J. OPERATOR THEORY
62:2(2009), 327–340

A REPRESENTATION THEOREM FOR COMPLETELY CONTRACTIVE DUAL BANACH ALGEBRAS

FARUK UYGUL

Communicated by Kenneth R. Davidson

ABSTRACT. In this paper, we prove that every completely contractive dual Banach algebra is completely isometric to a w*-closed subalgebra of the operator space of completely bounded linear operators on some reflexive operator space.

KEYWORDS: *Operator spaces, interpolations, representation.*

MSC (2000): 47L25, 46H15, 47L50, 46B70, 46L99, 43A10.

1. INTRODUCTION

A Banach algebra $\mathfrak{A}$ which is a dual Banach space is called a dual Banach algebra if the multiplication on $\mathfrak{A}$ is separately w*-continuous. All von Neumann algebras are dual Banach algebras, but so are all measure algebras $M(G)$, where G is a locally compact group, and all algebras $\mathcal{B}(E)$, where E is a reflexive Banach space. Of course, every w*-closed subalgebra of $\mathcal{B}(E)$ for a reflexive Banach space E is then also a dual Banach algebra. Surprisingly, as proven recently by Daws [4], every dual Banach algebra arises in this fashion.

A completely contractive dual Banach algebra is a Banach algebra which is a dual operator space in the sense of [5] such that multiplication is completely contractive and separately w*-continuous. Then von Neumann algebras are examples of completely contractive dual Banach algebras. Also, whenever $\mathfrak{A}$ is a dual Banach algebra, then $\max \mathfrak{A}$ [5] is a completely contractive dual Banach algebra. If G is a locally compact group, then the Fourier–Stieltjes algebra $B(G)$ [6] is a completely contractive dual Banach algebra which, in general, is neither a von Neumann algebra nor of the form $\max B(G)$. In the present paper, we prove an operator space analog of Daws' representation theorem: if $\mathfrak{A}$ is a completely contractive dual Banach algebra, then there is a reflexive operator space E and a w*-w*-continuous, completely isometric algebra homomorphism from $\mathfrak{A}$ to $\mathcal{CB}(E)$, where $\mathcal{CB}(E)$ stands for the completely bounded operators on E (see

[5]). We would like to stress that even when $\mathfrak{A}$ is of the form $\max \mathfrak{A}$ for some dual Banach algebra $\mathfrak{A}$, our result is not just a straightforward consequence of Daws' result, but requires a careful adaptation of his techniques to the operator space setting. The construction of such a reflexive operator space heavily relies on the theory of real and complex interpolation of operator spaces defined by Xu [15] and Pisier ([10] and [11]), respectively.

This representation theorem is somewhat related in spirit to results by Ghahramani [7] and Neufang, Ruan, and Spronk [9]: in [7], $M(G)$ is (completely) isometrically represented on $\mathcal{B}(L^2(G))$, and in [9], a similar representation is constructed for the completely contractive dual Banach algebra $M_{\rm cb}(A(G))$ consisting of the completely bounded multipliers of the Fourier algebra $A(G)$. We would like to emphasize, however, that our representation theorem neither implies nor is implied by those results: $\mathcal{B}(L^2(G))$ is a dual operator space, but not reflexive.

2. PRELIMINARIES

2.1. DUAL BANACH ALGEBRAS AND OPERATOR SPACES.

DEFINITION 2.1. A Banach algebra $\mathfrak{A}$ is called a *dual Banach algebra* if it is a dual Banach space and the multiplication on $\mathfrak{A}$ is separately w*-continuous.

EXAMPLES 2.2. (i) Every von Neumann algebra is a dual Banach algebra.

(ii) If E is a reflexive Banach space, then $\mathcal{B}(E)$ is a dual Banach algebra with the predual $E^* \otimes_\gamma E$, where $\otimes_\gamma$ represents the projective tensor product of Banach spaces.

(iii) If G is a locally compact group, then the measure algebra $M(G)$ and the Fourier–Stieltjes algebra $B(G)$ are dual Banach algebras with preduals $C_0(G)$ and $C^*(G)$, respectively.

DEFINITION 2.3. An *operator space* is a linear space E with a complete norm $\|\cdot\|_n$ on $M_n(E)$ for each $n \in \mathbb{N}$ such that

(R 1) $\left\| \begin{array}{c|c} x & 0 \\ \hline 0 & y \end{array} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\} \quad (n, m \in \mathbb{N},\ x \in M_n(E),\ y \in M_m(E)),$

(R 2) $\|\alpha x \beta\|_n \leqslant \|\alpha\| \|x\|_n \|\beta\| \quad (n \in \mathbb{N},\ x \in M_n(E),\ \alpha, \beta \in M_n).$

EXAMPLES 2.4. (i) Every closed subspace of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is an operator space.

(ii) If G is a locally compact group, then the group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$, and the Fourier–Stieltjes algebra $B(G)$ are operator spaces.

DEFINITION 2.5. Let E_1, E_2, and F be operator spaces. A bilinear map $T : E_1 \times E_2 \to F$ is called *completely contractive* if

$$\|T\|_{\mathrm{cb}} := \sup_{n_1,n_2\in\mathbb{N}} \|T^{(n_1,n_2)}\| \leqslant 1,$$

where

$$T^{(n_1,n_2)} : M_{n_1}(E_1) \times M_{n_2}(E_2) \to M_{n_1 n_2}(F), \quad ((x_{i,j}),(y_{k,l})) \mapsto (T(x_{i,j}, y_{k,l})).$$

DEFINITION 2.6. A *completely contractive Banach algebra* is a Banach algebra which is also an operator space such that multiplication is a completely contractive bilinear map.

EXAMPLES 2.7. (i) Every closed subalgebra of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is a completely contractive Banach algebra.

(ii) If G is a locally compact group, then the group algebra $L^1(G)$, the measure algebra $M(G)$, the Fourier algebra $A(G)$ and the Fourier–Stieltjes algebra $B(G)$ are completely contractive Banach algebras.

DEFINITION 2.8. Let E and F be operator spaces, and let $T \in \mathcal{B}(E,F)$. Then:

(i) T is *completely bounded* if

$$\|T\|_{\mathrm{cb}} := \sup_{n\in\mathbb{N}} \|T^{(n)}\|_{\mathcal{B}(M_n(E),M_n(F))} < \infty.$$

(ii) T is a *complete contraction* if $\|T\|_{\mathrm{cb}} \leqslant 1$.

(iii) T is a *complete isometry* if $T^{(n)}$ is an isometry for each $n \in \mathbb{N}$.

