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DIMENSION FORMULA FOR LOCALIZATION OF HILBERT MODULES

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ABSTRACT. This paper considers a conjecture of Douglas, Misra and Varughese about the dimension formula of the localization of an analytic Hilbert submodule generated by polynomials [8]. The conjecture states that there exists a relation between the dimension of the localization of an analytic submodule generated by an ideal of polynomials and the codimension of the zero variety of the ideal. It is shown that the conjecture is true in most natural cases. Some examples show that there are exceptions to this conjecture. The results apply here to compute curvature invariants of quotient submodules.

KEYWORDS: *Hilbert module, localization.*

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INTRODUCTION

The concept of tensor product of Hilbert modules was introduced in [9]. It is useful to analyze Šilov resolutions as well as to construct new Hilbert modules. Let $\mathcal{M}$ be a Hilbert module over a function algebra $\mathcal{A}$, which consists of holomorphic functions over a domain Ω of $\mathbb{C}^n$. Let $\mathrm{x} \in \mathcal{M}_{\mathcal{A}}$, the maximal ideal space of the function algebra $\mathcal{A}$, then the module tensor product of $\mathcal{M}$ with the one dimensional Hilbert module $\mathbb{C}_{\mathrm{x}}$ is called the localization of $\mathcal{M}$ at x. One can obtain much information about the Hilbert modules by using localization technique. Higher dimensional localization of Hilbert modules was considered in [5].

Note that $\mathcal{M} \otimes_{\mathcal{A}} \mathbb{C}_{\mathrm{x}}$ is isomorphic to the module direct sum of copies of the one dimensional module $\mathbb{C}_{\mathrm{x}}$, so the only invariant for the localization of $\mathcal{M}$ is $\dim(\mathcal{M} \otimes_{\mathcal{A}} \mathbb{C}_{\mathrm{x}})$. In [8], Douglas, Misra and Varughese made a conjecture about the following dimension formula for the localization of Hilbert module. Let us recall it briefly. Let Ω be a bounded, simply connected domain in $\mathbb{C}^n$ and let $\mathcal{A}(\Omega)$ denote the algebra consisting of holomorphic functions in Ω, which are continuous on $\overline{\Omega}$, the closure of Ω, then it is a closed algebra with respect to the supremum norm on Ω. Assume further that Ω is polynomially convex, then

$\mathcal{A}(\Omega)$ is the closure of polynomials with respect to the supremum norm on Ω. Let τ be an ideal of the polynomial ring $\mathcal{C} = \mathbb{C}[z_1, \dots, z_n]$, H be an analytic Hilbert module over Ω, and let $\mathcal{M}_0$ be a submodule of H formed by the closure of τ in H, then one can see that the localization of the submodule $\mathcal{M}_0$ at a point $\lambda = (\lambda_1, \dots, \lambda_n) \in \Omega$ is $\bigcap\limits_{l=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* \subseteq \mathcal{M}_0$. Douglas, Misra and Varughese conjectured that

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = \begin{cases} 1 & \text{for } \lambda \notin Z(\tau) \cap \Omega, \\ \operatorname{codim} Z(\tau) & \text{for } \lambda \in Z(\tau) \cap \Omega, \end{cases}$$

where $Z(\tau)$ is the common zero set of the ideal τ, and codim $Z(\tau)$ is the codimension of $Z(\tau)$, which is $n - \dim Z(\tau)$. Using the techniques of the characteristic spaces cf. [11], [12], [13], it is shown that the conjecture is equivalent to a problem from elementary algebraic geometry. The problem is: If τ is an ideal of $C[z_1, \dots, z_n]$, and $\lambda \in Z(\tau) \cap \Omega$, does it hold that

$$\dim \tau / \mathfrak{M}_\lambda \tau = \operatorname{codim} Z(\tau)\,?$$

where $\mathfrak{M}_\lambda = (z_1 - \lambda_1, \dots, z_n - \lambda_n)$ is the maximal ideal of $\mathbb{C}[z_1, \dots, z_d]$ at λ. In this paper, it is shown that if τ is prime, and λ is any smooth point in $Z(\tau)$, then above equality holds. However, some classic examples in algebraic geometry show that there are exceptions to this equality. Hence, the conjecture is true in most natural cases. When the ideal is generated by a single polynomial, then the conjecture is always true. Moreover, if we assume that the Hilbert module H is in the Cowen–Douglas class $B_1(\Omega^*)$ (cf. [3], [4]), then the Hilbert submodule $\mathcal{M}_0$ is locally free, which leads to some calculations about the curvature of the vector bundle associated with the Hilbert submodule $\mathcal{M}_0$. We apply this calculation to show that some Hilbert submodules are not equivalent.

1. PRELIMINARIES

In this section, we will give some preliminaries. First we recall the method of characteristic space, which is systematically developed by Guo (cf. [11], [12], [13]).

Let $\mathcal{C} = \mathbb{C}[z_1, \dots, z_n]$ denote the polynomial ring, and let

$$q = \sum a_{m_1,\dots,m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$$

be a polynomial in $\mathcal{C}$. We use $q(D)$ to denote the linear partial differential operator

$$q(D) = \sum a_{m_1,\dots,m_n} \frac{\partial^{m_1+m_2+\cdots+m_n}}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}}.$$

For an ideal J of $\mathcal{C}$, the characteristic space of J at $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ is defined to be

$$J_\alpha = \{q \in \mathcal{C} : q(D)p|_\alpha = 0, \forall p \in J\},$$

where $q(D)p|_\alpha$ denotes $(q(D)p)(\alpha)$. One can easily check the following identity:

$$q(D)(z_jf)\mid_\alpha = \alpha_j q(D)f\mid_\alpha + \frac{\partial q}{\partial z_j}(D)f\mid_\alpha, \quad j = 1, \dots, n.$$

Basic facts about J_α are that J_α is invariant under the action of the partial differential operators $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$, and that $J_\alpha \neq 0$ if and only if $\alpha \in Z(J)$.

