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# GEOMETRIC PRE-ORDERING ON $C^*$-ALGEBRAS

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ABSTRACT. It has been a successful practice to define a canonical pre-ordering on a normed space using the inclusion of faces of its closed dual unit ball. This pre-ordering reflects some geometric property in a natural way. In this article, we will give an algebraic description of this pre-ordering in the case of complex $C^*$-algebras as well as that of their self-adjoint parts. In developing our theory we introduce the *essential support* of an element, which is closely related to the notion of peak projections studied recently by Blecher and Hay. As applications, we give some interesting facts about weak*-closed faces, and will identify the quasi-maximal elements and the quasi-minimal elements with respects to this pre-ordering. They are closely related to the extreme points and the smooth points of the unit sphere of the $C^*$-algebra.

KEYWORDS: *Geometric pre-ordering, $C^*$-algebra, essential support, weak*-closed face, partial isometry, exposure, hermitian exposure.*

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## 1. INTRODUCTION

Suppose that $X$ is a (real or complex) normed space with Banach dual space $X^*$ and double dual space $X^{**}$. We set

$$Q_X(x) := \{f \in X^* : \|f\| \leqslant 1 \text{ and } f(x) = \|f\|\|x\|\}, \quad (x \in X^{**} \setminus \{0\}), \tag{1.1}$$

and $Q_X(0) := \{0\}$ by convention. One can understand the local geometric property of $x$ by looking at this set. In particular, a norm one element $x$ in $X$ is smooth if the weak* compact convex set $Q_X(x)$ is one-dimensional. On the other hand, when $X$ is a complex $C^*$-algebra, the maximality of $Q_X(x)$ is related to the notion of extreme points (see Section 4). In general, one can define a canonical pre-ordering on the set of non-zero elements of $X^{**}$ by declaring that

$$x \lesssim_X y \quad \text{if} \quad Q_X(x) \subseteq Q_X(y). \tag{1.2}$$

The main objective of this paper is to give an algebraic description for the above pre-ordering in the cases of a complex $C^*$-algebra $A$ as well as that of the self-adjoint part $A_h$ of $A$. There are already many papers devoted to the study of the facial structure of the closed dual unit ball $\mathfrak{B}_1(A^*)$ (respectively $\mathfrak{B}_1(A_h^*)$) of $A$ (respectively $A_h$) (see, e.g., [1], [4] and [5], as well as the references therein). In particular, the ordering on the set of partial isometries studied in [1] is the restriction of the pre-ordering in (1.2). In order to study this pre-ordering on general arbitrary elements, however, we need to put further attention on the weak* closed faces of $\mathfrak{B}_1(A^*)$ and $\mathfrak{B}_1(A_h^*)$. In particular, using the results in [1], one needs to find a way to obtain the canonical partial isometry (or self-adjoint partial isometry) $u$ that defines the same weak* closed face in (1.1); that is,

$$Q_A(x) = Q_A(u) \quad \text{or} \quad Q_{A_h}(x) = Q_{A_h}(u)$$

This gives rise to the interesting concept of the *essential support* $\mathrm{es}(x)$ of a given element $x$ in $A$, which is related to the so-called *peak projections* (see e.g. [6]) defined by Blecher and Hay, and will be studied in Section 2.

More precisely, if $X = A_h$ and $x \in A_h \setminus \{0\}$, then $Q_{A_h}(x) = Q_A(x) \cap A_h^*$ is supported by a difference of two orthogonal projections (by Theorem 3.11 of [1]) and we will show that these two projections are closely related to the essential supports of the positive part and the negative part of $x$ (Proposition 3.7). On the other hand, if $X = A$ and $a \in A \setminus \{0\}$, then $Q_A(a)$ is supported by a unique partial isometry and it will be shown that this partial isometry is given by the polar decomposition and the essential support of $a$. Finally, we will study, in Section 4, the quasi-maximal and quasi-minimal elements with respect to the geometric pre-orderings.

Throughout this article, $X$ is a (real or complex) normed space, $A$ is a complex $C^*$-algebra, $M$ is a complex von Neumann algebra with predual $M_*$, $A_h$ is the set of self-adjoint elements of $A$, and $A_+$ is the positive cone of $A$.

NOTATION 1.1. We denote by $\mathfrak{S}_1(X)$ and $\mathfrak{B}_1(X)$ the unit sphere and the closed unit ball of $X$. For any $a \in M$ and $f \in M_*$, we denote by $|a|$ and $|f|$ the absolute value of $a$ and that of $f$ respectively. Moreover, $\mathrm{s_r}(a)$ and $\mathrm{s_r}(f)$ (respectively, $\mathrm{s}(|a|)$ and $\mathrm{s}(|f|)$) are the right support projections (respectively, the support projections) of $a$ and $f$ (respectively, $|a|$ and $|f|$). The normal functionals $a \cdot f$ and $f \cdot a$ in $M_*$ are defined by

$$(a \cdot f)(x) = f(xa) \quad \text{and} \quad (f \cdot a)(x) = f(ax) \quad (x \in M).$$

Furthermore, if $E$ is a convex subset of a normed space $X$, then we denote by $\mathrm{ext}(E)$ the set of all extreme points of $E$.

## 2. THE ESSENTIAL SUPPORTS

Inspired by the statement of the main theorem of [9], we define and study the notion of the *essential support* of an element in a $C^*$-algebra. It turns out that the essential support plays an important role in the algebraic description of the geometric pre-ordering on a $C^*$-algebra. The first question that one needs to ask is whether such projections exist. Our first theorem answers this question affirmatively.

Before we present this result, let us first recall the definition of closed projection from 3.11.10 of [8]. A projection $p$ in the bidual $A^{**}$ of $A$ is said to be *closed* if $\{\phi \in A^*_+ : \phi(p) = \|\phi\| \leqslant 1\}$ is $\sigma(A^*, A)$-closed.

Moreover, we recall the following well-known lemma (for part (i), see e.g., the argument in Theorem III.4.2(i) of [11] while part (ii) follows from Corollary 3.5 of [4] or Proposition 4.4 of [2], and part (iii) is a direct application of Lemma III.3.2 of [11]).