The set of completely bounded operators from E to F is denoted by $\mathcal{CB}(E,F)$.

DEFINITION 2.9. A *completely contractive dual Banach algebra* is a Banach algebra which is a dual operator space such that multiplication is completely contractive and separately w*-continuous.

Note that there are operator spaces for which there exist predual Banach spaces, but not predual operator spaces ([3], Lemma 2.7.15).

EXAMPLES 2.10. (i) Every w*-closed subalgebra of $\mathcal{CB}(E)$, where E is a reflexive operator space, is a completely contractive dual Banach algebra.

(ii) If G is a locally compact group, then the measure algebra $M(G)$, the Fourier–Stieltjes algebra $B(G)$ and the reduced Fourier–Stieltjes algebra $B_{\mathrm{r}}(G)$ are completely contractive dual Banach algebras.

(iii) If $\mathbf{K}$ is a Kac algebra, then $M_0A(\mathbf{K})$, completely bounded multipliers of the Fourier algebra of $\mathbf{K}$, is a completely contractive dual Banach algebra [8].

2.2. COMPLEX INTERPOLATION OF BANACH SPACES. Let X_0, X_1 be two complex Banach spaces. The couple (X_0, X_1) is called *compatible* (in the sense of interpolation theory) if there is a Hausdorff complex topological vector space $\mathcal{X}$ and $\mathbb{C}$-linear continuous inclusions $X_0 \hookrightarrow \mathcal{X}$ and $X_1 \hookrightarrow \mathcal{X}$.

Now let X_0 and X_1 be two compatible normed spaces. Then we define a norm on the set $X_0 + X_1$ by $\|x\| := \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\}$. We denote this space by $X_0 +_1 X_1$. For $0 < \theta < 1$, let

$$X_{[\theta]} = (X_0, X_1)_\theta := \{x \in X_0 +_1 X_1 : \ x = f(\theta) \ \text{for some } f \text{ satisfying (2.1)}\}$$

where

$$f : \mathbb{C} \to X_0 +_1 X_1 \tag{2.1}$$

is a function which satisfies the following conditions:

(i) f is bounded and continuous on the strip $S := \{z \in \mathbb{C} : \ 0 \leqslant \mathrm{Re}(z) \leqslant 1\}$;
(ii) f is analytic on S_0, the interior of S;
(iii) $f(\mathrm{i}t) \in X_0$ and $f(1 + \mathrm{i}t) \in X_1$ $(t \in \mathbb{R})$.

Define a norm on $X_{[\theta]}$ via

$$\|x\|_{[\theta]} := \inf\{\|f\| : \ x = f(\theta), \ f \text{ satisfying (2.1)}\}$$

where the norm of f is defined to be

$$\|f\| := \max\{\sup\{\|f(\mathrm{i}t)\|_{X_0}\}, \sup\{\|f(1 + \mathrm{i}t)\|_{X_1}\} : \ t \in \mathbb{R}\}.$$

By this construction, $X_{[\theta]}$ becomes an interpolation space between X_0 and X_1. For more information on interpolation of Banach spaces, we refer the reader to [2].

2.3. INTERPOLATION OF OPERATOR SPACES. Suppose that E_0, E_1 are operator spaces such that (E_0, E_1) is a compatible couple of Banach spaces. Note that for each $n \in \mathbb{N}$, we have continuous inclusions $M_n(E_0) \hookrightarrow M_n(\mathcal{X})$ and $M_n(E_1) \hookrightarrow M_n(\mathcal{X})$ where $M_n(\mathcal{X})$ is identified with $\mathcal{X}^{n^2}$. Thus $(M_n(E_0), M_n(E_1))$ is a compatible couple of Banach spaces. Clearly $E_0 \oplus_\infty E_1$ is an operator space by setting

$$M_n(E_0 \oplus_\infty E_1) = M_n(E_0) \oplus_\infty M_n(E_1).$$

Then $E_0 \oplus_1 E_1$ becomes an operator space with the embedding $E_0 \oplus_1 E_1 \hookrightarrow (E_0^* \oplus_\infty E_1^*)^*$. Now, for each $1 < p < \infty$, $E_0 \oplus_p E_1$ becomes an operator space via

$$M_n(E_0 \oplus_p E_1) = (M_n(E_0 \oplus_1 E_1), M_n(E_0 \oplus_\infty E_1))_\theta, \quad \frac{1}{p} = 1 - \theta.$$

The complex and real interpolation of operator spaces are defined by G. Pisier in [10] and by Q. Xu in [15], respectively. The construction of the latter heavily uses the first one.

2.4. COMPLEX INTERPOLATION OF OPERATOR SPACES. Let $0 < \theta < 1$ and E_0, E_1 be operator spaces such that (E_0, E_1) is a compatible couple of Banach spaces. Then, for each $n \in \mathbb{N}$, the couple $(M_n(E_0), M_n(E_1))$ is also compatible. Now define

$$M_n(E_\theta) := (M_n(E_0), M_n(E_1))_\theta \tag{2.2}$$

in the sense of complex interpolation [2]. By this definition, $E_\theta = (E_0, E_1)_\theta$ becomes an operator space. This is called the complex interpolation of operator spaces E_0 and E_1 (see [10] and [11] for more information).

2.5. REAL INTERPOLATION OF OPERATOR SPACES. The construction of the interpolation of operator spaces by the real method is more complicated than by the complex method. This is because definition (2.2) does not work for the real interpolation $(E_0, E_1)_{\theta,p}$ if $p < \infty$. Now we introduce real interpolation of operator spaces by the discrete K-method as defined by Xu in [15].

