{\pi}{2}\hbar^2)^{-n} \int_{\mathbb{C}^n} F(u) I_{n-1}^{-}(\tfrac{2}{\hbar}\sqrt{\overline{u}\cdot z}) K_0\left(\tfrac{2}{\hbar}|u|\right) \mathrm{d}\nu(u) \tag{1.1}$$

for any function $F \in \mathcal{E}^0$. This formula can be shown explicitly by computations and we will give here a generalization of it. Before that, we need to extend the definition of weight functions.

For every $\nu, s \in \mathbb{R}^+$, we set:

$$I_\nu^-(z) := z^{-\nu} I_\nu(z) = 2^{-\nu} \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\Gamma(m+\nu+1)} \quad \text{for every } z \in \mathbb{C}, \tag{1.2}$$

$$K_\nu^+(x) := x^\nu K_\nu(x) \quad \text{for every } x > 0, \tag{1.3}$$

$$\varpi_s(z) := (\pi\hbar)^{-2n} 2^{-2s} K_{2s}^+(\tfrac{2}{\hbar}|z|) \quad \text{for every } z \in \mathbb{C}^n, \tag{1.4}$$

where I_ν is the modified Bessel function of the first kind and of order ν and K_ν is the modified Bessel function of the third kind of order ν also called Macdonald function cf. [9].

On the space $\mathcal{E}^s := L^2(\mathbb{C}^n; \varpi_s \mathrm{d}\nu) \cap \mathrm{Ker}\overline{\partial}$, we define the inner product:

$$\langle F, G \rangle_s := \int_{\mathbb{C}^n} F(z)\overline{G(z)}\varpi_s(z)\mathrm{d}\nu(z) \quad \text{for every } F, G \in \mathcal{E}^s. \tag{1.5}$$

REMARK 1.1. The well-known asymptotic formula for Macdonald functions shows that the definition of $\mathcal{E}^s$ is equivalent to the one given in [7].

LEMMA 1.2. *For every $A \geqslant 0$ and $s \in \mathbb{R}^+$, we have*

$$\int_0^\infty r^{2n-1+2A} K_{2s}^+\left(\tfrac{2}{\hbar}r\right) \mathrm{d}r = 2^{2s-2}\hbar^{2n+2A}\Gamma(A+2s+n)\Gamma(A+n). \tag{1.6}$$

Proof. This is an application of the Heaviside formula, cf. formula 13.21(8) in [9]. ∎

LEMMA 1.3. *For every multi-index $\alpha, \beta \in \mathbb{N}^n$ and $s \in \mathbb{R}^+$, we have*

$$\int_{\mathbb{C}^n} z^\alpha \overline{z}^\beta K_{2s}^+\left(\tfrac{2}{\hbar}|z|\right) \mathrm{d}\nu(z) = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ (\frac{\pi}{2}\hbar^2)^n 2^{n-1+2s}\hbar^{2|\alpha|}\alpha!\Gamma(|\alpha|+2s+n) & \text{if } \alpha = \beta. \end{cases} \tag{1.7}$$

Proof. We express the integral in polar coordinates and use formula (1.6). ∎

LEMMA 1.4. *Let $R, S \in \mathbb{R}^+$, $p(\overline{z}, z)$ a polynomial and $u \in \mathbb{C}^n$, then we have:*

$$\begin{aligned} &\int_{\mathbb{C}^n} p(\overline{z}, z) I_R^-\left(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u}\right) K_S^+\left(\tfrac{2}{\hbar}|z|\right) \mathrm{d}\nu(z) \\ &= 2^{-R} \sum_{m=0}^{\infty} \frac{\hbar^{-2m}}{m!\Gamma(m+R+1)} \Big(\sum_{|\gamma|=m} \frac{m!}{\gamma!} u^\gamma \int_{\mathbb{C}^n} \overline{z}^\gamma p(\overline{z}, z) K_S^+\left(\tfrac{2}{\hbar}|z|\right) \mathrm{d}\nu(z) \Big). \end{aligned} \tag{1.8}$$