For ideals I_1, I_2 of $\mathcal{C}$ satisfying $I_1 \supseteq I_2$, I_1 and I_2 are said to have the same multiplicity at α if $I_{1\alpha} = I_{2\alpha}$. We use the symbol $Z(I_2)\backslash Z(I_1)$ to denote the set $\{\alpha \in Z(I_2) : I_{2\alpha} \neq I_{1\alpha}\}$ (note that $I_{1\alpha} \subseteq I_{2\alpha}$). For $\alpha \in Z(I_2)\backslash Z(I_1)$, the multiplicity of I_2 relative to I_1 at α is defined to be $\dim I_{2\alpha}/I_{1\alpha}$, and the cardinality of $Z(I_2)\backslash Z(I_1)$, denoted by $\mathrm{card}(Z(I_2)\backslash Z(I_1))$, is defined by the equality

$$\mathrm{card}(Z(I_2)\backslash Z(I_1)) = \sum_{\alpha \in Z(I_2)\backslash Z(I_1)} \dim I_{2\alpha}/I_{1\alpha}.$$

We need the following two lemmas. The first lemma appeared in [11], which is a generalization of a result of Ahern and Clark [1].

LEMMA 1.1 ([11], Theorem 3.1). *Let H be an analytic Hilbert module over the domain $\Omega \subseteq \mathbb{C}^n$, and I_1, I_2 be two ideals of $\mathcal{C}$ satisfying $I_1 \supseteq I_2$, and $Z(I_2)\backslash Z(I_1) \subset \Omega$. Let $[I_1], [I_2]$ be the closures of I_1, I_2 respectively in H. Then*

$$\dim [I_1]/[I_2] = \dim I_1/I_2.$$

LEMMA 1.2 ([11], Corollary 2.5). *If I_1, I_2 are two ideals in $\mathcal{C}$, $I_1 \supseteq I_2$, and*

$$\dim I_1/I_2 = k < \infty.$$

Then we have

$$\dim I_1/I_2 = \mathrm{card}(Z(I_2)\backslash Z(I_1)).$$

Next we recall the concept of localization of Hilbert modules. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two Hilbert modules over the function algebra $\mathcal{A}$, let $\mathcal{M}_1 \otimes \mathcal{M}_2$ denote the Hilbert space tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\mathcal{M}_1 \otimes \mathcal{M}_2$ can be made into a Hilbert module over $\mathcal{A}$ using the action either on $\mathcal{M}_1$ or on $\mathcal{M}_2$. Let $\mathcal{N}$ denote the closure of the linear span of the vectors

$$\{fx \otimes y - x \otimes fy \mid f \in \mathcal{A}, x \in \mathcal{M}_1, y \in \mathcal{M}_2\},$$

then $\mathcal{N}$ is a Hilbert submodule of $\mathcal{M}_1 \otimes \mathcal{M}_2$. By forming the quotient of $\mathcal{M}_1 \otimes \mathcal{M}_2$ by $\mathcal{N}$, we see the two actions on the quotient are equal. The quotient with this action is called the module tensor product of $\mathcal{M}_1$ and $\mathcal{M}_2$, and denoted by $\mathcal{M}_1 \otimes_{\mathcal{A}} \mathcal{M}_2$.

Let $\mathrm{x} \in \mathcal{M}_{\mathcal{A}}$, then the one dimensional module $\mathbb{C}_\mathrm{x}$ is defined to be the space of complex numbers with the module action as follows:

$$f \cdot \zeta = f(\mathrm{x})\zeta, \quad f \in \mathcal{A}, \zeta \in \mathbb{C}.$$

As in the introduction, for an analytic Hilbert module H over the function algebra $\mathcal{A}(\Omega)$, and a submodule $\mathcal{M}_0$ of H, we see ([9], Theorem 5.14) that the localization of $\mathcal{M}_0$ at a point $\lambda = (\lambda_1, \ldots, \lambda_n)$ is identified with

$$\bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = \bigcap_{i=1}^{n} \big(\mathcal{M}_0 \ominus (z_i - \lambda_i)\mathcal{M}_0\big) = \mathcal{M}_0 \ominus \Big(\sum_{i=1}^{n} (z_i - \lambda_i)\mathcal{M}_0\Big).$$

Moreover, let τ be an ideal of the polynomial ring $\mathcal{C}$, and set $\mathcal{M}_0 = [\tau]$, where $[\tau]$ denotes the closure of τ in H. Then the above equality implies that

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = \dim\,[\tau]/[\mathfrak{M}_\lambda \tau] \tag{1.1}$$

where $\lambda \in \Omega$, and $\mathfrak{M}_\lambda = (z_1 - \lambda_1, \ldots, z_n - \lambda_n)$ is the maximal ideal of $\mathcal{C}$ at λ.

From the definition of characteristic space of an ideal, we have the following assertion.

PROPOSITION 1.3. $Z(\mathfrak{M}_\lambda \tau) \backslash Z(\tau) = \{\lambda\}$.

Using Proposition 1.3 and combining Lemma 1.1 and (1.1), we have

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = \dim \tau/\mathfrak{M}_\lambda \tau, \quad \lambda \in \Omega. \tag{1.2}$$

From the equality (1.2), the conjecture is equivalent to a problem from elementary algebraic geometry.