Note that if E is an operator space and $t > 0$, then tE denotes the operator space obtained by multiplying the norm on each matrix level by t. Now let μ denote a weighted counting measure on $\mathbb{Z}$ (That is: For $E \subseteq \mathbb{Z}$, $\mu(E)$ is defined by $\mu(E) := \sum_{n\in E} a_n$, where $\{a_n\}_{n\in\mathbb{Z}}$ is a sequence of non-negative reals) and $\{E_k\}_{k\in\mathbb{Z}}$ a sequence of operator spaces. Then for $1 \leqslant p \leqslant \infty$, we define

$$l_p(\{E_k\}_{k\in\mathbb{Z}};\mu) := \{(x_k)_{k\in\mathbb{Z}} : \ x_k \in E_k \ \text{and} \ (\|x_k\|)_{k\in\mathbb{Z}} \in l_p(\mu)\}.$$

Clearly $l_\infty(\{E_k\}_{k\in\mathbb{Z}};\mu)$ is an operator space with its natural operator space structure. Then $l_1(\{E_k\}_{k\in\mathbb{Z}};\mu)$ becomes an operator space when it is considered as a subspace of $(l_\infty(\{E_k^*\}_{k\in\mathbb{Z}};\mu))^*$. Finally $l_p(\{E_k\}_{k\in\mathbb{Z}};\mu)$ becomes an operator space by complex interpolation:

$$l_p(\{E_k\}_{k\in\mathbb{Z}};\mu) = (l_1(\{E_k\}_{k\in\mathbb{Z}};\mu), l_\infty(\{E_k\}_{k\in\mathbb{Z}};\mu))_\theta, \quad \frac{1}{p} = 1-\theta.$$

Let (E_0, E_1) be a compatible couple of operator spaces. For $1 \leqslant p \leqslant \infty$, we define $N_p(E_0, E_1) := \{(x, -x) : \ x \in E_0 \cap E_1\}$ regarded as a subspace of $E_0 \oplus_p E_1$. Then we define

$$E_0 +_p E_1 := \frac{E_0 \oplus_p E_1}{N_p(E_0, E_1)}.$$

$K_p(t; E_0, E_1)$ denotes the operator space $E_0 +_p tE_1$; for any $x \in E_0 + E_1$, we let $K_p(x, t; E_0, E_1) := \|x\|_{E_0 +_p tE_1}$. Now we may give the definition of $E_{\theta,p;\underline{K}}$, the real interpolation of the compatible couple (E_0, E_1) with the discrete K-method, as follows:

$$\begin{aligned} E_{\theta,p;\underline{K}} &= (E_0, E_1)_{\theta,p;\underline{K}} \\ &= \Big\{x \in E_0 + E_1 : \|x\|_{\theta,p;\underline{K}} := \Big[\sum_{k\in\mathbb{Z}} (2^{-k\theta} K_p(x, 2^k; E_0, E_1))^p\Big]^{1/p} < \infty\Big\}. \end{aligned}$$

Then $E_{\theta,p;\underline{K}}$ is a Banach space.

If $\alpha \in \mathbb{R}$, then $l_p(2^{k\alpha})$ is the weighted space

$$l_p(2^{k\alpha}) := \Big\{x = (x_k)_{k\in\mathbb{Z}} : \ \|x\|_{l_p(2^{k\alpha})} = \Big(\sum_{k\in\mathbb{Z}} |2^{k\alpha} x_k|^p\Big)^{1/p} < \infty\Big\}.$$

If E is an operator space, then we similarly define $l_p(E)$ and $l_p(E; 2^{k\alpha})$ of sequences with values in E. Then $l_p(E)$ and $l_p(E; 2^{k\alpha})$ are operator spaces.

For each $k \in \mathbb{Z}$, let $F_k := K_p(2^k; E_0, E_1)$. Then we define $E_{\theta,p;\underline{K}}$, *the operator space interpolation of the couple* (E_0, E_1) *by the discrete K-method,* as a subspace of $l_p(\{F_k\}_{k\in\mathbb{Z}}; 2^{-k\theta})$ consisting of the constant sequences. More explicitly, let $x = (x_{i,j}) \in M_n((E_0, E_1)_{\theta,p;\underline{K}})$ for some $n \in \mathbb{N}$. Then

$$\|x\|_{M_n((E_0,E_1)_{\theta,p;\underline{K}})} := \inf\{\|(u,v)\|_{M_n(l_p(E_0;2^{-k\theta})\oplus_p l_p(E_1;2^{k(1-\theta)}))} : (u,v) \text{ satisfying (2.3)}\}.$$

$$u = (u_{i,j}) \in M_n(l_p(E_0; 2^{-k\theta})), \quad v = (v_{i,j}) \in M_n(l_p(E_1; 2^{k(1-\theta)})) \tag{2.3}$$

where each

$$u_{i,j} = (u^k_{i,j})_{k\in\mathbb{Z}} \quad \text{and} \quad v_{i,j} = (v^k_{i,j})_{k\in\mathbb{Z}}$$

such that

$$x_{i,j} = u^k_{i,j} + v^k_{i,j}, \quad \text{for each } k \in \mathbb{Z}, \;\; i,j = 1,\ldots,n.$$

3. MAIN THEOREM

3.1. NOTATION. 1. Let $(E_\alpha)_{\alpha\in I}$ be a family of operator spaces where I is some index set. Then $l^2\text{-}\bigoplus_\alpha E_\alpha$ and $l^2(I, E_\alpha)$ will denote the l^2- direct sum of E_α's and the complex interpolation operator space $(l^\infty(I, E_\phi), l^1(I, E_\phi))_{1/2}$, respectively.

2. Let $\mathfrak{A}$ be a completely contractive Banach algebra and X be an operator (bi- or) left $\mathfrak{A}$-module. For $a = (a_{i,j})$ and $x = (x_{i,j})$ in $M_n(\mathfrak{A})$ and $M_m(X)$, respectively, for some $n, m \in \mathbb{N}$, $x \star y$ will represent the matrix

$$x \star y = (x_{i,j} \cdot y_{k,l}) \tag{3.1}$$

where "$\cdot$" represents the module action of $\mathfrak{A}$ on X.

DEFINITION 3.1. Let $\mathfrak{A}$ be a completely contractive dual Banach algebra, $\phi \in M_n(\mathfrak{A}_*)$ for some $n \geqslant 1$. Suppose that for each $m \geqslant 1$, there is a matricial norm $\|\cdot\|_{\phi,m}$ on $M_m(\mathfrak{A}\cdot\phi)$. Let E_ϕ denote the completion of $(\mathfrak{A}\cdot\phi, \|\cdot\|_{\phi,1})$. Suppose that

$$\|a \star b\|_{\phi,mk} \leqslant \|a\|_m \|b\|_{\phi,k}, \tag{3.2}$$

$$\|a \star \phi\|_{mn} \leqslant \|a \star \phi\|_{\phi,m} \leqslant \|a\|_m \|\phi\|_n, \tag{3.3}$$

for all $a \in M_m(\mathfrak{A})$ and $b \in M_k(E_\phi), m, k \in \mathbb{N}$.

Furthermore, suppose that E_ϕ is reflexive and the inclusion $\iota_\phi : E_\phi \to M_n(\mathfrak{A}_*)$ injective. Then $(\|\cdot\|_{\phi,m})_{m=1}^\infty$ is called an *admissible operator norm* for ϕ.