Proof. Following (1.2) the left hand side of (1.8) reads $2^{-R}\int_{\mathbb{C}^n}\Big(\sum_{m=0}^{\infty} f_m(\overline{z}, z)\Big)\mathrm{d}\nu(z)$ with

$$f_m(\overline{z}, z) = \frac{\hbar^{-2m} p(\overline{z}, z)}{m!\Gamma(m+R+1)} (\overline{z} \cdot u)^m K_S^+\left(\tfrac{2}{\hbar}|z|\right).$$

These functions are integrable since the Macdonald function $K_S^+(\frac{2}{\hbar}|z|)$ is exponentially decreasing at infinity. Moreover, the series $\left(\sum_{m=0}^{\infty} f_m\right)$ is simply converging to $2^R p(\overline{z},z) I_R^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u}) K_S^+(\frac{2}{\hbar}|z|)$ and the asymptotic formulas for Bessel functions show that, for every $M \in \mathbb{N}$, the finite sums $\sum_{m=0}^{M} f_m$ are dominated by

$$C|p(\overline{z},z)||z|^{-R/2-S-3/4}\exp(\tfrac{2}{\hbar}\phi(z)) \quad \text{with } \phi(z) = (|u|^{1/2} - |z|^{1/2})|z|^{1/2}$$

for some constant C depending on $\hbar$ and u. When $|z|$ is large enough the dominating function is exponentially decreasing, thus integrable, so that we may use the theorem of Lebesgue to interchange the integration and the summation and get formula (1.8). ∎

PROPOSITION 1.5. *For every $\alpha \in \mathbb{N}^n$, $s \in \mathbb{R}$, we have*

$$(\tfrac{\pi}{2}\hbar^2)^{-n} \int_{\mathbb{C}^n} z^\alpha I_{n-1+2s}^-(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u}) K_{2s}^+(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z) = u^\alpha. \tag{1.9}$$

Proof. We denote by J the left hand side of (1.9). By Lemma 1.4 we get

$$J = (\tfrac{\pi}{2}\hbar^2)^{-n} 2^{1-n-2s} \sum_{m=0}^{\infty} \frac{\hbar^{-2m}}{m!\Gamma(m+2s+n)} \Big(\sum_{|\beta|=m} \frac{m!}{\beta!} u^\beta \int_{\mathbb{C}^n} z^\alpha \overline{z}^\beta K_{2s}^+(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z)\Big).$$

Lemma 1.3 tells us that the last integral vanishes except for $\beta = \alpha$. Therefore, the only remaining term is for $m = |\alpha|$ and using formula (1.7) we find

$$J = (\tfrac{\pi}{2}\hbar^2)^{-n} 2^{1-n-2s} \frac{\hbar^{-2|\alpha|}}{m!\Gamma(|\alpha|+2s+n)} \frac{m!}{\alpha!} u^\alpha (\tfrac{\pi}{2}\hbar^2)^n 2^{n-1+2s} \hbar^{2|\alpha|} \alpha! \Gamma(|\alpha|+2s+n) = u^\alpha. \quad ∎$$

COROLLARY 1.6. *For every $s \in \mathbb{R}$ and $u \in \mathbb{C}^n$, we have*

$$(\tfrac{\pi}{2}\hbar^2)^{-n} \int_{\mathbb{C}^n} F(z) I_{n-1+2s}^-(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u}) K_{2s}^+\left(\tfrac{2}{\hbar}|z|\right) \mathrm{d}\nu(z) = F(u) \quad \text{for every } F \in \mathcal{E}^s. \tag{1.10}$$

This is a reproducing kernel for entire functions of exponential type in $\mathcal{E}^s$.