LEMMA 2.1. (i) *Let $a \in \mathfrak{S}_1(M)$ and $f \in M_*$ with $f(a) = \|f\|$. If $a^* = v|a^*|$ is the polar decomposition of $a^*$ and $p = \mathrm{s}(|f|)$, then $|f| = v^* \cdot f$, $p \leqslant v^*v$ and $f = vp \cdot |f|$ is the polar decomposition of $f$.*

(ii) *Let $q \in A^{**}$ be a closed projection and $u \in qA_+q$. There exists $\psi \in \mathfrak{S}_1(A^*) \cap A^*_+$ such that $\psi(u) = \|u\|$.*

(iii) *For any $u \in A^{**}_+ \setminus \{0\}$, we have $Q_A(u) = Q_{A_h}(u) = Q_A(u) \cap A^*_+$.*

For any $u \in A^{**}_+ \setminus \{0\}$, let $E : \sigma(u) \to A^{**}$ be the spectral measure induced by $u$. Denote $p_u := E(\{\|u\|\})$.

THEOREM 2.2. *Suppose that $A$ is a $C^*$-algebra and $x \in A \setminus \{0\}$. Let $\mathcal{E}(x)$ be the set of all closed projections $p$ in $A^{**}$ satisfying the following three conditions:*

(i) *$px^*x = x^*xp$;*

(ii) *$\|xp\| = \|x\|$;*

(iii) *$\|xq\| < \|x\|$ for any closed projection $q \leqslant 1 - p$.*

(a) *There exists a smallest element $\mathrm{es}(x)$ in $\mathcal{E}(x)$ which is the unique closed projection with*

$$Q_A(x^*x) = Q_A(\mathrm{es}(x)).$$

*Moreover, $\mathrm{es}(x) = p_{x^*x} \leqslant \mathrm{s_r}(x)$.*

(b) *If $x \in A_+ \setminus \{0\}$, then $\mathrm{es}(x) = \mathrm{es}(x^t)$ for any $t$ in $\mathbb{R}_+ \setminus \{0\}$, and $x\,\mathrm{es}(x) = \|x\|\mathrm{es}(x)$.*

(c) *$\mathrm{es}(x) = \mathrm{es}(x^*x) = \mathrm{es}(|x|)$.*

*Proof.* Without loss of generality, we can assume $\|x\| = 1$.

(a) We note first of all that $\mathrm{s_r}(x) \in \mathcal{E}(x)$ and $\mathrm{es}(x) \leqslant \mathrm{s_r}(x)$ if $\mathrm{es}(x)$ exists. As $Q_A(x^*x)$ is a weak* closed face of the quasi-state space $Q_A(1)$ of $A$, there exists, by Theorem 2.10 of [1], a non-zero closed projection $\mathrm{es}(x)$ such that $Q_A(x^*x) = Q_A(\mathrm{es}(x))$. Moreover, observe that

$$\|x\,\mathrm{es}(x)\|^2 = \|\mathrm{es}(x)x^*x\,\mathrm{es}(x)\| = \sup_{\phi \in Q_A(1)} \phi(\mathrm{es}(x)x^*x\,\mathrm{es}(x)) = \sup_{\varphi \in Q_A(x^*x)} \varphi(x^*x) = 1$$

(because of Lemma 2.1(ii)). On the other hand, suppose there is a closed projection $q \leqslant 1 - \mathrm{es}(x)$ with $\|xq\| = 1$. Then $\psi(qx^*xq) = 1$ for some state $\psi$ (by Lemma 2.1(ii)). This implies that $\varphi(\cdot) := \psi(q \cdot q) \in Q_A(\mathrm{es}(x)) \cap Q_A(q) =$

$\{0\}$ which contradicts $\varphi(x^*x) = 1$. Therefore, $\mathrm{es}(x)$ satisfies Conditions (ii) and (iii). In order to show that $\mathrm{es}(x)$ satisfies Condition (i), we first show that $\mathrm{es}(x) = p_{x^*x}$. Indeed, for any $\varphi$ in $A^*_+$, there is a Radon measure $\mu_\varphi$ on $\sigma(x^*x)$ such that $\varphi\Big(\int f \mathrm{d}E\Big) = \int f \mathrm{d}\mu_\varphi$ for any bounded Borel measurable function $f$. If $\varphi \in Q_A(x^*x)$, then

$$\int \lambda \mathrm{d}\mu_\varphi(\lambda) = \varphi(x^*x) = \|\varphi\| = \mu_\varphi(\sigma(x^*x)).$$

Thus, $\mu_\varphi$ is supported on $\{1\}$ and $\varphi(p_{x^*x}) = \mu_\varphi(\{1\}) = \|\varphi\|$. Conversely, if $\varphi \in Q_A(p_{x^*x})$, then $\mu_\varphi(\{1\}) = \|\varphi\| = \mu_\varphi(\sigma(x^*x))$ and we have $\varphi(x^*x) = \|\varphi\|$. Thus, $Q_A(x^*x) = Q_A(p_{x^*x})$ and $\mathrm{es}(x) = p_{x^*x}$ (by Theorem 2.5 of [1]). Finally, we show that $\mathrm{es}(x)$ is the smallest element in $\mathcal{E}(x)$. Assume on the contrary that there is an $r$ in $\mathcal{E}(x)$ such that $\mathrm{es}(x)$ is not a subprojection of $r$. Then $Q_A(r)$ does not contain all the extreme points of $Q_A(x^*x)$. As $Q_A(x^*x)$ is a weak* closed face of $Q_A(1)$, there exists a pure state $\varphi$ in $Q_A(x^*x)$ with $\varphi(r) < 1$. By Condition (i), we have

$$\begin{aligned} 1 = \varphi(x^*x) &= \varphi(rx^*xr) + \varphi((1-r)x^*x(1-r)) \\ &\leqslant \|x^*x\|\varphi(r) + \|x^*x\|\varphi(1-r) = \varphi(1) = 1. \end{aligned}$$

Consequently, $\varphi((1-r)x^*x(1-r)) = \varphi(1-r) > 0$. Define $\psi(a) := \frac{\varphi((1-r)a(1-r))}{\varphi(1-r)}$ for all $a$ in $A$. Let $(\pi_\varphi, H, \xi)$ be the GNS representation arising from the pure state $\varphi$. As

$$\|\pi_\varphi(1-r)\xi\|^2 = \varphi(1-r) > 0,$$

one can define $\xi' = \frac{\pi_\varphi(1-r)\xi}{\|\pi_\varphi(1-r)\xi\|}$. Since $\pi_\varphi$ is irreducible, $\psi(a) = \langle \pi_\varphi(a)\xi', \xi'\rangle$ is again a pure state of $A$. If $q$ is the closed projection such that $S_A(q) = \{\psi\} \subseteq S_A(1-r)$, then it follows from $q \leqslant 1-r$ and Condition (iii) that $\|xq\| < 1$. Since $q$ supports $\psi$, we have $\psi(x^*x) = \psi(qx^*xq) \leqslant \|qx^*xq\| = \|xq\|^2 < 1$, which is a contradiction.