Note that in the previous definition, the inequality (3.2) means that E_ϕ is an operator left $\mathfrak{A}$-module.

EXAMPLE 3.2. Let G_d be a discrete group and $\phi = (\phi_{i,j}) \in M_n(C^*(G_d))$ with $\|\phi\|_n = 1$ for some $n \geqslant 1$, where $C^*(G_d)$ is the group C^*-algebra of G_d.

Then each $\phi_{i,j}$ is a finite sum of the form

$$\phi_{i,j} = \sum_{g \in G_{\mathrm{d}}} \lambda_g^{i,j} \delta_g,$$

where each $\lambda_g^{i,j} \in \mathbb{C}$ and δ_g is the Dirac function. Consider E_ϕ with the usual norm on $M_n(C^*(G_{\mathrm{d}}))$. Clearly E_ϕ is a closed subspace of $M_n(C^*(G_{\mathrm{d}}))$. Hence, it is an operator space. Clearly E_ϕ is reflexive. Since E_ϕ is an operator $\mathfrak{A}$-module, (3.2) is satisfied. Since $B(G_{\mathrm{d}})$ is a completely contractive Banach algebra, (3.3) is also satisfied. Therefore, the usual norm on $M_n(C^*(G_{\mathrm{d}}))$ defines an admissible operator norm for ϕ.

THEOREM 3.3. *Let $\mathfrak{A}$ be a completely contractive dual Banach algebra and let $\phi = (\phi_{i,j}) \in M_n(\mathfrak{A}_*)$ have an admissible operator norm for some $n \geqslant 1$. Then there is a* w**-continuous, completely contractive representation of $\mathfrak{A}$ on $\mathcal{CB}(E_\phi)$.*

Proof. It is easy to see that E_ϕ is a left $\mathfrak{A}$-module. Moreover, ι_ϕ^* has dense range if and only if $\iota_\phi^{**} : E_\phi^{**} \to M_n(\mathfrak{A}^*)$ is injective. Since E_ϕ is reflexive, $\iota_\phi^{**} = \iota_\phi$. Hence, ι_ϕ^* has dense range. Note that

$$\iota_\phi^* : T_n(\mathfrak{A}) \to E_\phi^*$$

where $T_n(\mathfrak{A})$ is as defined in [5]. Now we define

$$S_\phi : E_\phi^* \widehat{\otimes} E_\phi \to M_{n^2}(\mathfrak{A}_*), \quad \iota_\phi^*(b) \otimes a \cdot \phi \mapsto a \cdot \phi \star b,$$

where $\widehat{\otimes}$ represents the projective tensor product of operator spaces.

Due to Definition 3.1, this map is well-defined. Then the map defined by

$$T_\phi := S_\phi^* : \ T_{n^2}(\mathfrak{A}) \to \mathcal{CB}(E_\phi)$$

is w*-continuous. Since $\mathfrak{A}$ is completely isometrically isomorphic to a closed subspace of $T_{n^2}(\mathfrak{A})$ by the map

$$\mathfrak{A} \to T_{n^2}(\mathfrak{A}), \quad a \mapsto (a_{i,j}) \tag{3.4}$$

where

$$a_{i,j} = \begin{cases} a & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise,} \end{cases}$$

T_ϕ induces a multiplicative representation from $\mathfrak{A}$ into $\mathcal{CB}(E_\phi)$. For simplicity, we will denote this representation again by T_ϕ. In order to see that T_ϕ is multiplicative on $\mathfrak{A}$, let $a, b \in \mathfrak{A}$, $c = (c_{i,j}) \in T_n(\mathfrak{A})$. Consider a as an element of $T_{n^2}(\mathfrak{A})$ via (3.4). Then

$$\begin{aligned} \langle T_\phi(a)(b \cdot \phi), \iota_\phi^*(c) \rangle &= \langle T_\phi(a), \iota_\phi^*(c) \otimes b \cdot \phi \rangle = \langle a, S_\phi(\iota_\phi^*(c) \otimes b \cdot \phi) \rangle \\ &= \langle a, b \cdot \phi \star c \rangle = \langle a, b \cdot \phi_{1,1} \cdot c_{1,1} \rangle. \end{aligned}$$

Hence $\langle T_\phi(a), b \cdot \phi \rangle = (x_{i,j}) \in M_n(\mathfrak{A}_*)$ where

$$x_{i,j} = \begin{cases} ab \cdot \phi_{1,1} & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Now let $a,b,d\in\mathfrak{A}$. Then

$$(T_\phi(d)T_\phi(a))(b\cdot\phi)=T_\phi(d)(T_\phi(a)(b\cdot\phi))=T_\phi(d)((x_{i,j}))=(y_{i,j})=T_\phi(da)(b\cdot\phi)\in E_\phi$$

where

$$y_{i,j}=\begin{cases} dab\cdot\phi_{1,1} & \text{if } (i,j)=(1,1),\\ 0 & \text{otherwise.}\end{cases}$$

Hence T_ϕ is multiplicative on $\mathfrak{A}$.

By using Effros and Ruan ([5], Proposition 7.1.2), S_ϕ is a complete contraction if and only if the induced map $\widetilde{S}_\phi\in B(E_\phi^*\times E_\phi, M_{n^2}(\mathfrak{A}_*))$ is a completely contractive bilinear mapping. Now

$$\begin{aligned}\|\widetilde{S}_\phi\|_{\mathrm{cb}}=\sup\{&\|\widetilde{S}_\phi^{(m,m)}(x,y)\| : x=(x_{i,j})\in M_m(E_\phi^*),\ y=(y_{i,j})\in M_m(E_\phi),\ \|x\|_{\phi,m},\\ &\|y\|_{\phi,m}\leqslant 1,\ m\in\mathbb{N}\}\\ =\sup\{&|\langle\langle\widetilde{S}_\phi^{(m,m)}(x,y),z\rangle\rangle| : x=(x_{i,j})\in M_m(E_\phi^*),\ y=(y_{i,j})\in M_m(E_\phi),\\ &z=(z_{i,j})\in M_{m^2}(M_{n^2}(\mathfrak{A})),\ \|x\|_{\phi,m},\ \|y\|_{\phi,m},\ \|z\|_{m^2n^2}\leqslant 1,\ m\in\mathbb{N}\}.\end{aligned}$$