Proof. We expand $F(z)$ into a power series: $F(z) = \sum_\beta a_\beta z^\beta$ and use (1.9). ∎

PROPOSITION 1.7. *For every $\alpha \in \mathbb{N}^n$, $s \in \mathbb{R}^+$ and $u \in \mathbb{C}^n$, we have*

$$(\tfrac{\pi}{2}\hbar^2)^{-n} \int_{\mathbb{C}^n} z^\alpha I_{n-1}^-(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u}) K_{2s}^+(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z) = 2^{2s} \frac{\Gamma(|\alpha|+2s+n)}{\Gamma(|\alpha|+n)} u^\alpha. \tag{1.11}$$

Proof. The proof is similar to the one of Proposition 1.5. ∎

PROPOSITION 1.8. *Let $\alpha, \beta \in \mathbb{N}^n, k, s \in \mathbb{R}^+$. Let $u \in \mathbb{C}^n$ such that $u^\alpha \neq 0$ and $\overline{u}^\beta \neq 0$.*

(i) *If $\beta \geqslant \alpha$ (i.e. $\beta_j \geqslant \alpha_j$, for every j) then we have:*

$$(1.12)\qquad (\tfrac{\pi}{2}\hbar^2)^{-n}\int_{\mathbb{C}^n} \overline{z}^\alpha z^\beta I^-_{n-1+k}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u})K^+_{2s}(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z) = 2^{2s-k}\hbar^{2|\alpha|}\frac{\Gamma(|\beta|+2s+n)}{\Gamma(|\beta|-|\alpha|+k+n)}\frac{\beta!}{(\beta-\alpha)!}\frac{u^\beta}{u^\alpha}.$$

If β is not $\geqslant \alpha$, then the integral vanishes.

(ii) *If $\alpha \geqslant \beta$ (i.e. $\alpha_j \geqslant \beta_j$, for every j) then we have:*

$$(1.13)\qquad (\tfrac{\pi}{2}\hbar^2)^{-n}\int_{\mathbb{C}^n} \overline{z}^\alpha z^\beta I^-_{n-1+k}(\tfrac{2}{\hbar}\sqrt{\overline{u}\cdot z})K^+_{2s}(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z) = 2^{2s-k}\hbar^{2|\beta|}\frac{\Gamma(|\alpha|+2s+n)}{\Gamma(|\alpha|-|\beta|+k+n)}\frac{\alpha!}{(\alpha-\beta)!}\frac{\overline{u}^\alpha}{\overline{u}^\beta}.$$

If α is not $\geqslant \beta$, then the integral vanishes.

Proof. (i) We denote by J the left hand side of (1.12). By Lemma 1.4 we get

$$J = (\tfrac{\pi}{2}\hbar^2)^{-n}2^{1-n-k}\sum_{m\geqslant 0}\frac{\hbar^{-2m}}{m!\Gamma(m+k+n)}u^\gamma\sum_{|\gamma|=m}\frac{m!}{\gamma!}\int_{\mathbb{C}^n} z^\beta\overline{z}^{\alpha+\gamma}K^+_{2s}(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z).$$

Lemma 1.3 tells us that the last integral vanishes if $\beta \neq \alpha+\gamma$. We suppose first that $\beta \geqslant \alpha$. The only remaining term is for $\gamma = \beta-\alpha$, therefore in the sum we keep only the term where $m = |\beta|-|\alpha|$ and $\gamma = \beta-\alpha$. Then we may use formula (1.7) to deduce the claimed formula.

Now if β is not $\geqslant \alpha$, then $\beta_j < \alpha_j$ for some j so that $\beta_j < \alpha_j+\gamma_j$ for any $\gamma_j \geqslant 0$. Therefore β will never be equal to $\alpha+\gamma$ and the integral will vanish.

(ii) The second assertion is proved in the same way. ∎

COROLLARY 1.9. *For every $\alpha,\beta \in \mathbb{N}^n, k,s \in \mathbb{R}^+$ and $u \in \mathbb{C}^n$, we have*

$$(1.14)\qquad (\tfrac{\pi}{2}\hbar^2)^{-n}\int_{\mathbb{C}^n} z^{\alpha+\beta}\overline{z}^\alpha I^-_{n-1+k}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot u})K^+_{2s}(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z) = 2^{2s-k}\hbar^{2|\alpha|}\frac{\Gamma(|\alpha|+|\beta|+2s+n)}{\Gamma(|\beta|+k+n)}\frac{(\alpha+\beta)!}{\beta!}u^\beta.$$

2. THE COVARIANT SYMBOLIC CALCULUS

The analysis by wave packet made in [7] and the Berezin quantization have led us to build a covariant symbolic calculus for Toeplitz operators of order zero on the sphere. Several formulas were given in [8] and we sum up here some of them:

- The packet transform: $\widehat{f}(z) := \int_X f(x)\mathrm{e}^{\overline{x}\cdot z/\hbar}\mathrm{d}\sigma(x)$ for every $f \in \mathcal{O}^s$ and $s \in \mathbb{R}$.
- The Szegő kernel: $S(x,\overline{y}) = (\pi\hbar)^{-2n}\int_{\mathbb{C}^n}\mathrm{e}^{(x\cdot\overline{z}+\overline{y}\cdot z)/\hbar}K_0(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z)$ as an oscillatory integral.

• The indicator of orthogonality: $(e_z, e_u) := \int\limits_X \mathrm{e}^{(x\cdot\overline{z}+\overline{x}\cdot u)/\hbar}\mathrm{d}\sigma(x) = (2\pi)^n I^-_{n-1}$ $(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})$.

This Bessel function plays a central role in our calculus and it is as important as the exponential function is important in Fourier analysis.

• The covariant symbol for a Toeplitz operator T: $\sigma_T(\overline{z}, u) := \frac{(Te_z, e_u)}{(e_z, e_u)}$ with $e_z(x) := \mathrm{e}^{(x\cdot\overline{z})/\hbar}$ and (z, u) closed enough to the diagonal $z = u$ in $\mathbb{C}^n \times \mathbb{C}^n$.

• Properties of these covariant symbols: $\sigma_{\mathrm{Id}}(\overline{z}, u) \equiv 1, \sigma_{T^*}(\overline{z}, u) = \overline{\sigma_T(\overline{u}, z)}$.

• Formula of composition: $\sigma_A \# \sigma_B(\overline{z}, u) := \sigma_{AB}(\overline{z}, u) =$

$$\frac{(\frac{\pi}{2}\hbar^2)^{-n}}{I^-_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}\int\limits_{\mathbb{C}^n} I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w})\sigma_B(\overline{z}, w) I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{w}\cdot u})\sigma_A(\overline{w}, u) K_0(\tfrac{2}{\hbar}|w|)\mathrm{d}\nu(w).$$

We need to extend these formulas to Toeplitz operators of any order.

From Theorem 1 in [7], the spaces $\mathcal{E}^s$ are isomorphic to $\mathcal{O}^s$. In order to make them isometric, we may define the inner product in $\mathcal{O}^s$ by: $(f, g)_s := \langle \widehat{f}, \widehat{g}\rangle_s$.

REMARK 2.1. If $s = 0$, then $(f, g)_0 = (f, g)$ and $\langle \widehat{f}, \widehat{g}\rangle_0 = \langle \widehat{f}, \widehat{g}\rangle$.

Since one main goal of our calculus is computations, we will first give some formulas involving these new inner products.

LEMMA 2.2. *For every $\alpha \in \mathbb{N}^n$ and $w \in \mathbb{C}^n$, we have:*

$$\widehat{x^\alpha}(w) = \frac{2\pi^n \hbar^{-|\alpha|}}{\Gamma(|\alpha| + n)} w^\alpha; \tag{2.1}$$

$$\widehat{e_z}(w) = (2\pi)^n I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w}). \tag{2.2}$$

Proof. (i)

$$\begin{aligned}\widehat{x^\alpha}(w) &= \int\limits_X x^\alpha \mathrm{e}^{\overline{x}\cdot w/\hbar}\mathrm{d}\sigma(x) = \int\limits_X x^\alpha \sum_{m=0}^\infty \frac{\hbar^{-m}}{m!}\sum_{|\beta|=m}\frac{m!}{\beta!}\overline{x}^\beta w^\beta \mathrm{d}\sigma(x)\\ &= \sum_{m=0}^\infty \frac{\hbar^{-m}}{m!}\sum_{|\beta|=m}\frac{m!}{\beta!} w^\beta \int\limits_X x^\alpha \overline{x}^\beta \mathrm{d}\sigma(x).\end{aligned}$$

The only non zero term is for $\beta = \alpha$: $\widehat{x^\alpha}(w) = \frac{\hbar^{-|\alpha|}}{|\alpha|!}\frac{|\alpha|!}{\alpha!} w^\alpha \frac{2\pi^n \alpha!}{\Gamma(|\alpha|+n)} = \frac{2\pi^n \hbar^{-|\alpha|}}{\Gamma(|\alpha|+n)} w^\alpha$.

