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TYPE II_1 FACTORS WITH A SINGLE GENERATOR

JUNHAO SHEN

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ABSTRACT. In the paper, we study the generator problem for type II_1 factors. By defining an invariant closely related to the number of generators of a von Neumann algebra, we are able to show that a large class of type II_1 factors are singly generated, i.e., generated by two self-adjoint elements.

KEYWORDS: *Generator problem, type II_1 factor, free entropy dimension.*

MSC (2000): Primary 46L10; Secondary 46L54.

1. INTRODUCTION

Let H be a separable complex Hilbert space, $\mathcal{B}(H)$ be the algebra consisting of all bounded linear operators from H to H. A von Neumann algebra $\mathcal{M}$ is defined to be a self-adjoint subalgebra of $\mathcal{B}(H)$ which is closed in the strong operator topology. Factors are the von Neumann algebras whose centers are scalar multiples of the identity. The factors are classified by means of a relative dimension function into type I, II, III factors. (see [10])

The generator problem for von Neumann algebras asks whether every von Neumann algebra acting on a separable Hilbert space can be generated by two self-adjoint elements (equivalently be singly generated). It is a long-standing open problem (see [9]), and is still unsolved. Many people (see [3], [4], [6], [10], [14], [15], [16], [22]) have contributed to this topic. For example, von Neumann [11] proved that every abelian von Neumann algebra is generated by one self-adjoint element and every type II_1 hyperfinite von Neumann algebra is singly generated. W. Wogen [22] showed that every properly infinite von Neumann algebra is singly generated. It follows that the generator problem for von Neumann algebras, except for the non hyperfinite type II_1 von Neumann algebras, is solved (see [18] for a good introduction of the history). Theorem 3.5 in [16] shows that a type II_1 factor with Cartan subalgebras is singly generated. Theorem 6.2 in [6] proves that certain type II_1 factors are singly generated. These type II_1 factors include the ones with property Γ, those that are not prime. In [4], Ge and the author proved that some type II_1 factors with property T, including $L(SL(\mathbb{Z}, 2m+1))$ ($m \geqslant 1$), are singly generated. This result answered a question proposed by Voiculescu.

In the early 1980s, D. Voiculescu began the development of the theory of free probability and free entropy. This new and powerful tool was crucial in solving some old open problems in the field of von Neumann algebras. In his influential paper [19], Voiculescu introduced $\delta_0(\mathcal{M})$, called "free entropy dimension" of a finite von Neumann algebra $\mathcal{M}$, by which Voiculescu was able to show that free group factors have no Cartan subalgebras [20]. To better understand the free entropy dimension of von Neumann algebras has become an urgent task for the subject.

On the other hand, it is believed that the free entropy dimension is closely related to the number of generators of a von Neumann algebra. It is expected that a type II_1 factor, whose free entropy dimension is equal to 1, is then singly generated. The goal of this paper is to consider the generator problem for type II_1 factors from the point of view of free entropy theory.

To measure the number of generators of a diffuse finite von Neumann algebra $\mathcal{M}$, we introduce a new von Neumann algebra invariant $\mathcal{G}(\mathcal{M})$, whose definition is motivated by Voiculescu's approach to Cartan subalgebra problems in [20]. This invariant $\mathcal{G}(\mathcal{M})$ enjoys many good properties, some of which are listed as follow:

(i) If $\mathcal{M}$ is a type II_1 factor and $\mathcal{G}(\mathcal{M}) < 1/4$, then $\mathcal{M}$ is singly generated.

(ii) If $\mathcal{M}$ is a diffuse hyperfinite von Neumann algebra with a tracial state τ, then $\mathcal{G}(\mathcal{M}) = 0$.

(iii) $\mathcal{G}(\mathcal{M}) = 0$ if the type II_1 factor $\mathcal{M}$ is generated by a family of von Neumann subalgebras $\{\mathcal{N}_j\}_{j=1}^{\infty}$ of $\mathcal{M}$ such that $\mathcal{G}(\mathcal{N}_j) = 0$ and $\mathcal{N}_j \cap \mathcal{N}_{j+1}$ is a diffuse von Neumann subalgebra for all $j \geqslant 1$;

(iv) $\mathcal{G}(\mathcal{M}) = 0$ if the type II_1 factor $\mathcal{M}$ is generated by $\{\mathcal{N}, u_1, \ldots, u_j, \ldots\}$, where $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with $\mathcal{G}(\mathcal{N}) = 0$ and $\{u_j\}_{j=1}^{\infty}$ is a family of unitary elements of $\mathcal{M}$ such that, for every $j \geqslant 1$, $u_j^* v_j u_j$ is in $\mathcal{N}$ for some Haar unitary element v_j in $\mathcal{N}$;

(v) $\mathcal{G}(\mathcal{M}) = 0$ if the type II_1 factor $\mathcal{M}$ is generated by an ascending sequence of subalgebras $\{\mathcal{N}_k\}_{k=1}^{\infty}$ such that $\mathcal{G}(\mathcal{N}_k) = 0$ for all $k \geqslant 1$.

Using the listed properties of $\mathcal{G}(\mathcal{M})$, we are able to compute its values for a large class of II_1 factors. In fact, we show that $\mathcal{G}(\mathcal{M}) = 0$ if $\mathcal{M}$ is one of the followings: type II_1 factors with Cartan subalgebras, those with property Γ, non-prime factors or some II_1 factors with property T. By property (i) this implies that all these type II_1 factors $\mathcal{M}$ are singly generated. (see Theorem 5.5, 5.9, 5.10, 5.11 and 5.14) (Unfortunately we are not able to compute $\mathcal{G}(L(F_n))$ where $L(F_n)$ is the free group factor on n ($n \geqslant 2$) generators.) Also, we show that $\mathcal{G}(\mathcal{M}) = 0$ if $\mathcal{M}$ is one of the type II_1 factors considered in [7]. As corollaries, we extend the results in [16], [6] and [4] (see Corollary 5.6, 5.7, 5.12 and 5.15) and provide new examples of II_1 factors which are singly generated (see Example 5.19).

The organization of the paper is as follows. In Section 2, we introduce our new invariant $\mathcal{G}(\mathcal{M})$ of a diffuse von Neumann algebra $\mathcal{M}$. The value of $\mathcal{G}(\mathcal{M})$, when $\mathcal{M}$ is a diffuse hyperfinite von Neumann algebra, is computed in Section 3. A cut-and-paste theorem is proved in Section 4. In Section 5, we prove our main results of the paper. We also give some examples of type II_1 factors which are singly generated in this section.

In the paper, for a subset $\mathcal{S}$ of $\mathcal{B}(H)$, we denote $W^*(S)$ the von Neumann algebra generated by the elements of $\mathcal{S} \cup \mathcal{S}^*$ in $\mathcal{B}(H)$.

2. DEFINITION OF $\mathcal{G}(\mathcal{M})$

In this section, we will introduce a von Neumann algebra invariant, which is closely related to the number of the generators of this von Neumann algebra.

DEFINITION 2.1. Suppose that $\mathcal{M}$ is a diffuse von Neumann algebra with a faithful normal tracial state τ. Let $\{p_j\}_{j=1}^k$ be a family of mutually orthogonal projections of $\mathcal{M}$ with $\tau(p_j) = 1/k$ for each $1 \leqslant j \leqslant k$. For each element x of $\mathcal{M}$, we define

$$\mathcal{I}(x; \{p_j\}_{j=1}^k) = \frac{|\{(i,j) \mid p_i x p_j \neq 0\}|}{k^2},$$

where $|\cdot|$ denotes the cardinality of the set. The *support of x on $\{p_j\}_{j=1}^k$* is defined by

$$\mathcal{S}(x; \{p_j\}_{j=1}^k) = \bigvee \{p_j \mid p_j x \neq 0, \text{ or } x p_j \neq 0,\ 1 \leqslant j \leqslant k\},$$

where $\bigvee$ denotes the union of the projections. For elements $x_1, \ldots, x_n$ in $\mathcal{M}$, we define

$$\mathcal{I}(x_1, \ldots, x_n; \{p_j\}_{j=1}^k) = \sum_{m=1}^{n} \mathcal{I}(x_m; \{p_j\}_{j=1}^k).$$

DEFINITION 2.2. For each positive integer k, let

$$\mathfrak{E}_k = \{\{p_j\}_{j=1}^k \mid \{p_j\}_{j=1}^k \text{ is a family of mutually orthogonal projections of } \mathcal{M}\ . \\ \text{with } \tau(p_j) = 1/k \text{ for each } 1 \leqslant j \leqslant k\}.$$