By the density of the range of ι_ϕ^*, for each $i,j=1,\dots,m$, without loss of generality we may suppose that

$$x_{i,j}=\iota_\phi^*(A_{i,j}),\quad y_{i,j}=b_{i,j}\cdot\phi,$$

where

$$A_{i,j}=(a_{i,j}^{k,l})\in T_n(\mathfrak{A})\quad\text{and}\quad b_{i,j}\in\mathfrak{A}.$$

Then we have

$$\langle\langle\widetilde{S}_\phi^{(m,m)}(x,y),z\rangle\rangle=(\langle\widetilde{S}_\phi(x_{i,j},y_{k,l}),z_{s,t}\rangle)=(\langle b_{k,l}\cdot\phi\star A_{i,j},z_{s,t}\rangle).$$

Since

$$\langle b_{k,l}\cdot\phi_{o,p}\cdot a_{i,j}^{q,r},z_{s,t}\rangle=\langle\phi_{o,p}\cdot a_{i,j}^{q,r},z_{s,t}b_{k,l}\rangle=\langle a_{i,j}^{q,r},z_{s,t}b_{k,l}\cdot\phi_{o,p}\rangle$$

for all indices i,j,k,l,m,n,o,p,q and r where

$$i,j,k,l=1,\dots,m,\quad o,p,q,r=1,\dots,n,\quad\text{and}\quad s,t=1,\dots,m^2n^2,$$

we conclude that

$$|\langle\langle\widetilde{S}_\phi^{(m,m)}(x,y),z\rangle\rangle|=|\langle\iota_\phi^*(A_{i,j}),z_{m,n}b_{k,l}.\phi\rangle|=|\langle\langle(\iota_\phi^*(A_{i,j})),z\star y\rangle\rangle|.$$

On the other hand,

$$\|z\star y\|_{\phi,m^3n^2}\leqslant\|z\|_{m^2n^2}\|y\|_{\phi,m}\leqslant 1.$$

Therefore,

$$|\langle\langle\widetilde{S}^{m,m}(x,y),z\rangle\rangle|\leqslant\|(\iota_\phi^*(A_{i,j}))\|_{\phi,m}=\|x\|_{\phi,m}\leqslant 1.$$

Thus, $\widetilde{S}_\phi$ is a complete contraction. ∎

Let $(E_\alpha)_{\alpha\in I}$ be a family of operator spaces and let $E = l_2(I, E_\alpha)$. We will need an approximation for the norm of an arbitrary element on each matrix level of E. To manage this, we will need the following two propositions.

PROPOSITION 3.4. *If $(E_\alpha)_{\alpha\in I}$ is a family of operator spaces, then:*
(i) $l^\infty(I, M_n(E_\alpha)) \cong M_n(l^\infty(I, E_\alpha))$ *for every* $n \in \mathbb{N}$.
(ii) *If* $A = (a_{i,j}) \in M_n(l^1(I, E_\alpha))$ *where* $a_{i,j} = (a^\alpha_{i,j})_\alpha$, $a^\alpha_{i,j} \in E_\alpha$ *for each i and j, then*

$$\|A\|_{M_n(l^1(I,E_\alpha))} \leqslant \sum_\alpha \|(a^\alpha_{i,j})\|_{M_n(E_\alpha)} \text{ for every } n \in \mathbb{N}.$$

Proof. The first identity is obvious. Hence, we will prove only the second one. Let A be as in the claim. Then we have

$$\begin{aligned}
\|A\|_{M_n(l^1(I,E_\alpha))} &= \sup\{|\langle\langle A, F\rangle\rangle| : F \in M_n(l^\infty(I, E^*_\alpha)), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leqslant 1\} \\
&= \sup\{|(\langle a_{i,j}, f_{k,l}\rangle)| : F \in M_n(l^\infty(I, E^*_\alpha)), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leqslant 1\} \\
&= \sup\Big\{\Big|\Big(\sum_\alpha \langle f^\alpha_{k,l}, a^\alpha_{i,j}\rangle\Big)\Big| : F \in M_n(l^\infty(I, E^*_\alpha)), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leqslant 1\Big\} \\
&\leqslant \sup\Big\{\sum_\alpha |(\langle f^\alpha_{k,l}, a^\alpha_{i,j}\rangle)| : F \in M_n(l^\infty(I, E^*_\alpha)), \|F\|_{M_n(l^\infty(I,E^*_\alpha))} \leqslant 1\Big\} \\
&\leqslant \sum_\alpha \|(a^\alpha_{i,j})\|_{M_n(E_\alpha)}. \quad \blacksquare
\end{aligned}$$

Next proposition will be needed to prove Lemma 3.6 and Theorem 3.9.

PROPOSITION 3.5. *Let (X, Z) be a compatible couple of Banach spaces in the sense of Banach space interpolation. Suppose that there is a contractive embedding from a Banach space Y into Z. Let $E_1 := (X, Y)_\theta$ and $E_2 := (X, Z)_\theta$ for some $0 < \theta < 1$. Then for every $a \in E_1$, we have $\|a\|_{E_1} \geqslant \|a\|_{E_2}$.*

Proof. Clearly, any function $f : \mathbb{C} \to X +_1 Y$ satisfying (2.1) can be viewed as a function from $\mathbb{C}$ to $X +_1 Z$, and it will satisfy analogous properties. To distinguish these two functions, we will denote them by $f_{X,Y}$ and $f_{X,Z}$, respectively. Recall that

$$\|f_{X,Y}\| = \max\{\sup\{\|f(\mathrm{i}t)\|_X\}, \sup\{\|f(1+\mathrm{i}t)\|_Y\} : t \in \mathbb{R}\},$$
$$\|f_{X,Z}\| = \max\{\sup\{\|f(\mathrm{i}t)\|_X\}, \sup\{\|f(1+\mathrm{i}t)\|_Z\} : t \in \mathbb{R}\}.$$

This shows that $E_1 \subset E_2$.

