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THE C^*-ALGEBRAS $qA \otimes \mathcal{K}$ AND $S^2A \otimes \mathcal{K}$ ARE ASYMPTOTICALLY EQUIVALENT

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ABSTRACT. Let A be a separable C^*-algebra. We prove that its stabilized second suspension $S^2A \otimes \mathcal{K}$ and the C^*-algebra $qA \otimes \mathcal{K}$ constructed by Cuntz in the framework of his picture of KK-theory are asymptotically equivalent. This means that there exists an asymptotic morphism from $S^2A \otimes \mathcal{K}$ to $qA \otimes \mathcal{K}$ and an asymptotic morphism from $qA \otimes \mathcal{K}$ to $S^2A \otimes \mathcal{K}$ whose compositions are homotopic to the identity maps. This result yields an easy description of the natural transformation from KK-theory to E-theory. Also by Loring's result any asymptotic morphism from $q\mathbb{C}$ to any C^*-algebra B is homotopic to a $*$-homomorphism. We prove that the same is true when $\mathbb{C}$ is replaced by any nuclear C^*-algebra A and when B is stable.

KEYWORDS: *C^*-algebra, asymptotic morphism, E-theory, KK-theory.*

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INTRODUCTION

Let A be a separable C^*-algebra. Its first suspension is the C^*-algebra $SA = C_0(\mathbb{R}) \otimes A$. There are two other C^*-algebras associated to A that are of importance in the KK-theory of Kasparov: the second suspension C^*-algebra $S^2A = C_0(\mathbb{R}^2) \otimes A$ and the C^*-algebra qA constructed by Cuntz [3] in the framework of his picture of KK-theory. Both C^*-algebras can replace A in the definition of the KK-groups: for the second suspension this is Bott periodicity and for qA this is Cuntz's picture for KK-theory. These C^*-algebras are E-equivalent, i.e. their stabilized suspensions $S^3A \otimes \mathcal{K}$ and $SqA \otimes \mathcal{K}$ are equivalent in the category of separable C^*-algebras with morphisms being homotopy classes of asymptotic morphisms, where $\mathcal{K}$ denotes the C^*-algebra of compact operators. In the present paper we show that they are equivalent in this category without taking the suspension of the stabilizations. More precisely we construct an asymptotic morphism from $S^2A \otimes \mathcal{K}$ to $qA \otimes \mathcal{K}$ and a $*$-homomorphism from $qA \otimes \mathcal{K}$ to $S^2A \otimes \mathcal{K}$ such that their compositions are homotopic to the identity maps. In general one says that two C^*-algebras are *asymptotically equivalent* if there exist asymptotic morphisms from each to the other whose compositions are homotopic to the identity maps. So the main result of this paper (Theorem 3.10) says that C^*-algebras $qA \otimes \mathcal{K}$ and $S^2A \otimes \mathcal{K}$ are asymptotically equivalent.

As a corollary (Corollary 3.11) we obtain a description of E-theory that is similar in form to Cuntz's description of KK-theory. Cuntz [3] proved that $KK(A, B) = [qA, B \otimes \mathcal{K}]$ (where $[\,\cdot\,]$ means homotopy classes of $*$-homomorphisms). We assert that $E(A, B) = [[qA, B \otimes \mathcal{K}]]$ (where $[[\,\cdot\,]]$ means homotopy classes of asymptotic morphisms) and that the well known natural transformation $KK(A, B) \to E(A, B)$ is then nothing but the map that sends any $*$-homomorphism $qA \to B \otimes \mathcal{K}$ to itself.

One more corollary (Corollary 3.12) concerns the question of when asymptotic morphisms are homotopic to $*$-homomorphisms. In [6] it was proved that any asymptotic morphism from $q\mathbb{C}$ to any C^*-algebra B is homotopic to a $*$-homomorphism. We prove that the same is true not only for $\mathbb{C}$ but for any nuclear (even K-nuclear) C^*-algebra A if B is assumed to be stable. Recall that a C^*-algebra B is called *stable* if $B \otimes \mathcal{K} \cong B$.

The plan of the paper is as follows. The first section contains all necessary information about C^*-algebra qA. In the second one we construct an asymptotic morphism $f^A : S^2A \otimes \mathcal{K} \to qA \otimes \mathcal{K}$ and a $*$-homomorphism $g^A : qA \otimes \mathcal{K} \to S^2A \otimes \mathcal{K}$ and show that f^A induces a natural transformation from the KK-functor to the E-functor. In the third section we prove that f^A and g^A provide an asymptotic equivalence of the C^*-algebras $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$ and obtain the corollaries described above.

1. NECESSARY INFORMATION ABOUT qA

Let A and B be two C^*-algebras. A C^*-algebra C is called *the free product of A and B* if there are $*$-homomorphisms $i^A : A \to C$ and $i^B : B \to C$ with the following (universal) property: given $*$-homomorphisms $\phi_A : A \to D$ and $\phi_B : B \to D$ mapping A and B into the same C^*-algebra D, there is a unique $*$-homomorphism $\phi : C \to D$ such that $\phi \circ i^A = \phi_A$ and $\phi \circ i^B = \phi_B$. The $*$-homomorphisms i^A and i^B are referred to as *the canonical inclusions*. The free product of A and B will be denoted by $A * B$.

Consider $A * A$. Let ${i_1}^A : A \to A * A$ and $i_2^A : A \to A * A$ denote the two canonical inclusions of A as a C^*-subalgebra of $A * A$. The C^*-algebra qA constructed by Cuntz [3] is the closed ideal in $A * A$ generated by the set $\{i_1(x) - i_2(x) : x \in A\}$. One can prove that elements of the form

$$(i_1^A(x_1) - i_2^A(x_1)) \cdots (i_1^A(x_N) - i_2^A(x_N)) \text{ and } i_1^A(x)(i_1^A(x_1) - i_2^A(x_1)) \cdots (i_1^A(x_N) - i_2^A(x_N)),$$

where $x_0, x_1, \ldots, x_N \in A$, $N \in \mathbb{N}$, span a dense $*$-subalgebra in qA.

Let $\phi, \psi : A \to B$ be two $*$-homomorphisms. By the universal property of $A * A$ there is a unique $*$-homomorphism $Q(\phi, \psi) : A * A \to B$ such that

$$Q(\phi, \psi) \circ i_1^A = \phi, \quad Q(\phi, \psi) \circ i_2^A = \psi.$$

Let $q(\phi, \psi)$ denote the restriction of $Q(\phi, \psi)$ to qA. Note that if J is an ideal in B, then $Q(\phi, \psi)$ maps qA into J if and only if $\phi(x) - \psi(x) \in J$ for all $x \in A$. So in this case, $q(\phi, \psi) \in \mathrm{Hom}(qA, J)$.

2. CONSTRUCTING AN ASYMPTOTIC EQUIVALENCE BETWEEN $S^2A \otimes \mathcal{K}$ AND $qA \otimes \mathcal{K}$

Below all C^*-algebras are assumed to be separable.