(b) Let $v = x^2$. Note that $\Omega \mapsto \Omega^{1/t} := \{\omega^{1/t} : \omega \in \Omega\}$ gives a bijection between the Borel subsets of $\sigma(v^t)$ and those of $\sigma(v)$. Let $E$ be the spectral measure for $v$ as in the paragraph preceding this theorem. Then

$$E_t : \Omega \mapsto E(\Omega^{1/t})$$

is a spectral measure on $\sigma(v^t)$. For any $f$ in $C(\sigma(v^t))$, define $\Phi_t(f)$ in $C(\sigma(v))$ by $\Phi_t(f)(\lambda) = f(\lambda^t)$. If $\Psi : C(\sigma(v)) \to A^{**}$ is given by the functional calculus for $v$, then $\Psi \circ \Phi_t$ is given by the functional calculus for $v^t$. Consequently, $E_t$ is the spectral measure for $v^t$. By part (a), we have $\mathrm{es}(x) = E(\{1\}) = E_t(\{1\}) = \mathrm{es}(x^t)$, and $x\mathrm{es}(x) = \mathrm{es}(x)$ as elements in the image of $C_0(\sigma(x^2))^{**}$.

(c) For any projection $p$ in $A^{**}$, we have

$$\|xp\|^2 = \|px^*xp\| \leqslant \|x^*xp\| \leqslant \|xp\| \leqslant 1.$$

Therefore, $\|xp\| = 1$ if and only if $\|x^*xp\| = 1$. On the other hand, a projection commutes with $x^*x$ if and only if it commutes with $(x^*x)^2$. Therefore, $\mathrm{es}(x) = \mathrm{es}(x^*x)$. By (c), we also have $\mathrm{es}(x) = \mathrm{es}(|x|)$. ∎

DEFINITION 2.3. Let $x \in A \setminus \{0\}$. The projection $\mathrm{es}(x)$ in Theorem 2.2 is called the *essential (right) support* of $x$.

EXAMPLE 2.4. (i) Let $A = C[0,1]$ and $x$ in $A$ be defined by

$$x(t) = \begin{cases} 1 & t \in [0, \frac{1}{2}]; \\ 2(1-t) & t \in (\frac{1}{2}, 1]. \end{cases}$$

In this case, one should expect the essential support of $x$ to be the characteristic function of $[0, \frac{1}{2}]$. Note however that $\|x(1 - \mathrm{es}(x))\| = \|x\| = 1$. This explains why we cannot use $\|x(1-p)\| < \|x\|$ in Condition (iii).

(ii) Let $A = M_3(\mathbb{C})$ and $x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. In this case, one should expect the essential support of $x$ to be $x$ itself (as it is a projection). However, if $r$ is the projection $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then it is easy to see that $r$ satisfies Conditions (ii) and (iii) but $x \not\lesssim r$. This explains why we need Condition (i).

REMARK 2.5. One might think that there could be a von Neumann algebra version for Theorem 2.2(a). However, if $M$ is a von Neumann algebra, there may not be a projection *in* $M$ satisfying similar conditions as in Theorem 2.2. For example, suppose that $M = l^\infty$ and $x = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right) \in M$. Then $E_0(\{1\}) = 0$ (where $E_0 : \sigma(x) \to M$ is the normal spectral measure induced by $x$) and $\{f \in l^1 : f(x) = \|f\|\} = \{0\}$.

Note that the concept of essential supports is closely related to the notion of *peak projections* as defined in [6]. However, it does not seem to us that one can use the results in [3] nor those in [6] to obtain Theorem 2.2. As a matter of fact, Theorem 2.2 states that any contraction $a$ in $A$ is a peak associated with a peak projection $p$ in $A^{**}$ for the inclusion $A \subseteq A$. Moreover, the following result gives a relationship between these two concepts.

PROPOSITION 2.6. *Suppose that $A$ is a unital $C^*$-algebra and $q \in A^{**}$ is a non-zero closed projection. Then $q$ is a peak projection for $A$ (considered as a subalgebra of itself) if and only if there exists $a \in A$ such that $q = \mathrm{es}(a)$.*

*Proof.* Suppose that $q = \mathrm{es}(a)$ for some $a \in A \setminus \{0\}$. Set $u := |a|/\|a\| \in \mathfrak{S}_1(A) \cap A_+$. Then $q = \mathrm{es}(u)$ (by parts (b) and (c) of Theorem 2.2). Now using parts (a) and (b) of Theorem 2.2, we know that $q$ is a peak projection for $A$ (see Definition 5.2 of [6]). Conversely, suppose that $a \in A$ is a peak associated with a peak projection $q$ and $u = a^*a$. Then as noted in [6], $a$ commutes with $q$ and thus $uq = q = qu$. Moreover, we have

$$\|u\| \leqslant 1 = \|q\| = \|uq\| \leqslant \|u\|.$$

From Theorem 5.1(6) of [6], $\|up\| \leqslant \|ap\| < 1$ for any closed projection $p \leqslant 1 - q$. Consequently, $q \in \mathcal{E}(u)$ and so $q \geqslant \mathrm{es}(u)$. For any $f$ in $Q_A(q)$, we have $\mathrm{s}(f) \leqslant q$ (by Lemma 2.1(i)) and so $f(u) = f(uq) = f(q) = \|f\|$. This shows that $Q_A(q) \subseteq Q_A(u) = Q_A(\mathrm{es}(u))$ and $q \leqslant \mathrm{es}(u)$. ∎

## 3. GEOMETRIC PRE-ORDERING ON NORMED SPACES

DEFINITION 3.1. Suppose that $X$ is a (real or complex) normed space and $x, y \in X^{**} \setminus \{0\}$. We define

$$x \lesssim_X y \quad \text{if} \quad Q_X(x) \subseteq Q_X(y), \quad \text{and} \quad x \sim_X y \quad \text{if} \quad Q_X(x) = Q_X(y).$$

We call $\lesssim_X$ the *canonical geometric pre-ordering* on $X^{**} \setminus \{0\}$.