Problem. If τ is an ideal of $\mathbb{C}[z_1, \ldots, z_n]$, and $\lambda \in Z(\tau) \cap \Omega$, does it hold that

$$\dim \tau/\mathfrak{M}_\lambda \tau = \operatorname{codim} Z(\tau)\,?$$

Some more definitions and facts from commutative algebra and algebraic geometry are also needed. We refer the reader to [6], [15], [18] for more details. If τ is prime, then the height of τ, denoted by height(τ), is defined to be the maximal length l among all the properly increasing chain of prime ideals

$$0 = \tau_0 \subset \tau_1 \subset \cdots \subset \tau_l = \tau.$$

Since $\mathcal{C}$ is Noetherian, every prime ideal has finite height. For an arbitrary ideal, its height is defined to be the minimum of the height of its associated prime ideals. For an ideal J of $\mathcal{C}$ with height$(J) = l$, we have

$$\dim Z(J) = n - l = n - \operatorname{codim} Z(J),$$

that is, $\operatorname{codim} Z(J) = \text{height}(J)$. So correspondingly, $\dim Z(J)$ is the maximal dimension of proper irreducible components of $Z(J)$.

Now assume that τ is prime, and is generated by polynomials $p_1, p_2, \ldots, p_t$, i.e., $\tau = (p_1, \ldots, p_t)$, and assume that codim $(\tau) = r$. Then for any point $\lambda \in Z(\tau)$, it holds that

$$\operatorname{rank}\Big(\frac{\partial p_i}{\partial z_j}\Big)_{\substack{1 \leqslant i \leqslant t \\ 1 \leqslant j \leqslant n}}\Big|_\lambda \leqslant r. \tag{1.3}$$

A point $\lambda \in Z(\tau)$ is said to be a smooth (nonsingular) point if the equality in (1.3) holds, otherwise, λ is said to be singular.

2. MAIN RESULT

In this section, we will give the main result on the dimension formula of the localization of Hilbert submodules.

First, we assert that $\dim \tau/\mathfrak{M}_\lambda \tau < \infty$.

In fact, assume that $\tau = (p_1, \ldots, p_t)$, i.e., τ is generated by the polynomials $p_1, \ldots, p_t$. Then for $\forall f \in \tau$, we have $f = \sum_{i=1}^{t} h_i p_i, h_i \in \mathcal{C}$. Write

$$h_i = (h_i - h_i(\lambda)) + h_i(\lambda) = \sum_{j=1}^{n} (z_j - \lambda_j) f_{ij} + h_i(\lambda),$$

where $f_{ij} \in \mathcal{C}$, it then follows that

$$f = \sum_{j=1}^{n} \sum_{i=1}^{t} (z_j - \lambda_j) p_i f_{ij} + \sum_{i=1}^{t} h_i(\lambda) p_i,$$

which gives that

$$\dim \tau/\mathfrak{M}_\lambda \tau \leqslant t < \infty. \tag{2.1}$$

By Lemma 1.2 and Proposition 1.3, it holds that

$$\dim\ \tau/\mathfrak{M}_\lambda \tau = \dim\, (\mathfrak{M}_\lambda \tau)_\lambda / \tau_\lambda.$$

Therefore, we need to compute $\dim\, (\mathfrak{M}_\lambda \tau)_\lambda / \tau_\lambda$.

PROPOSITION 2.1. *If $\lambda \notin Z(\tau)$, then $(\mathfrak{M}_\lambda \tau)_\lambda / \tau_\lambda = \mathbb{C}$.*

Proof. For $\lambda \notin Z(\tau)$, clearly, $\tau_\lambda = \{0\}$. Choose $f \in \tau$ such that $f(\lambda) \neq 0$, then $(z_j - \lambda_j) f \in \mathfrak{M}_\lambda \tau$, for $j = 1, \ldots, n$. Given $(a_1, \ldots, a_n) \in \mathbb{C}^n$, then for $j = 1, \ldots, n$, we have

$$\Big(\sum_{i=1}^{n} a_i z_i \Big)(D)((z_j - \lambda_j) f)|_\lambda = \sum_{i=1}^{n} a_i \frac{\partial}{\partial z_i} ((z_j - \lambda_j) f)|_\lambda = a_j f(\lambda).$$

If $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$, the above identity shows that $\sum_{i=1}^{n} a_i z_i \notin (\mathfrak{M}_\lambda \tau)_\lambda$. From the invariance of $(\mathfrak{M}_\lambda \tau)_\lambda$ under $\frac{\partial}{\partial z_i}$, $i = 1, 2, \ldots, n$, one concludes that $(\mathfrak{M}_\lambda \tau)_\lambda = \mathbb{C}$, and the result follows immediately. ∎

Thus, we have that

$$\dim \tau/\mathfrak{M}_\lambda \tau = \dim\, (\mathfrak{M}_\lambda \tau)_\lambda / \tau_\lambda = 1,$$

if $\lambda \notin Z(\tau)$. From (1.2) we conclude that:

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = 1, \quad \text{for } \lambda \notin Z(\tau). \tag{2.2}$$

This shows that the conjecture of Douglas, Misra and Varughese is true in the case $\lambda \notin Z(\tau)$.

Now we assume that $\lambda \in Z(\tau)$. If τ is generated by a single polynomial p, i.e., $[\tau] = [p]$, then $\forall f \in \tau$, we have $f = ph$ for some $h \in \mathcal{C}$. By the equality $pf = p(f - f(\lambda)) + pf(\lambda)$, a simple reasoning shows that

$$\dim \tau / \mathfrak{M}_\lambda \tau = 1.$$

This gives the following.