For any two C^*-algebras A and B Connes and Higson define $E(A, B)$ to be the abelian group $[[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$ of homotopy classes of asymptotic morphisms from $SA \otimes \mathcal{K}$ to $SB \otimes \mathcal{K}$ [2]. Recall that an *asymptotic morphism from A to B* is a family of maps $(\phi_t)_{t\in[0,\infty)} : A \to B$ satisfying the following conditions:

(i) for any $a \in A$ the function $t \mapsto \phi_t(a)$ is continuous;
(ii) for any $a, b \in A$, $\lambda \in \mathbb{C}$

- $\lim\limits_{t\to\infty} \|\phi_t(a^*) - \phi_t(a)^*\| = 0,$
- $\lim\limits_{t\to\infty} \|\phi_t(a + \lambda b) - \phi_t(a) - \lambda\phi_t(b)\| = 0,$
- $\lim\limits_{t\to\infty} \|\phi_t(ab) - \phi_t(a)\phi_t(b)\| = 0.$

In [2] it was also shown that $[[SA \otimes \mathcal{K}, SB \otimes \mathcal{K}]] \cong [[S^2A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ and we shall always mean by the E-group the group $[[S^2A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ of homotopy classes of asymptotic morphisms from $S^2A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$.

Let $\beta^{\mathbb{C}} : C_0(\mathbb{R}^2) \otimes \mathcal{K} \to \mathcal{K}$ be the Bott asymptotic morphism. In fact it is the tensor product of the identity map $\mathrm{id}_{\mathcal{K}} : \mathcal{K} \to \mathcal{K}$ with the restriction to $C_0(\mathbb{R}^2) \subset C(\mathbb{T}^2)$ of the family of maps from $C(\mathbb{T}^2)$ to $\mathcal{K}^+$ constructed in the Voiculescu's example of almost commuting unitaries [8], but here we shall not use an explicit form of $\beta^{\mathbb{C}}$ but only the fact that it induces the identity map in the K-groups. Let

$$\beta^A = \beta^{\mathbb{C}} \otimes \mathrm{id}_A : S^2A \otimes \mathcal{K} \to A \otimes \mathcal{K}.$$

Obviously $\beta^A \in E(A, A)$. Note that since we always consider asymptotic morphisms up to homotopy we denote in the same way a class of homotopy equivalent asymptotic morphisms and any its representative.

For the KK-groups we will use Cuntz's approach [3] in which, as already was written, one regards $KK(A, B)$ as the group $[qA \otimes \mathcal{K}, B \otimes \mathcal{K}]$ of homotopy classes of $*$-homomorphisms from $qA \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$. Let

$$\gamma^A = q(\mathrm{id}_A, 0) \otimes \mathrm{id}_{\mathcal{K}} : qA \otimes \mathcal{K} \to A \otimes \mathcal{K}.$$

Then $\gamma^A \in KK(A,A)$ is a unit element for the associative product $KK(A,B)\times KK(B,C)\to KK(A,C)$. Namely there exists a bilinear pairing $KK(A,B)\times KK(B,C) \to KK(A,C)$ such that $x\times\gamma^B = x = \gamma^A \times x$ for any $x\in KK(A,B)$ [3].

Let A be a C^*-algebra. By [2] there exists a natural transformation from the functor $KK(A,-)$ into the functor $E(A,-)$ which is unique up to its value on $\gamma^A \in KK(A,A)$. Let

$$I_{A,B} : KK(A,B) \to E(A,B)$$

be such a natural transformation that $I_{A,A}(\gamma^A) = \beta^A$. Define an asymptotic morphism $f^A : S^2A\otimes\mathcal{K} \to qA\otimes\mathcal{K}$ by

$$f^A = I_{A,qA}(\mathrm{id}_{qA\otimes\mathcal{K}}).$$

The following easy theorem asserts that the asymptotic morphism f^A induces the natural transformation $I_{A,B}$.

THEOREM 2.1. *$I_{A,B}(\phi) = \phi\circ f^A$ for any $\phi\in KK(A,B)$.*

Proof. Since $\phi\in KK(A,B)$ is a $*$-homomorphism from $qA\otimes\mathcal{K}$ to $B\otimes\mathcal{K}$ it induces the maps $\phi_{KK} : KK(A,qA) \to KK(A,B)$ and $\phi_E : E(A,qA) \to E(A,B)$ in the KK-groups and the E-groups respectively. By the definition of a natural transformation of covariant functors the following diagram commutes

$$\begin{array}{ccc} KK(A,B) & \xrightarrow{I_{A,B}} & E(A,B) \\ \phi_{KK}\big\uparrow & & \big\uparrow\phi_E \\ KK(A,qA) & \xrightarrow{I_{A,qA}} & E(A,qA)\,. \end{array}$$

Hence for the element $\mathrm{id}_{qA\otimes\mathcal{K}} \in KK(A,qA)$ we get

$$\phi_E(I_{A,qA}(\mathrm{id}_{qA\otimes\mathcal{K}})) = I_{A,B}(\phi_{KK}(\mathrm{id}_{qA\otimes\mathcal{K}})).$$

But $\phi_E(I_{A,qA}(\mathrm{id}_{qA\otimes\mathcal{K}})) = \phi\circ I_{A,qA}(\mathrm{id}_{qA\otimes\mathcal{K}}) = \phi\circ f^A$ and $I_{A,B}(\phi_{KK}(\mathrm{id}_{qA\otimes\mathcal{K}})) = I_{A,B}(\phi\circ\mathrm{id}_{qA\otimes\mathcal{K}}) = I_{A,B}(\phi)$. ∎

COROLLARY 2.2. *$\gamma^A\circ f^A = \beta^A$.*

Proof. By Theorem 2.1 $\gamma^A\circ f^A = I_{A,A}(\gamma^A)$. Since we have chosen a natural transformation to be equal to β^A on the element γ^A we get $\gamma^A\circ f^A = \beta^A$. ∎

Now we define a $*$-homomorphism $g^A : qA\otimes\mathcal{K} \to S^2A\otimes\mathcal{K}$ in the following way. Let $\pi_1,\pi_2 : \mathbb{C}\to C_0(\mathbb{R}^2)^+\otimes M_2$ be two $*$-homomorphisms given by

$$\pi_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_2(1) = p_{\mathrm{Bott}} = \frac{1}{1+z\bar z}\begin{pmatrix} z\bar z & z \\ \bar z & 1 \end{pmatrix}$$

(we identify $\mathbb{R}^2$ with $\mathbb{C}$). Fix once and for all some inclusion $j : M_2 \to \mathcal{K}$ and some isomorphism $i : \mathcal{K}\otimes\mathcal{K} \to \mathcal{K}$. Define $\widetilde\pi_1,\widetilde\pi_2 : A \to A\otimes C_0(\mathbb{R}^2)^+\otimes\mathcal{K}$ by

$$\widetilde\pi_1 = (j\otimes\mathrm{id}_{A\otimes C_0(\mathbb{R}^2)^+})\circ(\mathrm{id}_A\otimes\pi_1), \quad \widetilde\pi_2 = (j\otimes\mathrm{id}_{A\otimes C_0(\mathbb{R}^2)^+})\circ(\mathrm{id}_A\otimes\pi_2),$$

respectively. Since

$$\widetilde{\pi}_1(a) - \widetilde{\pi}_2(a) \in C_0(\mathbb{R}^2) \otimes \mathcal{K} \otimes A = S^2A \otimes \mathcal{K}$$

for any $a \in A$, the $*$-homomorphism $q(\widetilde{\pi}_1, \widetilde{\pi}_2) : qA \to S^2A \otimes \mathcal{K}$ is defined. Set

$$g^A = (\mathrm{id}_{S^2A} \otimes i) \circ (q(\widetilde{\pi}_1, \widetilde{\pi}_2) \otimes \mathrm{id}_{\mathcal{K}}).$$

In the next section we show that f^A and g^A provide an asymptotic equivalence between $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$.