REMARK 3.2. (i) A complex normed space $X$ can be regarded as a real normed space, denoted by $X_{\mathbb{R}}$. In this case, $\Gamma : f \mapsto \mathrm{Re}(f)$ is an isometric isomorphism from $X^*$ onto $(X_{\mathbb{R}})^*$ such that $\Gamma(Q_X(x)) = Q_{X_{\mathbb{R}}}(x)$ for any $x$ in $X^{**}$. Therefore, $\lesssim_X$ and $\lesssim_{X_{\mathbb{R}}}$ are the same.

(ii) Note that $Q_X(x) \cap \mathfrak{S}_1(X^*)$ is either an empty set or a closed face of the dual closed unit ball $\mathfrak{B}_1(X^*)$, and it is a weak* closed face if $x \in X$.

(iii) Note that $x \lesssim_X y$ if and only if $\frac{x}{\|x\|} \lesssim_X \frac{y}{\|y\|}$. Therefore, we consider mainly the pre-ordering on the unit sphere $\mathfrak{S}_1(X^{**})$.

The following lemma is easy to check.

LEMMA 3.3. *Let $X$ and $Y$ be normed spaces, $J : X \to Y$ be an isometry and $u, v \in \mathfrak{S}_1(X^{**})$. Then $u \lesssim_X v$ if and only if $J^{**}(u) \lesssim_Y J^{**}(v)$.*

In this article, we will mainly consider the cases when $X = A$ as well as that of $X = A_h$. Note that by Remark 3.2(i), the two pre-ordering $\lesssim_{A_{\mathbb{R}}}$ and $\lesssim_A$ are the same. Moreover, since the canonical injection from $A_h$ to $A_{\mathbb{R}}$ is an isometry, Lemma 3.3 implies the following result.

PROPOSITION 3.4. *Suppose that $x, y \in \mathfrak{S}_1(A_h)$. Then*

$$x \lesssim_A y \quad \textit{if and only if} \quad x \lesssim_{A_h} y.$$

In the following, we will give algebraic descriptions of the geometric pre-orderings $\lesssim_{A_h}$ and $\lesssim_A$.

### 3.1. ALGEBRAIC DESCRIPTION OF THE GEOMETRIC PRE-ORDERING ON $A_h \setminus \{0\}$.

We begin this subsection with the following result which tells us that if $u \in \mathfrak{S}_1(A) \cap A_+$, the unique closed projection that associated with $Q_{A_h}(u)$ (as given by Theorem 2.10 of [1]) is precisely $\mathrm{es}(u)$.

PROPOSITION 3.5. *Let $u, v \in A_+ \setminus \{0\}$, and let $\mathrm{es}(u)$ and $\mathrm{es}(v)$ be the essential supports of $u$ and $v$, respectively. Then*

$$Q_{A_h}(u) \subseteq Q_{A_h}(v) \quad \textit{if and only if} \quad \mathrm{es}(u) \leqslant \mathrm{es}(v).$$

*Proof.* The argument of Theorem 2.2 tells us that $\mathrm{es}(u^{1/2}) = \mathrm{es}(u)$ is the unique projection with $Q_{A_h}(\mathrm{es}(u)) = Q_{A_h}(u)$. Thus, the result follows from Theorem 2.10 of [1]. ∎

For any $Q_1, Q_2 \subseteq \mathfrak{B}_1(A^*) \cap A_+^*$, we denote

$$Q_1 \widehat{\times} Q_2 := \{(f,g) \in Q_1 \times Q_2 : \|f\| + \|g\| \leqslant 1\}, \quad Q_1 \widehat{-} Q_2 := \{f - g : (f,g) \in Q_1 \widehat{\times} Q_2\}.$$

We recall from Lemma 2.1(iii) that $Q_{A_h}(u) = Q_A(u) \cap A_+^*$ if $u \in A_+^{**} \setminus \{0\}$.

LEMMA 3.6. *Let $x \in \mathfrak{S}_1(A_h^{**})$.*

(i) *The map $(f,g) \mapsto f - g$ defines a bijection from $(Q_{A_h}(x) \cap A_+^*) \widehat{\times} (Q_{A_h}(-x) \cap A_+^*)$ onto $Q_{A_h}(x)$. In particular,*

$$Q_{A_h}(x) = (Q_{A_h}(x) \cap A_+^*) \widehat{-} (Q_{A_h}(-x) \cap A_+^*).$$

(ii) *If $x = x_+ - x_-$ is the Jordan decomposition of $x$, then*

$$Q_{A_h}(x) \cap A_+^* = \begin{cases} Q_{A_h}(x_+) & \text{if} \|x_+\| = 1, \\ \{0\} & \text{if} \|x_+\| < 1. \end{cases}$$

*Proof.* (i) If $(f,g) \in (Q_{A_h}(x) \cap A_+^*) \widehat{\times} (Q_{A_h}(-x) \cap A_+^*)$ and $h = f - g$, then

$$\|h\| \leqslant \|f\| + \|g\| = f(x) - g(x) = h(x) \leqslant \|h\|$$

and $h \in Q_{A_h}(x)$. Conversely, if $h \in Q_{A_h}(x)$ and $h = f - g$ is the Jordan decomposition of $h$, then

$$\|h\| = h(x) = f(x) - g(x) \leqslant \|f\| + \|g\| = \|h\| \leqslant 1.$$

Thus, $g(-x) = \|g\|$ and $f(x) = \|f\|$ as required. To show the injectivity, suppose that $h = f' - g'$ where $f' \in Q_{A_h}(x) \cap A_+^*$ and $g' \in Q_{A_h}(-x) \cap A_+^*$. Then

$$\|f'\| + \|g'\| = f'(x) - g'(x) = h(x) \leqslant \|h\| \leqslant \|f'\| + \|g'\|$$

which implies $\|h\| = \|f'\| + \|g'\|$. Thus, $f = f'$ and $g = g'$ by the uniqueness of the Jordan decomposition.