(ii) Following formula (13) in [8], we have: $\widehat{e_z}(w) = (e_z, e_w) = (2\pi)^n I^-_{n-1}$ $(\frac{2}{\hbar}\sqrt{\overline{z}\cdot w})$. ∎

PROPOSITION 2.3. *For every $\alpha, \beta \in \mathbb{N}^n, z, u \in \mathbb{C}^n$ and $s \in \mathbb{R}^+$, we have:*

$$(x^\alpha, x^\beta)_s = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \frac{2\pi^n \alpha!}{\Gamma(|\alpha|+n)} \frac{\Gamma(|\alpha|+2s+n)}{\Gamma(|\alpha|+n)} & \text{if } \alpha = \beta; \end{cases} \tag{2.3}$$

$$(e_z, e_u)_s = 2\pi^n \sum_{m=0}^{\infty} \frac{\hbar^{-2m}(\overline{z}\cdot u)^m}{m!\Gamma(m+n)} \frac{\Gamma(m+2s+n)}{\Gamma(m+n)}. \tag{2.4}$$

Proof. We show the first equation:

If $\alpha \neq \beta$ then following (1.7) we get $(x^\alpha, x^\beta)_s = \langle \widehat{x^\alpha}, \widehat{x^\beta} \rangle_s = 0$, otherwise

$$(x^\alpha, x^\alpha)_s = \langle \widehat{x^\alpha}, \widehat{x^\alpha} \rangle_s = (\pi\hbar)^{-2n} 2^{-2s} \Big(\frac{(2\pi)^n \hbar^{-|\alpha|}}{\Gamma(|\alpha|+n)} \Big)^2 \int_{\mathbb{C}^n} |w^\alpha|^2 K^+_{2s}(\tfrac{2}{\hbar}|w|) \mathrm{d}\nu(w)$$

and the conclusion follows again from formula (1.7). Now we show the second equation:

$$(e_z, e_u)_s = \langle \widehat{e_z}, \widehat{e_u} \rangle_s = \int_{\mathbb{C}^n} \widehat{e_z}(w) \cdot \overline{\widehat{e_u}(w)} \varpi_s(w) \mathrm{d}\nu(w).$$

Following (2.2) and (1.4), we have

$$\begin{aligned}(e_z, e_u)_s &= (2\pi)^{2n} \int_{\mathbb{C}^n} I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w}) I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{w}\cdot u}) \varpi_s(w) \mathrm{d}\nu(w) \\ &= (\tfrac{2}{\hbar})^{2n} 2^{1-n-2s} \int_{\mathbb{C}^n} \sum_{m=0}^{\infty} \tfrac{\hbar^{-2m}}{m!} \tfrac{1}{\Gamma(m+n)} \sum_{|\gamma|=m} \tfrac{m!}{\gamma!} \overline{z}^\gamma w^\gamma I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{w}\cdot u}) K^+_{2s}(\tfrac{2}{\hbar}|w|) \mathrm{d}\nu(w).\end{aligned}$$

Following Lemma 1.4 we may interchange the integration and the summation and we get for the last expression

$$(\tfrac{2}{\hbar})^{2n} 2^{1-n-2s} \sum_{m=0}^{\infty} \tfrac{\hbar^{-2m}}{m!} \tfrac{1}{\Gamma(m+n)} \sum_{|\gamma|=m} \tfrac{m!}{\gamma!} \overline{z}^\gamma \int_{\mathbb{C}^n} w^\gamma I^-_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{w}\cdot u}) K^+_{2s}(\tfrac{2}{\hbar}|w|) \mathrm{d}\nu(w).$$

The claimed formula is then obtained thanks to formula (1.11). ∎

Let $\alpha \in \mathbb{N}^n$, we denote by X^α the operator of mutiplication by x^α. Let $D_j = \frac{\partial}{\partial x_j}$ and $D^\alpha = (\frac{\partial}{\partial x})^\alpha$ which is a differential Toeplitz operator of order $|\alpha|$.