3. PROOF OF THE MAIN ASSERTION

To prove that f^A and g^A provide an asymptotic equivalence between $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$ we are going to show that their compositions induce the identity maps in E-functor and in the functor G that will be introduced in Subsection 3.2.

3.1. THE MAPS INDUCED BY f^A AND g^A IN E-FUNCTOR.

LEMMA 3.1. $\beta^A \circ g^A \sim \gamma^A$.

Proof. Note first of all that $g^A : qA \otimes \mathcal{K} \to S^2A \otimes \mathcal{K}$ and $\gamma^A : qA \otimes \mathcal{K} \to A \otimes \mathcal{K}$ factorize through the C^*-algebra $q\mathbb{C} \otimes A \otimes \mathcal{K}$. Namely let $\eta_1, \eta_2 : A \to (\mathbb{C} * \mathbb{C}) \otimes A$ be given by formulas

$$\eta_1(a) = i_1^{\mathbb{C}}(1) \otimes a, \quad \eta_2(a) = i_2^{\mathbb{C}}(1) \otimes a,$$

for any $a \in A$. Set

$$s^A = q(\eta_1, \eta_2) : qA \to q\mathbb{C} \otimes A.$$

It is easy to see that the diagrams

$$\begin{tikzcd} qA \otimes \mathcal{K} \arrow[rr, "\gamma^A"] \arrow[dr, "s^A \otimes \mathrm{id}_K"'] & & A \otimes \mathcal{K} \\ & q\mathbb{C} \otimes A \otimes \mathcal{K} \arrow[ur, "\gamma^{\mathbb{C}} \otimes \mathrm{id}_A"'] & \end{tikzcd}$$

and

$$\begin{tikzcd} qA \otimes \mathcal{K} \arrow[rr, "g^A"] \arrow[dr, "s^A \otimes \mathrm{id}_K"'] & & S^2A \otimes \mathcal{K} \\ & q\mathbb{C} \otimes A \otimes \mathcal{K} \arrow[ur, "g^{\mathbb{C}} \otimes \mathrm{id}_A"'] & \end{tikzcd}$$

commute, that is

$$\gamma^A = (\gamma^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (s^A \otimes \mathrm{id}_{\mathcal{K}}), \quad g^A = (g^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (s^A \otimes \mathrm{id}_{\mathcal{K}}).$$

Since $\beta^A = \beta^{\mathbb{C}} \otimes \mathrm{id}_A$ we have to establish the homotopy equivalence

$$(\gamma^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (s^A \otimes \mathrm{id}_{\mathcal{K}}) \sim (\beta^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (g^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (s^A \otimes \mathrm{id}_{\mathcal{K}})$$

or, equivalently,

$$\gamma^{\mathbb{C}} \sim \beta^{\mathbb{C}} \circ g^{\mathbb{C}}.$$

For that we use K-theory. Let $\gamma^{\mathbb{C}}_*$ and $(\beta^{\mathbb{C}} \circ g^{\mathbb{C}})_*$ be the induced homomorphisms from $K_0(q\mathbb{C})$ to $K_0(\mathbb{C})$. For the generator $[i_1^{\mathbb{C}}(1)] - [i_2^{\mathbb{C}}(1)]$ of $K_0(q\mathbb{C})$ we have

$$\begin{aligned}
&(\beta^{\mathbb{C}} \circ g^{\mathbb{C}})_*([i_1^{\mathbb{C}}(1)] - [i_2^{\mathbb{C}}(1)]) = \beta^{\mathbb{C}}_*([\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)] - [p_{\mathrm{Bott}}]) = [1],\\
&\gamma^{\mathbb{C}}_*([i_1^{\mathbb{C}}(1)] - [i_2^{\mathbb{C}}(1)]) = [1] - [0] = [1].
\end{aligned}$$

We used here that $[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)] - [p_{\mathrm{Bott}}]$ is a generator of $K_0(S^2\mathbb{C})$ and that the Bott map $\beta^{\mathbb{C}}$ induces the identity homomorphism in K-theory. So $\gamma^{\mathbb{C}}$ and $\beta^{\mathbb{C}} \circ g^{\mathbb{C}}$ induce the same homomorphisms in K-theory. This implies that these asymptotic homomorphisms are homotopic because, by Universal coefficients theorem,

$$\mathrm{Hom}(K_0(q\mathbb{C}), K_0(\mathcal{K})) \oplus \mathrm{Hom}(K_1(q\mathbb{C}), K_1(\mathcal{K})) \cong KK(q\mathbb{C}, \mathcal{K}) \oplus KK(Sq\mathbb{C}, \mathcal{K}),$$

and since

$$K_1(q\mathbb{C}) = K_1(\mathcal{K}) = 0, \quad KK(Sq\mathbb{C}, \mathcal{K}) = 0, \quad KK(q\mathbb{C}, \mathcal{K}) = [q\mathbb{C}, \mathcal{K}] \overset{[3]}{=} [[q\mathbb{C}, \mathcal{K}]],$$

we get

$$\mathrm{Hom}(K_0(q\mathbb{C}), K_0(\mathbb{C})) \cong [[q\mathbb{C} \otimes \mathcal{K}, \mathcal{K}]]. \quad \blacksquare$$

Let B be any C^*-algebra. Let

$$f_E^A : E(B, S^2A) \to E(B, qA) \quad \text{and} \quad g_E^A : E(B, qA) \to E(B, S^2A)$$

be the maps induced by f^A and g^A respectively.

PROPOSITION 3.2. (i) $f_E^A \circ g_E^A = \mathrm{id}$;
(ii) $g_E^A \circ f_E^A = \mathrm{id}$.

Here *id* means both the identity map from $E(B, S^2A)$ into itself and the identity map from $E(B, qA)$ into itself.

Proof. Consider the following diagram

$$\begin{tikzcd}
 & E(B,A) & \\
E(B,S^2A) \arrow[ur, "\beta_E^A"] \arrow[rr, shift right, "f_E^A"'] & & E(B,qA) \arrow[ul, "\gamma_E^A"'] \arrow[ll, shift right, "g_E^A"']
\end{tikzcd}.$$

Here β_E^A and γ_E^A are the maps induced by β^A and γ^A respectively. It is proved in [2] that β_E^A is an isomorphism. Furthermore γ_E^A also is an isomorphism. Indeed by [3] the map induced by γ^A in any covariant, homotopy invariant, split exact and stable functor is an isomorphism. Since the functor $E(B, -)$ has all these properties γ_E^A is an isomorphism.

By Lemma 3.1, $\beta_E^A \circ g_E^A = \gamma_E^A$ whence

$$g_E^A = (\beta_E^A)^{-1} \circ \gamma_E^A. \tag{3.1}$$

By Corollary 2.2 $\gamma_E^A \circ f_E^A = \beta_E^A$ whence

$$f_E^A = (\gamma_E^A)^{-1} \circ \beta_E^A. \tag{3.2}$$

The assertions of the proposition follow from (3.1) and (3.2). ∎

3.2. THE MAPS INDUCED BY f^A AND g^A IN G-FUNCTOR. Now instead of E-functor we are going to consider another bifunctor $G(B, A)$ and prove the result similar to Lemma 3.2 for the maps, induced by f^A and g^A in the functor $G(B, -)$, where B is fixed. Namely let $G(B, A)$ be the semigroup $[[qB \otimes \mathcal{K}, A \otimes \mathcal{K}]]$ of the classes of homotopy equivalent asymptotic homomorphisms from $qB \otimes \mathcal{K}$ to $A \otimes \mathcal{K}$. Obviously this is a contravariant functor in the first variable and a covariant functor in the second one. We need two results about this bifunctor — the Bott periodicity and the isomorphism $G(B, A) \cong G(B, qA)$. To prove them we need first of all a construction which produces an asymptotic morphism $q\psi : qD_1 \to qD_2$ out of an asymptotic morphism $\psi : D_1 \to D_2$, where D_1, D_2 are any C^*-algebras.