Let $x_1, \ldots, x_n$ be the elements in $\mathcal{M}$. We define

$$\mathcal{I}(x_1, \ldots, x_n; k) = \inf \{\mathcal{I}(x_1, \ldots, x_n; \{p_j\}_{j=1}^k) \mid \{p_j\}_{j=1}^k \in \mathfrak{E}_k\}; \text{ and}$$

$$\mathcal{G}(\mathcal{M}; k) \\ = \begin{cases} \inf\{\mathcal{I}(x_1, \ldots, x_n; k) \mid x_1, \ldots, x_n \text{ generate } \mathcal{M} \text{ as a von Neumann algebra}\}, \\ \infty \quad \text{if } \mathcal{M} \text{ is not finitely generated.} \end{cases}$$

Finally, we define

$$\mathcal{G}(\mathcal{M}) \; = \; \liminf_{k\to\infty} \; \mathcal{G}(\mathcal{M};k).$$

REMARK 2.3. By the definition, for every $k>1$, we know that $\mathcal{G}(\mathcal{M};k^n)$ is a decreasing function as n increases. Thus, $\mathcal{G}(\mathcal{M}) \leqslant \mathcal{G}(\mathcal{M};k) \leqslant \mathcal{G}(x_1,\ldots,x_n;k)$ for each $k \geqslant 1$, each family of generators $\{x_1,\ldots,x_n\}$ of $\mathcal{M}$.

3. $\mathcal{G}(\mathcal{M})$ WHEN $\mathcal{M}$ IS A DIFFUSE FINITE HYPERFINITE VON NEUMANN ALGEBRA

In this section, we are going to compute $\mathcal{G}(\mathcal{M})$ when $\mathcal{M}$ is a diffuse hyperfinite von Neumann algebra with a faithful normal tracial state τ.

LEMMA 3.1. *Suppose $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$ is a von Neumann algebra with a faithful normal tracial state τ, where $\mathcal{M}_1,\mathcal{M}_2$ are the von Neumann subalgebras of $\mathcal{M}$. Then $\mathcal{G}(\mathcal{M}) \leqslant \mathcal{G}(\mathcal{M}_1)+\mathcal{G}(\mathcal{M}_2)$.*

Proof. The inequality is trivial when one of $\mathcal{G}(\mathcal{M}_1),\mathcal{G}(\mathcal{M}_2)$ is infinite. Assume that both $\mathcal{G}(\mathcal{M}_1)$ and $\mathcal{G}(\mathcal{M}_2)$ are finite. Let $c_i=\mathcal{G}(\mathcal{M}_i)$ for $i=1,2$. By the definitions of $\mathcal{G}(\mathcal{M}_1)$ and $\mathcal{G}(\mathcal{M}_2)$, for each positive ε, we know there exist a large positive integer k, elements $\{p_j\}_{j=1}^k$, $\{q_j\}_{j=1}^k$, $\{x_1,\ldots,x_n\}$ and $\{y_1,\ldots,y_m\}$ of $\mathcal{M}$ such that:

(i) $\{p_j\}_{j=1}^k$, or $\{q_j\}_{j=1}^k$, is a family of mutually orthogonal projections of $\mathcal{M}_1$, or $\mathcal{M}_2$ respectively, with $\tau(p_j)=\tau(I_{\mathcal{M}_1})/k, \tau(q_j)=\tau(I_{\mathcal{M}_2})/k$, $\sum_j p_j = I_{\mathcal{M}_1}$ and $\sum_j q_j = I_{\mathcal{M}_2}$.

(ii) $\{x_1,\ldots,x_n\}$, or $\{y_1,\ldots,y_m\}$, is a family of generators of $\mathcal{M}_1$, or $\mathcal{M}_2$ respectively.

(iii)

$$\mathcal{I}(x_1,\ldots,x_n;\{p_j\}_{j=1}^k) \leqslant c_1+\varepsilon$$
$$\mathcal{I}(y_1,\ldots,y_m;\{q_j\}_{j=1}^k) \leqslant c_2+\varepsilon.$$

Note that $\mathcal{M}=\mathcal{M}_1\oplus\mathcal{M}_2$. A little computation shows

$$\mathcal{I}(x_1,\ldots,x_n,y_1,\ldots,y_m;\{p_j+q_j\}_{j=1}^k) \; \leqslant c_1+c_2+2\varepsilon.$$

Hence, by definitions, we have $\mathcal{G}(\mathcal{M}) \leqslant c_1+c_2+2\varepsilon$; whence $\mathcal{G}(\mathcal{M}) \leqslant \mathcal{G}(\mathcal{M}_1)+\mathcal{G}(\mathcal{M}_2)$. ∎

The following two propositions are obvious.

PROPOSITION 3.2. *Suppose M_k is a factor of type I_k and $\{e_{ij}\}_{i,j=1}^k$ is a system of matrix units of M_k. Let $x_1=e_{11}$ and $x_2=\sum_{i=1}^{k-1}(e_{i,i+1}+e_{i,i+1}^*)$. Then x_1,x_2 are two self-adjoint elements that generate M_k as a von Neumann algebra.*

PROPOSITION 3.3. *Suppose $\mathcal{M} \simeq \mathcal{A} \otimes \mathcal{N}$ is a von Neumann algebra with a tracial state τ, where $\mathcal{A}, \mathcal{N}$ are finitely generated von Neumann subalgebras of $\mathcal{M}$. If $\mathcal{A}$ is a von Neumann subalgebra with $\mathcal{G}(\mathcal{A}) = 0$, then $\mathcal{G}(\mathcal{M}) = 0$. In particular, if $\mathcal{A}$ is a diffuse abelian von Neumann subalgebra of $\mathcal{M}$, then $\mathcal{G}(\mathcal{M}) = 0$.*

The following theorem can be obtained as a corollary of Theorem 5.11. But the proof we present here motivates the proof of Proposition 5.4, our main technical result in the paper. So we include it here.

THEOREM 3.4. *Suppose that $\mathcal{R}$ is the hyperfinite type* II$_1$ *factor. Then $\mathcal{G}(\mathcal{R}) = 0$.*

Proof. Let $\{n_k\}_{k=1}^{\infty}$ be a sequence of positive integers with $n_k \geqslant 3$ for $k = 1, 2, \ldots$. It is well-known that $\mathcal{R} \simeq \bigotimes_{k=1}^{\infty} M_{n_k}(\mathbb{C})$ where $M_{n_k}(\mathbb{C})$ is the algebra of $n_k \times n_k$ matrices with complex entries. Assume that $\{e_{i,j}^{(k)}\}_{i,j=1}^{n_k}$ is the canonical system of matrix units of $M_{n_k}(\mathbb{C})$. We should identify $M_{n_k}(\mathbb{C})$ with its canonical image in $\bigotimes_{k=1}^{\infty} M_{n_k}(\mathbb{C})$ if it causes no confusion. Let

$$x_1 = e_{11}^{(1)} + \sum_{k=1}^{\infty} \frac{1}{3^k} e_{22}^{(1)} \otimes e_{22}^{(2)} \otimes \cdots \otimes e_{22}^{(k)} \otimes e_{11}^{(k+1)},$$

$$x_2 = \sum_{j=2}^{n_1} (e_{j-1,j}^{(1)} + e_{j,j-1}^{(1)}) + \sum_{k=1}^{\infty} \sum_{j=2}^{n_{k+1}} \frac{1}{3^k} e_{22}^{(1)} \otimes e_{22}^{(2)} \otimes \cdots \otimes e_{22}^{(k)} \otimes (e_{j-1,j}^{(k+1)} + e_{j,j-1}^{(k+1)}).$$