PROPOSITION 2.2. *If τ is singly generated and $\lambda \in Z(\tau)$, then*

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = 1 = \operatorname{codim} Z(\tau). \tag{2.3}$$

Now we reach the main result in this paper, which shows when τ is prime, and $\lambda \in Z(\tau)$ is a smooth point, that the conjecture of Douglas, Misra and Varughese is true.

THEOREM 2.3. *If $\tau = (p_1, \ldots, p_t)$ is prime with* codim $Z(\tau) = r$, *for a smooth point $\lambda \in Z(\tau)$, we have*

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = \dim \tau / \mathfrak{M}_\lambda \tau = r. \tag{2.4}$$

Proof of Theorem 2.3. First we make an assertion which follows from the proof of Theorem 5.1 in [15], we give the details for the reader's convenience.

Assertion. $\dim (\tau + \mathfrak{M}_\lambda^2)/\mathfrak{M}_\lambda^2 = r$.

In fact, construct a linear map

$$\Phi : \mathfrak{M}_\lambda \mapsto \mathbb{C}^n, \Phi(f) = \Big(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\Big)|_\lambda, \quad f \in \mathfrak{M}_\lambda.$$

Then Φ is surjective, moreover, $\Phi(z_1 - \lambda_1), \ldots, \Phi(z_n - \lambda_n)$ is a linear basis in $\mathbb{C}^n$, and $\ker\Phi = \mathfrak{M}_\lambda^2$. It yields that

$$\widetilde{\Phi} : \mathfrak{M}_\lambda / \mathfrak{M}_\lambda^2 \mapsto \mathbb{C}^n$$

is a linear isomorphism. Since $\lambda \subset Z(\tau)$, it follows that $\tau \subset \mathfrak{M}_\lambda$. Moreover, since $B = \{\overline{p}_1, \ldots, \overline{p}_t\}$ is a linear generating set for $(\tau + \mathfrak{M}_\lambda^2)/\mathfrak{M}_\lambda^2$, we obtain that the cardinal number of the maximal linear independence subset of B equals that of the set $\{\widetilde{\Phi}(\overline{p}_1), \ldots, \widetilde{\Phi}(\overline{p}_t)\}$, which is, rank $\left(\frac{\partial p_i}{\partial z_j}\right)_{\substack{1 \leqslant i \leqslant t \\ 1 \leqslant j \leqslant n}}\Big|\lambda = r$. This completes the proof of the assertion.

Since $(\tau + \mathfrak{M}_\lambda^2)/\mathfrak{M}_\lambda^2$ is isomorphic to $\tau/\tau \cap \mathfrak{M}_\lambda^2$ as linear vector space, it suffices to show that

$$\dim \tau/\mathfrak{M}_\lambda \tau = \dim \tau/\tau \cap \mathfrak{M}_\lambda^2.$$

Since $\tau \subseteq \mathfrak{M}_\lambda$, it follows that $\mathfrak{M}_\lambda \tau \subseteq \tau \cap \mathfrak{M}_\lambda^2$. Combining this inclusion and (2.1) gives that

$$t \geqslant \dim \tau/\mathfrak{M}_\lambda \tau \geqslant \dim \tau/\tau \cap \mathfrak{M}_\lambda^2. \tag{2.5}$$

Since $\mathfrak{M}_\lambda \tau \subseteq \tau \cap \mathfrak{M}_\lambda^2 \subseteq \tau$, then for $\forall \alpha \in \mathbb{C}^n$, it holds that

$$\tau_\alpha \subseteq (\tau \cap \mathfrak{M}_\lambda^2)_\alpha \subseteq (\mathfrak{M}_\lambda \tau)_\alpha.$$

By Proposition 1.3, if $\alpha \neq \lambda$, then $\tau_\alpha = (\mathfrak{M}_\lambda \tau)_\alpha$, and thus

$$\tau_\alpha = (\tau \cap \mathfrak{M}_\lambda^2)_\alpha = (\mathfrak{M}_\lambda \tau)_\alpha, \quad \alpha \neq \lambda.$$

Using Lemma 1.2, we have

$$\dim \tau/\mathfrak{M}_\lambda \tau = \dim (\mathfrak{M}_\lambda \tau)_\lambda/\tau_\lambda, \quad \dim \tau/\tau \cap \mathfrak{M}_\lambda^2 = \dim (\tau \cap \mathfrak{M}_\lambda^2)_\lambda/\tau_\lambda.$$

Thus it suffices to prove that

$$(\mathfrak{M}_\lambda \tau)_\lambda = (\tau \cap \mathfrak{M}_\lambda^2)_\lambda. \tag{2.6}$$

To prove (2.6), we need some technique of localization. First, we recall some notions and facts which can be found in [10], [16]. Let $\mathcal{O}_\lambda$ denote the ring of all germs of analytic functions at λ. Then $\mathcal{O}_\lambda$ is a unique factorization domain (UFD), and is a Noetherian local ring of Krull dimension n. Let J be an ideal of $\mathcal{C}$, and $f \in J$. The restriction of f to a neighborhood of λ represents a germ in $\mathcal{O}_\lambda$, which is denoted by f_λ. Let $J^{(\lambda)}$ be the ideal of $\mathcal{O}_\lambda$ generated by $\{f_\lambda : f \in J\}$, and $J^{(\lambda)}$ is called the localized ideal of J at λ. Obviously, the characteristic spaces of J and $J^{(\lambda)}$ at λ are the same [13], i.e., $J_\lambda = (J^{(\lambda)})_\lambda$. We denote $\tau \cap \mathfrak{M}_\lambda^2$ by K for simplicity. Thus to prove (2.6), it suffices to show that $\mathfrak{M}_\lambda \tau$ and K have the same localized ideal at λ, that is,

$$(\mathfrak{M}_\lambda \tau)^{(\lambda)} = K^{(\lambda)}. \tag{2.7}$$

Recall that $\operatorname{codim}(\tau) = r$ and λ is a smooth point in $Z(\tau)$. We claim first that:

Claim. $\tau^{(\lambda)}$ can be generated by r germs $f_{1\lambda}, \ldots, f_{r\lambda} \in \tau^{(\lambda)}$, which are represented by r holomorphic functions $f_1, \ldots, f_r$ in a neighborhood of λ with

$$\operatorname{rank} \begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_r} \\ \cdots & \cdots & \cdots \\ \frac{\partial f_r}{\partial z_1} & \cdots & \frac{\partial f_r}{\partial z_r} \end{pmatrix} \Bigg|_\lambda = r. \tag{2.8}$$

In fact, since the rank of $\left(\frac{\partial p_i}{\partial z_j}\right)_{\substack{1\leqslant i\leqslant t\\ 1\leqslant j\leqslant n}}\Big|_\lambda$ equals r, without loss of generality, we assume that

$$\operatorname{rank}\begin{pmatrix}\frac{\partial p_1}{\partial z_1} & \cdots & \frac{\partial p_1}{\partial z_r}\\ \multicolumn{3}{c}{\dotfill}\\ \frac{\partial p_r}{\partial z_1} & \cdots & \frac{\partial p_r}{\partial z_r}\end{pmatrix}\Bigg|_\lambda = r,$$

and $\lambda = \mathbf{0} = (0,\dots,0)$. Now we define a holomorphic map $F : \mathbb{C}^n \mapsto \mathbb{C}^n$ by

$$F(z_1,\dots,z_n) = (\omega_1,\dots,\omega_n),$$

where

$$\omega_1 = p_1(z_1,\dots,z_n),\dots,\omega_r = p_r(z_1,\dots,z_n),\omega_{r+1} = z_{r+1},\dots,\omega_n = z_n.$$

Since $\operatorname{rank}\left(\frac{\partial \omega_i}{\partial z_j}\right)\Big|_{\mathbf{0}} = n$, it follows from the Inverse Mapping Theorem that there exist neighborhoods U, W of $\mathbf{0}$ such that F is a biholomorphic mapping from U onto W. Let $\widehat{\tau}^{(\mathbf{0})}$ denote the ideal of $\mathcal{O}_{\mathbf{0}}$ generated by $\{f \circ F^{-1}|_W : f \in \tau\}$, then it suffices to prove that $\widehat{\tau}^{(\mathbf{0})}$ can be generated by r germs of $\{f \circ F^{-1}|_W : f \in \tau\}$ at 0.

Observe that

$$Z(\tau) \cap U = \{z \in U : p_1 = \cdots = p_t = 0\}.$$

On the neighborhood W of $\mathbf{0}$, let

$$\begin{aligned}\widetilde{p}_1(\omega_1,\dots,\omega_n) &= p_1 \circ F^{-1}(\omega_1,\dots,\omega_n) = \omega_1,\\ &\cdots\\ \widetilde{p}_r(\omega_1,\dots,\omega_n) &= p_r \circ F^{-1}(\omega_1,\dots,\omega_n) = \omega_r,\\ &\cdots\\ \widetilde{p}_t(\omega_1,\dots,\omega_n) &= p_t \circ F^{-1}(\omega_1,\dots,\omega_n),\end{aligned}$$

which are holomorphic functions in W. So

$$z = (z_1,\dots,z_n) \in \{z \in U : p_i(z) = 0,\ i = 1,\dots,t\} = Z(\tau) \cap U$$

if and only if

$$\omega = (\omega_1,\dots,\omega_n) \in V$$

where $V = \{\omega \in W : \omega_i = 0,\ i = 1,\dots,r;\ \widetilde{p}_j(\omega) = 0,\ j = r+1,\dots,t\}$.

By page 109 of [17], when the algebraic variety $Z(\tau)$ is viewed as a holomorphic subvariety, then their smooth points sets are the same, and the dimensions are also the same. So we can work with the algebraic variety $Z(\tau)$ in the category of holomorphic varieties. Since F maps $Z(\tau) \cap U$ biholomorphically onto V, it follows that

$$\dim V = \dim Z(\tau) \cap U = n - r.$$

Let $W' = \{(w_{r+1}, \dots, w_n) \in \mathbb{C}^{n-r} : (0, \dots, 0, w_{r+1}, \dots, w_n) \in W\}$, and

$$p'_{r+1}(\omega_{r+1}, \dots, \omega_n) = \widetilde{p}_{r+1}(0, \dots, 0, \omega_{r+1}, \dots, \omega_n)$$
$$\cdots$$
$$p'_t(\omega_{r+1}, \dots, \omega_n) = \widetilde{p}_t(0, \dots, 0, \omega_{r+1}, \dots, \omega_n).$$