An asymptotic morphism ψ gives rise to a genuine $*$-homomorphism

$$F : D_1 \to C_b([0, \infty), D_2)/C_0([0, \infty), D_2)$$

given by

$$F(x) = \psi_t(x) + C_0([0, \infty), D_2),$$

for any $x \in D_1$. There are two $*$-homomorphisms $\bar{i}_1, \bar{i}_2 : C_b([0, \infty), D_2) \to C_b([0, \infty), D_2 * D_2)$ given by formulas

$$\bar{i}_1(f)(t) = i_1^{D_2}(f(t)), \quad \bar{i}_2(f)(t) = i_2^{D_2}(f(t)),$$

$f \in C_b([0, \infty), D_2)$. Since these $*$-homomorphisms send $C_0([0, \infty), D_2)$ to $C_0([0, \infty), D_2 * D_2)$ we have two $*$-homomorphisms

$$\widehat{i}_1, \widehat{i}_2 : C_b([0, \infty), D_2)/C_0([0, \infty), D_2) \to C_b([0, \infty), D_2 * D_2)/C_0([0, \infty), D_2 * D_2).$$

Set

$$\Phi = Q(\widehat{i}_1 \circ F, \widehat{i}_2 \circ F) : D_1 * D_1 \to C_b([0, \infty), D_2 * D_2)/C_0([0, \infty), D_2 * D_2).$$

Let $p : C_b([0, \infty), D_2 * D_2) \to C_b([0, \infty), D_2 * D_2)/C_0([0, \infty), D_2 * D_2)$ be the canonical surjection. Since

$$\Phi(i_1^{D_1}(a)) = p(i_1^{D_2}(\psi_t(a))), \quad \Phi(i_2^{D_1}(a)) = p(i_2^{D_2}(\psi_t(a))),$$

for any $a \in D_1$, and since qD_1 is the closed ideal generated by the set $\{i_1^{D_1}(a) - i_2^{D_1}(a) : a \in D_1\}$, we get

$$\Phi(qD_1) \subset p(C_b([0, \infty), qD_2)).$$

We shall denote the restriction of Φ to qD_1 also by Φ. Define a $*$-homomorphism

$$\tau : p(C_b([0, \infty), qD_2)) \to C_b([0, \infty), qD_2)/C_0([0, \infty), qD_2)$$

by

$$\tau(p(f)) = f + C_0([0,\infty), qD_2),$$

$f \in C_{\mathrm{b}}([0,\infty), qD_2)$. It is well-defined because for any $f \in C_{\mathrm{b}}([0,\infty), qD_2)$ the condition $f \in C_0([0,\infty), D_2 * D_2)$ implies $f \in C_0([0,\infty), qD_2)$. So we have $\tau \circ \Phi : qD_1 \to C_{\mathrm{b}}([0,\infty), qD_2)/C_0([0,\infty), qD_2)$. Choose a continuous section

$$s : C_{\mathrm{b}}([0,\infty), qD_2)/C_0([0,\infty), qD_2) \to C_{\mathrm{b}}([0,\infty), qD_2)$$

(it exists by Bartle–Graves theorem, [1], [5]) and define an asymptotic morphism $q\psi$ by

$$(q\psi)_t(x) = (s(\tau \circ \Phi(x)))(t).$$

Thus we get an asymptotic morphism $q\psi : qD_1 \to qD_2$ out of an asymptotic morphism $\psi : D_1 \to D_2$.

For any C^*-algebra D let

$$\rho^D = q(i_1^D \otimes \mathrm{id}_{\mathcal{K}}, i_2^D \otimes \mathrm{id}_{\mathcal{K}}) : q(D \otimes \mathcal{K}) \to qD \otimes \mathcal{K}$$

and let $\theta_D : qD \otimes \mathcal{K} \to q^2 D \otimes \mathcal{K}$ denote the isomorphism constructed in [3].

LEMMA 3.3. *The diagram*

$$\begin{tikzcd}
q(A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "\gamma^{A\otimes\mathcal{K}}"] \arrow[d, "\rho^A \otimes \mathrm{id}_K"] & A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[r, "\mathrm{id}_A \otimes i"] & A \otimes \mathcal{K} \\
qA \otimes \mathcal{K} \otimes \mathcal{K} \arrow[d, "\mathrm{id}_{qA} \otimes i"] & & \\
qA \otimes \mathcal{K} \arrow[uurr, "\gamma^A"'] & &
\end{tikzcd}$$

is commutative, namely $\gamma^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) = (\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}}$.

Proof. Since elements of the form

$$(i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T)) \otimes S$$

and

$$(i_1^{A\otimes\mathcal{K}}(a_0 \otimes T_0)(i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T))) \otimes S,$$

where $T, S, T_0 \in \mathcal{K}$, $a, a_0 \in A$, span a dense subspace of $q(A \otimes \mathcal{K}) \otimes \mathcal{K}$ (see [7], for example) it is enough to check that $\gamma^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}})$ and $(\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}}$ coincide on elements of such form. For any $T, S \in \mathcal{K}$, $a \in A$ we have

$$\begin{aligned}
\gamma^A \circ &(\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}})((i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T)) \otimes S) \\
&= a \otimes i(T \otimes S) = (\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}}((i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T)) \otimes S),
\end{aligned}$$

for another pair $T_0 \in \mathcal{K}$, $a_0 \in A$ we have

$$\begin{aligned}
&\gamma^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}})((i_1^{A\otimes\mathcal{K}}(a_0 \otimes T))(i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T))) \otimes S) \\
&\qquad = a_0 a \otimes i(T_0 T \otimes S) \\
&\qquad = (\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}}((i_1^{A\otimes\mathcal{K}}(a_0 \otimes T_0)(i_1^{A\otimes\mathcal{K}}(a \otimes T) - i_2^{A\otimes\mathcal{K}}(a \otimes T))) \otimes S)
\end{aligned}$$

and we are done. ∎

LEMMA 3.4. *Let $\phi \in [[qB, A \otimes \mathcal{K}]]$. Then the diagram*

$$\begin{tikzcd}
q^2B \otimes \mathcal{K} \arrow[r, "q\phi\otimes\mathrm{id}_{\mathcal{K}}"] \arrow[d, "\gamma^{qB}"] & q(A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "\gamma^{A\otimes\mathcal{K}}"] & A \otimes \mathcal{K} \otimes \mathcal{K} \\
qB \otimes \mathcal{K} \arrow[urr, bend right=10, "\phi\otimes\mathrm{id}_K"'] & &
\end{tikzcd}$$

commutes, that is $\gamma^{A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) = (\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB}$.