Note

$$\{e_{11}^{(1)}, e_{22}^{(1)} \otimes e_{11}^{(2)}, \ldots, e_{22}^{(1)} \otimes e_{22}^{(2)} \otimes \cdots \otimes e_{22}^{(k)} \otimes e_{11}^{(k+1)}, \ldots\}$$

is a family of mutually orthogonal projections in $\mathcal{R}$. By functional calculus, we get that

$$\{e_{11}^{(1)}, e_{22}^{(1)} \otimes e_{11}^{(2)}, \ldots, e_{22}^{(1)} \otimes e_{22}^{(2)} \otimes \cdots \otimes e_{22}^{(k)} \otimes e_{11}^{(k+1)}, \ldots\}$$

is in the von Neumann subalgebra generated by x_1. Thus $e_{11}^{(1)} x_2 = e_{12}^{(1)}$ and $x_2 e_{11}^{(1)} = e_{21}^{(1)}$ are $W^*(\{x_1, x_2\})$. Hence $e_{22}^{(1)} = e_{21}^{(1)} e_{12}^{(1)}$ is in $W^*(\{x_1, x_2\})$. It follows that $\sum_{j=2}^{n_1} (e_{j-1,j}^{(1)} + e_{j,j-1}^{(1)}) = x_2 - e_{22}^{(1)} x_2 e_{22}^{(1)}$ is in $W^*(\{x_1, x_2\})$. Now

$$e_{32}^{(1)} = \Big(\sum_{j=2}^{n_1} (e_{j-1,j}^{(1)} + e_{j,j-1}^{(1)}) \Big) e_{22}^{(1)} - e_{11}^{(1)} \Big(\sum_{j=2}^{n_1} (e_{j-1,j}^{(1)} + e_{j,j-1}^{(1)}) \Big) e_{22}^{(1)}$$

is in $W^*(\{x_1, x_2\})$. Repeating this process, we get that $\{e_{j,j-1}^{(1)}, e_{j-1,j}^{(1)}\}_{j=2}^{n_k}$ are in $W^*(\{x_1, x_2\})$. Similarly, for each $k \geqslant 1$, $\{e_{ij}^{(k)}\}_{i,j=1}^{n_k}$ is in the von Neumann subalgebra generated by x_1, x_2 in $\mathcal{R}$. Thus x_1, x_2 are two self-adjoint elements that generate $\mathcal{R}$. Moreover, we have $\mathcal{I}(x_1, x_2; \{e_{jj}^{(1)}\}_{j=1}^{n_1}) \leqslant 3/n_1$. Therefore, $\mathcal{G}(\mathcal{R}) \leqslant 3/n_1$. Since n_1 can be arbitrarily large, $\mathcal{G}(\mathcal{R}) = 0$. ∎

Now we are able to compute $\mathcal{G}(\mathcal{M})$ for a diffuse hyperfinite von Neumann algebra $\mathcal{M}$.

THEOREM 3.5. *Suppose $\mathcal{M}$ is a diffuse hyperfinite von Neumann algebra with a tracial state τ. Then $\mathcal{G}(\mathcal{M}) = 0$.*

Proof. Note a diffuse hyperfinite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state can always be decomposed as

$$\mathcal{M} \simeq \mathcal{A}_0 \otimes \mathcal{R} \oplus \Big(\bigoplus_{k=1}^{\infty} \mathcal{A}_k \otimes \mathcal{M}_{n_k}(\mathbb{C}) \Big),$$

where $\mathcal{R}$ is the hyperfinite type II_1 factor, $\mathcal{A}_0$ is an abelian von Neumann subalgebra of $\mathcal{M}$, and $\mathcal{A}_k$ is a diffuse abelian von Neuamnn subalgebra of $\mathcal{M}$. The rest follows from Lemma 3.1, Proposition 3.2 and 3.3, and Theorem 3.4. ∎

4. CUT-AND-PASTE THEOREM

The proof of following theorem, needed in Section 5, is based on a "cut-and-paste" trick from [6] or [4].

THEOREM 4.1. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a tracial state τ. Suppose $\{e_{ij}\}_{i,j=1}^{k}$ is a system of matrix units of a type I_k subfactor in $\mathcal{M}$ with $\sum_{j=1}^{k} e_{jj} = I$. If $x_1, \ldots, x_n$ are the elements in $\mathcal{M}$ such that*

$$\mathcal{I}(x_1, \ldots, x_n; \{e_{jj}\}_{j=1}^{k}) = c^2$$

with $c \leqslant 1/2 - 1/k$, then there exists a projection q in $W^*(\{x_1, \ldots, x_n, e_{ij}; 1 \leqslant i, j \leqslant k\})$ so that

$$\tau(\mathcal{S}(q; \{e_{jj}\}_{j=1}^{k})) \leqslant 2c + \frac{2}{k}$$

and

$$W^*(\{q, e_{ij}; 1 \leqslant i, j \leqslant k\}) = W^*(\{x_1, \ldots, x_n, e_{ij}; 1 \leqslant i, j \leqslant k\}),$$

where

$$\mathcal{I}(x_1, \ldots, x_n; \{e_{jj}\}_{j=1}^{k}), \quad \text{and} \quad \mathcal{S}(q; \{e_{jj}\}_{j=1}^{k})$$

are as defined in Definition 2.1 and 2.2.

Proof. Let

$$\mathcal{T} = \{(i, j, p) \mid e_{ii} x_p e_{jj} \neq 0,\ 1 \leqslant i, j \leqslant k, 1 \leqslant p \leqslant n\}.$$

Note that

$$|\mathcal{T}| = k^2 \cdot \mathcal{I}(x_1, \ldots, x_n; \{e_{jj}\}_{j=1}^{k}) = c^2 k^2,$$

and the cardinality of the set

$$\{(s, t) \mid 1 \leqslant s \leqslant [ck] + 1,\ [ck] + 2 \leqslant t \leqslant 2[ck] + 2\}$$

is equal to $([ck]+1)^2 \geqslant c^2k^2$. There exists an injective mapping from $(i,j,p)\in\mathcal{T}$ to

$$(s,t) \in \{(s,t) \mid 1 \leqslant s \leqslant [ck]+1,\ [ck]+2 \leqslant t \leqslant 2[ck]+2\},$$

and denote this map by $(i,j,p) \mapsto (s(i,j,p), t(i,j,p))$. Then each $e_{ii}x_pe_{jj}$ may be replaced by $e_{s(i,j,p)i}\, x_p e_{jt(i,j,p)}$ for all $(i,j,p) \in \mathcal{T}$. Let

$$y = \sum_{(i,j,p)\in\mathcal{T}} (e_{s(i,j,p)i}x_pe_{jt(i,j,p)} + (e_{s(i,j,p)i}x_pe_{jt(i,j,p)})^*),$$

$$q_1 = \sum_{s=1}^{[ck]+1} e_{ss} \quad \text{and} \quad q_2 = \sum_{t=[ck]+2}^{2[ck]+2} e_{tt}.$$

Without loss of generality, we can assume that $\|y\| \leqslant 1$. Then let

$$q = \frac{1}{2}q_1(1+(1-y^2)^{1/2})q_1 + \frac{1}{2}y + \frac{1}{2}q_2(1-(1-y^2)^{1/2})q_2.$$

Note that

$$y = q_1yq_2 + q_2yq_1 \quad \text{and} \quad y^2 = q_1yq_2yq_1 + q_2yq_1yq_2.$$

Let $u = q_1(1-y^2)^{1/2}q_1 + y - q_2(1-y^2)^{1/2}q_2$. Thus, $u = u^*$ and

$$\begin{aligned} u^2 &= (q_1(1-y^2)^{1/2}q_1 + y - q_2(1-y^2)^{1/2}q_2)^2 \\ &= ((q_1 - q_1yq_2yq_1)^{1/2} + q_1yq_2 + q_1yq_2 - (q_2 - q_2yq_1yq_2)^{1/2})^2 \\ &= q_1 + q_2. \end{aligned}$$

Now it is not hard to check that

$$q = \frac{q_1+q_2+u}{2} = \frac{(q_1+q_2) + (q_1(1-y^2)^{1/2}q_1 + y - q_2(1-y^2)^{1/2}q_2)}{2}$$

is a projection in $\mathcal{M}$ with $\tau(\mathcal{S}(q; \{e_{jj}\}_{j=1}^k)) \leqslant 2c + 2/k$. By the construction of q, we know that $W^*(\{q, e_{ij}; 1 \leqslant i,j \leqslant k\}) = W^*(\{x_1,\ldots,x_n, e_{ij}; 1 \leqslant i,j \leqslant k\})$. ∎

The following theorem indicates the relationship between $\mathcal{G}(\mathcal{M})$ and singly generated type II_1 factors.