Define a map Π from $W' \subseteq \mathbb{C}^{n-r}$ into W as follows:

$$\Pi(\omega_{r+1}, \dots, \omega_n) = (0, \dots, 0, \omega_{r+1}, \dots, \omega_n),$$

then Π is a biholomorphic map from W' onto $\Pi(W')$. So the dimension of

$$V' \triangleq \{\omega' = (\omega_{r+1}, \dots, \omega_n) \in W' : p'_{r+1}(\omega') = \cdots = p'_t(\omega') = 0\}$$

is equal to the dimension of $\Pi(V') = V$, and is equal to $n - r$. Noticing that $V' \subseteq W' \subseteq \mathbb{C}^{n-r}$, and $\dim \mathbb{C}^{n-r} = n - r$, there exists a neighborhood of $\mathbf{0}$ contained in W', denoted by $B_{n-r}(\mathbf{0})$, such that

$$p'_i(\omega') \equiv 0, \tag{2.9}$$

for $i = r + 1, \dots, t$, and $\omega' = (\omega_{r+1}, \dots, \omega_n) \in B_{n-r}(\mathbf{0})$. Otherwise, without loss of generality, we assume that p'_{r+1} will not vanish identically on any neighborhood of $\mathbf{0}$ in $\mathbb{C}^{n-r}$. Since

$$V' \subseteq \{\omega' = (\omega_{r+1}, \dots, \omega_n) \in W' : p'_{r+1}(\omega') = 0\},$$

if there is no neighborhood in W' on which p'_{r+1} vanishes identically, then $\{\omega' = (\omega_{r+1}, \dots, \omega_n) \in W' : p'_{r+1}(\omega') = 0\}$ is an analytic hypersurface in $\mathbb{C}^{n-r}$, with dimension $n - r - 1$, from which it follows that $\dim V' \leqslant n - r - 1$. This yields a contradiction.

Now by (2.9), we have

$$\widetilde{p}_i(0, \dots, 0, \dots, 0, \omega_{r+1}, \dots, \omega_n) \equiv 0,$$

for $i = r + 1, \dots, t,$ and $(\omega_{r+1}, \dots, \omega_n) \in B_{n-r}(\mathbf{0})$. Using the product topology of $\mathbb{C}^r \times \mathbb{C}^{n-r} = \mathbb{C}^n$, we can find a neighborhood $B_r(\mathbf{0})$ of $\mathbf{0}$ in $\mathbb{C}^r$ such that

$$B_r(\mathbf{0}) \times B_{n-r}(\mathbf{0}) \triangleq \Sigma \subseteq W.$$

Now for $\omega = (\omega_1, \dots, \omega_r, \omega_{r+1}, \dots, \omega_n) \in \Sigma$, and $j = r + 1, \dots, t$, it holds that

$$\widetilde{p}_j(\omega) = \widetilde{p}_j(\omega) - \widetilde{p}_j(0, \dots, 0, \omega_{r+1}, \dots, \omega_n) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{p}_j(t\omega_1, \dots, t\omega_r, \omega_{r+1}, \dots, \omega_n) \mathrm{d}t$$

$$= \sum_{k=1}^{r} \omega_k \int_0^1 D_k \widetilde{p}_j(t\omega_1, \dots, t\omega_r, \omega_{r+1}, \dots, \omega_n) \mathrm{d}t.$$

Since $\widehat{\tau}^{(\mathbf{0})}$ is generated by $\omega_{1\mathbf{0}}, \dots, \omega_{r\mathbf{0}}, \widetilde{p}_{r+1,\mathbf{0}}, \dots, \widetilde{p}_{t,\mathbf{0}}$, the above equality implies that $\widehat{\tau}^{(\mathbf{0})}$ is generated by $\omega_{1\mathbf{0}}, \dots, \omega_{r\mathbf{0}}$, r holomorphic germs in $\widehat{\tau}^{(\mathbf{0})}$. Moreover, (2.8) is obviously true for $\omega_1, \dots, \omega_r$ in the coordinate $\{\omega_1, \dots, \omega_n.\}$ We thus complete the proof of the claim.

Now we continue to prove Theorem 2.3. For any $f \in K^{(\mathbf{0})}$, we have $f = \sum_{i=1}^{t} p_i h_i$ in some neighborhood of $\mathbf{0}$, where h_i is analytic in some neighborhood of $\mathbf{0}$ for $i = 1, \ldots, t$. Since

$$p_i h_i = p_i(h_i - h_i(\mathbf{0})) + p_i h_i(\mathbf{0}) \quad \text{and} \quad (p_i(h_i - h_i(\mathbf{0})))_{\mathbf{0}} \in K^{(\mathbf{0})},$$

we conclude that $\Big(\sum_{i=1}^{t} p_i h_i(\mathbf{0})\Big)_{\mathbf{0}} \in K^{(\mathbf{0})}$. By the claim, $\tau^{(\mathbf{0})}$ is generated by r germs $f_{1\mathbf{0}}, \ldots, f_{r\mathbf{0}}$, so we can write $p_i = \sum_{k=1}^{r} g_{ik} f_k$ in some neighborhood of $\mathbf{0}$, for $i = 1, \ldots, t$, and g_{ik} is analytic in some neighborhood of $\mathbf{0}$, for $i = 1, \ldots, t$, $k = 1, \ldots, r$. Thus,

$$\begin{aligned}\sum_{i=1}^{t} p_i h_i(\mathbf{0}) &= \sum_{i=1}^{t} \Big(\sum_{k=1}^{r} g_{ik} f_k\Big) h_i(\mathbf{0}) = \sum_{k=1}^{r} \Big(\sum_{i=1}^{t} g_{ik} h_i(\mathbf{0})\Big) f_k \\ &= \sum_{k=1}^{r} \Big[\sum_{i=1}^{t} (g_{ik} - g_{ik}(\mathbf{0})) h_i(\mathbf{0})\Big] f_k + \sum_{k=1}^{r} \Big[\sum_{i=1}^{t} (g_{ik}(\mathbf{0})) h_i(\mathbf{0})\Big] f_k.\end{aligned}$$

Since $(g_{ik} - g_{ik}(\mathbf{0}))_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}})^{(\mathbf{0})}$, it follows that

$$\Big\{\sum_{k=1}^{r} \Big[\sum_{i=1}^{t} (g_{ik} - g_{ik}(\mathbf{0})) h_i(\mathbf{0})\Big] f_k\Big\}_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}} \tau)^{(\mathbf{0})} \subseteq K^{(\mathbf{0})},$$

thus

$$\Big\{\sum_{k=1}^{r} \Big[\sum_{i=1}^{t} g_{ik}(\mathbf{0}) h_i(\mathbf{0})\Big] f_k\Big\}_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}}^2)^{(\mathbf{0})}. \tag{2.10}$$