Proof. Let $x \in qB$, $T \in \mathcal{K}$. By the definition of $q\phi$ we have

$$(q\phi)_t(i_1^{qB}(x) - i_2^{qB}(x)) - (i_1^{A\otimes\mathcal{K}}(\phi_t(x)) - i_2^{A\otimes\mathcal{K}}(\phi_t(x))) \to 0$$

when $t \to \infty$. Hence

$$\begin{aligned}
&\lim_{t\to\infty}[\gamma^{A\otimes\mathcal{K}} \circ ((q\phi)_t \otimes \mathrm{id}_{\mathcal{K}})((i_1^{qB}(x) - i_2^{qB}(x)) \otimes T) - (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB}((i_1^{qB}(x) - i_2^{qB}(x)) \otimes T)] \\
&\qquad = \lim_{t\to\infty}[\gamma^{A\otimes\mathcal{K}}((i_1^{A\otimes\mathcal{K}}(\phi_t(x)) - i_2^{A\otimes\mathcal{K}}(\phi_t(x))) \otimes T) - \phi_t(x) \otimes T] = 0.
\end{aligned}$$

In a similar way we find that $\gamma^{A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}})$ and $(\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB}$ asymptotically agree on elements $(i_1^{qB}(x_0)(i_1^{qB}(x) - i_2^{qB}(x))) \otimes T$ when $x_0, x \in qB$, $T \in \mathcal{K}$. Since elements of the form

$$(i_1^{qB}(x) - i_2^{qB}(x)) \otimes T \quad \text{and} \quad (i_1^{qB}(x_0)(i_1^{qB}(x) - i_2^{qB}(x))) \otimes T$$

span a dense subspace of $qB \otimes \mathcal{K}$ we see that $\gamma^{A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) = (\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB}$. ∎

LEMMA 3.5. *Let $\phi \in [[qB, qA \otimes \mathcal{K}]]$. Then the diagram*

$$\begin{tikzcd}
q^2B \arrow[r, "q\phi"] \arrow[d, "q(\mathrm{id}_{qB}, 0)"] & q(qA \otimes \mathcal{K}) \arrow[r, "\rho^{qA}"] & q^2A \otimes \mathcal{K} \arrow[r, "\gamma^{qA}"] & qA \otimes \mathcal{K} \\
qB \arrow[urrr, bend right=10, "\phi"'] & & &
\end{tikzcd}$$

is commutative, that is $\gamma^{qA} \circ \rho^{qA} \circ q\phi = \phi \circ q(\mathrm{id}_{qB}, 0)$.

Proof. Let $x \in qB$, $t \in [0, \infty)$. Writing $\phi_t(x)$ in the form

$$\phi_t(x) = \lim_{k\to\infty} \sum_{i=1}^{N_k} z_i^{(k)}(t) \otimes T_i^{(k)}(t),$$

where $z_i^{(k)}(t) \in qA$, $T_i^{(k)}(t) \in \mathcal{K}$, we get

$$\begin{aligned}
&\lim_{t\to\infty}[\gamma^{qA}\circ\rho^{qA}\circ(q\phi)_t(i_1^{qB}(x)-i_2^{qB}(x))-\phi_t\circ q(\mathrm{id}_{qB},0)(i_1^{qB}(x)-i_2^{qB}(x))]\\
&\quad=\lim_{t\to\infty}[\gamma^{qA}\circ\rho^{qA}(i_1^{qA\otimes\mathcal{K}}(\phi_t(x))-i_2^{qA\otimes\mathcal{K}}(\phi_t(x)))-\phi_t(x)]\\
&\quad=\lim_{t\to\infty}\Big[\lim_{k\to\infty}\sum_{i=1}^{N_k}\gamma^{qA}(i_1^{qA}(z_i^{(k)})\otimes T_i^{(k)}-i_2^{qA}(z_i^{(k)})\otimes T_i^{(k)})-\lim_{k\to\infty}\sum_{i=1}^{N_k}z_i^{(k)}\otimes T_i^{(k)}\Big]=0.
\end{aligned}$$

In a similar way we find that $\gamma^{qA}\circ\rho^{qA}\circ q\phi$ and $\phi\circ q(\mathrm{id}_{qB},0)$ asymptotically agree on elements $i_1^{qB}(x_0)(i_1^{qB}(x)-i_2^{qB}(x))$, where $x_0, x\in qB$. Since elements of the form $i_1^{qB}(x)-i_2^{qB}(x)$ and $i_1^{qB}(x_0)(i_1^{qB}(x)-i_2^{qB}(x))$ span a dense subspace of qB we conclude that the asymptotic morphisms $\gamma^{qA}\circ\rho^{qA}\circ q\phi$ and $\phi\circ q(\mathrm{id}_{qB},0)$ coincide. ∎

Let $\psi\in G(B,A)$. There is an asymptotic morphism $\phi: qB\to A\otimes\mathcal{K}$ such that $(\mathrm{id}_A\otimes i)\circ(\phi\otimes\mathrm{id}_{\mathcal{K}})\sim\psi$ [2]. Define an asymptotic morphism $\Gamma(\psi)\in G(B,qA)$ by the following composition

$$qB\otimes\mathcal{K}\xrightarrow{\theta_B}q^2B\otimes\mathcal{K}\xrightarrow{q\phi\otimes\mathrm{id}_{\mathcal{K}}}q(A\otimes\mathcal{K})\otimes\mathcal{K}\xrightarrow{\rho^A\otimes\mathrm{id}_{\mathcal{K}}}qA\otimes\mathcal{K}\otimes\mathcal{K}\xrightarrow{\mathrm{id}_{qA}\otimes i}qA\otimes\mathcal{K}.$$

Thus a map $\Gamma: G(B,A)\to G(B,qA)$ is defined by formula

$$\Gamma(\psi)=(\mathrm{id}_{qA}\otimes i)\circ(\rho^A\otimes\mathrm{id}_{\mathcal{K}})\circ(q\phi\otimes\mathrm{id}_{\mathcal{K}})\circ\theta_B,$$

for any $\psi\in G(B,A)$. Let $\gamma_G^A: G(B,qA)\to G(B,A)$ be the map induced by γ^A.

PROPOSITION 3.6. *$\Gamma: G(B,A)\to G(B,qA)$ is a semigroup isomorphism with inverse γ_G^A.*

Proof. Obviously Γ and γ_G^A are semigroup homomorphisms so we have to check only the following:
(i) $\Gamma(\gamma_G^A(\psi))\sim\psi$ for any $\psi\in G(B,qA)$;
(ii) $\gamma_G^A(\Gamma(\psi))\sim\psi$ for any $\psi\in G(B,A)$.

(i) Let $\psi\in G(B,qA)$ and $\phi: qB\to qA\otimes\mathcal{K}$ be such an asymptotic morphism that $(\mathrm{id}_{qA}\otimes i)\circ(\phi\otimes\mathrm{id}_{\mathcal{K}})\sim\psi$. Then

$$\begin{aligned}
\Gamma(\gamma_G^A(\psi))&=(\mathrm{id}_{qA}\otimes i)\circ(\rho^A\otimes\mathrm{id}_{\mathcal{K}})\circ(q(\gamma^A\circ\phi)\otimes\mathrm{id}_{\mathcal{K}})\circ\theta_B\\
&=(\mathrm{id}_{qA}\otimes i)\circ(\rho^A\otimes\mathrm{id}_{\mathcal{K}})\circ(q\gamma^A\otimes\mathrm{id}_{\mathcal{K}})\circ(q\phi\otimes\mathrm{id}_{\mathcal{K}})\circ\theta_B,
\end{aligned}$$

because clearly $q(\gamma^A\circ\phi)\otimes\mathrm{id}_K=(q\gamma^A\otimes\mathrm{id}_K)\circ(q\phi\otimes\mathrm{id}_K)$.