(ii) If $f \in Q_{A_h}(x) \cap A_+^*$, we have $\|f\| = f(x) = f(x_+) - f(x_-) \leqslant \|f\| - f(x_-)$ and so $f(x_-) = 0$ and $f(x_+) = \|f\|$. Thus, if $\|x_+\| < 1$, then $f = 0$. Conversely, suppose that $f \in Q_{A_h}(x_+)$ and $\|x_+\| = 1$. Since $x_+ + x_-$ has norm one, the positivity of $f$ and the inequalities

$$\|f\| \geqslant f(x_+ + x_-) = \|f\| + f(x_-)$$

implies $f(x_-) = 0$. Consequently, $f \in Q_{A_h}(x) \cap A_+^*$. ∎

The above lemma tells us that if $x \in \mathfrak{S}_1(A_h^{**})$ with $\|x_+\| < 1$ (respectively, $\|x_-\| < 1$), then $Q_{A_h}(x) = -Q_{A_h}(x_-)$ (respectively, $Q_{A_h}(x) = Q_{A_h}(x_+)$).

For any $u$ in $A_+ \cap \mathfrak{B}_1(A)$, we set

$$q_u := \begin{cases} \mathrm{es}(u) & \text{if} \|u\| = 1, \\ 0 & \text{if} \|u\| < 1. \end{cases} \tag{3.1}$$

Compare with Theorem 3.11 of [1], the following theorem means that for any $x$ in $\mathfrak{S}_1(A_h)$, the two compact orthogonal projections that associated with the weak* closed face $Q_{A_h}(x)$ are precisely $q_{x_+}$ and $q_{x_-}$. We call $q_{x_+} - q_{x_-}$ the *hermitian exposure* of $x$.

THEOREM 3.7. *Let $x, y \in \mathfrak{S}_1(A_h)$. Then*

$$Q_{A_h}(x) = Q_{A_h}(q_{x_+} - q_{x_-}),$$

*and $x \lesssim_{A_h} y$ if and only if $q_{x_+} \leqslant q_{y_+}$ and $q_{x_-} \leqslant q_{y_-}$.*

*Proof.* Note that by Lemma 3.6(ii) and the argument of Proposition 3.5, we have $Q_{A_h}(q_{x_\pm}) = Q_{A_h}(\pm x) \cap A_+^*$. Therefore, if $q_{x_+} \leqslant q_{y_+}$ and $q_{x_-} \leqslant q_{y_-}$, then Lemma 3.6(i) tells us that

$$Q_{A_h}(x) = Q_{A_h}(q_{x_+}) \widehat{-} Q_{A_h}(q_{x_-}) \subseteq Q_{A_h}(y).$$

Conversely, suppose that $Q_{A_h}(x) \subseteq Q_{A_h}(y)$. Then $Q_{A_h}(x) \cap A_+^* \subseteq Q_{A_h}(y)$. Since the decomposition in Lemma 3.6(i) is given by the Jordan decomposition, we have $Q_{A_h}(x) \cap A_+^* \subseteq Q_{A_h}(y) \cap A_+^*$ and so $q_{x_+} \leqslant q_{y_+}$. Similarly, we have $q_{x_-} \leqslant q_{y_-}$. ∎

3.2. ALGEBRAIC DESCRIPTION OF THE GEOMETRIC PRE-ORDERING ON $A \setminus \{0\}$. Let us start this subsection with the following easy lemma. Again, we recall from Lemma 2.1(iii) that $Q_A(u) \subseteq A_+^*$ if $u \in A_+^{**} \setminus \{0\}$.

LEMMA 3.8. *Let $A$ be a $C^*$-algebra, $a \in \mathfrak{S}_1(A^{**})$ and $h \in A^*$.*

(i) *$f \mapsto f^*$ (where $f^*(b) := \overline{f(b^*)}$) defines an isometric affine bijection from $Q_A(a)$ onto $Q_A(a^*)$.*

(ii) *$\Lambda : \varphi \mapsto |\varphi|$ is an isometric affine bijection from $Q_A(a)$ onto $Q_A(|a^*|)$.*

*Proof.* (i) This part is clear.

(ii) Let $a^* = u \cdot |a^*|$ be the polar decomposition of $a^*$ and $\varphi \in Q_A(a)$. By Lemma 2.1(i), we have $\Lambda(\varphi) = u^* \cdot \varphi$ and $\Lambda$ is an affine isometry. Now, $|\varphi|(|a^*|) = \varphi(|a^*|u^*) = \varphi(a) = \|\varphi\|$. This shows that $|\varphi| \in Q_A(|a^*|)$. Moreover, Lemma 2.1(i) tells us that if $\psi \in Q_A(a)$ satisfies $u^* \cdot \psi = u^* \cdot \varphi$, then $\psi = uu^* \cdot \psi = uu^* \cdot \varphi = \varphi$ and $\Lambda$ is injective. Finally, for any $\omega \in Q_A(|a^*|)$, if we set $\varphi := u \cdot \omega$, then $\|\varphi\| \leqslant \|\omega\| = \omega(|a^*|) = \omega(au) = \varphi(a)$ which shows that $\|\omega\| = \|\varphi\|$ and $\varphi \in Q_A(a)$. As $\|\omega\| = \|\varphi\| \leqslant 1$ and $|\varphi(b)|^2 \leqslant \omega(bb^*)$ for all $b$ in $A$, we know from Proposition III.4.6 of [11] that $\omega = |\varphi|$ as required. ∎

This, together with Theorem 4.11 of [1], gives the following corollary.

COROLLARY 3.9. *Let $F$ be a proper closed (respectively, weak* closed) face of $\mathfrak{B}_1(A^*)$. Then*

$$|F| := \{|f| : f \in F\}$$

*is a closed (respectively, weak* closed) face of the quasi-state space $Q_A(1)$ of $A$ and the map $f \mapsto |f|$ is an isometric affine bijection from $F$ onto $|F|$.*

The following is more or less the same as Proposition 5.10(a) of [1] but since there seems to be a slight confusion in that statement, we decide to state it clearly here.

LEMMA 3.10. *Let $u, v \in A^{**} \setminus \{0\}$ be partial isometries. Then $Q_A(u) \subseteq Q_A(v)$ if and only if $u = uu^*v$. Consequently, $Q_A(u) = Q_A(v)$ if and only if $u = v$.*

For any $x \in A$, we denote the polar decomposition of $x$ by

$$x = u(x)|x|.$$

The following tells us that the unique partial isometry that associated with the face $Q_A(x) \cap \mathfrak{S}_1(A^*)$ in Theorem 4.6 of [1] is precisely $u(x)\mathrm{es}(x)$. We call $u(x)\mathrm{es}(x)$ the *exposure* of $x$ in $A \setminus \{0\}$.