THEOREM 4.2. *Suppose $\mathcal{M}$ is a type II_1 factor with the tracial state τ. If $\mathcal{G}(\mathcal{M}) < 1/4$, then $\mathcal{M}$ is singly generated.*

Proof. Note that $\mathcal{M}$ is a type II_1 factor. From the preceding theorem and the definition of $\mathcal{G}(\mathcal{M})$, for a sufficiently large integer k, there exist a system of matrix units, $\{e_{ij}\}_{i,j=1}^k$, of a I_k subfactor of $\mathcal{M}$ and a projection q in $\mathcal{M}$ so that the following hold:

(i) $\{q\} \cup \{e_{ij}\}_{i,j=1}^k$ generates $\mathcal{M}$; and

(ii) $\tau(\mathcal{S}(q; \{e_{jj}\}_{j=1}^k)) \leqslant 2\sqrt{\mathcal{G}(\mathcal{M})} + 2/k < 1 - 1/k$.

Therefore we can assume that e_{11} and q are two mutually orthogonal projections of $\mathcal{M}$. Let $x_1 = e_{11} + 2q$ and $x_2 = \sum_{i=1}^{k-1}(e_{i,i+1} + e_{i,i+1}^*)$. Since e_{11} and q are mutually orthogonal, we know that e_{11} and q are in the von Neumann subalgebra generated by x_1. Thus $\{e_{ij}\}_{i,j=1}^{k}$ is in the von Neumann algebra generated by x_1 and x_2. Combining with the fact that $\{q\} \cup \{e_{ij}\}_{i,j=1}^{k}$ generates $\mathcal{M}$, we obtain that x_1, x_2 are two self-adjoint elements of $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. ∎

REMARK 4.3. Instead of constructing a projection q in Theorem 4.1, if we are interested in constructing a self-adjoint element, then the result in Theorem 4.2 can be improved as follows. Suppose $\mathcal{M}$ is a type II_1 factor with the tracial state τ. If $\mathcal{G}(\mathcal{M}) < 1/2$, then $\mathcal{M}$ is singly generated.

5. MAIN RESULTS

The following lemma essentially comes from Popa's remarkable paper [17].

LEMMA 5.1. *Suppose $\mathcal{M}$ is a type II_1 factor with the tracial state τ. Suppose $\{p_j\}_{j=1}^{k}$ is a family of mutually orthogonal projections in $\mathcal{M}$ with each $\tau(p_j) = 1/k$. Then there exists a hyperfinite type II_1 subfactor $\mathcal{R}$ of $\mathcal{M}$ such that $\mathcal{R}' \cap M = \mathbb{C}I$ and $\{p_j\}_{j=1}^{k} \subseteq \mathcal{R}$.*

Proof. By [17], there exists a hyperfinite subfactor $\mathcal{R}_0$ of $\mathcal{M}$ such that $\mathcal{R}_0' \cap \mathcal{M} = \mathbb{C}I$. Since $\mathcal{M}$ is a type II_1 factor, there exists a unitary element w in $\mathcal{M}$ such that $\{p_j\}_{j=1}^{k} \subset w^*\mathcal{R}_0 w$. Let $\mathcal{R} = w^*\mathcal{R}_0 w$. Then $\mathcal{R}$ is a hyperfinite type II_1 subfactor of $\mathcal{M}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathbb{C}I$ and $\{p_j\}_{j=1}^{k} \subseteq \mathcal{R}$. ∎

Recall $\mathcal{M}_1$ is called an irreducible subfactor of a type II_1 factor $\mathcal{M}$ if $\mathcal{M}_1 \subset \mathcal{M}$ and $\mathcal{M}_1' \cap \mathcal{M} = \mathbb{C}I$.

LEMMA 5.2. *Suppose that $\mathcal{M}$ is a type II_1 factor with the tracial state τ. Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ with $\mathcal{G}(\mathcal{N}) = c$. Then for each $\varepsilon > 0$, there exists an irreducible subfactor $\mathcal{M}_\varepsilon$ of $\mathcal{M}$ such that $\mathcal{N} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}$ and $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant c + \varepsilon$.*

Proof. Since $\mathcal{G}(\mathcal{N}) = c$, there exist some positive integer $k > 8/\varepsilon$, a family of mutually orthogonal projections $\{p_j\}_{j=1}^{k}$ in $\mathcal{N}$ with $\tau(p_j) = 1/k$ for $1 \leqslant j \leqslant k$, and a family of generators $\{x_1, \ldots, x_n\}$ of $\mathcal{N}$, such that

$$\mathcal{I}(x_1, \ldots, x_n; \{p_j\}_{j=1}^{k}) \leqslant c + \frac{\varepsilon}{2}.$$

By Lemma 5.1, we can find an irreducible hyperfinite type II_1 subfactor $\mathcal{R}$ of $\mathcal{M}$ such that $\{p_j\}_{j=1}^{k} \subset \mathcal{R}$. Thus there exists a system of matrix units $\{e_{ij}\}_{i,j=1}^{k}$ of a I_k subfactor M_k of $\mathcal{R}$ such that $e_{jj} = p_j$ for each $j = 1, \ldots, k$. Note $\mathcal{R} \simeq \mathcal{R}_1 \otimes M_k$ for some hyperfinite type II_1 subfactor $\mathcal{R}_1$ of $\mathcal{R}$. By Theorem 3.4 and Theorem 4.2, we know the hyperfinite subfactor $\mathcal{R}_1$ is generated by two self-adjoint elements

y_1, y_2 that commute with M_k. By Proposition 3.2, M_k is generated by two self-adjoint elements $z_1 = e_{11} = p_1$ and $z_2 = \sum_{j=1}^{k-1}(e_{j,j+1} + e_{j+1,j})$ as a von Neumann algebra. A little computation shows that

$$\mathcal{I}(x_1, \dots, x_n, y_1, y_2, z_1, z_2; \{p_j\}_{j=1}^{k}) \leqslant c + \frac{\varepsilon}{2} + \frac{2}{k} + \frac{2}{k} \leqslant c + \varepsilon.$$

Let $\mathcal{M}_\varepsilon$ be the von Neumann subalgebra generated by $\mathcal{R}$ and $\mathcal{N}$ in $\mathcal{M}$, which is also generated by $x_1, \dots, x_n, y_1, y_2, z_1, z_2$ in $\mathcal{M}$ as a von Neumann algebra. Since $\mathcal{R}$ is an irreducible type II$_1$ subfactor of $\mathcal{M}$, $\mathcal{M}_\varepsilon$ is also an irreducible type II$_1$ subfactor of $\mathcal{M}$. Moreover $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant \mathcal{I}(x_1, \dots, x_n, y_1, y_2, z_1, z_2; \{p_j\}_{j=1}^{k}) \leqslant c + \varepsilon$. ∎

5.1. THE CASE WHEN THE INTERSECTION OF TWO VON NEUMANN SUBALGEBRAS IS DIFFUSE. We start this subsection with the following definition which is just for our convenience.

DEFINITION 5.3. The family of elements $\{e_{ij}\}_{i,j=1}^{k}$ is called *a subsystem of matrix units of a von Neumann algebra $\mathcal{M}$* if the following hold:

(i) $\{e_{ij}\}_{i,j=1}^{k} \subset \mathcal{M}$;

(ii) there exists a projection p in $\mathcal{M}$ such that $\sum_{j=1}^{k} e_{jj} = p$;

(iii) $e_{ij}^* = e_{ji}$ for $1 \leqslant i, j \leqslant k$;

(iv) $e_{il}e_{lj} = e_{ij}$ for $1 \leqslant i, l, j \leqslant k$.

Next proposition is our main technical result in the paper.

PROPOSITION 5.4. *Suppose that $\mathcal{M}$ is a type* II$_1$ *factor with the tracial state τ. Suppose $\{\mathcal{N}_k\}_{k=1}^{\infty}$ is a sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\{\mathcal{N}_k\}_{k=1}^{\infty}$ generates $\mathcal{M}$ as a von Neumann algebra and $\mathcal{N}_k \cap \mathcal{N}_{k+1}$ is a diffuse von Neumann subalgebra of $\mathcal{M}$ for all $k \geqslant 1$. Suppose, for each $k \geqslant 1$, $\varepsilon > 0$, there is an irreducible subfactor $\mathcal{M}_{k,\varepsilon}$ of $\mathcal{M}$ such that $\mathcal{N}_k \subseteq \mathcal{M}_{k,\varepsilon} \subseteq \mathcal{M}$ and $\mathcal{G}(\mathcal{M}_{k,\varepsilon}) \leqslant \varepsilon$. Then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

Proof. Let $\varepsilon < 1/8$ be a positive number. From the assumption on $\mathcal{N}_1$, there exists an irreducible type II$_1$ subfactor $\mathcal{M}_1$ of $\mathcal{M}$ such that $\mathcal{N}_1 \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ and $\mathcal{G}(\mathcal{M}_1) \leqslant \varepsilon$. By Theorem 4.1 and the definition of $\mathcal{G}(\mathcal{M}_1)$, for a sufficiently large integer $m_1 > 3/\varepsilon$, there exist a projection q_1 in $\mathcal{M}_1$ and a system of matrix units $\{e_{ij}^{(1)}\}_{i,j=1}^{m_1}$ of $\mathcal{M}_1$ such that $\sum_{j=1}^{m_1} e_{jj}^{(1)} = I$, $\tau(\mathcal{S}(q_1; \{e_{jj}^{(1)}\}_{j=1}^{m_1})) \leqslant 3\varepsilon$, and $\{q_1\} \cup \{e_{ij}^{(1)}\}_{i,j=1}^{m_1}$ generates $\mathcal{M}_1$ as a von Neumann algebra. Without loss of generality, we can assume that $e_{11}^{(1)}, e_{22}^{(1)}, q$ are mutually orthogonal projections in $\mathcal{M}_1$.