By Lemma 5.1.11 of [7] $\rho^A\circ q\gamma^A\sim\gamma^{qA}\circ\rho^{qA}$ and we have

$$\begin{aligned}
\Gamma(\gamma_G^A(\psi))&=(\mathrm{id}_{qA}\otimes i)\circ(\gamma^{qA}\otimes\mathrm{id}_{\mathcal{K}})\circ(\rho^{qA}\otimes\mathrm{id}_{\mathcal{K}})\circ(q\phi\otimes\mathrm{id}_{\mathcal{K}})\circ\theta_B\\
&\overset{\text{Lemma 3.5}}{=}(\mathrm{id}_{qA}\otimes i)\circ(\phi\otimes\mathrm{id}_{\mathcal{K}})\circ\gamma^{qB}\circ\theta_B\sim\psi\circ\gamma^{qB}\circ\theta_B\overset{[1]}{\sim}\psi.
\end{aligned}$$

(ii) Now let $\psi \in G(B, A)$ and $\phi : qB \to A \otimes \mathcal{K}$ be such an asymptotic morphism that $(\mathrm{id}_A \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}}) \sim \psi$. Then

$$\begin{aligned}\gamma_G^A(\Gamma(\psi)) &= \gamma^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\ &\overset{\text{Lemma 3.3}}{=} (\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\ &\overset{\text{Lemma 3.4}}{=} (\mathrm{id}_A \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB} \circ \theta_B \sim \psi \circ \gamma^{qB} \circ \theta_B \overset{[1]}{\sim} \psi. \quad \blacksquare\end{aligned}$$

LEMMA 3.7. *The following diagram commutes:*

$$\begin{tikzcd} q(S^2A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "q\beta^A \otimes \mathrm{id}_{\mathcal{K}}"] \arrow[dd, "\gamma^{S^2A\otimes\mathcal{K}}"'] & q(A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "\rho^A \otimes \mathrm{id}_{\mathcal{K}}"] & qA \otimes \mathcal{K} \otimes \mathcal{K} \arrow[d, "\mathrm{id}_{qA\otimes i}"] \\ & & qA \otimes \mathcal{K} \arrow[d, "S^2A\otimes\mathcal{K}"] \\ S^2A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[rr, "\mathrm{id}_{S^2A}\otimes i"] & & S^2A \otimes \mathcal{K}. \end{tikzcd}$$

Namely $g^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\beta^A \otimes \mathrm{id}_{\mathcal{K}}) \sim (\mathrm{id}_{S^2A} \otimes i) \circ \gamma^{S^2A\otimes\mathcal{K}}$.

Proof. We will prove the assertion by establishing the commutativity of the left and right triangles of the diagram

$$\begin{tikzcd} q(S^2A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "q\beta^A \otimes \mathrm{id}_{\mathcal{K}}"] \arrow[dd, "\gamma^{S^2A\otimes\mathcal{K}}"'] & q(A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[r, "\rho^A \otimes \mathrm{id}_{\mathcal{K}}"] & qA \otimes \mathcal{K} \otimes \mathcal{K} \arrow[d, "\mathrm{id}_{qA\otimes i}"] \arrow[ddll, dashed, "g^A\otimes\mathrm{id}_{\mathcal{K}}"'] \\ & & qA \otimes \mathcal{K} \arrow[d, "S^2A\otimes\mathcal{K}"] \\ S^2A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[rr, "\mathrm{id}_{S^2A}\otimes i"] & & S^2A \otimes \mathcal{K}. \end{tikzcd}$$

To prove the commutativity of the right triangle we have to prove

$$g^A \circ (\mathrm{id}_{qA} \otimes i) \sim (\mathrm{id}_{S^2A} \otimes i) \circ (g^A \otimes \mathrm{id}_{\mathcal{K}}). \tag{3.3}$$

Let $h_1, h_2 : \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K} \to \mathcal{K}$ be the isomorphisms which send $T_1 \otimes T_2 \otimes T_3$ to $i(T_1 \otimes i(T_2 \otimes T_3))$ and $i(i(A \otimes B) \otimes C)$ respectively for any operators $T_1, T_2, T_3 \in \mathcal{K}$. Then for any $T, S \in \mathcal{K}$, $a \in A$ we have

$$\begin{aligned}&(\mathrm{id}_{S^2A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (\mathrm{id}_{qA} \otimes i)((i_1^A(a) - i_2^A(a)) \otimes T \otimes S) \\ &= (\mathrm{id}_{S^2A} \otimes (h_2 \circ h_1^{-1}))(a \otimes i(j\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \otimes i(T \otimes S))) \\ &= a \otimes i(i(j\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right) \otimes T) \otimes S) = (\mathrm{id}_{S^2A} \otimes i) \circ (g^A \otimes \mathrm{id}_{\mathcal{K}})((i_1^A(a) - i_2^A(a)) \otimes T \otimes S)\end{aligned}$$

and for another $a_0 \in A$

$$\begin{aligned}
&(\mathrm{id}_{S^2A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (\mathrm{id}_{qA} \otimes i)(i_1^A(a_0)(i_1^A(a) - i_2^A(a)) \otimes T \otimes S) \\
&= (\mathrm{id}_{S^2A} \otimes (h_2 \circ h_1^{-1}))(a_0 a \otimes i(j(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)((\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) - p_{\mathrm{Bott}})) \otimes i(T \otimes S))) \\
&= a_0 a \otimes i(i(j(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)((\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) - p_{\mathrm{Bott}})) \otimes T) \otimes S) \\
&= (\mathrm{id}_{S^2A} \otimes i) \circ (g^A \otimes \mathrm{id}_{\mathcal{K}})(i_1^A(a_0)(i_1^A(a) - i_2^A(a)) \otimes T \otimes S).
\end{aligned}$$

Since elements of the form

$$(i_1^A(a) - i_2^A(a)) \otimes T \otimes S \quad \text{and} \quad i_1^A(a_0)(i_1^A(a) - i_2^A(a)) \otimes T \otimes S$$

span a dense subspace of $qA \otimes \mathcal{K} \otimes \mathcal{K}$ we get

$$(\mathrm{id}_{S^2A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (\mathrm{id}_{qA} \otimes i) = (\mathrm{id}_{S^2A} \otimes i) \circ (g^A \otimes \mathrm{id}_{\mathcal{K}}).$$

As well known any two isomorphisms from $\mathcal{K}$ to itself are homotopic, hence $h_2 \circ h_1^{-1} \sim \mathrm{id}_{\mathcal{K}}$ and we obtain (3.3). Now to prove the commutativity of the left triangle of the diagram we have to prove that

$$(g^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\beta^A \otimes \mathrm{id}_{\mathcal{K}}) \sim \gamma^{S^2A \otimes \mathcal{K}}. \tag{3.4}$$