THEOREM 3.11. *Suppose that $x, y \in \mathfrak{S}_1(A)$.*
(i) *$u(x)\mathrm{es}(x) = \mathrm{es}(x^*)u(x^*)^*$ and is a partial isometry with $Q_A(x) = Q_A(u(x)\mathrm{es}(x))$.*
(ii) *$x \lesssim_A y$ if and only if $u(x)\mathrm{es}(x) = u(y)\mathrm{es}(x)$.*

*Proof.* (i) Let $v = u(x^*)$. Note that $\mathrm{es}(x^*) = \mathrm{es}(|x^*|) \leqslant \mathrm{s}(|x^*|) = v^*v$, and $\mathrm{es}(x^*)v^*$ is a partial isometry. By the argument of Lemma 3.8(ii), we see that $f \in Q_A(x)$ if and only if $v^* \cdot f = |f| \in Q_{A_h}(|x^*|) = Q_{A_h}(\mathrm{es}(x^*))$ which is equivalent to $f \in Q_A(\mathrm{es}(x^*)v^*)$. On the other hand, $f \in Q_A(x)$ if and only if $f^* \in Q_A(x^*) = Q_A(\mathrm{es}(x)u(x)^*)$ (by Lemma 3.8(i) and the above) which is equivalent to $f \in Q_A(u(x)\mathrm{es}(x))$ (Lemma 3.8(i)). Now, the uniqueness of the partial isometry associated with a face tells us that $\mathrm{es}(x^*)u(x^*)^* = u(x)\mathrm{es}(x)$.

(ii) This part follows from part (i) and Lemma 3.10. ∎

Note that although by Proposition 3.4, the two preorderings $\lesssim_A$ and $\lesssim_{A_h}$ coincide on $A_h$, the algebraic description given by Theorem 3.11 and the one given by Theorem 3.7 is not in general the same. It is because in general, $Q_A(x) \neq Q_{A_h}(x)$ ($x \in A_h$). Let us first give a relation between them.

PROPOSITION 3.12. *Let $x \in \mathfrak{S}_1(A_h^{**})$ and $x = u|x|$ be the polar decomposition. We have*

$$Q_{A_h}(x) = \{h \in Q_A(x) : u \cdot h = h \cdot u\}, \quad \{|h| : h \in Q_{A_h}(x)\} = \{f \in Q_A(|x|) : u \cdot f = f \cdot u\}.$$

*Proof.* Let $p_\pm = \mathrm{s}(x_\pm)$. Then $u = p_+ - p_-$. If $g \in Q_{A_h}(x)$ and $g = g_+ - g_-$ is the Jordan decomposition, then $g_\pm \in Q_{A_h}(x_\pm) = Q_{A_h}(\mathrm{es}(x_\pm))$ (by Lemma 3.6) and so $s(g_\pm) \leqslant p_\pm$ (by Lemma 2.1(i)). Therefore, $u \cdot g = |g| = g \cdot u$. Conversely, suppose that $h \in Q_A(x)$ such that $u \cdot h = h \cdot u$. Then by Lemma 2.1(i), we have $|h| = u \cdot h = h \cdot u$ and $h = u \cdot |h| = |h| \cdot u$ which implies that $h \in A_h^*$. The above also shows that $u \cdot |h| = |h| \cdot u$ for any $h$ in $Q_{A_h}(x)$. Finally, if $f \in Q_{A_h}(|x|)$ with $u \cdot f = f \cdot u$, we put $g = u \cdot f \in Q_A(x)$. Then $g \in A_h^*$ (as $g = u \cdot f = f \cdot u$ and $f$ is positive) and $f = u \cdot g$ (by Lemma 2.1(i)). ∎

As a consequence of Proposition 3.12, if $A$ is commutative, then $Q_{A_h}(x) = Q_A(x)$ and thus, $|Q_{A_h}(x)| = |Q_A(x)|$. In the non-commutative case, although we always have $Q_{A_h}(x) = Q_A(x) \cap A_h^*$, it is possible that $|Q_{A_h}(x)| \subsetneq |Q_A(x)|$ which implies $Q_{A_h}(x) \subsetneq Q_A(x)$ (by the injectivity of $\Lambda$ in Lemma 3.8(ii)).

EXAMPLE 3.13. Let $u = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\xi = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $\eta = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Define $f(x) = \langle x\xi, \xi \rangle$. Then $u$ is a symmetry and $f \in Q_A(1)$. However, we have $u \cdot f(x) = \langle x\eta, \xi \rangle$ and $f \cdot u(x) = \langle x\xi, \eta \rangle$ which shows that $u \cdot f \neq f \cdot u$. This means $f \in |Q_A(u)| \setminus |Q_{A_h}(u)|$.

## 4. MAXIMAL AND MINIMAL ELEMENTS

For any $x \in \mathfrak{S}_1(X)$ we set

$$S_X(x) := Q_X(x) \cap \mathfrak{S}_1(X^*).$$

4.1. MINIMAL ELEMENTS AND SMOOTHNESS. Recall that $x \in \mathfrak{S}_1(X)$ is said to be *smooth* in $X$ if $S_X(x)$ is a singleton set. We call an element $x$ in $\mathfrak{S}_1(A^{**})$ *quasi-minimal* if for any $y$ in $\mathfrak{S}_1(A^{**})$, $y \lesssim_A x$ implies that $y \sim_A x$.

By Theorem 2.5 of [1], there is a canonical one to one correspondence between non-zero projections of $A^{**}$ and (norm) closed faces of $Q_A(1)$ not containing zero. Therefore, a projection $p$ in $A^{**}$ is (quasi-)minimal if and only if $S_A(p)$ is a singleton. In the case when $p$ is a closed projection in $A^{**}$, this is the same as the minimality of $p$ among all closed projections in $A^{**}$.

We can apply the above to give the following slightly different form of Theorem 3.1 of [10]. Note that our Condition (B) in part (i) is different from Theorem 3.1(c) of [10] (and their equivalence is not obvious). Moreover, by the argument of Theorem 4.1, we also know that the (unique) minimal projection in Theorem 3.1(c) of [10] is a closed projection. On the other hand, Theorem 4.1(ii) seems to be new.