PROPOSITION 2.4. *Let $\alpha \in \mathbb{N}^n$ and A a Toeplitz operator on X, we have:*

(i) *If $T = X^\alpha$ then $\sigma_T(\overline{z}, u) = (\frac{2}{\hbar})^{|\alpha|} u^\alpha \frac{I^-_{n-1+|\alpha|}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I^-_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}$.*

(ii) *If $T = A \circ X^\alpha$ then $\sigma_T(\overline{z}, u) = \frac{\hbar^{|\alpha|}}{I^-_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})} \big(\frac{\partial}{\partial \overline{z}}\big)^\alpha \big(\sigma_A(\overline{z}, u) I^-_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})\big)$.*

(iii) *If $T = D^\alpha$ then $\sigma_T(\overline{z}, u) = \hbar^{-|\alpha|}\overline{z}^\alpha$.*

(iv) *If $T = A \circ D^\alpha$ then $\sigma_T(\overline{z}, u) = \hbar^{-|\alpha|}\overline{z}^\alpha \sigma_A(\overline{z}, u)$.*

Proof. The first two equations were shown in [8]. The two other formulas are shown by the equation $\big(\frac{\partial}{\partial x}\big)^\alpha \mathrm{e}^{x\cdot\overline{z}/\hbar} = \hbar^{-|\alpha|}\overline{z}^\alpha \mathrm{e}^{x\cdot\overline{z}/\hbar}$ and the definition of a covariant symbol. ∎

Therefore, we can compute easily the covariant symbol of any differential Toeplitz operator. These covariant symbols are finite sums of

$$c_{\alpha,\beta}\overline{z}^{\beta}u^{\alpha}\frac{I^{-}_{n-1+|\alpha|}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I^{-}_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}\quad \text{with } \alpha,\beta\in\mathbb{N}^n \text{ and } c_{\alpha,\beta}\in\mathbb{C}.$$

3. A CONDITION OF POSITIVITY

DEFINITION 3.1. A covariant symbol is said to be of *positive type* when it is a finite sum of terms of the form

$$c_{\alpha,k}\overline{z}^{\alpha}u^{\alpha}\frac{I^{-}_{n-1+k}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I^{-}_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}\quad \text{with } \alpha\in\mathbb{N}^n, c_{\alpha,k}>0 \text{ and } k\in\mathbb{N}.$$

The following theorem is our main result:

THEOREM 3.2. *Let T be a differential Toeplitz operator on X. If its covariant symbol is of positive type then the operator T is positive.*

Proof. Since polynomials are dense in each $\mathcal{O}^s$ it is sufficient to prove that $(Tf,f)\geqslant 0$ for any polynomial f. Following formula (15) in [8], we have

$$\begin{aligned}(Tf,f) &= \langle\widehat{Tf},\widehat{f}\rangle\\ &= c_n\int\limits_{\mathbb{C}^n}\Big[\int\limits_{\mathbb{C}^n}\sigma_T(\overline{z},w)I^{-}_{n-1}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w})\widehat{f}(z)K_0(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z)\Big]\overline{\widehat{f}(w)}K_0(\tfrac{2}{\hbar}|w|)\mathrm{d}\nu(w)\end{aligned}$$

with $c_n=(\frac{\pi}{2}\hbar^2)^{-n}\times(\pi\hbar)^{-2n}$. We may suppose that $\sigma_T(\overline{z},w)=c_{\alpha,k}\overline{z}^{\alpha}w^{\alpha}\frac{I^{-}_{n-1+k}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot w})}{I^{-}_{n-1}(\frac{2}{\hbar}\sqrt{\overline{z}\cdot w})}$ for some $\alpha\in\mathbb{N}^n, c_{\alpha,k}>0$ and $k\in\mathbb{N}$. Now according to Lemma 2.2 $\widehat{f}(z)$ is also a polynomial in the variable z: $\widehat{f}(z)=\sum\limits_{\beta}a_\beta z^\beta$. The integral J in the brackets reads