Like in Lemma 3.1 we will reduce the general case to the case $A = \mathbb{C}$ using the map $s^A : qA \to q\mathbb{C} \otimes A$ that was introduced in the proof of Lemma 3.1. The right-hand side of (3.4) is

$$\gamma^{S^2A \otimes \mathcal{K}} = (\gamma^{\mathbb{C}} \otimes \mathrm{id}_{S^2A \otimes \mathcal{K}}) \circ (s^{S^2A \otimes \mathcal{K}} \otimes \mathrm{id}_{\mathcal{K}}), \tag{3.5}$$

that is the diagram

$$\begin{tikzcd}
q(S^2A \otimes \mathcal{K}) \otimes \mathcal{K} \arrow[rr, "\gamma^{S^2A \otimes \mathcal{K}}"] \arrow[dr, "s^{S^2A \otimes \mathcal{K}} \otimes \mathrm{id}_{\mathcal{K}}"'] & & S^2A \otimes \mathcal{K} \otimes \mathcal{K} \\
& q\mathbb{C} \otimes S^2A \otimes \mathcal{K} \otimes \mathcal{K} \arrow[ur, "\gamma^{\mathbb{C}} \otimes \mathrm{id}_{S^2A \otimes \mathcal{K}}"'] &
\end{tikzcd}$$

commutes. It can be easily checked by comparing of the left-hand side and the right-hand side of (3.5) on elements of $q(S^2A \otimes \mathcal{K}) \otimes \mathcal{K}$ of the form

$$(i_1^{S^2A \otimes \mathcal{K}}(\phi \otimes a \otimes S) - i_2^{S^2A \otimes \mathcal{K}}(\phi \otimes a \otimes S)) \otimes T$$

and

$$i_1^{S^2A \otimes \mathcal{K}}(\phi_0 \otimes a_0 \otimes S_0)(i_1^{S^2A \otimes \mathcal{K}}(\phi \otimes a \otimes S) - i_2^{S^2A \otimes \mathcal{K}}(\phi \otimes a \otimes S)) \otimes T,$$

$\phi, \phi_0 \in S^2\mathbb{C}, a, a_0 \in A, T, S, S_0 \in \mathcal{K}$, that span a dense subspace in $q(S^2A \otimes \mathcal{K}) \otimes \mathcal{K}$. Clearly the left-hand side of (3.4) is equal to $(g^A \circ \rho^A \circ q\beta^A) \otimes \mathrm{id}_{\mathcal{K}}$. We assert that

$$g^A \circ \rho^A \circ q\beta^A = (g^{\mathbb{C}} \otimes \mathrm{id}_A) \circ (\mathrm{id}_{q\mathbb{C}} \otimes \beta^{\mathbb{C}} \otimes \mathrm{id}_A) \circ s^{S^2A \otimes \mathcal{K}}, \tag{3.6}$$

that is that the diagram

$$\begin{tikzcd}
q(S^2A \otimes \mathcal{K}) \arrow[r, "q\beta^A"] \arrow[d, "s^{S^2A\otimes\mathcal{K}}"] & q(A \otimes \mathcal{K}) \arrow[r, "\rho^A"] & qA \otimes \mathcal{K} \arrow[d, "g^A"] \\
q\mathbb{C} \otimes S^2A \otimes \mathcal{K} \arrow[dr, "\mathrm{id}_{q\mathbb{C}}\otimes\beta^{\mathbb{C}}\otimes\mathrm{id}_A"'] & & S^2A \otimes \mathcal{K} \\
& q\mathbb{C} \otimes A \otimes \mathcal{K} \arrow[ur, "g^{\mathbb{C}}\otimes\mathrm{id}_A"'] &
\end{tikzcd}$$

commutes. Indeed it is straightforward to show that the left-hand side and the right-hand side of (3.6) asymptotically agree on elements of $q(S^2A \otimes \mathcal{K})$ of the form

$$i_1^{S^2A\otimes\mathcal{K}}(\phi \otimes a \otimes T) - i_2^{S^2A\otimes\mathcal{K}}(\phi \otimes a \otimes T)$$

and

$$i_1^{S^2A\otimes\mathcal{K}}(\phi_0 \otimes a_0 \otimes T_0)(i_1^{S^2A\otimes\mathcal{K}}(\phi \otimes a \otimes T) - i_2^{S^2A\otimes\mathcal{K}}(\phi \otimes a \otimes T)),$$

$\phi, \phi_0 \in S^2\mathbb{C}$, $a, a_0 \in A$, $T, T_0 \in \mathcal{K}$, that span a dense subspace in $q(S^2A \otimes \mathcal{K})$. Now, by (3.5), (3.6), to get (3.4) it remains to prove that

$$(\gamma^{\mathbb{C}}\otimes\mathrm{id}_{S^2A\otimes\mathcal{K}})\circ(s^{S^2A\otimes\mathcal{K}}\otimes\mathrm{id}_{\mathcal{K}})\sim(g^{\mathbb{C}}\otimes\mathrm{id}_{A\otimes\mathcal{K}})\circ(\mathrm{id}_{q\mathbb{C}}\otimes\beta^{\mathbb{C}}\otimes\mathrm{id}_{A\otimes\mathcal{K}})\circ(s^{S^2A\otimes\mathcal{K}}\otimes\mathrm{id}_{K})$$

or, equivalently,

$$\gamma^{\mathbb{C}} \otimes \mathrm{id}_{S^2\mathbb{C}} \sim g^{\mathbb{C}} \otimes (\mathrm{id}_{q\mathbb{C}} \otimes \beta^{\mathbb{C}}).$$

For that note that $\gamma^{\mathbb{C}} \otimes \mathrm{id}_{S^2\mathbb{C}}$ and $g^{\mathbb{C}} \otimes (\mathrm{id}_{q\mathbb{C}} \otimes \beta^{\mathbb{C}})$ induce the same homomorphisms in K-theory. Indeed they both send the generator

$$([\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)] - [p_{\mathrm{Bott}}]) \otimes ([i_1^{\mathbb{C}}(1)] - [i_2^{\mathbb{C}}(1)])$$

of $K_0(S^2\mathbb{C} \otimes q\mathbb{C})$ to the generator

$$[\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)] - [p_{\mathrm{Bott}}]$$

of $K_0(S^2\mathbb{C})$. This implies that $\gamma^{\mathbb{C}} \otimes \mathrm{id}_{S^2\mathbb{C}}$ and $g^{\mathbb{C}} \otimes (\mathrm{id}_{q\mathbb{C}} \otimes \beta^{\mathbb{C}})$ are homotopic because, as is well known,

$$[[S^2\mathbb{C} \otimes q\mathbb{C} \otimes \mathcal{K}, S^2\mathbb{C} \otimes \mathcal{K}]] \cong Z \cong \mathrm{Hom}(K_0(S^2\mathbb{C} \otimes q\mathbb{C}), K_0(S^2\mathbb{C})). \quad \blacksquare$$

Let $\psi \in G(B, A)$ and as before $\phi : qB \to A \otimes \mathcal{K}$ be an asymptotic morphism such that $(\mathrm{id}_A \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}}) \sim \psi$. Define an asymptotic morphism $b(\psi) \in G(B, S^2A)$ by the composition

$$\begin{aligned}
qB \otimes \mathcal{K} \xrightarrow{\theta_B} q^2B \otimes \mathcal{K} &\xrightarrow{q\phi\otimes\mathrm{id}_{\mathcal{K}}} q(A \otimes \mathcal{K}) \otimes \mathcal{K} \\
&\xrightarrow{\rho^A\otimes\mathrm{id}_{\mathcal{K}}} qA \otimes \mathcal{K} \otimes \mathcal{K} \xrightarrow{\mathrm{id}_{qA}\otimes i} qA \otimes \mathcal{K} \xrightarrow{g^A} S^2A \otimes \mathcal{K}.
\end{aligned}$$

Thus the map $b : G(B, A) \to G(B, S^2A)$ is defined by formula

$$b(\psi) = g^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B,$$

for any $\psi \in G(B, A)$. Let $\beta^A_G : G(B, S^2A) \to G(B, A)$ be the map induced by β^A.