THEOREM 4.1. *Let $A$ be a $C^*$-algebra.*

(i) *If $x \in \mathfrak{S}_1(A)$, the following statements are equivalent:*

(A) *$x$ is a quasi-minimal element in $\mathfrak{S}_1(A^{**})$ with respect to $\lesssim_A$.*

(B) $\mathrm{es}(x)$ *is a minimal projection in $A^{**}$.*

(C) *$x$ is a smooth point of $A$.*

*The above are also equivalent to the corresponding statements with $x$ replaced by $x^*$, $|x|$ or $|x^*|$.*

(ii) *Suppose that $x \in \mathfrak{S}_1(A_h)$ and $x = x_+ - x_-$ is the Jordan decomposition. Then statements* (A)–(C) *are equivalent to the following two statements:*

(D) *$x$ is a smooth point in $A_h$.*

(E) *Either* $\|x_+\| < 1$ *and* $x_-$ *is smooth in* $A$, *or* $\|x_-\| < 1$ *and* $x_+$ *is smooth in* $A$.

*Proof.* (i) (A) $\Rightarrow$ (B). Take any $f$ in $\text{ext}(S_A(x))$. There exists a partial isometry $v$ in $A^{**}$ such that $S_A(v) = \{f\}$ (by Theorem 4.6 of [1]). Thus, the quasi-minimality of $x$ implies that $S_A(x) = \{f\}$. Therefore, Lemma 3.8(i) tells us that $S_A(\text{es}(|x|))$ is a singleton and so, $\text{es}(x) = \text{es}(|x|)$ (Theorem 2.2(iii)) is a minimal projection.

(B) $\Rightarrow$ (C). As $\text{es}(x)$ is a minimal projection, $S_A(|x|) = S_A(\text{es}(x))$ is a singleton and so is $S_A(x)$ (because of Lemma 3.8).

(C) $\Rightarrow$ (A). This is clear.

The last statement follows from Lemma 3.8.

(ii) (C) $\Rightarrow$ (D). This is clear.

(D) $\Rightarrow$ (E). If $\|x_+\| = 1 = \|x_-\|$, then $\{0\} \neq Q_A(x) \cap A^*_+ \subsetneq Q_{A_h}(x)$ (by Lemma 3.6) which contradicts $S_{A_h}(x)$ being a singleton set. Without loss of generality, suppose that $\|x_+\| < 1$. By Lemmas 3.6 and 3.8, we see that $Q_{A_h}(x) = -Q_A(x_-)$. This shows that $x_-$ is smooth in $A$ and hence in $A_h$.

(E) $\Rightarrow$ (C). Suppose that $\|x_-\| < 1$. Then by the argument of Theorem 2.2, we have $\text{es}(|x|) = \text{es}(x_+)$. Now the minimality of $\text{es}(x_+)$ (by part (i)) implies that of $\text{es}(|x|)$. Therefore, $|x|$ is smooth in $A$ and so is $x$ (by part (i)). ∎

REMARK 4.2. Note that the above are also equivalent to the corresponding Statement (E$'$) when one replaces in Statement (E), the smoothness in $A$ with the smoothness in $A_h$. Furthermore, by parts (i) and (ii), Statement (C) is also equivalent to the following statement:

(F) $|x|$ is a smooth point in $A_h$.

On the other hand, we have the following interesting application of Theorem 4.1(ii) (note that $(a_1 - a_2)_+ = a_1$, $(a_1 - a_2)_- = a_2$ and $|a_1 - a_2| = a_1 + a_2$).

COROLLARY 4.3. *Let* $a_1, a_2 \in A_+$ *be disjoint elements in the sense that* $a_1 a_2 = 0$. *Then the following are equivalent:*

(i) $a_1 - a_2$ *is a smooth point in* $A$ *(or equivalently in* $A_h$*).*

(ii) $a_1 + a_2$ *is a smooth point in* $A$ *(or equivalently in* $A_h$*).*

(iii) $\|a_1\| \neq \|a_2\|$; *and if* $\|a_i\| = \max\{\|a_1\|, \|a_2\|\}$ *(*$i = 1$ *or* $2$*), then* $a_i$ *is a smooth point in* $A$.

*Moreover* $a + tb$ *is smooth if* $|t| < 1$ *and* $a$ *is smooth in* $A$, *but* $a \pm b$ *is not smooth in any case; while the smoothness of* $a + tb$, *in the case when* $|t| > 1$, *depends on the smoothness of* $b$.

The smoothness of an element in a $C^*$-algebra will not imply that the norm is Fréchet differentiable at that element. The following example shows that this implication does not hold even for abelian $C^*$-algebras of real rank zero.

EXAMPLE 4.4. Let $A$ be the $C^*$-algebra of all complex convergent sequences. Then $A^* = \ell_1 \oplus_1 \mathbb{C} f_\infty$ where $f_\infty((\lambda_n)) = \lim_{n\to\infty} \lambda_n$. Consider $x = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right)$

in $A$. It is clear that $S_A(x) = \{f_\infty\}$ and so $x$ is a smooth point in $A$. Note that $g_\infty := \mathrm{Re}(f_\infty)$ in $(A_{\mathbb{R}})^*$ is the unique norm one supporting real functional of $x$ (see Remark 3.2(i)). Thus, $g_\infty$ is the Gâteaux derivative of $\|\cdot\|$ at $x$. However, 1 is clearly not an isolated point of $\sigma(x) = \{0, \frac{1}{2}, \frac{2}{3}, \dots, 1\}$ and the norm on $A$ is not Fréchet differentiable at $x$. In fact, let $y_n$ in $A$ be the sequence whose $n$-th coordinate is $\frac{2}{n}$ and all others are zero. Then $\|y_n\| \to 0$ (as $n \to \infty$) but the following will not converge to 0:

$$\frac{\|x + y_n\| - 1 - g_\infty(y_n)}{\|y_n\|} = \frac{1}{2}.$$

On the other hand, we can use the above materials to obtain very easily the following result which is similar to (yet different from) Theorem 4.1 of [10]. Indeed, (iii)⇒(ii) and (ii)⇔(v) follow directly from the main theorem of [9] while the implication (v)⇒(iv) follows from the main theorem of [9] as well as the fact that $p_{|x|} = p_{x^*x} = \mathrm{es}(x)$ (see Theorem 2.2) is the corresponding element of the characteristic function $\chi_{\{1\}}$ which is inside the $C^*$-algebra generated by $|x|$. Moreover, (iv)⇒(i) is obvious, and by the arguments of Theorems 4.1(i) and Theorem 2.2, one can show that (i)⇒(iii) (notice that in this case, $q = 1 - \mathrm{es}(x)$ is also a closed projection).