Now let $a \in E_1$. Recall that

$$\|a\|_{E_1} := \inf\{\|f_{X,Y}\| : a = f(\theta),\ f \text{ satisfying (2.1) }\}.$$

($\|a\|_{E_2}$ is defined in a similar fashion). Since for each f satisfying (2.1) we have, $\|f_{X,Y}\| \geqslant \|f_{X,Z}\|$, we conclude that $\|a\|_{E_1} \geqslant \|a\|_{E_2}$. $\blacksquare$

Proposition 3.4 and Proposition 3.5 show that:

LEMMA 3.6. *Let $(E_\alpha)_{\alpha\in I}$ be a family of operator spaces and let $A = (a_{i,j})$ be in $M_n(E)$ where $E = l_2(I, E_\alpha)$, for some $n \in \mathbb{N}$. If each $a_{i,j}$ is of the form $a_{i,j} = (a^\alpha_{i,j})_\alpha$, then $\|A\|_{M_n(E)} \leqslant \sqrt{\sum_\alpha \|A_\alpha\|^2_{M_n(E_\alpha)}}$ where each $A_\alpha = (a^\alpha_{i,j}) \in M_n(E_\alpha)$. That is, the canonical inclusion from $l_2(I, M_n(E_\alpha))$ into $M_n(l_2(I, E_\alpha))$ is a contraction.*

THEOREM 3.7. *Let $\mathfrak{A}$ be a completely contractive dual Banach algebra. Then for each $n \in \mathbb{N}$, every non-zero element in the unit ball of $M_n(\mathfrak{A}_*)$ has an admissible operator norm.*

Proof. Suppose that $\mathfrak{A}$ is a completely contractive dual Banach algebra and $\phi \in M_n(\mathfrak{A}_*)$, $\phi \neq 0$, $\|\phi\|_n \leqslant 1$, for some $n \geqslant 1$. The map

$$R_\phi : \mathfrak{A} \to \mathfrak{A}\cdot\phi, \quad a \mapsto a\cdot\phi,$$

is a complete contraction. Then the induced map $\pi : \mathfrak{A}/\ker R_\phi \to A\cdot\phi$ is a complete isometry. For each $m \geqslant 1$, define a norm $\|\cdot\|_{\mathfrak{A}.\phi,m}$ on $M_m(\mathfrak{A}\cdot\phi)$ via

$$\|x\|_{\mathfrak{A}\cdot\phi,m} := \inf\{\|a\|_m : x = a\star\phi,\ a \in M_m(\mathfrak{A})\}.$$

Then $\mathfrak{A}\cdot\phi$ becomes an operator space with this matricial norm.

Clearly $(\mathfrak{A}\cdot\phi, M_n(\mathfrak{A}_*))$ is a compatible couple of operator spaces. Now define E_ϕ to be the space of constant sequences in $l_2(\{F_k\}_{k\in\mathbb{N}}; 2^{-k/2})$, where $F_k = K_2(2^k; \mathfrak{A}\cdot\phi, M_n(\mathfrak{A}_*))$ for each $k \in \mathbb{N}$. By Section 2.3, Proposition 1 of [1], we know that

$$\begin{aligned}
&(\mathfrak{A}\cdot\phi, M_n(\mathfrak{A}_*))_{1/2,2;\underline{K}} \text{ is reflexive}\\
&\quad\iff \text{the inclusion } \mathfrak{A}\cdot\phi \to M_n(\mathfrak{A}_*) \text{ is weakly compact}\\
&\quad\iff \text{the map } R_\phi : \mathfrak{A} \to M_n(\mathfrak{A}^*),\ a \mapsto a\cdot\phi \text{ is weakly compact.}
\end{aligned}$$

However, $\mathrm{Im}(R_\phi) \subseteq M_n(\mathfrak{A}_*)$ and $M_n(\mathfrak{A}_*) \subseteq WAP(M_n(\mathfrak{A}^*))$, by [14]. (Recall that, if $\mathfrak{A}$ is a Banach algebra, then $WAP(\mathfrak{A}^*)$ denotes the space of weakly almost periodic functionals on $\mathfrak{A}$). Hence R_ϕ is weakly compact. Therefore, as a closed subspace of $E = (\mathfrak{A}\cdot\phi, M_n(\mathfrak{A}_*))_{1/2,2;\underline{K}}$, E_ϕ is reflexive too.

Let $\|\cdot\|_{\phi,m}$ denote the norm on $M_m(E_\phi)$, for every $m \in \mathbb{N}$. If $f \in E_\phi$, then

$$\begin{aligned}
\|f\|_{\phi,1} &= \Big[\sum_{k\in\mathbb{N}} 2^{-k}\|f\|^2_{F_k}\Big]^{1/2} = \Big[\sum_{k\in\mathbb{N}} 2^{-k} \inf_{b\in\mathfrak{A}} \Big\{\sqrt{\|b\cdot\phi\|^2_{\mathfrak{A}\cdot\phi,1} + 2^{2k}\|f - b\cdot\phi\|^2_n}\Big\}^2\Big]^{1/2}\\
&= \Big[\sum_{k\in\mathbb{N}} 2^{-k} \inf_{b\in\mathfrak{A}} \{\|b\cdot\phi\|^2_{\mathfrak{A}\cdot\phi,1} + 2^{2k}\|f - b\cdot\phi\|^2_n\}\Big]^{1/2}\\
&= \Big[\sum_{k\in\mathbb{N}} \inf_{b\in\mathfrak{A}} \{2^{-k}\|b\cdot\phi\|^2_{\mathfrak{A}\cdot\phi,1} + 2^{k}\|f - b\cdot\phi\|^2_n\}\Big]^{1/2}.
\end{aligned}$$

Hence,

$$f \in E_\phi \iff \sum_{k\in\mathbb{N}} \inf\{2^{-k}\|b\cdot\phi\|^2_{\mathfrak{A}\cdot\phi,1} + 2^k\|f - b\cdot\phi\|^2_n : b \in \mathfrak{A}\} < \infty.$$

Thus there exists a sequence $(b_k)_k$ in $\mathfrak{A}$ such that $2^{2k}\|f - b_k \cdot \phi\|_n^2 \to 0$. Hence $\mathfrak{A} \cdot \phi$ is dense in E_ϕ. This shows that $\|a \cdot \phi\|_n \leqslant \|a \cdot \phi\|_{\phi,1}$. Now to prove (3.3), we will use the following claim.

CLAIM 3.8. *If $\mathfrak{A} \cdot \phi$ is dense in E_ϕ, then $M_m(\mathfrak{A} \cdot \phi)$ is dense in $M_m(E_\phi)$ for every $m \geqslant 1$.*

Proof. Let $\varepsilon > 0$ and $F = (f_{i,j}) \in M_m(E_\phi)$ for some $m \geqslant 1$. Then for each $i, j = 1, \ldots, m$, there exists a sequence $(b_{i,j}^k)_k$ in $\mathfrak{A}$ such that $b_{i,j}^k \cdot \phi \to f_{i,j}^k$ in $\|\cdot\|_n$. Consider the sequence $(F_k)_k$ in $M_m(\mathfrak{A} \cdot \phi)$ where each $F_k = (b_{i,j}^k)_k$. Then we have

$$\|F - f_k\|_{mn} \leqslant \sum_{i,j=1}^{m} \|f_{i,j} - b_{i,j}^k \cdot \phi\|_n \to 0 \quad \text{as } k \to \infty. \quad \blacksquare$$

Hence, by Claim 3.8, we have $\|a \star \phi\|_{mn} \leqslant \|a \star \phi\|_{\phi,m}$ for all $a \in M_m(\mathfrak{A})$ and $m \in \mathbb{N}$.