CLAIM 1. *There is a sequence of positive integers* $\{m_k\}_{k=1}^{\infty}$, *a sequence of irreducible type* II_1 *subfactors* $\mathcal{M}_k$ *of* $\mathcal{M}$, *subsystems of matrix units* $\{\{e_{ij}^{(k)}\}_{i,j=1}^{m_k}\}_{k=1}^{\infty}$, *and a family of projections* $\{q_k\}_{k=1}^{\infty}$, *such that:*

(i) $\mathcal{N}_k \subseteq \mathcal{M}_k \subseteq \mathcal{M}$ *for* $k \geqslant 1$;

(ii) $\sum_{j=1}^{m_{k+1}} e_{jj}^{(k+1)} = e_{22}^{(k)}$ *for* $k \geqslant 1$;

(iii) $q_{k+1} = e_{22}^{(k)} q_{k+1} e_{22}^{(k)}$, $q_{k+1} e_{11}^{(k+1)} = 0$, $q_{k+1} e_{22}^{(k+1)} = 0$ *for* $k \geqslant 1$;

(iv) $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k) = W^*(\{q_1, \ldots, q_k, e_{ij}^{(p)};\ 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k\})$ *for* $k \geqslant 1$.

Proof of Claim. We have already finished the construction when $k = 1$. Suppose that we have finished the construction till k-step. Note that, by the assumption on $\mathcal{N}_{k+1}$, there exists an irreducible subfactor $\mathcal{M}_{k+1}$ of $\mathcal{M}$ such that

$$\mathcal{G}(\mathcal{M}_{k+1}) \leqslant \Big(\frac{1}{8m_1 \cdots m_k}\Big)^2,$$

and $\mathcal{N}_{k+1} \subseteq \mathcal{M}_{k+1} \subseteq \mathcal{M}$, i.e., (i) holds.

By the definition of $\mathcal{G}(\mathcal{M}_{k+1})$, there exist a sufficiently large integer $m_{k+1} > 4m_1 \cdots m_k + 6$, a family of mutually orthogonal projections $\{p_j\}_{j=1}^{m_1 \cdots m_{k+1}}$ in $\mathcal{M}_{k+1}$ with each $\tau(p_j) = 1/m_1 \cdots m_{k+1}$ and a family of generators $\{x_1, \ldots, x_n\}$ of $\mathcal{M}_{k+1}$ such that

$$(*) \qquad \mathcal{I}(x_1, \ldots, x_n; \{p_j\}_{j=1}^{m_1 \cdots m_{k+1}}) \leqslant \Big(\frac{1}{4m_1 \cdots m_k}\Big)^2.$$

From the induction hypothesis on each $\mathcal{M}_j$, we know that $\{\mathcal{M}_j\}_{j=1}^{k}$ are a family of irreducible type II_1 subfactors of $\mathcal{M}$, which implies $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$ is a type II_1 subfactor of $\mathcal{M}$. And

$$(**) \quad W^*(\{q_1, \ldots, q_k, e_{ij}^{(p)};\ 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k\}) = W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k).$$

Let $\{e_{ij}^{(k+1)}\}_{i,j=1}^{m_{k+1}}$ be a subsystem of matrix units in $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$ such that $e_{22}^{(k)} = \sum_{j=1}^{m_{k+1}} e_{jj}^{(k+1)}$, i.e., (ii) holds.

Then

$$\mathcal{T}_{k+1} = \{e_{i_1 2}^{(1)} \cdots e_{i_k,2}^{(k)} e_{st}^{(k+1)} e_{2,j_k}^{(k)} \cdots e_{2j_1}^{(1)} \mid 1 \leqslant i_p, j_p \leqslant m_p, 1 \leqslant p \leqslant k, 1 \leqslant s,t \leqslant m_{k+1}\}$$

is a system of matrix units of a $\mathrm{I}_{m_1 m_2 \cdots m_k m_{k+1}}$ subfactor of $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$; and

$$\mathcal{P}_{k+1} = \{e_{i_1 2}^{(1)} \cdots e_{i_k,2}^{(k)} e_{ss}^{(k+1)} e_{2,i_k}^{(k)} \cdots e_{2i_1}^{(1)} \mid 1 \leqslant i_p \leqslant m_p, 1 \leqslant p \leqslant k, 1 \leqslant s \leqslant m_{k+1}\}$$

is a family of mutually orthogonal equivalent projections in $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$ with sum $I_{\mathcal{M}}$. Note the following facts: (1) $\mathcal{M}_k \cap \mathcal{M}_{k+1}$ is a diffuse von Neumann subalgebra; (2) $\mathcal{P}_{k+1}$ is a family of mutually orthogonal equivalent projections with sum $I_{\mathcal{M}}$ in the type II$_1$ subfactor $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$; (3) $\{p_j\}_{j=1}^{m_1 \cdots m_{k+1}}$ is a family of mutually orthogonal equivalent projections with sum $I_{\mathcal{M}}$ in the type II$_1$ subfactor $\mathcal{M}_{k+1}$; (4) The cardinalities of $\{p_j\}_{j=1}^{m_1 \cdots m_{k+1}}$ and $\mathcal{P}_{k+1}$ are equal to $m_1 \cdots m_{k+1}$. Thus there exist unitary elements v_{k+1} in $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$ and w_{k+1} in $\mathcal{M}_{k+1}$ such that $w_{k+1}v_{k+1}$ maps $\mathcal{P}_{k+1}$, one to one, onto $\{p_j\}_{j=1}^{m_1 \cdots m_{k+1}}$. By $(*)$, we have that

$$\mathcal{I}(v_{k+1}^* w_{k+1}^* x_1 w_{k+1} v_{k+1}, \ldots, v_{k+1}^* w_{k+1}^* x_n w_{k+1} v_{k+1}; \mathcal{P}_{k+1}) \leqslant \Big(\frac{1}{4m_1 \cdots m_k}\Big)^2.$$

By Theorem 4.1, there exists a projection q_{k+1} in $\mathcal{M}$ so that

$$\begin{aligned} & W^*(\{v_{k+1}^* w_{k+1}^* x_1 w_{k+1} v_{k+1}, \ldots, v_{k+1}^* w_{k+1}^* x_n w_{k+1} v_{k+1}, \mathcal{T}_{k+1}\}) \\ (***) \qquad & \qquad = W^*(\{q_{k+1}\} \cup \mathcal{T}_{k+1}) \end{aligned}$$

and

$$\tau(\mathcal{S}(q_{k+1}; \mathcal{P}_{k+1})) \leqslant \frac{1}{2m_1 \cdots m_k} + \frac{2}{m_{k+1}} < \frac{1}{m_1 \cdots m_k} - \frac{3}{m_1 \cdots m_{k+1}}.$$

Because

$$\tau(e_{22}^{(k)}) = \frac{1}{m_1 \cdots m_k}, \quad \tau(e_{11}^{(k+1)}) = \tau(e_{22}^{(k+1)}) = \frac{1}{m_1 \cdots m_k m_{k+1}},$$

we might assume that $q_{k+1} = e_{22}^{(k)} q_{k+1} e_{22}^{(k)}$, $q_{k+1} e_{11}^{(k+1)} = 0$, $q_{k+1} e_{22}^{(k+1)} = 0$, i.e., (iii) holds.