PROPOSITION 3.8. *$b : G(B, A) \to G(B, S^2A)$ is a semigroup isomorphism with inverse β^A_G.*

Proof. Obviously b and β^A_G are semigroup homomorphisms so we have to check only the following:

(i) $(\beta^A_G \circ b)(\psi) \sim \psi$ for any $\psi \in G(B, A)$;

(ii) $(b \circ \beta^A_G)(\psi) \sim \psi$ for any $\psi \in G(B, S^2A)$.

(i) Let $\psi \in G(B, A)$ and $\phi : qB \to A \otimes \mathcal{K}$ be such an asymptotic morphism that $\psi \sim (\mathrm{id}_A \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}})$. Then

$$\begin{aligned}
(\beta^A_G \circ b)(\psi) &= \beta^A \circ g^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&\overset{\text{Lemma 3.1}}{\sim} \gamma^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&\overset{\text{Lemma 3.3}}{=} (\mathrm{id}_A \otimes i) \circ \gamma^{A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&\overset{\text{Lemma 3.4}}{=} (\mathrm{id}_A \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB} \circ \theta_B \sim \psi \circ \gamma^{qB} \circ \theta_B \overset{[1]}{\sim} \psi.
\end{aligned}$$

(ii) Let $\psi \in G(B, S^2A)$ and $\phi : qB \to S^2A \otimes \mathcal{K}$ be such an asymptotic morphism that $\psi \sim (\mathrm{id}_{S^2A} \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}})$. Then

$$\begin{aligned}
(b \circ \beta^A_G)(\psi) &= g^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q(\beta^A \circ \phi) \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&= g^A \circ (\mathrm{id}_{qA} \otimes i) \circ (\rho^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\beta^A \otimes \mathrm{id}_{\mathcal{K}}) \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&\overset{\text{Lemma 3.7}}{\sim} (\mathrm{id}_{S^2A} \otimes i) \circ \gamma^{S^2A\otimes\mathcal{K}} \circ (q\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \theta_B \\
&\overset{\text{Lemma 3.4}}{=} (\mathrm{id}_{S^2A} \otimes i) \circ (\phi \otimes \mathrm{id}_{\mathcal{K}}) \circ \gamma^{qB} \circ \theta_B \sim \psi \circ \gamma^{qB} \circ \theta_B \overset{[1]}{\sim} \psi. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 3.9. $f^A_G \circ g^A_G = \mathrm{id}, g^A_G \circ f^A_G = \mathrm{id}$.

Here *id* means both the identity map from $G(B, S^2A)$ into itself and the identity map from $G(B, qA)$ into itself.

Proof. Consider the following diagram

$$\begin{tikzcd}
& G(B,A) & \\
G(B,S^2A) \arrow[ur, "\beta^A_G"] \arrow[rr, shift right, "f^A_E"'] & & G(B,qA) \arrow[ul, "\gamma^A_G"'] \arrow[ll, shift right, "g^A_G"']
\end{tikzcd}.$$

We shall prove that it commutes and this will imply the statement of the proposition. By Propositions 3.8 and 3.6 β^A_G and γ^A_G are isomorphisms. By Lemma 3.1 $\beta^A_G \circ g^A_G = \gamma^A_G$ whence

$$g^A_G = (\beta^A_G)^{-1} \circ \gamma^A_G. \tag{3.7}$$

By Corollary 2.2 $\gamma_G^A \circ f_G^A = \beta_G^A$ whence

$$f_G^A = (\gamma_G^A)^{-1} \circ \beta_G^A. \tag{3.8}$$

From (3.7) and (3.8) we obtain the assertions of the proposition. ∎

3.3. MAIN RESULT.

THEOREM 3.10. (i) $g^A \circ f^A \sim \mathrm{id}_{S^2A\otimes\mathcal{K}}$;
(ii) $f^A \circ g^A \sim \mathrm{id}_{qA\otimes\mathcal{K}}$.

Proof. (i) By Proposition 3.2 $g_E^A \circ f_E^A = \mathrm{id}$ whence $g^A \circ f^A \circ \phi \sim \phi$ for any $\phi \in E(B, S^2A \otimes \mathcal{K})$. Set $B = A \otimes \mathcal{K}$, $\phi = \mathrm{id}_{S^2A\otimes\mathcal{K}}$. Then

$$\mathrm{id}_{S^2A\otimes\mathcal{K}} \sim g^A \circ f^A \circ \mathrm{id}_{S^2A\otimes\mathcal{K}} = g^A \circ f^A.$$

(ii) By Proposition 3.9 $f_G^A \circ g_G^A = \mathrm{id}$ whence $f^A \circ g^A \circ \phi \sim \phi$ for any $\phi \in [[qB \otimes \mathcal{K}, qA \otimes \mathcal{K}]]$. Setting $B = A$, $\phi = \mathrm{id}_{qA\otimes\mathcal{K}}$ we get

$$\mathrm{id}_{qA\otimes\mathcal{K}} \sim f^A \circ g^A. \quad ∎$$

So C^*-algebras $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$ are asymptotically equivalent and we obtain immediately

COROLLARY 3.11. *$E(A, B) = [[qA, B \otimes \mathcal{K}]]$ for every C^*-algebras A and B.*

COROLLARY 3.12. *Let A be a nuclear C^*-algebra and B be any C^*-algebra. Then every asymptotic morphism from qA to $B \otimes \mathcal{K}$ is homotopic to a $*$-homomorphism from qA to $B \otimes \mathcal{K}$.*

Proof. Let $\phi_t \in [[qA, B \otimes \mathcal{K}]]$. Since A is nuclear $I_{A,B}$ is an isomorphism [2]. Define a $*$-homomorphism $\psi_0 : qA \otimes \mathcal{K} \to B \otimes \mathcal{K}$ by

$$\psi_0 = I_{A,B}^{-1}((\mathrm{id}_B \otimes i) \circ (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A). \tag{3.9}$$

By [7] there exists a $*$-homomorphism $\psi : qA \to B \otimes \mathcal{K}$ such that

$$(\mathrm{id}_B \otimes i) \circ (\psi \otimes \mathrm{id}_{\mathcal{K}}) \sim \psi_0. \tag{3.10}$$

We will prove that $\phi_t \sim \psi$. By Theorem 2.1

$$I_{A,B}(\psi_0) = \psi_0 \circ f^A. \tag{3.11}$$

By (3.9) the left-hand side of (3.11) is $I_{A,B}(\psi_0) = (\mathrm{id}_B \otimes i) \circ (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A$. By (3.10) the right-hand side of (3.11) is $\psi_0 \circ f^A \sim (\mathrm{id}_B \otimes i) \circ (\psi \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A$. So

$$(\mathrm{id}_B \otimes i) \circ (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A \sim (\mathrm{id}_B \otimes i) \circ (\psi \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A,$$

$$(\mathrm{id}_B \otimes i) \circ (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A \circ g^A \sim (\mathrm{id}_B \otimes i) \circ (\psi \otimes \mathrm{id}_{\mathcal{K}}) \circ f^A \circ g^A,$$

and by Theorem 3.10 we obtain

$$(\mathrm{id}_B \otimes i) \circ (\phi_t \otimes \mathrm{id}_{\mathcal{K}}) \sim (\mathrm{id}_B \otimes i) \circ (\psi \otimes \mathrm{id}_{\mathcal{K}})$$

whence $\phi_t \sim \psi$. ∎

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