$$\begin{aligned}J &= c_{\alpha,k}\int\limits_{\mathbb{C}^n}\overline{z}^{\alpha}w^{\alpha}I^{-}_{n-1+k}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w})\Big(\sum_{\beta}a_\beta z^\beta\Big)K_0(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z)\\ &= \sum_{\beta}c_{\alpha,k}a_\beta w^{\alpha}\int\limits_{\mathbb{C}^n}\overline{z}^{\alpha}z^{\beta}I^{-}_{n-1+k}(\tfrac{2}{\hbar}\sqrt{\overline{z}\cdot w})K_0(\tfrac{2}{\hbar}|z|)\mathrm{d}\nu(z).\end{aligned}$$

Following Proposition 1.8, if β is not $\geqslant\alpha$ then the integral vanishes.

If $\beta\geqslant\alpha$ then the integral alone is equal to $(\frac{\pi}{2}\hbar^2)^n2^{-k}\hbar^{2|\alpha|}\frac{(|\beta|+n-1)!}{(|\beta|-|\alpha|+k+n-1)!}\frac{\beta!}{(\beta-\alpha)!}\frac{w^\beta}{w^\alpha}$ so that $J=c_{\alpha,k}2^{-k}(\frac{\pi}{2})^n\hbar^{2n+2|\alpha|}\sum\limits_{\beta\geqslant\alpha}\frac{(|\beta|+n-1)!}{(|\beta|-|\alpha|+k+n-1)!}\frac{\beta!}{(\beta-\alpha)!}a_\beta w^\beta$.

Let us notice that the coefficient of each $a_\beta w^\beta$ is positive which will be denoted by $\kappa_{\alpha,\beta,k}$.

Now we have to integrate with respect to $\mathrm{d}\nu(w)$ and we get:

$$(Tf,f) = c_n \sum_{\beta\geqslant\alpha}\sum_{\gamma} \kappa_{\alpha,\beta,k} a_\beta \overline{a}_\gamma \int\limits_{\mathbb{C}^n} w^\beta \overline{w}^\gamma K_0(\tfrac{2}{\hbar}|w|)\mathrm{d}\nu(w).$$

Following formula (1.7), the only remaining term in the last sum is when $\gamma = \beta$ and we get

$$(Tf,f) = c_n \sum_{\beta\geqslant\alpha} \kappa_{\alpha,\beta,k} |a_\beta|^2 \int\limits_{\mathbb{C}^n} |w^\beta|^2 K_0(\tfrac{2}{\hbar}|w|)\mathrm{d}\nu(w).$$

which is positive. ∎

A covariant symbol of positive type is thus a covariant symbol which is nonnegative on the diagonal of $\mathbb{C}^n \times \mathbb{C}^n$. This idea may be used to choose an adequate definition for the algebra of covariant symbols for general Toeplitz operators.

EXAMPLE 3.3. (i) The covariant symbol of $X_j^* \circ X_j$ is $2\frac{I_n^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I_{n-1}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})} + (\frac{2}{\hbar})^2 \overline{z}_j u_j \frac{I_{n+1}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I_{n-1}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}$ which is of positive type, therefore this operator is positive.

(ii) The covariant symbol of the radial derivative operator $\sum\limits_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$ is $\frac{1}{2}(\frac{2}{\hbar})^2 \overline{z}\cdot u \frac{I_n^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I_{n-1}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}$ which is of positive type, therefore this operator is positive.

(iii) The covariant symbol of $x^\alpha\big(\frac{\partial}{\partial x}\big)^\alpha$ is $2^{-|\alpha|}\big(\frac{2}{\hbar}\big)^{2|\alpha|}\overline{z}^\alpha u^\alpha \frac{I_{n-1+|\alpha|}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}{I_{n-1}^-(\frac{2}{\hbar}\sqrt{\overline{z}\cdot u})}$ which is of positive type, therefore this operator is positive.

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DONARA NGUON, 78 RUE RAMBUTEAU, F-75001 PARIS, FRANCE
E-mail address: donara.nguon@laposte.net

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