Let $a \in M_m(\mathfrak{A})$ for some $m \geqslant 1$. Then by the definition of $\|\cdot\|_{\mathfrak{A}\cdot\phi,m}$, it is clear that $\|a \star \phi\|_{\mathfrak{A}\cdot\phi,m} \leqslant \|a\|_m$. Since $\mathfrak{A}$ is a completely contractive dual Banach algebra, we also have $\|a \star \phi\|_{mn} \leqslant \|a\|_m \|\phi\|_n \leqslant \|a\|_m$. By Lemma 3.6, we have

$$\|a \star \phi\|_{F_k} \leqslant \inf\{[\|b \star \phi\|_{\mathfrak{A}\cdot\phi,m}^2 + 2^{2k}\|a \star \phi - b \star \phi\|_{mn}^2]^{1/2} : b \in M_m(\mathfrak{A})\}.$$

By choosing $b = a$, we see that

$$\|a \star \phi\|_{F_k} \leqslant \|a \star \phi\|_{\mathfrak{A}\cdot\phi,m} \quad \text{for each } k.$$

Then we have

$$\|a \star \phi\|_{\phi,m} \leqslant \Big[\sum_{k\in\mathbb{N}} 2^{-k}\|a \star \phi\|_{F_k}^2\Big]^{1/2} \leqslant \|a \star \phi\|_{\mathfrak{A}\cdot\phi,m}\Big[\sum_{k\in\mathbb{N}} 2^{-k}\Big]^{1/2} = \|a \star \phi\|_{\mathfrak{A}\cdot\phi,m} \leqslant \|a\|_m.$$

Now let $a \in M_m(\mathfrak{A}), b \in M_t(\mathfrak{A})$ for some $m, t \geqslant 1$. Since the map π is a complete isometry, we have

$$\begin{aligned}\|b \star (a \star \phi)\|_{\mathfrak{A}\cdot\phi,mt} &= \|b \star a + M_{mt}(\ker R_\phi)\| = \inf\{\|b \star a + x\|_{mt} : x \in M_{mt}(\ker R_\phi)\} \\ &\leqslant \inf\{\|b \star a + b \star x\|_{mt} : x \in M_m(\ker R_\phi)\} \\ &\leqslant \|b\|_t \inf\{\|a + x\|_m : x \in M_m(\ker R_\phi)\} \\ &= \|b\|_t \|a + \ker R_\phi\| = \|b\|_t \|a \star \phi\|_{\mathfrak{A}\cdot\phi,m}.\end{aligned}$$

This shows that $\mathfrak{A} \cdot \phi$ is an operator left $\mathfrak{A}$-module. Since $M_n(\mathfrak{A}_*)$ is also an operator left $\mathfrak{A}$-module, so is E_ϕ. Therefore,

$$\|b \star d\|_{\phi,mt} \leqslant \|b\|_t \|d\|_{\phi,m} \quad \text{for every } d \in M_m(E_\phi),\ b \in M_m(\mathfrak{A}),\ m, t \in \mathbb{N}. \quad \blacksquare$$

THEOREM 3.9. *Let $\mathfrak{A}$ be a completely contractive dual Banach algebra. Then there is a* w*-continuous complete isometric representation of $\mathfrak{A}$ on $\mathcal{CB}(E)$ for some reflexive operator space E.*

Proof. Let $\dot{E} := l^2 - \bigoplus_{\phi\in\mathfrak{I}} E_\phi$. For $n \geqslant 1$, equip $M_n(\dot{E})$ with

$$\|A\|_{M_n(\dot{E})} = \Big(\sum_{\phi} \|(a^{\phi}_{i,j})\|^2_{\phi,n}\Big)^{1/2}$$

where $A = (a_{i,j}) \in M_n(\dot{E})$, $a_{i,j} = (a^{\phi}_{i,j})_\phi$, $a^{\phi}_{i,j} \in E_\phi$. Note that $\dot{E}$ is not an operator space. There is a natural map

$$S : \mathfrak{A} \to B(\dot{E}) \quad \text{defined by } S(a)((x_\phi)) := (T_\phi(a)(x_\phi)),$$

where $T_\phi : \mathfrak{A} \to \mathcal{CB}(E_\phi)$ is the w*-continuous complete contraction as defined in Theorem 3.3.

Note that Daws ([4], Theorem 4.5) proved that $S : \mathfrak{A} \to B(\dot{E})$ is an isometry. For an arbitrary $n \geqslant 1$, any element $(a_{i,j})$ of $M_n(\mathfrak{A})$ can be viewed as a map

$$S_n : \dot{E} \to M_n(\dot{E}).$$

We claim that S_n is a contraction. Let $\|(a_{i,j})\|_n \leqslant 1$ and $(x_\phi) \in \dot{E}$. Then

$$\begin{aligned}\|S_n(x_\phi)\|_{M_n(\dot{E})} &= \|(S(a_{i,j})((x_\phi)))\|_{M_n(\dot{E})} = \|((T_\phi(a_{i,j})(x_\phi)))\|_{M_n(\dot{E})} \\ &= \Big[\sum_{\phi} \|((T_\phi(a_{i,j})(x_\phi)))\|^2_{\phi,n}\Big]^{1/2} \leqslant \Big[\sum_{\phi} \|(a_{i,j})\|^2_n \|x_\phi\|^2_{\phi,1}\Big]^{1/2} \\ &= \|(a_{i,j})\|_n \Big[\sum_{\phi} \|x_\phi\|^2_{\phi,1}\Big]^{1/2} = \|(a_{i,j})\|_n \|(x_\phi)\|_{\dot{E}} \leqslant \|(x_\phi)\|_{\dot{E}}.\end{aligned}$$

Thus, S_n is a contraction.