Note v_{k+1} is in $W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k)$, which, by $(**)$, is in the von Neumann algebra generated by $\{q_1, \ldots, q_k\} \cup \{\{e_{ij}^{(p)}\}_{i,j=1,\ldots,m_p; 1 \leqslant p \leqslant k}\}$. On the other hand,

$$W^*(\mathcal{T}_{k+1}) = W^*(\{e_{ij}^{(p)};\ 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k+1\}).$$

Together with $(***)$, we get that $\{w_{k+1}^* x_1 w_{k+1}, \ldots, w_{k+1}^* x_n w_{k+1}\}$ is contained in the von Neumann subalgebra generated by $\{q_1, \ldots, q_k, q_{k+1}\} \cup \mathcal{T}_{k+1}$ in $\mathcal{M}$.

However$\{w_{k+1}^* x_1 w_{k+1}, \ldots, w_{k+1}^* x_n w_{k+1}\}$ is also a family of generators of $\mathcal{M}_{k+1}$, because w_{k+1} is unitary element in $\mathcal{M}_{k+1}$. Hence, $\mathcal{M}_{k+1}$ is in the von Neumann algebra generated by $\{q_1, \ldots, q_k, q_{k+1}\} \cup \mathcal{T}_{k+1}$.

Combining with the facts that

$$W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k) = W^*(\{q_1, \ldots, q_k, e_{ij}^{(p)}; 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k\}) \supseteq \mathcal{T}_{k+1},$$

and

$$\begin{aligned} q_{k+1} &\in W^*(\{v_{k+1}^* w_{k+1}^* x_1 w_{k+1} v_{k+1}, \ldots, v_{k+1}^* w_{k+1}^* x_n w_{k+1} v_{k+1}, \mathcal{T}_{k+1}\}) \\ &\subseteq W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k \cup \mathcal{M}_{k+1} \cup \mathcal{T}_{k+1}), \end{aligned}$$

we know that

$$\begin{aligned} W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k \cup \mathcal{M}_{k+1}) &\subseteq W^*(\mathcal{M}_{k+1} \cup \{q_1, \ldots, q_k, e_{ij}^{(p)};\ 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k+1\}) \\ &\subseteq W^*(\mathcal{M}_{k+1} \cup \{q_1, \ldots, q_k, \mathcal{T}_{k+1}\}) \subseteq W^*(\{q_1, \ldots, q_{k+1}\} \cup \mathcal{T}_{k+1}) \\ &\subseteq W^*(\{q_1, \ldots, q_{k+1}, e_{ij}^{(p)};\ 1 \leqslant i,j \leqslant m_p, 1 \leqslant p \leqslant k+1\}) \\ &\subseteq W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k \cup \mathcal{M}_{k+1} \cup \mathcal{T}_{k+1}) \\ &\subseteq W^*(\mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k \cup \mathcal{M}_{k+1}); \end{aligned}$$

whence (iv) holds. This finishes the construction at $(k+1)$-th step. ∎

Let

$$x_1 = \Big(\sum_{k=1}^{\infty} \frac{1}{2^k} e_{11}^{(k)}\Big) + \Big(\sum_{k=1}^{\infty} \frac{1}{3^k} q_k\Big), \quad x_2 = \sum_{k=1}^{\infty}\sum_{j=2}^{m_k} \frac{1}{2^k}(e_{j-1,j}^{(k)} + e_{j,j-1}^{(k)}).$$

Note that, by induction hypothesis (iii), we know $\{e_{11}^{(k)}, q_k; k \geqslant 1\}$ is a family of mutually orthogonal projections in $\mathcal{M}$. Thus, $\{e_{11}^{(k)}, q_k; k \geqslant 1\}$ is in the von Neumann subalgebra generated by x_1. By the construction of x_2 and the fact that $\{e_{11}^{(k)}; k \geqslant 1\}$ is in the von Neumann subalgebra generated by x_1, we get that $\{e_{ij}^{(k)};\ 1 \leqslant i,j \leqslant m_k, k \geqslant 1\}$ is in the von Neumann subalgebra generated by $\{x_1, x_2\}$. Hence, by induction hypothesis (iv), $\{\mathcal{M}_k\}_{k=1}^{\infty}$ is in the von Neumann subalgebra generated by $\{x_1, x_2\}$, i.e., x_1, x_2 are self-adjoint elements in $\mathcal{M}$ that generate $\mathcal{M}$ as a von Neumann algebra. Moreover, a little computation shows that

$$\begin{aligned} \mathcal{I}(x_1, x_2; \{e_{ii}^{(1)}\}_{i=1}^{m_1}) &\leqslant \mathcal{I}\Big(\frac{q_1}{3}; \{e_{ii}^{(1)}\}_{i=1}^{m_1}\Big) + \mathcal{I}\Big(\sum_{k=1}^{\infty} \frac{1}{2^k} e_{11}^{(k)}, \sum_{k=2}^{\infty} \frac{1}{3^k} q_k,\ x_2; \{e_{ii}^{(1)}\}_{i=1}^{m_1}\Big) \\ &\leqslant 3\varepsilon + \frac{3}{m_1} \leqslant 4\varepsilon. \end{aligned}$$

Therefore, $\mathcal{G}(\mathcal{M}) \leqslant 4\varepsilon$, for all $\varepsilon > 0$. It follows that $\mathcal{G}(\mathcal{M}) = 0$. ∎

Now we are ready to show our main result in this subsection.

THEOREM 5.5. *Suppose that $\mathcal{M}$ is a type* II_1 *factor with the tracial state τ. Suppose $\{\mathcal{N}_k\}_{k=1}^{\infty}$ is a sequence of von Neumann subalgebras of $\mathcal{M}$ that generates $\mathcal{M}$ as a von Neumann algebra and $\mathcal{N}_k \cap \mathcal{N}_{k+1}$ is a diffuse von Neumann subalgebra of $\mathcal{M}$ for each $k \geqslant 1$. If $\mathcal{G}(\mathcal{N}_k) = 0$ for $k \geqslant 1$, then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

Proof. The result follows easily from Lemma 5.2 and Proposition 5.4. ∎

The following corollaries follow easily from Theorem 5.5 (also see [4], [6]). Recall a unitary element v in $\mathcal{M}$ is called a Haar unitary element if $\tau(v^m) = 0$ for all $m \neq 0$. It is observed that a Haar unitary element v generates a diffuse abelian von Neumann subalgebra in $\mathcal{M}$.

COROLLARY 5.6. *Suppose* $\mathcal{M} = L(SL(\mathbb{Z}, 2m+1))$ $(m \geqslant 1)$ *is the group von Neumann algebra associated with* $SL(\mathbb{Z}, 2m+1)$*, the special linear group with integer entries. Then* $\mathcal{G}(\mathcal{M}) = 0$*. In particular,* $\mathcal{M}$ *is singly generated.*

Proof. By the structure of $L(SL(\mathbb{Z}, 2m+1))$, there is a sequence of Haar unitary elements $u_1, \ldots, u_n$ that generate $L(SL(\mathbb{Z}, 2m+1))$ as a von Neumann algebra and satisfy $u_k u_{k+1} = u_{k+1} u_k$ for all $1 \leqslant k \leqslant n-1$. Let $\mathcal{N}_k$ be the von Neumann subalgebra generated by u_k, u_{k+1} for $1 \leqslant k \leqslant n-1$. Now the result follows from Theorem 5.5. ∎

COROLLARY 5.7. *Suppose* $\mathcal{M}$ *is a nonprime type* II$_1$ *factor, i.e.* $\mathcal{M} \simeq \mathcal{M}_1 \otimes \mathcal{M}_2$ *for some type* II$_1$ *subfactors* $\mathcal{N}_1, \mathcal{N}_2$ *of* $\mathcal{M}$*. Then* $\mathcal{G}(\mathcal{M}) = 0$*. In particular,* $\mathcal{M}$ *is singly generated.*

Proof. We can assume that $\mathcal{M}_1$, or $\mathcal{M}_2$, is generated by a sequence of Haar unitary elements $u_1, \ldots, u_n, \ldots$, or $v_1, \ldots, v_m, \ldots$ respectively. Let $\mathcal{N}_{2k-1}$ be the von Neumann subalgebra generated by $\{u_k, v_k\}$ in $\mathcal{M}$ and $\mathcal{N}_{2k}$ be the von Neumann subalgebra generated by u_{k+1}, v_k in $\mathcal{M}$ for all $k \geqslant 1$. Now the result follows from Theorem 5.5. ∎

5.2. THE CASE WHEN A VON NEUMANN ALGEBRA IS GENERATED BY THE NORMALIZERS OF A VON NEUMANN SUBALGEBRA.

Suppose that $\mathcal{M}$ is a diffuse von Neumann subalgebra with a tracial state τ.