COROLLARY 4.5. *Suppose that $x \in \mathfrak{S}_1(A)$. The following are equivalent:*

(i) *$x$ is a smooth point in $A$ and* $\mathrm{es}(x) \in A$.

(ii) 1 *is an isolated point of $\sigma(|x|)$ and the corresponding spectral projection $p_{|x|}$ in $A^{**}$ is a minimal projection.*

(iii) *There exists $p$ in $A$ which is a minimal projection in $A^{**}$ such that*

$$\|xp\| = 1 \quad \text{and} \quad \|x(1-p)\| < 1.$$

(iv) *The norm of $A$ is Fréchet differentiable at $x$ and* $\mathrm{es}(x) \in A$.

(v) *$x$ is a smooth point of $A^{**}$.*

*The above are also equivalent to the corresponding statements when $x$ is replaced by $|x|$, $x^*$, or $|x^*|$.*

PROPOSITION 4.6. *Let $A$ be a $C^*$-algebra, $x$ be a norm one element in $A$ and $A_x$ be the smallest hereditary $C^*$-subalgebra containing $x$. Then $x$ is smooth in $A$ if and only if $x$ is smooth in $A_x$.*

*Proof.* The necessity is obvious. Assume that $x$ (and hence $|x|$) is smooth in $A_x$. Notice that $A_{|x|} \subseteq A_x$ and so $|x|$ is smooth in $A_{|x|}$. Let $\phi$ be the unique element in $S_{A_{|x|}}(|x|)$. Then $\phi$ is a state of $A_{|x|}$ (by Lemma III.3.2 of [11]). As $A_{|x|}$ is a hereditary subalgebra of $A$, there is only one state in $A$ extending $\phi$ (see e.g. Theorem 3.3.9 of [7]). Thus $|x|$ is smooth in $A$ and so is $x$. ∎

4.2. MAXIMAL ELEMENTS, EXTREME POINTS AND UNITARIES. We say that $u$ is a *maximal partial isometry in $A^{**}$* if for any partial isometry $v \in A^{**}$, the equality

$u = uu^*v$ implies that $v = u$. Moreover, an element $x$ in $A^{**}$ is said to be *quasi-maximal* (respectively, *maximal*) if for any $y$ in $A^{**}$, we have $x \lesssim_A y$ implies that $x \sim_A y$ (respectively, $x = y$).

For any norm one element $v$ in $\mathfrak{S}_1(A^{**})$, we denote (as in [1]),

$$S_A(v)^f = \{x \in \mathfrak{S}_1(A^{**}) : g(x) = 1; g \in S_A(v)\},$$
$$S_A(v)_f = \{x \in \mathfrak{S}_1(A) : g(x) = 1; g \in S_A(v)\}.$$

By Theorem 5.3 of [1], for any $v$ in $\mathfrak{S}_1(A^{**})$ (respectively $\mathfrak{S}_1(A)$), the set $S_A(v)^f$ (respectively $S_A(v)_f$) is the smallest weak* closed face in $\mathfrak{B}_1(A^{**})$ (respectively the smallest closed face in $\mathfrak{B}_1(A)$) containing $v$. Therefore, $v$ is an extreme point of $\mathfrak{B}_1(A^{**})$ (respectively $\mathfrak{B}_1(A)$) if and only if $S_A(v)^f = \{v\}$ (respectively $S_A(v)_f = \{v\}$). These, together with the results in Section 3, imply the following.

THEOREM 4.7. *Let $v$ be a norm one element in a $C^*$-algebra $A$, i.e., $v \in \mathfrak{S}_1(A)$.*

(a) *The following statements are equivalent:*

(i) *$v$ is a quasi-maximal element in $\mathfrak{S}_1(A^{**})$ with respect to $\lesssim_A$;*

(i′) *$v$ is a maximal element in $\mathfrak{S}_1(A^{**})$ with respect to $\lesssim_A$;*

(ii) *$v$ is an extreme point of $\mathfrak{B}_1(A^{**})$;*

(iii) *$v$ is a maximal partial isometry in $A^{**}$.*

(b) *The following statements are equivalent:*

(iv) *$v$ is a quasi-maximal element in $\mathfrak{S}_1(A)$ with respect to $\lesssim_A$ and* $\mathrm{ext}(S_A(v)_f) \neq \emptyset$;

(iv′) *$v$ is a maximal element in $\mathfrak{S}_1(A)$ with respect to $\lesssim_A$ and* $\mathrm{ext}(S_A(v)_f) \neq \emptyset$;

(v) *$v$ is an extreme point of $\mathfrak{B}_1(A)$.*

(c) *If $v$ is normal, then statements* (1)–(5) *are equivalent to the following:*

(vi) *$v$ is a unitary.*

In the case when $v$ is a self-adjoint element in $A$, we also have the following corollary of results in Section 3.

THEOREM 4.8. *Let $v \in \mathfrak{S}_1(A_h)$. Then statements* (1)–(6) *in Theorem* 4.7 *are equivalent to the following:*

($\mathrm{i}_h$) *$v$ is a quasi-maximal element in $\mathfrak{S}_1(A_h^{**})$ with respect to $\lesssim_{A_h}$.*

($\mathrm{ii}_h$) *$v$ is an extreme point of $\mathfrak{B}_1(A_h^{**})$.*

($\mathrm{iii}_h$) *$v$ is a maximal partial isometry in $A_h^{**}$.*

($\mathrm{iv}_h$) *$v$ is a quasi-maximal element in $\mathfrak{S}_1(A_h)$ with respect to $\lesssim_{A_h}$ and* $\mathrm{ext}(S_{A_h}(v)_f \cap A_h) \neq \emptyset$.

($\mathrm{v}_h$) *$v$ is an extreme point of $\mathfrak{B}_1(A_h)$.*

Again, Statements ($\mathrm{i}_h$) and ($\mathrm{iv}_h$) are also equivalent to the corresponding statements ($\mathrm{i}'_h$) and ($\mathrm{iv}'_h$) with the quasi-maximality being replaced by maximality.

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