By (2.8), we know that the vectors of the first order terms in the Taylor expansion of $f_k, k = 1, \ldots, r$ at $\mathbf{0}$ are linearly independent, thus from (2.10), we conclude that

$$\sum_{i=1}^{t} (g_{ik}(\mathbf{0})) h_i(\mathbf{0}) = 0, \quad \text{for } k = 1, \ldots, r.$$

This gives that

$$(p_i h_i(\mathbf{0}))_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}} \tau)^{(\mathbf{0})},$$

and hence $(p_i h_i)_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}} \tau)^{(\mathbf{0})}$. This shows that $f_{\mathbf{0}} \in (\mathfrak{M}_{\mathbf{0}} \tau)^{(\mathbf{0})}$. It follows that $K^{(\mathbf{0})} \subseteq (\mathfrak{M}_{\mathbf{0}} \tau)^{(\mathbf{0})}$ and we complete the proof of Theorem 2.3. ∎

It is interesting to understand the roles of primality and smoothness in Theorem 2.3. In dimension $n \geqslant 3$, the examples in Section 3 will show that the hypothesis of smoothness is necessary. However, it is not entirely clear if the hypothesis of primality in Theorem 2.3 is necessary.

Below, we show that in dimension $n = 2$, (2.4) remains true for $\forall \lambda \in Z(\tau)$.

PROPOSITION 2.4. *Let τ be a prime ideal of $\mathbb{C}[z_1, z_2]$, and* codim $Z(\tau) = r$. *Then for $\forall\lambda \in Z(\tau)$, it holds that*

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = r.$$

Proof. Since $\tau \subseteq \mathbb{C}[z_1, z_2]$, we have $\operatorname{codim} Z(\tau) \leqslant 2$. If $\operatorname{codim} Z(\tau) = 2$, then τ is a maximal ideal, and hence $\tau = (z_1 - \alpha_1, z_2 - \alpha_2)$ for some $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$. This means that

$$\dim \tau/\mathfrak{M}_\alpha \tau = 2.$$

Combining this equality with (1.2) gives that

$$\dim \bigcap_{i=1}^{n} \ker(M_{z_i}|_{\mathcal{M}_0} - \lambda_i)^* = 2,$$

where $\lambda = \alpha$ is the unique zero point of τ, and hence in this case, the desired conclusion holds. If $\operatorname{codim} Z(\tau) = 1$, then by Proposition 1.13 of [15], τ is generated by a single polynomial. Proposition 2.4 follows from Proposition 2.2 immediately. ∎

3. EXAMPLES AND APPLICATIONS

Now we give some examples to show that there exist exceptions for the conjecture of Douglas, Misra and Varughese in some cases.

EXAMPLE 3.1. Consider the ideal $\tau = (z_1^2, z_1z_2, z_2^2)$ of $\mathbb{C}[z_1, z_2]$, then obviously, τ is primary, but not prime, and $Z(\tau) = \{(0,0)\}$, hence

$$\operatorname{codim} Z(\tau) = 2.$$

However, any linear combination of z_1^2, z_1z_2, z_2^2 does not belong to the ideal $\mathfrak{M}_\mathbf{0}\tau = z_1\tau + z_2\tau$. Thus, applying (2.1) easily shows that

$$\dim\tau/\mathfrak{M}_\mathbf{0}\tau = \dim\tau/(z_1\tau + z_2\tau) = 3.$$

So in this case $\operatorname{codim}Z(\tau) \neq \dim \tau/\mathfrak{M}_\mathbf{0}\tau$.

EXAMPLE 3.2. Consider the ideal $\tau = (z_1^4 - z_2^3, z_1^5 - z_3^3, z_2^5 - z_3^4)$ in $\mathbb{C}[z_1, z_2, z_3]$, then τ is prime with $\operatorname{codim} Z(\tau) = 2$ (cf. Exercise 1.11 of [15]), and $(0,0,0)$ is a singular point in $Z(\tau)$. Moreover, by a careful computation, one gets that any nonzero linear combination of $z_1^4 - z_2^3, z_1^5 - z_3^3, z_2^5 - z_3^4$ does not belong to $\mathfrak{M}_\mathbf{0}\tau = z_1\tau + z_2\tau + z_3\tau$. So applying (2.1) shows that $\dim \tau/\mathfrak{M}_\mathbf{0}\tau = 3$. Thus we have $\operatorname{codim}Z(\tau) \neq \dim\tau/\mathfrak{M}_\mathbf{0}\tau$.

EXAMPLE 3.3. Consider the ideal $\tau = (z_2^2 - z_1z_3, z_1z_4 - z_2z_3, z_3^2 - z_1z_4)$ in $\mathbb{C}[z_1, z_2, z_3, z_4]$, then τ is a prime ideal in $\mathbb{C}[z_1, z_2, z_3, z_4]$ and $(0,0,0,0)$ is a singular point in $Z(\tau)$. It is clear that any nonzero linear combination of the generators $z_2^2 - z_1z_3,\ z_1z_4 - z_2z_3\ z_3^2 - z_1z_4$ does not belong to $\mathfrak{M}_\mathbf{0}\tau = z_1\tau + z_2\tau + z_3\tau + z_4\tau$.