LEMMA 5.8. *Suppose that* $\mathcal{M}$ *is a type* II$_1$ *factor with the tracial state* τ*. Suppose* $\mathcal{N}$ *is a von Neumann subalgebra of* $\mathcal{M}$ *such that* $\mathcal{G}(\mathcal{N}) = c$*. Suppose* u *is a unitary element in* $\mathcal{M}$ *such that, for some Haar unitary element* v *in* $\mathcal{N}$*,* u^*vu *is contained in* $\mathcal{N}$*. Then, for every* $\varepsilon > 0$*, there exists an irreducible type* II$_1$ *subfactor* $\mathcal{M}_\varepsilon$ *such that* $W^*(\mathcal{N} \cup \{u\}) \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}$ *and* $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant c + \varepsilon$*.*

Proof. By Lemma 5.2, there exists an irreducible type II$_1$ subfactor $\mathcal{N}_\varepsilon$ of $\mathcal{M}$ such that $\mathcal{N} \subseteq \mathcal{N}_\varepsilon \subseteq \mathcal{M}$ and $\mathcal{G}(\mathcal{N}_\varepsilon) \leqslant c + \varepsilon/2$. Thus, by the definition of $\mathcal{G}(\mathcal{N}_\varepsilon)$, there exist some positive integer $k > 8/\varepsilon$, a family of mutually orthogonal projections $\{p_j\}_{j=1}^k$ in $\mathcal{N}_\varepsilon$ with $\tau(p_j) = 1/k$ for $1 \leqslant j \leqslant k$, and a family of generators $\{x_1, \ldots, x_n\}$ of $\mathcal{N}_\varepsilon$, such that

$$\mathcal{I}(x_1, \ldots, x_n; \{p_j\}_{j=1}^k) \leqslant c + \frac{\varepsilon}{2}.$$

Note u is a unitary element in $\mathcal{M}$ such that, for some Haar unitary element v in $\mathcal{N}$, u^*vu is contained in $\mathcal{N}$. It follows that there exist two families of mutually orthogonal projections, $\{e_j\}_{j=1}^k$, $\{f_j\}_{j=1}^k$, in $\mathcal{N}$ with $\tau(e_j) = \tau(f_j) = 1/k$ such that $u^*e_ju = f_j$ for $j = 1, \ldots, k$. Note $\mathcal{N}_\varepsilon$ is a type II$_1$ subfactor that contains $\mathcal{N}$. There exist two unitary elements w_1, w_2 in $\mathcal{N}_\varepsilon$ such that $p_j = w_1^* e_j w_1 = w_2^* f_j w_2$

for $j = 1, \dots, k$. Thus $w_1^* u w_2 p_j = p_j w_1^* u w_2$ for $j = 1, \dots, k$. It follows

$$\mathcal{I}(x_1, \dots, x_n, w_1^* u w_2; \{p_j\}_{j=1}^k) \leqslant c + \frac{\varepsilon}{2} + \frac{1}{k} \leqslant c + \varepsilon.$$

Let $\mathcal{M}_\varepsilon$ be the von Neumann subalgebra generated by $x_1, \dots, x_n, w_1^* u w_2$ in $\mathcal{M}$; whence $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant c + \varepsilon$. Note $\mathcal{N}_\varepsilon$ is contained in $\mathcal{M}_\varepsilon$, so are w_1, w_2. Thus u is also contained in $\mathcal{M}_\varepsilon$, whence $W^*(\mathcal{N} \cup \{u\}) \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}$. From the fact that $\mathcal{N}_\varepsilon' \cap \mathcal{M} = \mathbb{C}I$, it follows that $\mathcal{M}_\varepsilon$ is an irreducible type II_1 subfactor of $\mathcal{M}$. ∎

THEOREM 5.9. *Suppose that $\mathcal{M}$ is a type II_1 factor with the tracial state τ. Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $\{u_k\}$ is a family of unitary elements in $\mathcal{M}$ such that $\{\mathcal{N}, u_1, u_2, \dots\}$ generates $\mathcal{M}$ as a von Neumann algebra and there exists a family of Haar unitary elements $\{v_k\}_{k=1}^\infty$ in $\mathcal{N}$ satisfying $u_k^* v_k u_k$ in $\mathcal{N}$ for $k \geqslant 1$. If $\mathcal{G}(\mathcal{N}) = 0$, then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

Proof. Let $\mathcal{N}_k$ be the von Neumann subalgebra generated by $\mathcal{N}$ and u_k in $\mathcal{M}$ for $k \geqslant 1$. Using Lemma 5.8 and Proposition 5.4, we easily obtain the result. ∎

The following theorem is the generalization of Proposition 1 of [4].

THEOREM 5.10. *Suppose that $\mathcal{M}$ is a type II_1 factor with the tracial state τ. Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $\{u_k\}$ is a family of Haar unitary elements in $\mathcal{M}$ such that $\{\mathcal{N}, u_1, u_2, \dots\}$ generates $\mathcal{M}$ as a von Neumann algebra. Suppose u_1 is in $\mathcal{N}$ and $u_{k+1}^* u_k u_{k+1}$ is in the von Neumann subalgebra generated by $\mathcal{N} \cup \{u_1, \dots, u_k\}$ for $k \geqslant 1$. If $\mathcal{G}(\mathcal{N}) = 0$, then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

Proof. Let $\mathcal{N}_k$ be the von Neumann subalgebra generated by $\mathcal{N}$ and $u_1, \dots, u_k$ in $\mathcal{M}$ for $k \geqslant 1$. Using Lemma 5.8, inductively, and Proposition 5.4, we can easily obtain the result. ∎

Using Theorem 3.2 and 5.3, we have the following result.

THEOREM 5.11. *Suppose that $\mathcal{M}$ is a type II_1 factor. Suppose that $\{u_k\}_{k=1}^\infty$ is a family of Haar unitary elements in $\mathcal{M}$ that generate $\mathcal{M}$ and $u_{k+1}^* u_k u_{k+1}$ is contained in the von Neumann subalgebra generated by $\{u_1, \dots, u_k\}$ for $k \geqslant 1$. Then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

As another corollary of Theorem 3.5 and Theorem 5.9, we obtain the following result from [16].

COROLLARY 5.12. *Suppose $\mathcal{M}$ is a type II_1 factor with Cartan subalgebras. Then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

5.3. THE CASE WHEN A TYPE II_1 FACTOR HAS "PROPERTY Γ".

In this subsection, we study the von Neumann algebras with "Property Γ" in the sense of Murry and von Neumann.

LEMMA 5.13. *Suppose that $\mathcal{M}$ is a type II_1 factor with the tracial state τ. Suppose $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ such that $\mathcal{G}(\mathcal{N}) = 0$. Suppose u is a* *unitary element in* $\mathcal{M}$ *such that, for a family of Haar unitary elements* $\{v_n, w_n\}_{n=1}^{\infty}$ *in* $\mathcal{N}$, $\lim_{n\to\infty} \|u^* v_n u - w_n\|_2 = 0$. *Then, for every* $\varepsilon > 0$, *there exists an irreducible type* II_1 *factor* $\mathcal{M}_\varepsilon$ *such that* $W^*(\mathcal{N} \cup \{u\}) \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}$ *and* $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant \varepsilon$.

Proof. Let ω be a free ultrafilter in $\beta(\mathbb{N}) \setminus \mathbb{N}$ and $\mathcal{N}^\omega$ be the ultra-product von Neumann algebras of $\mathcal{N}$ along the ultrafilter ω, i.e. the quotient of the C^*-algebra $\prod_{m=1}^{\infty} \mathcal{N}$ by the norm closed ideal $\mathcal{I} = \left\{(x_n)_{n=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{N} \mid \tau_\omega((x_n^* x_n)_{n=1}^{\infty}) = 0\right\}$, where τ_ω is defined by $\tau_\omega((x_n)_{n=1}^{\infty}) = \lim_{n\to\omega} \tau(x_n)$ for each $(x_n)_{n=1}^{\infty} \in \prod_{m=1}^{\infty} \mathcal{N}$ (see [12]).