Now let $E = l^2(\mathfrak{I}, E_\phi)$. We define

$$T : \mathfrak{A} \to \mathcal{CB}(E) \quad \text{by } T(a)((x_\phi)) = (T_\phi(a)(x_\phi)). \tag{3.5}$$

Since T_ϕ is a representation for each $\phi \in \mathfrak{I}$, so is T. For each n, we have $T^{(n)} : M_n(\mathfrak{A}) \to \mathcal{CB}(E, M_n(E))$. On the other hand, by its definition

$$M_n(E) = \big(M_n(l^\infty(\mathfrak{I}, E_\phi)), M_n(l^1(\mathfrak{I}, E_\phi))\big)_{1/2} = \big(l^\infty(\mathfrak{I}, M_n(E_\phi)), M_n(l^1(\mathfrak{I}, E_\phi))\big)_{1/2}.$$

Each $(a_{i,j}) \in M_n(\mathfrak{A})$ defines a map from E into

$\big(l^\infty(\mathfrak{I}, M_n(E_\phi)), l^1(\mathfrak{I}, M_n(E_\phi))\big)_{1/2} = l^2 - \bigoplus_{\phi\in\mathfrak{I}} M_n(E_\phi)$ (on the Banach space level).

Hence, we have a natural map (which we will denote by $\widetilde{T}^n$)

$$\widetilde{T}^n : M_n(\mathfrak{A}) \to B(E, l^2 - \bigoplus_{\phi\in\mathfrak{I}} M_n(E_\phi)).$$

However, this map is a contraction for every $n \geqslant 1$. On the other hand, by Proposition 3.5 we have $\|\widetilde{T}^n\| \geqslant \|T^{(n)}\|$. Hence, $T^{(n)}$ is a contraction for every $n \geqslant 1$. Thus T is a complete contraction.

Note that without loss of generality we may suppose that $\mathfrak{A}$ is unital; let e denote its identity. For each $n \geqslant 1$, we have

$$T^{(n)} : M_n(\mathfrak{A}) \to M_n(\mathcal{CB}(E)) = M_n((E^* \widehat{\otimes} E)^*) = \mathcal{CB}(E^* \widehat{\otimes} E, M_n).$$

Let $a = (a_{i,j}) \in M_n(\mathfrak{A})$. Then for every $\varepsilon > 0$, there is $\phi = (\phi_{i,j}) \in M_n(\mathfrak{A}_*)$ such that

$$\|\phi\|_n \leqslant 1 \quad \text{and} \quad |\langle\langle a, \phi \rangle\rangle| \geqslant (1-\varepsilon)\|a\|_n.$$

For simplicity, set $\overline{T} := T^{(n)}(a) \in \mathcal{CB}(E^* \widehat{\otimes} E, M_n)$. Define $x = (x_{i,j}) \in M_n(E^* \widehat{\otimes} E)$ by

$$x_{i,j} = \begin{cases} (\ldots, \iota_\phi^*(B), \ldots) \otimes (\ldots, e \cdot \phi, \ldots) & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$B = (b_{i,j}) \in T_n(\mathfrak{A}) \quad \text{is defined by } b_{i,j} = \begin{cases} e & \text{if } (i,j) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

Now we have

$$\|x\|_{M_n(E^* \widehat{\otimes} E)} = \|x_1\|_{E^* \widehat{\otimes} E} \leqslant \|(\ldots, \iota_\phi^*(B), \ldots)\|_{E^*} \|(\ldots, e \cdot \phi, \ldots)\|_E.$$

On the other hand,

$$\|(\ldots, \iota_\phi^*(B), \ldots)\|_{E^*} \leqslant 1 \quad \text{since } \|B\|_{T_n(\mathfrak{A})} \leqslant 1 \text{ and } \iota_\phi^* \text{ is a contraction.}$$

Clearly

$$\|(\ldots, e \cdot \phi, \ldots)\|_E \leqslant \|\phi\|_n \leqslant 1.$$

Then

$$|\overline{T}^{(n)}(x)| = |(\overline{T}(x_{1,1}))| = |(T(a_{i,j})(x_{k,l}))| = |\langle\langle a, \phi \rangle\rangle| \geqslant (1-\varepsilon)\|a\|_n. \quad \blacksquare$$

Acknowledgements. This paper is part of the author's Ph.D. Dissertation written at the University of Alberta under the supervision of Volker Runde. While being in the final stages of the preparation of this manuscript, we learned that our representation theorem had also been independently proven by Daws.

I would like to thank my supervisor Volker Runde for his valuable recommendations and for giving me the reference to the paper [15].

REFERENCES

[1] B. BEAUZAMY, *Espaces d'Interpolation Réels: Topologie et Géométrie* [French], Lecture Notes in Math., vol. 666, Springer Verlag, Berlin 1978.

[2] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces. An introduction.*, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin-New York 1976.

[3] D.P. BLECHER, C. LE MERDY, *Operator Algebras and their Modules — An operator space approach*, Oxford Univ. Press, Oxford 2004.

[4] M. DAWS, Dual Banach algebras: representations and injectivity, to appear.

[5] E.G. EFFROS, Z.-J. RUAN, *Operator Spaces*, London Math. Soc. Monographs (N. S.), vol. 23, The Clarendon Press, Oxford Univ. Press, New York 2000.

[6] P. EYMARD, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92**(1964), 181–236.

[7] F. GHAHRAMANI, Isometric representations of $M(G)$ on $B(H)$, *Glasgow Math. J.* **23**(1982), 119–122.

[8] J. KRAUS, Z.-J. RUAN, Multipliers of Kac algebras, *International J. Math.* **8**(1997), 213-248.

[9] M. NEUFANG, Z.-J. RUAN, N. SPRONK, Completely isometric representations of $M_{\mathrm{cb}}A(G)$ and $UCB(\widehat{G})^*$, *Trans. Amer. Math. Soc.*, to appear.

[10] G. PISIER, The operator Hilbert space OH, complex interpolation and tensor norms, *Mem. Amer. Math. Soc.* **585**(1996).

[11] G. PISIER, Non-commutative vector valued L_p-spaces and completely p-summing maps, *Astérisque* **247**(1998).

[12] Z.-J. RUAN, Subspaces of C^*-algebras, *J. Funct. Anal.* **76**(1988), 217–230.

[13] V. RUNDE, Amenability for dual Banach algebras, *Studia Math.* **148**(2001), 47–66.

[14] V. RUNDE, Dual Banach algebras: Connes-amenability, normal, virtual diagonals, and injectivity of the predual bimodule, *Math. Scand.* **95**(2004), 124–144.

[15] Q. XU, Interpolation of operator spaces, *J. Funct. Anal.* **139**(1996), 500–539.

FARUK UYGUL, DEPARTMENT OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA, EDMONTON, T6G 2G1, CANADA
E-mail address: fuygul@ualberta.ca

Received February 5, 2007.