By (2.1) again, we have $\dim \tau/\mathfrak{M}_0\tau = 3$. However, the rank of the Jacobian matrix at the smooth points of $Z(\tau)$ is 2, thus one has that

$$\operatorname{codim} Z(\tau) = 2 \neq \dim \tau/\mathfrak{M}_0\tau.$$

Note that the algebraic variety determined by such an ideal is called twisted cubic (cf. Example 1.10 of [14]), see Example 1.14 of [14] for a generalization of this ideal.

The last two examples show that when $n > 2$, it is necessary to assume that λ is a smooth point in $Z(\tau)$ in Theorem 2.3. The dimension formula of the localization of Hilbert modules not only gives an invariant for the Hilbert modules but also enables one to compute the curvature invariant (cf. [3], [4]) when the Hilbert module is generated by one polynomial. Thus, we can prove that some Hilbert modules are not unitarily isomorphic.

EXAMPLE 3.4. For $\alpha, \beta > 0$, let $\mathcal{H}^{(\alpha,\beta)}$ be the reproducing kernel Hilbert module over the bidisk $\mathbb{D}^2$ with the reproducing kernel

$$\mathrm{K}((z_1, z_2), (\lambda_1, \lambda_2)) = \frac{1}{(1 - z_1\overline{\lambda}_1)^{\alpha}(1 - z_2\overline{\lambda}_2)^{\beta}},$$

where $(z_1, z_2), (\lambda_1, \lambda_2) \in \mathbb{D}^2$. Let $\mathcal{M}^{(\alpha,\beta)}$ be the Hilbert submodule of $\mathcal{H}^{(\alpha,\beta)}$ generated by $z_1^m - z_2^n$, where m, n are positive integers, and $m \neq n$. let

$$\mathcal{Z} = \{z_1^m - z_2^n = 0, (z_1, z_2) \in \mathbb{D}^2\}$$

be the common zero set of the ideal generated by $z_1^m - z_2^n$. Note that $z_1^m - z_2^n$ is a quasihomogeneous polynomial with weight (n, m), then one can prove that $\mathcal{M}^{(\alpha,\beta)}$ is the largest set of functions in $\mathcal{H}^{(\alpha,\beta)}$ which vanish on $\mathcal{Z}$. Let $\mathcal{N}^{(\alpha,\beta)}$ be the quotient module of $\mathcal{H}^{(\alpha,\beta)}$ by $\mathcal{M}^{(\alpha,\beta)}$, then the Hilbert quotient module $\mathcal{N}^{(\alpha,\beta)}$ is isomorphic to $\mathcal{H}^{(\alpha,\beta)}_{\text{res}}$(cf. [2], [7]). Let $\Omega = \mathbf{B}(\eta_0; \frac{1}{4})$ be a disk of $\mathbb{C}$ which centeres at $\eta_0 = (\frac{1}{2}, 0)$ with radius $\frac{1}{4}$. Let $U = \Omega \times \Omega$ be the open subset of $\mathbb{D}^2$. Let Ψ be the biholomorphic map

$$U \mapsto V = \Psi(U) : u_1 = z_1^m - z_2^n, u_2 = z_2.$$

Then on $U \cap \mathcal{Z}$, one will get $z_1 = u_2^{n/m}, z_2 = u_2$. The curvature invariant of the Hilbert quotient module $\mathcal{N}^{(\alpha,\beta)}$, denoted by $\mathcal{K}_{\mathcal{N}}^{(\alpha,\beta)}$ is computed (cf. (1.6) of [7]) as follows:

$$\begin{aligned}
&\frac{\partial^2}{\partial u_2 \partial \overline{u}_2} \log \mathrm{K}_{\psi}((0, u_2), (0, u_2))\psi\Big|^*_{\text{res}}(\mathrm{d}u_2 \wedge \mathrm{d}\overline{u}_2) \\
&= \frac{\partial^2}{\partial u_2 \partial \overline{u}_2} \log \frac{1}{(1 - |u_2|^{2n/m})^{\alpha}(1 - |u_2|^2)^{\beta}} \psi\Big|^*_{\text{res}}(\mathrm{d}u_2 \wedge \mathrm{d}\overline{u}_2)
\end{aligned}$$

$$= \frac{\partial}{\partial u_2}\Big\{\frac{\alpha\frac{n}{m}u_2(|u_2|^2)^{n/m-1}}{(1-(|u_2|^2)^{n/m})} + \frac{\beta u_2}{(1-|u_2|^2)}\Big\}\psi\Big|^*_{\text{res}}(\mathrm{d}u_2\wedge \mathrm{d}\overline{u}_2)$$

$$= \Big\{\alpha\Big(\frac{n}{m}\Big)^2\Big[\frac{(|u_2|^2)^{n/m-1}}{(1-|u_2|^{2n/m})}+\frac{(|u_2|^2)^{2n/m-1}}{(1-|u_2|^{2n/m})^2}\Big]+\beta\frac{1}{(1-|u_2|^2)^2}\Big\}\psi\Big|^*_{\text{res}}(\mathrm{d}u_2\wedge \mathrm{d}\overline{u}_2).$$

Now let $|u_2| = \frac{\sqrt{2}}{2}$ and $\frac{1}{2}$ respectively, then by an easy computation one gets that $\mathcal{K}_{\mathcal{N}}^{(\alpha_1,\beta_1)} = \mathcal{K}_{\mathcal{N}}^{(\alpha_2,\beta_2)}$ only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Since the curvature invariant is a complete invariant for the Hilbert module which belongs to the first Cowen–Douglas Class, i.e., $B_1(\Omega^*)$, (cf. [3], [4]), we conclude that $\mathcal{N}^{(\alpha_1,\beta_1)}$ is isomorphic to $\mathcal{N}^{(\alpha_2,\beta_2)}$ only if $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

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