Let

$$U = [(u, u, \ldots, u, \ldots)], \quad V = [(v_1, v_2, \ldots, v_n, \ldots)], \quad W = [(w_1, w_2, \ldots, w_n, \ldots)],$$

be unitary elements in $\mathcal{N}^\omega$. Thus V, W are two Haar unitary elements so that $U^* V U = W$. Let k be a positive integer and $\varepsilon = 1/k$. There is a family of mutually orthogonal projections $\{P_i\}_{i=1}^k$, or $\{Q_i\}_{i=1}^k$, in the abelian von Neumann subalgebra generated by V, or W respectively, in $\mathcal{N}^\omega$ such that $\tau_\omega(P_j) = \tau_\omega(Q_i) = 1/k$ and $U^* P_i U = Q_i$ for each $1 \leqslant i \leqslant k$. Or, $U = \sum_{i=1}^k P_i U = \sum_{i=1}^k P_i U Q_i$. Therefore, we can assume that there exist families of mutually orthogonal projections $\{p_j\}_{j=1}^k$, $\{q_j\}_{j=1}^k$ of $\mathcal{N}$ with each $\tau(p_j) = \tau(q_j) = 1/k$, such that $\left\| u - \sum_{j=1}^k p_j u q_j \right\|_2 < \varepsilon$. Let $x_k = \sum_{j=1}^k p_j u q_j$ and $\mathcal{N}_k = W^*(\mathcal{N} \cup \{x_k\})$. Thus $x_k \xrightarrow{\|\cdot\|_2} u$. A straightforward adaption of the proofs of Lemma 5.3 and Proposition 5.4 shows that there exist a subsequence $\{k_p\}_{p=1}^{\infty}$ of $\{k\}_{k=1}^{\infty}$ and an irreducible subfactor $\mathcal{M}_\varepsilon$ of $\mathcal{M}$ such that $\{\mathcal{N}_{k_p}\}_{p=1}^{\infty} \subseteq \mathcal{M}_\varepsilon \subseteq \mathcal{M}$ and $\mathcal{G}(\mathcal{M}_\varepsilon) \leqslant \varepsilon$. But $x_{k_p} \in \mathcal{N}_{k_p}$ and $x_{k_p} \xrightarrow{\|\cdot\|_2} u$, as $p \to \infty$. Thus $u \in \mathcal{M}_\varepsilon$. This completes the proof. ∎

Using Lemma 5.13 and Proposition 5.4, we can easily obtain the following theorem.

THEOREM 5.14. *Suppose that* $\mathcal{M}$ *is a type* II_1 *factor with the tracial state* τ. *Suppose* $\mathcal{N}$ *is a von Neumann subalgebra of* $\mathcal{M}$ *and* $\{u_k\}$ *is a family of unitary elements in* $\mathcal{M}$ *such that* $\{\mathcal{N}, u_1, u_2, \ldots\}$ *generates* $\mathcal{M}$ *as a von Neumann algebra. Suppose there exists a family of Haar unitary elements* $\{v_{k,n}, w_{k,n}\}_{k,n=1}^{\infty}$ *in* $\mathcal{N}$ *such that* $\lim_{n\to\infty} \|u_k^* v_{k,n} u_k - w_{k,n}\|_2 = 0$ *for* $k \geqslant 1$. *If* $\mathcal{G}(\mathcal{N}) = 0$, *then* $\mathcal{G}(\mathcal{M}) = 0$. *In particular,* $\mathcal{M}$ *is singly generated.*

Using Theorem 5.3 in [1] and Theorem 3.5 and Theorem 5.14, we have the following result from [6].

COROLLARY 5.15. *Suppose $\mathcal{M}$ is a type* II_1 *factor with property Γ. Then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.*

Proof. It follows from Theorem 5.3 in [1] that there exist a hyperfinite II_1 factor $\mathcal{R}$ and a family of Haar unitary elements $\{v_n\}_{n=1}^{\infty}$ of $\mathcal{R}$ such that $\|xv_n - v_nx\|_2 \to 0$ as $n \to 0$ for all x in $\mathcal{M}$. The rest follows from Theorem 5.14 by letting $\mathcal{N}$ to be $\mathcal{R}$. ∎

5.4. A SHORT SUMMARY AND SOME COROLLARIES. As a summary of the results in this section, we have the following corollary.

COROLLARY 5.16. *The following statements are true:*

(i) *$\mathcal{G}(\mathcal{M}) = 0$, if $\mathcal{M}$ is a diffuse hyperfinite von Neumann algebra with a tracial state τ.*

(ii) *$\mathcal{G}(\mathcal{M}) = 0$ if the type* II_1 *factor $\mathcal{M}$ is generated by a family of von Neumann subalgebras $\{\mathcal{N}_j\}_{j=1}^{\infty}$ of $\mathcal{M}$ such that $\mathcal{G}(\mathcal{N}_j) = 0$ and $\mathcal{N}_j \cap \mathcal{N}_{j+1}$ is a diffuse von Neumann subalgebra for all $j \geqslant 1$;*

(iii) *$\mathcal{G}(\mathcal{M}) = 0$ if the type* II_1 *factor $\mathcal{M}$ is generated by $\{\mathcal{N}, u_1, \ldots, u_j, \ldots\}$, where $\mathcal{N}$ is a von Neumann subalgebra $\mathcal{N}$ of $\mathcal{M}$ with $\mathcal{G}(\mathcal{N}) = 0$ and $\{u_j\}_{j=1}^{\infty}$ is a family of unitary elements of $\mathcal{M}$ such that, for every $j \geqslant 1$, $u_j^* v_j u_j$ is in $\mathcal{N}$ for some Haar unitary element v_j in $\mathcal{N}$;*

(iv) *$\mathcal{G}(\mathcal{M}) = 0$ if a type* II_1 *factor $\mathcal{M}$ is generated by an ascending sequence of subalgebras $\{\mathcal{N}_k\}_{k=1}^{\infty}$ such that $\mathcal{G}(\mathcal{N}_k) = 0$;*

(v) *If $\mathcal{M}$ is a type* II_1 *factor and $\mathcal{G}(\mathcal{M}) < 1/4$, then $\mathcal{M}$ is singly generated.*

Proof. (i) follows from Theorem 3.5. (ii) is from Theorem 5.5. (iii) is from Theorem 5.9. (iv) follows from Theorem 5.5. (v) is from Theorem 4.2 ∎

Using Theorem 3.5, 5.10 and 5.11, we have the following result.

THEOREM 5.17. *Suppose $\mathcal{M}$ is a type* II_1 *factor generated by a family $\{u_{ij}\}_{i,j=1}^{\infty}$ of Haar unitary elements in $\mathcal{M}$ such that:*

(i) *for each i, j, $u_{i+1,j}^* u_{ij} u_{i+1,j}$ is in the von Neumann subalgebra generated by $\{u_{1j}, \ldots, u_{ij}\}$;*

(ii) *for each $j \geqslant 1$, $\{u_{1j}, u_{2j}, \ldots\} \cap \{u_{1,j+1}, u_{2,j+1}, \ldots\} \neq \varnothing$.*

Then $\mathcal{G}(\mathcal{M}) = 0$. In particular, $\mathcal{M}$ is singly generated.

REMARK 5.18. Combining with the results in [4], [7], [8], we have shown that most of the type II_1 factors, whose free entropy dimensions are known to be less than or equal to one, are singly generated.

EXAMPLE 5.19. New examples of singly generated II_1 factors can be constructed by considering the group von Neumann algebras associated with some countable discrete groups. The following are a few of them:

(i) Let G be the group $\langle g_1, g_2, \ldots \mid g_i g_{i+1} = g_{i+1} g_i, i = 1, \ldots \rangle$. Then $\mathcal{G}(L(G)) = 0$ and $L(G)$ is singly generated, where $L(G)$ is the group von Neumann algebra associated with G.

(ii) Let G be the group $\langle a, b, c \mid ab^2a^{-1} = b^3, ac^2a^{-1} = c^3 \rangle$. Then $\mathcal{G}(L(G)) = 0$ and $L(G)$ is singly generated.

(iii) Let $\mathcal{R}$ is the hyperfinite II_1 factor and $\mathcal{B}$ is a diffuse von Neumann subalgebra of $\mathcal{B}$. Let

$$\mathcal{M} = \mathcal{R} *_{\mathcal{B}} *\mathcal{R} *_{\mathcal{B}} *\mathcal{R} *_{\mathcal{B}} * \cdots$$

be the amalgamated free product of $\mathcal{R}$ over $\mathcal{B}$. Then by Theorem 5.5, we know that $\mathcal{G}(\mathcal{M}) = 0$ and $\mathcal{M}$ is singly generated.

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JUNHAO SHEN, DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NH, 03824, U.S.A
E-mail address: junhao.shen@unh.edu

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